Existence des solutions et approximation volumes finis pour certains systèmes de diffusion croisée

B. Andreianov[1] M. Bendahmane [2] R. Ruiz Baier[3]

[1]Institut Denis Poisson / Université de Tours
[2]Université de Bordeaux
[3]Oxford Math. Institute

CANUM au Cap d'Agde, 28 mai – 1er juin 2018

Plan of the talk

Model and main results

Model and motivations

Standard population dynamics models : self-diffusion for each species.
Predicts a weak resistance of one species to the invasion of another one that has a slight competitive advantage.
Classical example: the "squirrel war". The endemic UK red squirrel was considerably but not totally displaced by the invasion of the grey squirrel.

Among others, Shigesada-Kawasaki-Teramoto proposed models with cross-diffusion in which the displaced species shows a better resistance to the invading species.

Goal : existence for a class of SKT-like systems featuring "weak" cross-diffusion (=positive definite matrix) + their FV numerical approximation.
The class is identified by the fact that our existence proof works :-)))

The existence proof is a rather usual combination of a priori estimates + compactness + passage-to-the limit arguments .
Along the same lines, convergence of a finite volume method is justified.

Model and motivations

Standard population dynamics models : self-diffusion for each species.
Predicts a weak resistance of one species to the invasion of another one that has a slight competitive advantage.
Classical example: the “squirrel war”. The endemic UK red squirrel was considerably but not totally displaced by the invasion of the grey squirrel.

Among others, Shigesada-Kawasaki-Teramoto proposed models with cross-diffusion in which the displaced species
shows a better resistance to the invading species.

Goal : existence for a class of SKT-like systems featuring “weak” cross-diffusion (=positive definite matrix) + their FV numerical approximation.
The class is identified by the fact that our existence proof works :-)))

The existence proof is a rather usual combination of
a priori estimates + compactness + passage-to-the limit arguments .
Along the same lines, convergence of a finite volume method is justified.

Model and motivations (cont[d])

$$\partial_t u - D_1 \Delta u - \mathrm{div}\left(\mathcal{A}_{11}(u,v)\nabla u + \mathcal{A}_{12}(u,v)\nabla v\right) = u(a_1 - b_1 u - c_1 v),$$

$$\partial_t v - D_2 \Delta v - \mathrm{div}\left(\mathcal{A}_{21}(u,v)\nabla u + \mathcal{A}_{22}(u,v)\nabla v\right) = v(a_2 - b_2 u - c_2 v)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0,\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x);$$

here $\mathcal{A} = \left(\mathcal{A}_{ij}\right)_{1 \le i,j \le 2}$ is the cross-diffusion matrix, a typical example being

$$\mathcal{A}(u,v) = \begin{pmatrix} u+v & u \\ v & u+v \end{pmatrix},$$

while a_1, a_2 are Malthusian growth coefficients
b_1, c_2 and b_2, c_1 are those of intra- and inter-species competition, respectively.
The coefficients are assumed (strictly) positive.

Model and motivations (cont[d])

$$\partial_t u - D_1 \Delta u - \mathrm{div}\left(\mathcal{A}_{11}(u,v)\nabla u + \mathcal{A}_{12}(u,v)\nabla v\right) = u(a_1 - b_1 u - c_1 v),$$

$$\partial_t v - D_2 \Delta v - \mathrm{div}\left(\mathcal{A}_{21}(u,v)\nabla u + \mathcal{A}_{22}(u,v)\nabla v\right) = v(a_2 - b_2 u - c_2 v)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0,\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x);$$

here $\mathcal{A} = \left(\mathcal{A}_{ij}\right)_{1\le i,j\le 2}$ is the cross-diffusion matrix, a typical example being

$$\mathcal{A}(u,v) = \begin{pmatrix} u+v & u \\ v & u+v \end{pmatrix},$$

while a_1, a_2 are Malthusian growth coefficients
b_1, c_2 and b_2, c_1 are those of intra- and inter-species competition, respectively.
The coefficients are assumed (strictly) positive.

Model and motivations (cont^{d})

$$\partial_t u - D_1 \Delta u - \mathrm{div}\left(\mathcal{A}_{11}(u,v)\nabla u + \mathcal{A}_{12}(u,v)\nabla v\right) = u(a_1 - b_1 u - c_1 v),$$

$$\partial_t v - D_2 \Delta v - \mathrm{div}\left(\mathcal{A}_{21}(u,v)\nabla u + \mathcal{A}_{22}(u,v)\nabla v\right) = v(a_2 - b_2 u - c_2 v)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \; u(x,0) = u_0(x), \quad v(x,0) = v_0(x);$$

here $\mathcal{A} = \left(\mathcal{A}_{ij}\right)_{1\le i,j\le 2}$ is the cross-diffusion matrix, a typical example being

$$\mathcal{A}(u,v) = \begin{pmatrix} u+v & u \\ v & u+v \end{pmatrix},$$

while a_1, a_2 are Malthusian growth coefficients
b_1, c_2 and b_2, c_1 are those of intra- and inter-species competition, respectively.
The coefficients are assumed (strictly) positive.

