Lipschitz universal Banach spaces

joint work with Luis Sánchez González

Tony Procházka

Université de Franche-Comté

February 7, 2014
Function Theory on Infinite Dimensional Spaces XIII
Madrid

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

...$C([0,1])$ is isometrically universal.

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

  ...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

  ...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

  ...c_0 is Lipschitz universal.

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

  ...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

  ...c_0 is Lipschitz universal.

- In fact $\exists D > 0\ \forall (X,d)\ \exists f : X \to c_0$ such that
  $$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

...c_0 is Lipschitz universal.

- In fact $\exists D > 0 \; \forall (X,d) \; \exists f : X \to c_0$ such that

$$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

...c_0 is D-Lip-universal.

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

...c_0 is Lipschitz universal.

- In fact $\exists D > 0 \; \forall (X,d) \; \exists f : X \to c_0$ such that

$$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

...c_0 is D-Lip-universal.

- $D \geq 2$ (Aharoni, 1974),

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

  ...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

  ...c_0 is Lipschitz universal.

- In fact $\exists D > 0\ \forall (X,d)\ \exists f : X \to c_0$ such that
$$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

  ...c_0 is D-Lip-universal.

- $D \geq 2$ (Aharoni, 1974), $D = 2$ (Kalton-Lancien, 2008)

Overview

- Every separable Banach space is linearly isometric to a subspace of $C([0,1])$ (Banach-Mazur, 1934),

  ...$C([0,1])$ is isometrically universal.

- Every separable Banach space is Lipschitz equivalent to a subset of c_0 (Aharoni, 1974),

  ...c_0 is Lipschitz universal.

- In fact $\exists D > 0\ \forall (X,d)\ \exists f : X \to c_0$ such that
$$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

  ...c_0 is D-Lip-universal.

- $D \geq 2$ (Aharoni, 1974), $D = 2$ (Kalton-Lancien, 2008)
- Baudier (2013) asks what happens if c_0 is replaced by $C(K)$ and constructs metric spaces $(\Delta_k)_{k=1}^{\infty}$ such that $\Delta_k \underset{D}{\hookrightarrow} C[0,\omega^k]$ implies $D \geq \frac{k+1}{k}$.

Main result

Aharoni: $\ell_1 \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2$

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Main result

Aharoni: $\exists$ separable (M, d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M, d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M,d) *such that*

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ *be a* D*-Lip-universal Banach space with* $D < 2$*.*

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M,d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ be a D-Lip-universal Banach space with $D < 2$. Then X is not Asplund.

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2$.

Theorem

a) There exists separable (M,d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ be a D-Lip-universal Banach space with $D < 2$. Then X is not Asplund.

Recall

- A separable Banach X is Asplund iff X^* is separable.

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M,d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ be a D-Lip-universal Banach space with $D < 2$. Then X is not Asplund.

Recall

- A separable Banach X is Asplund iff X^* is separable.
- e.g. c_0, ℓ_2, reflexive spaces, …

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M,d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ be a D-Lip-universal Banach space with $D < 2$. Then X is not Asplund.

Recall

- A separable Banach X is Asplund iff X^* is separable.
- e.g. c_0, ℓ_2, reflexive spaces, …
- A $C(K)$ Banach space is Asplund iff K is scattered.

Main result

Aharoni: $\exists$ separable (M,d) such that $M \underset{D}{\hookrightarrow} c_0 \Rightarrow D \geq 2.$

Theorem

a) There exists separable (M,d) such that

$$M \underset{D}{\hookrightarrow} X \text{ and } X \text{ Asplund} \quad \Rightarrow \quad D \geq 2.$$

b) Let $(X, \|\cdot\|)$ be a D-Lip-universal Banach space with $D < 2$. Then X is not Asplund.

Recall

- A separable Banach X is Asplund iff X^* is separable.
- e.g. c_0, ℓ_2, reflexive spaces, …
- A $C(K)$ Banach space is Asplund iff K is scattered.
- i.e. $K^{(\alpha)} = \emptyset$ for some ordinal α.

