Analysis and approximation of particle-in-Burgers model

B. Andreianov[1]

joint work with
F. Lagoutière (Orsay), N. Seguin (Paris VI) and T. Takahashi (Nancy)

[1] Laboratoire de Mathématiques CNRS UMR6623
Université de Franche-Comté
Besançon, France

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Plan of the talk

MODEL AND MOTIVATION

Model and motivation

D'Alembert paradox : a solid immersed in an inviscid fluid is not submitted to any resultant force; in other words, birds (and planes...) could not fly with a model where viscosity is neglected !
Yet, inviscid (hyperbolic !) models are desirable for some fluids.

1D case, scalar (playground?) :
the Lagoutière-Seguin-Takahashi 'JDE07 model for interaction, via a drag force , of a massive point particle with a Burgers fluid:

$$\begin{cases} \partial_t u + \partial_x(u^2/2) = -D(u - h'(t))\,\delta_0(x - h(t)), \\ mh''(t) = D(u|_{(t,h(t))} - h'(t)). \end{cases}$$

Call it B+P model. Here

- u, the velocity of the fluid, is unknown
- h, the position of the solid particle, is unknown.

Main focus: $D(v) = \lambda v$ (the linear case)
A variant: $D(v) = \lambda v|v|$ (the quadratic case).

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Delicate and important points

- Sense to give the the product of distributions $(u - h'(t))\,\delta_0(x - h(t))$ in the RHS of the PDE ? Describe u for a *frozen* h?
- Sense to give to the RHS $u|_{(t,h(t))} - h'(t)$ of the ODE ? Describe h for a fixed u?
- Resolve the coupling: fixed-point or splitting approach. Well-posedness, convergence of approximating schemes.
- Numerical approximation: a cheap scheme ? (good resolution at the particle location is essential, but full Riemann solver is not welcome).
- (is being improved) Particle path approximation: difficulty to keep particle(s!) at mesh interfaces. Re-meshing? Projection?
- (unsolved) uniqueness for L^∞ (non BV) data ???

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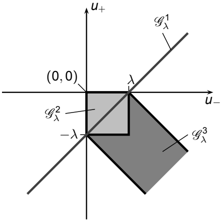
Giving sense to the PDE: description of interface coupling in Burgers

Interpret the singular source term as the limit of $(u - h'(t))\partial_x H_\varepsilon(x)$

$\Rightarrow$ description (LST'07) of the set of all couples $(u_-, u_+) \in \mathbb{R}^2$ that can be connected across the particle:



This is $\mathcal{G}_\lambda$ for particle at rest ($h' = 0$) ; one has $\mathcal{G}_\lambda(V) = \mathcal{G}_\lambda + (V, V)$ for $h'(t) = V$, due to invariance .

Hence: postulate as modeling assumption:
u is an admissible solution of the PDE in B+P model if

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- and it takes traces such that $\left(u|_{(t,h(t)^-)} , u|_{(t,h(t)^+)} \right) \in \mathcal{G}_\lambda(h'(t))$.

NB (decoupling) if the path h fixed $\Rightarrow$ existence, uniqueness of an admissible solution u for any given L^∞ initial datum.
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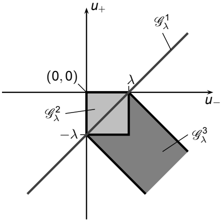

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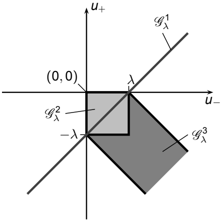

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Giving sense to the ODE: particle moved by conservation of momentum

Following LST'07 , we consider that particle moves according to

$$t \mapsto \int_{\mathbb{R}} u(t,x)\,dx + mh'(t) \;\equiv\; const.$$

Thus, the particle is driven by the "lack of conservativity" of the PDE, i.e., by the jump of the normal to the interface flux component:

$$mh''(t) = \left((u_-)^2/2 - h'(t)u_-\right) \quad - \left((u_+)^2/2 - h'(t)u_+\right).$$

We prefer to see it as follows (use Green-Gauss):
given u a Kruzhkov solution to Burgers away from $x = h(t)$,

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$\forall \xi \in \mathcal{D}([0,T))$,
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FIXED PARTICLE ($h \equiv 0$; THEN $h(\cdot)$ FROZEN)

The case $h \equiv 0$: the playground ?