Model and motivations (cont[d])

Written in matrix form with the unknown $\mathbf{u} := (u, v)$ (understood as a column vector), the system reads

$$\partial_t \mathbf{u} - \operatorname{div}\left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \nabla \mathbf{u}\right] - \operatorname{div}\left[\mathcal{A}(\mathbf{u})\nabla \mathbf{u}\right] = \begin{pmatrix} F(\mathbf{u}) \\ G(\mathbf{u}) \end{pmatrix},$$

where the reaction terms are given by

$$F(u,v) := u\,(a_1 - b_1 u - c_1 v), \quad G(u,v) := v\,(a_2 - b_2 u - c_2 v).$$

The method we apply works if the diffusion matrix fulfills

$$\forall u, v \geq 0 \qquad \mathcal{A}_{12}(0,v) = 0, \text{ and } \mathcal{A}_{21}(u,0) = 0,$$

$$\forall u, v \geq 0 \quad \forall \mathbf{w} \in \mathbb{R}^2 \qquad \Big(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w}\Big) \geq C\,\|\mathcal{A}(u,v)\|\,\|\mathbf{w}\|^2,$$

where $\Big(\cdot,\cdot\Big)$ is the usual scalar product on $\mathbb{R}^2$, and

$$\forall u, v \geq 0 \quad \|\mathcal{A}(u,v)\| \leq C\left(1 + u^r + v^r\right) \quad \text{with } r < \begin{cases} 4, & \text{if } d = 2 \\ 10/3, & \text{if } d = 3. \end{cases}$$

The first two assumptions allow for nonnegative solutions ; the second assumption expresses the positivity of the cross-diffusion matrix ; and the third assumption is a kind of growth assumption on $\mathcal{A}$.

Model and motivations (cont^{d})

Written in matrix form with the unknown $\mathbf{u} := (u, v)$ (understood as a column vector), the system reads

$$\partial_t \mathbf{u} - \operatorname{div}\left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \nabla \mathbf{u}\right] - \operatorname{div}\left[\mathcal{A}(\mathbf{u})\nabla \mathbf{u}\right] = \begin{pmatrix} F(\mathbf{u}) \\ G(\mathbf{u}) \end{pmatrix},$$

where the reaction terms are given by

$$F(u,v) := u\,(a_1 - b_1 u - c_1 v), \quad G(u,v) := v\,(a_2 - b_2 u - c_2 v).$$

The method we apply works if the diffusion matrix fulfills

$$\forall u, v \geq 0 \qquad \mathcal{A}_{12}(0,v) = 0, \ \text{ and } \ \mathcal{A}_{21}(u,0) = 0,$$

$$\forall u, v \geq 0 \quad \forall \mathbf{w} \in \mathbb{R}^2 \qquad \Big(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w}\Big) \geq C\,\|\mathcal{A}(u,v)\|\,\|\mathbf{w}\|^2,$$

where $\Big(\cdot\,,\,\cdot\Big)$ is the usual scalar product on $\mathbb{R}^2$, and

$$\forall u, v \geq 0 \quad \|\mathcal{A}(u,v)\| \leq C\left(1 + u^r + v^r\right) \quad \text{with } r < \begin{cases} 4, & \text{if } d = 2 \\ 10/3, & \text{if } d = 3. \end{cases}$$

The first two assumptions allow for nonnegative solutions ; the second assumption expresses the positivity of the cross-diffusion matrix ; and the third assumption is a kind of growth assumption on $\mathcal{A}$.

Definition and main result

Definition

A pair $\mathbf{u} = (u, v)$ of functions is a *weak solution* of the above cross-diffusion system if $u, v \in L^2(0, T; H^1(\Omega))$, $\mathcal{A}\nabla\mathbf{u} \in L^1(Q)$, and the following identities hold for all test functions $\varphi, \xi \in \mathcal{D}([0, T) \times \bar{\Omega})$:

$$\iint_Q \left\{ -u\partial_t\varphi + D_1\nabla u \cdot \nabla\varphi + [\mathcal{A}_{11}(u, v)\nabla u + \mathcal{A}_{12}(u, v)\nabla v] \cdot \nabla\varphi \right\} dx\, dt - \int_\Omega u_0(x)\varphi(0, x)\, dx = \iint_Q F(u, v)\varphi\, dx\, dt,$$

$$\iint_Q \left\{ -v\partial_t\xi + D_2\nabla v \cdot \nabla\xi + [\mathcal{A}_{21}(u, v)\nabla u + \mathcal{A}_{22}(u, v)\nabla v] \cdot \nabla\xi \right\} dx\, dt - \int_\Omega v_0(x)\xi(0, x)\, dx = \iint_Q G(u, v)\xi\, dx\, dt.$$

To get a nonnegative solution , we replace $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$, and we replace $F(u, v), G(u, v)$ by $F(u^+, v^+), G(u^+, v^+)$.