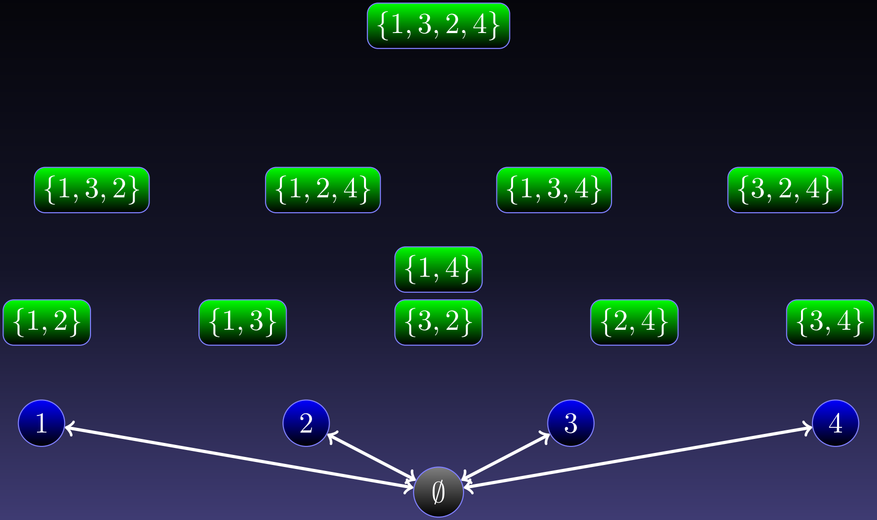
The unwieldy metric space M

The unwieldy metric graph M

The unwieldy metric graph M

$M = \{\emptyset\} \cup \mathbb{N} \cup F$ where $F = \{A \subset \mathbb{N} : 2 \leq |A| < \infty\}$

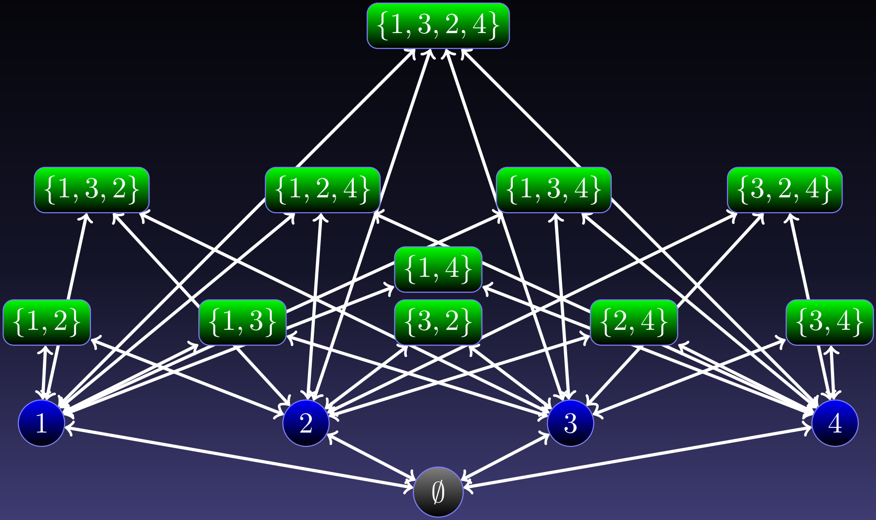
The unwieldy metric graph M



$M = \{\emptyset\} \cup \mathbb{N} \cup F$ where $F = \{A \subset \mathbb{N} : 2 \leq |A| < \infty\}$

(a, b) is an edge iff $a = \emptyset$ and $b \in \mathbb{N}$

The unwieldy metric graph M



$M = \{\emptyset\} \cup \mathbb{N} \cup F$ where $F = \{A \subset \mathbb{N} : 2 \leq |A| < \infty\}$

(a, b) is an edge iff $a = \emptyset$ and $b \in \mathbb{N}$ or $a \in \mathbb{N}$, $b \in F$ and $a \in b$

Lemma

Let $X = C(K)$ for some compact space K and assume that there exist $D \in [1, 2)$ and $f : M \to X$ such that

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

Then K is not scattered.

Lemma

Let $X = C(K)$ for some compact space K and assume that there exist $D \in [1, 2)$ and $f : M \to X$ such that

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

Then K is not scattered.

Proof.