Playground: freeze the particle at $x = 0$, i.e., work with

$$u_t + \Big(\frac{u^2}{2}\Big)_x = -u\,\delta_0(x).$$

Dissipative structure (source=singular absorption term) $\Rightarrow$ an extension of Kruzhkov theory applies A., Seguin DCDS-A'12 .

Our viewpoint: application of the theory of "discontinuous-flux conservation laws" (Risebro, Towers, Karlsen, Seguin-Vovelle, Adimurthi, Audusse-Perthame and many others). Framework of A., Karlsen, Risebro ARMA'11 guarantees well-posedness because of a specific property of $\mathcal{G}_\lambda$ (it is a "maximal L^1-dissipative germ").

Moreover: convergence of monotone finite volume schemes – with Godunov flux at $x = 0$! – is guaranteed.

CV is based: on well-balance (equilibrium) property of the scheme ; and on compactness due to stability:

- L^∞ stability (simple, by a kind of maximum principle)
- BV stability (involved: Wave-Front Tracking A.,L.,S.,T. SIMA'14 or fine observation on a special FV scheme Aguillon, L., S.'14)

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Particle: from $h \equiv 0$ to arbitrary frozen path $h(\cdot)$

Actually the case $h = 0$ is easily upgraded to $h = h(t)$ (arbirtary but frozen, decoupled from u). The arguments are:

- If $h'(t) = V = const$ (straight particle path), change $(x - Vt) \to \tilde{x}$, $(u - V) \to \tilde{u}$ yields $\tilde{u}_t + (\frac{\tilde{u}^2}{2})_x = -\tilde{u}\delta_0(x)$.
- Stop-and-restart $\Rightarrow$ well-posedness for piecewise affine h
- Approximation of given h by a sequence of piecewise affine paths h_n, compactness of $(u_n)_n$, passage to the limit $\Rightarrow$ existence for any C^1 path h.
- Numerical counterpart: work on $u_t + \left(\frac{u^2}{2} - h'(t)u\right)_x = -u\delta_0(t)$ or, equivalently, at each time step incline the mesh following the particle Aguillon, L.,S.'14 .
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RESOLVING THE COUPLING

Existence for B+P, L^∞ data

We have just resolved the PDE on u, given h:

$$A : h \mapsto u.$$

Conservation of momentum permits to resolve the ODE on h, given u solution to Burgers:

$$B : u \mapsto h.$$

Fixed point strategy (Schauder) applies in appropriate setting:

a ball of C^1 for h,

a ball of L^∞ considered in a weighted L^1 space, for u.

One shows that

- $C = B \circ A$ is continuous
  (quite delicate point: check the continuity of A !)
- C is compact (simple: compactification since $Image(B) \subset W^{2,\infty}$).

Thus C admits a fixed point. Existence follows. Uniqueness: open !
Details: A.,Lagoutière, Seguin, Takahashi SIMA'14 .

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Well-posedness for B+P, *BV* data

The essential ingredients here are:

- *BV* estimate on u (depending only on u_0 and on $\|h\|$ in $W^{2,\infty}$)
- results on Lipschitz dependence of solutions to $u_t + f(t,u)_x = \ldots$ on the flux f, here $f(t,u) = \frac{u^2}{2} - h'(t)u$.

As a result, one easily proves that A is Lipschitz.

Possibility to replace Schauder by Banach-Picard (contractive) fixed-point on short time intervals $\Rightarrow$ well-posedness, for *BV* data.