Definition and main result

Definition

A pair $\mathbf{u} = (u, v)$ of functions is a *weak solution* of the above cross-diffusion system if $u, v \in L^2(0, T; H^1(\Omega))$, $\mathcal{A}\nabla\mathbf{u} \in L^1(Q)$, and the following identities hold for all test functions $\varphi, \xi \in \mathcal{D}([0, T) \times \bar{\Omega})$:

$$\iint_Q \left\{ -u\partial_t\varphi + D_1\nabla u \cdot \nabla\varphi + [\mathcal{A}_{11}(u, v)\nabla u + \mathcal{A}_{12}(u, v)\nabla v] \cdot \nabla\varphi \right\} dx\, dt - \int_\Omega u_0(x)\varphi(0, x)\, dx = \iint_Q F(u, v)\varphi\, dx\, dt,$$

$$\iint_Q \left\{ -v\partial_t\xi + D_2\nabla v \cdot \nabla\xi + [\mathcal{A}_{21}(u, v)\nabla u + \mathcal{A}_{22}(u, v)\nabla v] \cdot \nabla\xi \right\} dx\, dt - \int_\Omega v_0(x)\xi(0, x)\, dx = \iint_Q G(u, v)\xi\, dx\, dt.$$

To get a nonnegative solution , we replace $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$, and we replace $F(u, v), G(u, v)$ by $F(u^+, v^+), G(u^+, v^+)$.

Definition and main result

Definition

A pair $\mathbf{u} = (u, v)$ of functions is a *weak solution* of the above cross-diffusion system if $u, v \in L^2(0, T; H^1(\Omega))$, $\mathcal{A}\nabla \mathbf{u} \in L^1(Q)$, and the following identities hold for all test functions $\varphi, \xi \in \mathcal{D}([0, T) \times \bar{\Omega})$:

$$\iint_Q \left\{ -u\partial_t\varphi + D_1\nabla u \cdot \nabla\varphi + [\mathcal{A}_{11}(u, v)\nabla u + \mathcal{A}_{12}(u, v)\nabla v] \cdot \nabla\varphi \right\} dx\, dt$$

$$- \int_\Omega u_0(x)\varphi(0, x)\, dx = \iint_Q F(u, v)\varphi\, dx\, dt,$$

$$\iint_Q \left\{ -v\partial_t\xi + D_2\nabla v \cdot \nabla\xi + [\mathcal{A}_{21}(u, v)\nabla u + \mathcal{A}_{22}(u, v)\nabla v] \cdot \nabla\xi \right\} dx\, dt$$

$$- \int_\Omega v_0(x)\xi(0, x)\, dx = \iint_Q G(u, v)\xi\, dx\, dt.$$

To get a nonnegative solution , we replace $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$, and we replace $F(u, v)$, $G(u, v)$ by $F(u^+, v^+)$, $G(u^+, v^+)$.

Definition and main result

Definition

A pair $\mathbf{u} = (u, v)$ of functions is a *weak solution* of the above cross-diffusion system if $u, v \in L^2(0, T; H^1(\Omega))$, $\mathcal{A}\nabla\mathbf{u} \in L^1(Q)$, and the following identities hold for all test functions $\varphi, \xi \in \mathcal{D}([0, T) \times \bar{\Omega})$:

$$\iint_Q \left\{ -u\partial_t\varphi + D_1\nabla u \cdot \nabla\varphi + [\mathcal{A}_{11}(u, v)\nabla u + \mathcal{A}_{12}(u, v)\nabla v] \cdot \nabla\varphi \right\} dx\, dt$$

$$- \int_\Omega u_0(x)\varphi(0, x)\, dx = \iint_Q F(u, v)\varphi\, dx\, dt,$$

$$\iint_Q \left\{ -v\partial_t\xi + D_2\nabla v \cdot \nabla\xi + [\mathcal{A}_{21}(u, v)\nabla u + \mathcal{A}_{22}(u, v)\nabla v] \cdot \nabla\xi \right\} dx\, dt$$

$$- \int_\Omega v_0(x)\xi(0, x)\, dx = \iint_Q G(u, v)\xi\, dx\, dt.$$

To get a nonnegative solution , we replace $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$, and we replace $F(u, v), G(u, v)$ by $F(u^+, v^+), G(u^+, v^+)$.

Definition and main result (cont^{d})

Our main result is

Theorem

Assume that the cross-diffusion matrix $\mathcal{A}$ fulfills the requirements imposed in the previous slides. Then for all $u_0, v_0 \in L^2(\Omega)$ there exists non-negative weak solution of the cross-diffusion system .

Moreover, this solution can be obtained as a limit (along a subseq.) of discrete solutions constructed by the ad hoc finite volume method .

Definition and main result (cont^{d})

Our main result is

Theorem

Assume that the cross-diffusion matrix $\mathcal{A}$ fulfills the requirements imposed in the previous slides. Then for all $u_0, v_0 \in L^2(\Omega)$ there exists non-negative weak solution of the cross-diffusion system .

Moreover, this solution can be obtained as a limit (along a subseq.) of discrete solutions constructed by the ad hoc finite volume method .