- Put $X_{a,b} = \{k \in K : |\langle f(a) - f(b), k\rangle| \geq 4 - 2D\}$ for $a, b \in \mathbb{N}$

Lemma

Let $X = C(K)$ for some compact space K and assume that there exist $D \in [1, 2)$ and $f : M \to X$ such that

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

Then K is not scattered.

Proof.

- Put $X_{a,b} = \{k \in K : |\langle f(a) - f(b), k\rangle| \geq 4 - 2D\}$ for $a, b \in \mathbb{N}$
- Showing that $X_{a,b} \cap K^{(\alpha)} \neq \emptyset$ for some $a, b \in \mathbb{N}$ and all ordinals α will finish the proof.

Lemma

Let $X = C(K)$ for some compact space K and assume that there exist $D \in [1,2)$ and $f : M \to X$ such that

$$d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y).$$

Then K is not scattered.

Proof.

- Put $X_{a,b} = \{k \in K : |\langle f(a) - f(b), k\rangle| \geq 4 - 2D\}$ for $a, b \in \mathbb{N}$
- Showing that $X_{a,b} \cap K^{(\alpha)} \neq \emptyset$ for some $a, b \in \mathbb{N}$ and all ordinals α will finish the proof.
- A bit more: $\forall\, A, B \in F$ with $A \cap B = \emptyset$, $\forall\, \alpha$

$$K^{(\alpha)} \cap \bigcap_{a\in A, b\in B} X_{a,b} \neq \emptyset$$

$K^{(\alpha)} \cap \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset$ by induction

$K^{(\alpha)} \cap \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.

$K^{(\alpha)} \cap \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.

$K^{(\alpha)} \cap \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.
- $\alpha \rightsquigarrow \alpha + 1$:

$K^{(\alpha)} \cap \bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.
- $\alpha \rightsquigarrow \alpha + 1$: We know that $\forall\, i, j > \max A \cup B$

$$\exists\, k_{i,j} \in K^{(\alpha)} \cap \bigcap_{a \in A \cup \{i\}, b \in B \cup \{j\}} X_{a,b}.$$

$K^{(\alpha)} \cap \bigcap_{a\in A, b\in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.
- $\alpha \rightsquigarrow \alpha + 1$: We know that $\forall\, i, j > \max A \cup B$

$$\exists\, k_{i,j} \in K^{(\alpha)} \cap \bigcap_{a\in A\cup\{i\}, b\in B\cup\{j\}} X_{a,b}.$$

- Put $\Gamma = \{k_{i,j} : i, j > \max A \cup B\}$, then Γ is infinite.

$K^{(\alpha)} \cap \bigcap_{a\in A, b\in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.
- $\alpha \rightsquigarrow \alpha + 1$: We know that $\forall\, i, j > \max A \cup B$

$$\exists\, k_{i,j} \in K^{(\alpha)} \cap \bigcap_{a\in A\cup\{i\}, b\in B\cup\{j\}} X_{a,b}.$$

- Put $\Gamma = \{k_{i,j} : i, j > \max A \cup B\}$, then Γ is infinite.
- Indeed, $\ell_\infty(\Gamma)$ contains $(4 - 2D)$-separated bounded set $\{f(i) \restriction_\Gamma : i \geq \max A \cup B\}$.

$K^{(\alpha)} \cap \bigcap_{a\in A, b\in B} X_{a,b} \neq \emptyset$ by induction

- Good for $\alpha = 0$.
- α limit ordinal – compactness of $\bigcap X_{a,b}$.
- $\alpha \rightsquigarrow \alpha + 1$: We know that $\forall\, i, j > \max A \cup B$

$$\exists\, k_{i,j} \in K^{(\alpha)} \cap \bigcap_{a\in A\cup\{i\}, b\in B\cup\{j\}} X_{a,b}.$$

- Put $\Gamma = \{k_{i,j} : i, j > \max A \cup B\}$, then Γ is infinite.
- Indeed, $\ell_\infty(\Gamma)$ contains $(4 - 2D)$-separated bounded set $\{f(i) \restriction_\Gamma : i \geq \max A \cup B\}$.
- So $K^{(\alpha+1)} \cap \bigcap_{a\in A, B\in B} X_{a,b} \neq \emptyset$.