Another version: Gronwall for uniqueness. Splitting for existence:

- freezing h' on $[t^{n-1}, t^n[$,
  resolve the problem $h \mapsto u$ on every time interval;
  then update h' at $t = t^n$ by conservation of momentum.
- get existence from stability (L^∞, *BV* for u, $W^{2,\infty}$ for h) using compactness due to flux nonlinearity, Lions-Perthame-Tadmor'94, Panov'94 .
- numerical counterpart for splitting: use a FV scheme (GODUNOV??) to resolve $h \mapsto u$ approximatively

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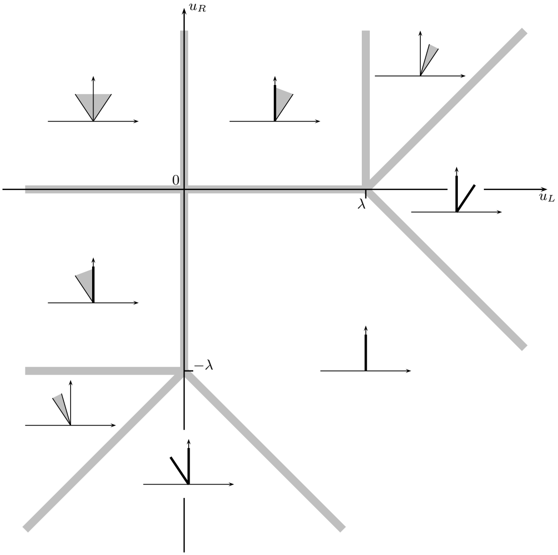
Case study to describe the Riemann solver, $h = 0$... Who wants to use Godunov ?



Interface Coupling Conditions and Role of Equilibria

A point of view on the Kruzhkov theory

Consider homogeneous scalar conservation law $u_t + f(u)_x = 0$.

- collective ("herd") behaviour: if $u, \hat{u}$ solutions, $\|u - \hat{u}\|_{L^1(\mathbb{R})}(t) \downarrow$ with t. More precisely, Kato inequality holds:

$$|u - \hat{u}|_t + \Phi(u, \hat{u})_x \leqslant 0 \ \text{ in } \mathcal{D}'$$

  ($|u - k|$ being Kruzhkov entropy, $\Phi(u, k)$ being the entropy flux).
- Kruzhkov: to describe admissible solutions it's enough to know
  - Kato ineq. holds for any couple of adm. solutions (inherited from approx.! Kruzhkov: from vanishing viscosity)
  - $\hat{u} \equiv k$ are trivially admissible (inherited from approx. as well)
- The basis of convergence proof of well-balanced monotone Finite Volume schemes is :
  - compactness of families of approximate solutions (TVD, or weak BV, or CompComp....)
  - monotonicity of the scheme $\Rightarrow$ discrete Kato inequality
  - well-balance property $\Rightarrow$ equilibria $\hat{u} \equiv k$ are trivial limits
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A point of view on the Kruzhkov theory

Consider homogeneous scalar conservation law $u_t + f(u)_x = 0$.

- collective ("herd") behaviour: if $u, \hat{u}$ solutions, $\|u - \hat{u}\|_{L^1(\mathbb{R})}(t) \downarrow$ with t. More precisely, Kato inequality holds:

$$|u - \hat{u}|_t + \Phi(u, \hat{u})_x \leqslant 0 \text{ in } \mathcal{D}'$$

  ($|u - k|$ being Kruzhkov entropy, $\Phi(u, k)$ being the entropy flux).
- Kruzhkov: to describe admissible solutions it's enough to know
  - Kato ineq. holds for any couple of adm. solutions (inherited from approx.! Kruzhkov: from vanishing viscosity)
  - $\hat{u} \equiv k$ are trivially admissible (inherited from approx. as well)
- The basis of convergence proof of well-balanced monotone Finite Volume schemes is :
  - compactness of families of approximate solutions (TVD, or weak BV, or CompComp....)
  - monotonicity of the scheme $\Rightarrow$ discrete Kato inequality
  - well-balance property $\Rightarrow$ equilibria $\hat{u} \equiv k$ are trivial limits
- We replicate this idea for $u_t + (\frac{u^2}{2})_x = -u\delta_0$. Seeking equilibria!