FINITE VOLUME APPROXIMATION

Discretization framework (builds on [Eymard, Gallouët, Herbin'00])

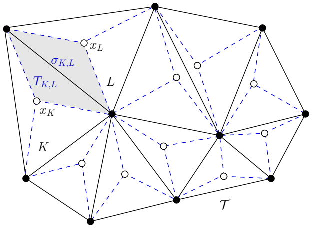

Figure: Orthogonal mesh with diamonds for gradient reconstruction.

DOFs u_K are attached to centers x_K of mesh volumes K.
Balance equations are integrated on each volume ;
the fluxes across edges are reconstructed from DOFs "by hands",
using in particular the orthogonality of the mesh .

The construction aims at "structure preservation at the discrete level"
(analogues of arguments of the stability analysis are needed).

The scheme

Notation: K for control volumes, $L \in \mathcal{N}(K)$ for neighbours, $\sigma_{K,L}$ for edges, and $d_{K,L}$ for distance between centers x_K, x_L.

Scheme for the 1st equation

$$\partial_t u - D_1 \Delta u - \mathrm{div}\left(\mathcal{A}_{11}(u,v)\nabla u + \mathcal{A}_{12}(u,v)\nabla v \right) = F(u,v),$$

is as follows: determine $(u_K^{n+1})_{K\in\mathcal{T}_h}$, $(v_K^{n+1})_{K\in\mathcal{T}_h}$ such that

$$|K|\frac{u_K^{n+1}-u_K^n}{\Delta t} - D_1 \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(u_L^{n+1}-u_K^{n+1})$$

$$-\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left[\mathcal{A}_{11,K,L}^{n+1}(u_L^{n+1}-u_K^{n+1}) + \mathcal{A}_{12,K,L}^{n+1}(v_L^{n+1}-v_K^{n+1}) \right] = |K| F_K^{n+1}$$

for all $K \in \mathcal{T}_h$ and $n \in [0, N_h]$

(and the same for discreizing the second equation).

The choices for approximation of cross-diffusion

The nonlinearities are approximated in the implicit way , namely:

$$F_K^{n+1} = F\left(u_K^{n+1\,+}, v_K^{n+1\,+}\right), \quad G_K^{n+1} = G\left(u_K^{n+1\,+}, v_K^{n+1\,+}\right);$$

and (the key choice for the scheme)

$$\mathcal{A}_{ij,K,L}^{n+1} := \mathcal{A}_{ij}\Big(\min\{u_K^{n+1\,+}, u_L^{n+1\,+}\}, \min\{v_K^{n+1\,+}, v_L^{n+1\,+}\}\Big) \qquad (2.1)$$

The choice of the minimum in the discretization of $\mathcal{A}_{12,K,L}^{n+1}$, $\mathcal{A}_{21,K,L}^{n+1}$ enforces the non-negativity . Then the analogous choice in $\mathcal{A}_{ii,K,L}^{n+1}$, $i = 1, 2$ is needed to preserve the coercivity of $\mathcal{A}$.

In practice, if one takes e.g. the simple "centered" scheme for the SKT model

$$\mathcal{A}_{12,K,L}^{n+1} := \frac{u_K^{n+1} + u_L^{n+1}}{2}, \quad \mathcal{A}_{21,K,L}^{n+1} := \frac{v_K^{n+1} + v_L^{n+1}}{2},$$

$$\mathcal{A}_{ii,K,L}^{n+1} := \frac{u_K^{n+1} + u_L^{n+1} + v_K^{n+1} + v_L^{n+1}}{2}, \quad i = 1, 2,$$

then no negative values appear in the numerical tests we made.

The study of convergence utilizes the "weak discrete gradients" defined per diamond :

$$\left(\nabla_h w_h\right)\Big|_{T_{K,L}} = \nabla_{K,L} w_h := 2\,\frac{w_L - w_K}{d_{K,L}}\eta_{K,L}.$$

The summation-by-parts is used to write a "weak formulation of the scheme", which reads like (e.g, for the 1st eqn)

$$\iint_Q \Big\{ -u_h \partial_t^h \varphi_h + D_1 \nabla_h u_h \cdot (\nabla\varphi)_h$$

$$+ \left[\mathcal{A}_{h11}(u,v)\nabla_h u_h + \mathcal{A}_{h12}(u,v)\nabla_h v_h\right] \cdot (\nabla\varphi)_h \Big\}\, dx\, dt$$

$$- \int_\Omega u_{0,h}(x)\varphi_h(0,x)\, dx = \iint_Q F(u_h, v_h)\varphi_h\, dx\, dt,$$

where integrals can be also seen as summation per volume (for evolution and reaction terms) or per diamond (for diffusion terms).

Convergence analysis for the scheme closely follows the stability analysis of weak solutions (forthcoming !) .

NUMERICAL RESULTS

Numerical example: cross-diffusion

The cross-diffusion system behaves in a qualitatively similar fashion as the self-diffusion system.