$\square$

Proof of the main theorem

Proof of the main theorem

Theorem (Zippin, 1977)

Let X be a separable Asplund space. Let $\varepsilon > 0$. Then there exist a Banach space Z, a countable Hausdorff compact S, a subspace Y of Z isometric to $C(S)$ and a linear embedding $i : X \to Z$ with $\|i\| \left\|i^{-1}\right\| < 1 + \varepsilon$ such that for any $x \in X$ we have

$$\mathrm{dist}_Z(i(x), Y) \leq \varepsilon \left\|i(x)\right\|_Z .$$

Proof of the main theorem

Theorem (Zippin, 1977)

Let X be a separable Asplund space. Let $\varepsilon > 0$. Then there exist a Banach space Z, a countable Hausdorff compact S, a subspace Y of Z isometric to $C(S)$ and a linear embedding $i : X \to Z$ with $\|i\| \left\|i^{-1}\right\| < 1 + \varepsilon$ such that for any $x \in X$ we have

$$\operatorname{dist}_Z(i(x), Y) \leq \varepsilon \left\|i(x)\right\|_Z .$$

Proof of the main theorem.

- If $M \underset{D}{\hookrightarrow} X$, $D < 2$, then $M \underset{D'}{\hookrightarrow} C(S)$ for some $D' < 2$.

Proof of the main theorem

Theorem (Zippin, 1977)

Let X be a separable Asplund space. Let $\varepsilon > 0$. Then there exist a Banach space Z, a countable Hausdorff compact S, a subspace Y of Z isometric to $C(S)$ and a linear embedding $i : X \to Z$ with $\|i\| \left\|i^{-1}\right\| < 1 + \varepsilon$ such that for any $x \in X$ we have

$$\operatorname{dist}_Z(i(x), Y) \leq \varepsilon \left\|i(x)\right\|_Z .$$

Proof of the main theorem.

- If $M \underset{D}{\hookrightarrow} X$, $D < 2$, then $M \underset{D'}{\hookrightarrow} C(S)$ for some $D' < 2$.
- Contradiction.

□

Corollary

$M \underset{D}{\hookrightarrow} C(K)$, $D < 2$, if and only if $C(K)$ is isometrically universal.

Corollary

$M \underset{D}{\hookrightarrow} C(K)$, $D < 2$, if and only if $C(K)$ is isometrically universal.

Proof. (“$\Rightarrow$”)

- By Lemma, K is not scattered.

Corollary

$M \underset{D}{\hookrightarrow} C(K)$, $D < 2$, if and only if $C(K)$ is isometrically universal.

Proof. ("$\Rightarrow$")

- By Lemma, K is not scattered.
- Then there is a continuous surjection from K onto $[0, 1]$. (Pełczyński, Semadeni, 1959)

Corollary

$M \underset{D}{\hookrightarrow} C(K)$, $D < 2$, if and only if $C(K)$ is isometrically universal.

Proof. ("$\Rightarrow$")

- By Lemma, K is not scattered.
- Then there is a continuous surjection from K onto $[0, 1]$. (Pełczyński, Semadeni, 1959)
- So $C([0, 1])$ embeds isometrically into $C(K)$.

$\square$

Questions, remarks

- Converse to Aharoni's theorem.

Questions, remarks

- Converse to Aharoni's theorem.
- Is it true that any Banach space that is D-Lip-universal for some $D < 2$, is necessarily (isometrically) universal?

Questions, remarks

- Converse to Aharoni's theorem.
- Is it true that any Banach space that is D-Lip-universal for some $D < 2$, is necessarily (isometrically) universal?
- Is it true that $\ell_1 \underset{D}{\hookrightarrow} C(K)$, $D < 2$, implies that K is not scattered?

Autumn 2014: Thematic trimester at the Université de Franche-Comté

“Geometric and noncommutative methods in functional analysis”

Functional calculus and Harmonic analysis of semigroups October 1-3 Besançon	**Autumn school on Nonlinear geometry of Banach spaces and Applications** October 20-24 Métabief	**Conference on Geometric functional analysis and its Applications** October 27-31 Besançon
Annual meeting of the French research network (GDR) in Noncommutative geometry Novermber 27-29 Besançon	**Winter school on Operator spaces, Noncommutative probability and Quantum groups** December 1-12 Métabief	**Conference on Operator spaces, Quantum probability and Applications** December 15-19 Besançon