Blackboard talk (15 min?)

- Piecewise constants $k(x) = k_- 1\!\!1_{x<0} + k_+ 1\!\!1_{x>0}$ are the simplest equilibria. Blow-up argument identifies the set of all admissible $k(x)$ with "$(k_-, k_+) \in \mathcal{G}_\lambda$".
- The set $\mathcal{G}_\lambda$ has a structure of L^1D germ ($(u_-, u_+), (\hat{u}_-, \hat{u}_+) \in \mathcal{G}_\lambda \Rightarrow$ dissipation of the entropy flux: $\Phi(u_+, \hat{u}^+) - \Phi(u_-, \hat{u}_-) \leqslant 0$; the maximality of $\mathcal{G}_\lambda$). $\exists, !$ from theory of A.,Karlsen,Risebro'11 . Numerics: Godunov at interface converges...
- Forget traces, use adapted entropy ineq. ($\equiv$ Kato ineq. with $\hat{u}(t,x) = k(x)$).

- The line $\mathcal{G}^1_\lambda = \{(u_-, u_+) \,|\, u_- - u_+ = 1\}$ is a "preferential part" of $\mathcal{G}_\lambda$. Idea: use this graph to "predict" $u_\pm$ from $u_\mp$. This leads to "prediction" interface numerical fluxes $F^\pm$ (non-conservativity $\Rightarrow$ one-sided fluxes at interface!):

$$F^-(u_{-1/2}, u_{1/2}) := F(u_{-1/2}, u_{1/2} - \lambda), \quad F^+(u_{-1/2}, u_{1/2}) := F(u_{-1/2} + \lambda, u_{1/2})$$

- The resulting "prediction" scheme is well-balanced w.r.t. $\mathcal{G}^1_\lambda$ but not w.r.t. $\mathcal{G}_\lambda$. Difficulty: $\mathcal{G}_\lambda$ is a *maximal* L^1D germ; its part $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda$ is a *definite* L^1D germ, but $\mathcal{G}^1_\lambda$ is *not definite*. Therefore, convergence of the scheme on some Riemann problems (with endstates in $\mathcal{G}^2_\lambda$) has to be proved "by hands" (delicate).

- Infinitely many maximal L^1-D germs! Example: conservative transmission maps A.,Cancès JHDE'15 . Extension to transmission-dissipation maps: germs $\mathcal{G}[\varphi, \psi]$. $\varphi, \psi \Rightarrow$ "transmission" interface num. fluxes $F^\pm_{\varphi,\psi}$ (via an implicit relation).
- The graph $\mathcal{G}^1_\lambda$ is a transmission map, moreover, the flux dissipation across the interface equals $\lambda \frac{u_+ + u_-}{2}$. A.,Cancès'15 : one finds $\mathcal{G}_\lambda = \mathcal{G}[\varphi, \psi]$ for $\varphi = \mathrm{Id} - \lambda$, $\psi = \lambda \mathrm{Id} \Rightarrow$ new well-balanced "hybrid" scheme (1 implicit unknown at interface) .

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Finite Volume Schemes for B+P

Numerical Finite Volume schemes (first order) ≡ CONCLUSION

Four basic schemes for $h \equiv 0$ (all of them convergent)
which only differ by the choice of numerical flux at the particle location:

- "totally well-balanced" : Godunov, with the Riemann solver of L.,S.,T.'07
- "(partially) well-balanced" prediction interface flux of A.,S.'12
- improves on the previous: "very well-balanced" flux of Aguillon,L.,S.'14
- "almost well balanced" (totally WB, if based on Godunov for Burgers) hybridized transmission interface flux of A.,Cancès JHDE'15 with one nonlinear equation to solve (scalar, monotone: easy!) per time step.