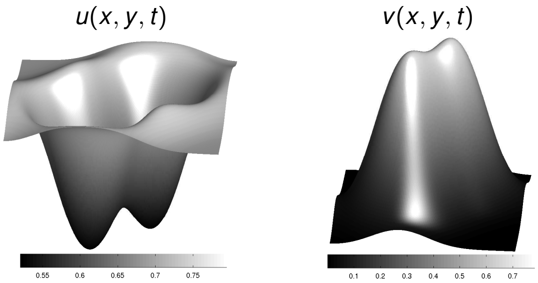

Figure: Model with cross-diffusion:
Spread at time $t = 20$ for species u (resident) and v (invading).

The snapshot of the u-species (near the gradient of density) shows that cross-diffusion allows the resident species to compete with the invaders. Although the invading population also increases, invaders disperse away from the center of resident outbreaks since they diffuse faster than residents.

Numerical comparison cross / pure self diffusion

Comparison of the species' behaviour without/with cross-diffusion : profiles of numerical solutions at time $t = 20$ in a 1D slice of the domain.

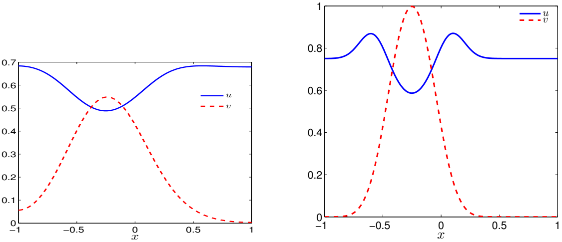

Figure: Profiles at time $t = 20$ on slice $y = 0$ for species u, v.
Left : self-diffusion only. Right: self + cross-diffusion (SKT model).

The concentration of resident species (in the left plot) in two small regions driven by cross diffusion suggest the creation of spatial niches that allows the survival of residents that would otherwise be completely depleted.

Numerical results: convergence rates for the cross-diffusion

The method provides an experimental rate of convergence of $\approx h^{1.8}$.

Number of control volumes $\mathcal{N}$, meshsize h, approximate L^1-errors for u and observed convergence rates r at different simulated times.

Time		time $t=20$		time $t=10$		time $t=1$	
$\mathcal{N}$	h	L^1-error	*rate*	L^1-error	*rate*	L^1-error	*rate*
1024	4×10^{-3}	5.68×10^{-3}	–	2.34×10^{-3}	–	9.12×10^{-4}	–
4096	1×10^{-1}	4.97×10^{-4}	1.74	1.91×10^{-4}	1.81	7.57×10^{-5}	1.80
16384	2.5×10^{-4}	3.96×10^{-5}	1.84	1.54×10^{-5}	1.83	6.24×10^{-6}	1.80
65536	6×10^{-5}	3.20×10^{-6}	1.82	1.27×10^{-6}	1.80	5.21×10^{-7}	1.79

NB: The cross-diffusion system being non-linear and the cross-diffusion discretization being fully implicit, the resolution of the discrete system at each time step is relatively expensive.

Further numerical results and applications of the scheme :
Ricardo Ruiz-Baier, Canrong Tian,...

WEAK CROSS-DIFFUSION EXISTENCE PROOF

Existence proof : construction of solutions, positivity

We can use uniformly parabolic approximation to get a sequence of approximate solutions $\mathbf{u}_h = (u_h, v_h)$ (but we also think of discrete solutions constructed by a finite volume method).

Recall that in order to get a nonnegative solution, we "force" the problem by replacing $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$; we also truncate.

Since $u_0, v_0 \geq 0$, using the test functions $-u_h^-$,$-v_h^-$ in the weak formulation and the assumptions

$$\mathcal{A}_{12}(0, v^+) = 0, \quad \mathcal{A}_{21}(u^+, 0) = 0,$$
$$\mathcal{A}_{11}(u^+, v^+) \geq 0, \quad \mathcal{A}_{22}(u^+, v^+) \geq 0$$

one easily shows the non-negativity of approx. solutions u_h and v_h for the "forced" problem. Hence an approx. solution of the forced cross-diffusion system is a non-negative approx. solution of the initial ("non-forced") system.

Existence proof : construction of solutions, positivity

We can use uniformly parabolic approximation to get a sequence of approximate solutions $\mathbf{u}_h = (u_h, v_h)$ (but we also think of discrete solutions constructed by a finite volume method).

Recall that in order to get a nonnegative solution, we "force" the problem by replacing $\mathcal{A}_{ij}(u, v)$ by $\mathcal{A}_{ij}(u^+, v^+)$; we also truncate.

Since $u_0, v_0 \geq 0$, using the test functions $-u_h^-, -v_h^-$ in the weak formulation and the assumptions

$$\mathcal{A}_{12}(0, v^+) = 0, \;\; \mathcal{A}_{21}(u^+, 0) = 0,$$
$$\mathcal{A}_{11}(u^+, v^+) \geq 0, \;\; \mathcal{A}_{22}(u^+, v^+) \geq 0$$

one easily shows the non-negativity of approx. solutions u_h and v_h for the "forced" problem. Hence an approx. solution of the forced cross-diffusion system is a non-negative approx. solution of the initial ("non-forced") system.

Existence proof : estimates

We have

$$(u_h, v_h) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

$$\|\mathcal{A}(u_h, v_h)\| \left(|\nabla u_h|^2 + |\nabla v_h|^2 \right) \in L^1(Q)$$

with a uniform estimate of the norms.

Indeed, multiplication of the first and the second equation in the system by u_h and v_h, respectively, gives

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|v_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C,$$
$$\|\nabla u_h\|_{L^2(Q)} + \|\nabla v_h\|_{L^2(Q)} \leq C,$$
$$\iint_Q \|\mathcal{A}(u_h, v_h)\| \left(|\nabla u_h|^2 + |\nabla v_h|^2 \right) dx\, dt \leq C,$$

where C is independent of h. Here we use the assumption

$$\left(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w} \right) \geq C \, \|\mathcal{A}(u,v)\| \, \|\mathbf{w}\|^2.$$

Existence proof : estimates

We have

$$(u_h, v_h) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

$$\|\mathcal{A}(u_h, v_h)\| \left(|\nabla u_h|^2 + |\nabla v_h|^2\right) \in L^1(Q)$$

with a uniform estimate of the norms.

Indeed, multiplication of the first and the second equation in the system by u_h and v_h, respectively, gives

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|v_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C,$$
$$\|\nabla u_h\|_{L^2(Q)} + \|\nabla v_h\|_{L^2(Q)} \leq C,$$
$$\iint_Q \|\mathcal{A}(u_h, v_h)\| \left(|\nabla u_h|^2 + |\nabla v_h|^2\right) dx\,dt \leq C,$$

where C is independent of h. Here we use the assumption

$$\left(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w}\right) \;\geq\; C\,\|\mathcal{A}(u,v)\|\,\|\mathbf{w}\|^2.$$

Existence proof : compactness

Then we use the following version
of the compactness lemma of S.N. Kruzhkov .

Lemma

Let Ω be an open domain in $\mathbb{R}^d$, $T > 0$, $Q = (0,T) \times \Omega$. Assume that families of functions $(w^h)_h, (F^h_\alpha)_{h,\alpha}$ are bounded in $L^1(Q)$ and satisfy

$$\frac{\partial}{\partial t} w^h = \sum_{|\alpha| \leq m D^\alpha F^h_\alpha} \quad in \quad \mathcal{D}'(Q).$$

Assume that w^h can be extended outside Q, and one has

$$\sup_{|dx| \leq \Delta} \iint_Q |w^h(t,x+dx) - w^h(t,x)|\, dxdt \leq \omega(\Delta), \text{ with } \lim_{\Delta \to 0} \omega(\Delta) = 0.$$

Then $(w^h)_h$ is relatively compact in $L^1(Q)$.

Existence proof : compactness

Then we use the following version
of the compactness lemma of S.N. Kruzhkov .

Lemma

Let Ω be an open domain in $\mathbb{R}^d$, $T > 0$, $Q = (0,T) \times \Omega$. Assume that families of functions $(w^h)_h, (F^h_\alpha)_{h,\alpha}$ are bounded in $L^1(Q)$ and satisfy

$$\frac{\partial}{\partial t} w^h = \sum_{|\alpha| \leq m D^\alpha F^h_\alpha} \quad in \quad \mathcal{D}'(Q).$$

Assume that w^h can be extended outside Q, and one has

$$\sup_{|dx| \leq \Delta} \iint_Q |w^h(t, x + dx) - w^h(t,x)|\, dxdt \leq \omega(\Delta), \text{ with } \lim_{\Delta \to 0} \omega(\Delta) = 0.$$

Then $(w^h)_h$ is relatively compact in $L^1(Q)$.

Existence proof : compactness (cont^{d})

Getting bounds required in the Kruzhkov lemma:

From the $L^2(0,T;H^1(\Omega))$ estimate of u_h, v_h it is easy to derive a uniform space translation estimate of the form required in the Kruzhkov lemma, for u_h and v_h .

Further, uniform L^1 estimates on u_h, v_h,
on the reaction terms and on the self-diffusion fluxes $\nabla u_h, \nabla v_h$
readily follow from the $L^2(0,T;H^1(\Omega))$ estimate of u_h, v_h.

Next, by Sobolev embedding u_h, v_h are bounded in $L^2(0,T;L^{2^*}(\Omega))$.
By interpolation with $L^\infty(0,T;L^2(\Omega))$ and growth assumption on $\mathcal{A}$,
we get uniform L^1 bound on $\mathcal{A}_{ij}(u_h, v_h)$.

Uniform L^1 estimates on the cross-diffusion fluxes $\mathcal{A}(u_h, v_h)\nabla \mathbf{u}_h$ follow
from the $L^1(Q)$ estimate of $|\mathcal{A}_{ij}(u_h, v_h)|\left(|\nabla u_h|^2 + |\nabla v_h|^2\right)$ and of $|\mathcal{A}_{ij}(u_h, v_h)|$
(details will be given below).

Existence proof : compactness (cont^{d})

Getting bounds required in the Kruzhkov lemma:

From the $L^2(0,T;H^1(\Omega))$ estimate of u_h, v_h it is easy to derive a uniform space translation estimate of the form required in the Kruzhkov lemma, for u_h and v_h .