Adapting these schemes to the coupled problem: main ideas.

- Splitting : update (u^n, h^n, V^n) to (u^{n+1}, h^n, V^n) by one of the above; then update the particle velocity V^n to V^{n+1} by conservation of moment, finally update its position h^n to h^{n+1} by integration of the velocity.
- All methods require to localize h^{n+1} at a mesh interface ⇒
  - making the mesh follow the particle ? Aguillon,L.,S.'14 , 1 particle only (but the only scheme with convergence proof !) ⇒ serious difficulties if more than 1 particle:
  - use projection by random sampling (like Glimm) ? A.,L.,S.,T.'10 trapezoid cells around the particle used for update, then erased
  - or shift the mesh by a multiple of Δx ? Towers'15

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Coupled problem: a well-balanced random-choice numerical scheme

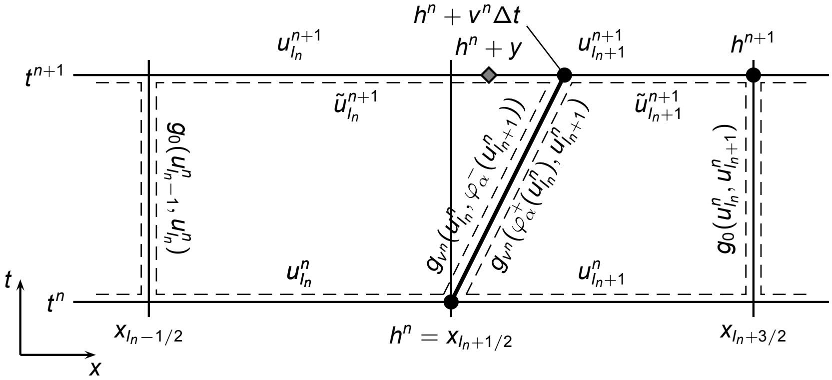

Figure: Representation of the algorithm based on the well-balanced scheme.

Numerics: Glimm scheme versus well-balanced random-choice scheme

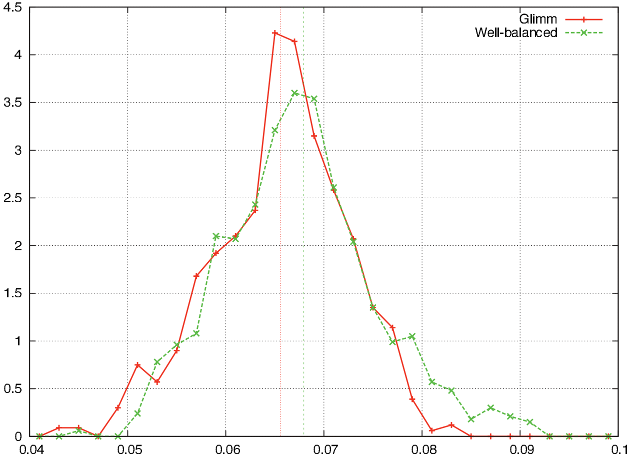

Figure: Probability distributions of velocity of the particle for the two schemes (1000 runs, Van der Corput equidistributed sequences used)

Numerics: drafting-kissing-tumbling

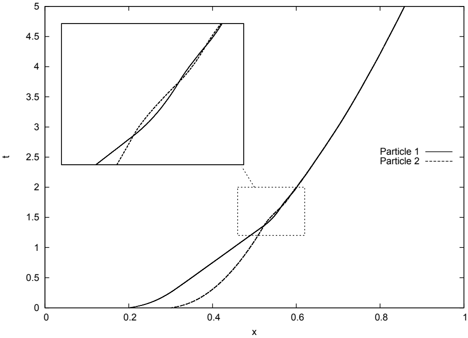

Figure: Trajectories of two particles

Thx !!!

MERCI – THANK YOU !