Further, uniform L^1 estimates on u_h, v_h,
on the reaction terms and on the self-diffusion fluxes ∇u_h, ∇v_h
readily follow from the $L^2(0,T;H^1(\Omega))$ estimate of u_h, v_h.

Next, by Sobolev embedding u_h, v_h are bounded in $L^2(0,T;L^{2^*}(\Omega))$.
By interpolation with $L^\infty(0,T;L^2(\Omega))$ and growth assumption on $\mathcal{A}$,
we get uniform L^1 bound on $\mathcal{A}_{ij}(u_h,v_h)$.

Uniform L^1 estimates on the cross-diffusion fluxes $\mathcal{A}(u_h,v_h)\nabla\mathbf{u}_h$ follow from the $L^1(Q)$ estimate of $|\mathcal{A}_{ij}(u_h,v_h)|\left(|\nabla u_h|^2+|\nabla v_h|^2\right)$ and of $|\mathcal{A}_{ij}(u_h,v_h)|$
(details will be given below).

Existence proof : passage to the limit

Then using the weak form of the system,
by the Kruzhkov Lemma we infer that u_h and v_h converge a.e. in Q (up to extraction of a subsequence) to some limits u and v .

Using the a.e. convergence of u_h, v_h to u, v, using again the Sobolev embeddings and the “strict growth assumption” on $\mathcal{A}$, we infer that $\mathcal{A}_{ij}(u_h, v_h)$ converge to $\mathcal{A}_{ij}(u, v)$ strongly in $L^1(Q)$.

Similarly, $u_h \to u$ and $v_h \to v$ strongly in $L^2(Q)$.

Next, we rewrite the cross-diffusion terms under the form

$$\sqrt{\mathcal{A}_{11}}\left(\sqrt{\mathcal{A}_{11}}\,\nabla u\right), \quad \operatorname{sign}\mathcal{A}_{12}\sqrt{|\mathcal{A}_{12}|}\left(\sqrt{|\mathcal{A}_{12}|}\,\nabla v\right)$$

and so on. Then we consider the terms

$$\begin{array}{ll}\sqrt{\mathcal{A}_{11}(u_h, v_h)}\,\nabla u_h, & \sqrt{|\mathcal{A}_{12}(u_h, v_h)|}\,\nabla v_h,\\ \sqrt{|\mathcal{A}_{21}(u_h, v_h)|}\,\nabla u_h, & \sqrt{\mathcal{A}_{22}(u_h, v_h)}\,\nabla v_h\end{array}$$

in the approximate weak formulation; from the the uniform $L^1(Q)$ estimate on $|\mathcal{A}_{ij}(u_h, v_h)|\left(|\nabla u_h|^2 + |\nabla v_h|^2\right)$, these terms are bounded in $L^2(Q_T)$.

Existence proof : passage to the limit

Then using the weak form of the system, by the Kruzhkov Lemma we infer that u_h and v_h converge a.e. in Q (up to extraction of a subsequence) to some limits u and v .

Using the a.e. convergence of u_h, v_h to u, v, using again the Sobolev embeddings and the “strict growth assumption” on $\mathcal{A}$, we infer that $\mathcal{A}_{ij}(u_h, v_h)$ converge to $\mathcal{A}_{ij}(u, v)$ strongly in $L^1(Q)$.

Similarly, $u_h \to u$ and $v_h \to v$ strongly in $L^2(Q)$.

Next, we rewrite the cross-diffusion terms under the form

$$\sqrt{\mathcal{A}_{11}}\left(\sqrt{\mathcal{A}_{11}}\,\nabla u\right), \quad \operatorname{sign}\mathcal{A}_{12}\sqrt{|\mathcal{A}_{12}|}\left(\sqrt{|\mathcal{A}_{12}|}\,\nabla v\right)$$

and so on. Then we consider the terms

$$\begin{array}{ll} \sqrt{\mathcal{A}_{11}(u_h, v_h)}\,\nabla u_h, & \sqrt{|\mathcal{A}_{12}(u_h, v_h)|}\,\nabla v_h, \\ \sqrt{|\mathcal{A}_{21}(u_h, v_h)|}\,\nabla u_h, & \sqrt{\mathcal{A}_{22}(u_h, v_h)}\,\nabla v_h \end{array}$$

in the approximate weak formulation; from the the uniform $L^1(Q)$ estimate on $|\mathcal{A}_{ij}(u_h, v_h)|\left(|\nabla u_h|^2 + |\nabla v_h|^2\right)$, these terms are bounded in $L^2(Q_T)$.

Existence proof : passage to the limit

Then using the weak form of the system,
by the Kruzhkov Lemma we infer that u_h and v_h converge a.e. in Q (up to extraction of a subsequence) to some limits u and v .

Using the a.e. convergence of u_h, v_h to u, v, using again the Sobolev embeddings and the “strict growth assumption” on $\mathcal{A}$, we infer that $\mathcal{A}_{ij}(u_h, v_h)$ converge to $\mathcal{A}_{ij}(u, v)$ strongly in $L^1(Q)$.

Similarly, $u_h \to u$ and $v_h \to v$ strongly in $L^2(Q)$.

Next, we rewrite the cross-diffusion terms under the form

$$\sqrt{\mathcal{A}_{11}}\left(\sqrt{\mathcal{A}_{11}}\,\nabla u\right), \quad \operatorname{sign}\mathcal{A}_{12}\,\sqrt{|\mathcal{A}_{12}|}\left(\sqrt{|\mathcal{A}_{12}|}\,\nabla v\right)$$

and so on. Then we consider the terms

$$\begin{array}{ll}\sqrt{\mathcal{A}_{11}(u_h, v_h)}\,\nabla u_h, & \sqrt{|\mathcal{A}_{12}(u_h, v_h)|}\,\nabla v_h, \\ \sqrt{|\mathcal{A}_{21}(u_h, v_h)|}\,\nabla u_h, & \sqrt{\mathcal{A}_{22}(u_h, v_h)}\,\nabla v_h\end{array}$$

in the approximate weak formulation; from the the uniform $L^1(Q)$ estimate on $|\mathcal{A}_{ij}(u_h, v_h)|\left(|\nabla u_h|^2 + |\nabla v_h|^2\right)$, these terms are bounded in $L^2(Q_T)$.

Existence proof : passage to the limit

Their weak L^2 limits (along a subsequence) can be identified , via the weak $L^1(Q)$ convergence, with the terms

$$\sqrt{\mathcal{A}_{11}(u,v)}\,\nabla u, \qquad \sqrt{|\mathcal{A}_{12}(u,v)|}\,\nabla v,$$
$$\sqrt{|\mathcal{A}_{21}(u,v)|}\,\nabla u, \qquad \sqrt{\mathcal{A}_{22}(u,v)}\,\nabla v,$$

Here we use the strong $L^2(Q)$ compactness of $\left(\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$ and the weak $L^2(0,T;H^1(\Omega))$ compactness of $(u_h)_h$, $(v_h)_h$.

Using again strong L^2 compactness of $\left(\operatorname{sign}\mathcal{A}_{ij}(u_h,v_h)\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$, we pass to the limit in the cross-diffusion terms of the weak formulation.

Finally, the $L^2(Q)$ convergence of u_h, v_h to u, v is enough in order to pass to the strong limit in the reaction terms $F(u_h,v_h)$ and $G(u_h,v_h)$.

Conclusion: (u,v) is a nonnegative weak solution of the system .

Existence proof : passage to the limit

Their weak L^2 limits (along a subsequence) can be identified , via the weak $L^1(Q)$ convergence, with the terms

$$\sqrt{\mathcal{A}_{11}(u,v)}\,\nabla u, \qquad \sqrt{|\mathcal{A}_{12}(u,v)|}\,\nabla v,$$
$$\sqrt{|\mathcal{A}_{21}(u,v)|}\,\nabla u, \qquad \sqrt{\mathcal{A}_{22}(u,v)}\,\nabla v,$$

Here we use the strong $L^2(Q)$ compactness of $\left(\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$ and the weak $L^2(0,T;H^1(\Omega))$ compactness of $(u_h)_h$, $(v_h)_h$.

Using again strong L^2 compactness of $\left(\operatorname{sign}\mathcal{A}_{ij}(u_h,v_h)\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$, we pass to the limit in the cross-diffusion terms of the weak formulation.

Finally, the $L^2(Q)$ convergence of u_h, v_h to u, v is enough in order to pass to the strong limit in the reaction terms $F(u_h,v_h)$ and $G(u_h,v_h)$.

Conclusion: (u,v) is a nonnegative weak solution of the system .

Existence proof : passage to the limit

Their weak L^2 limits (along a subsequence) can be identified , via the weak $L^1(Q)$ convergence, with the terms

$$\sqrt{\mathcal{A}_{11}(u,v)}\,\nabla u, \qquad \sqrt{|\mathcal{A}_{12}(u,v)|}\,\nabla v,$$
$$\sqrt{|\mathcal{A}_{21}(u,v)|}\,\nabla u, \qquad \sqrt{\mathcal{A}_{22}(u,v)}\,\nabla v,$$

Here we use the strong $L^2(Q)$ compactness of $\left(\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$ and the weak $L^2(0,T;H^1(\Omega))$ compactness of $(u_h)_h$, $(v_h)_h$.

Using again strong L^2 compactness of $\left(\operatorname{sign}\mathcal{A}_{ij}(u_h,v_h)\sqrt{|\mathcal{A}_{ij}(u_h,v_h)|}\right)_h$, we pass to the limit in the cross-diffusion terms of the weak formulation.

Finally, the $L^2(Q)$ convergence of u_h, v_h to u, v is enough in order to pass to the strong limit in the reaction terms $F(u_h,v_h)$ and $G(u_h,v_h)$.

Conclusion: (u,v) is a nonnegative weak solution of the system .

STRONG CROSS-DIFFUSION

A family of (stronger) cross-diffusions.

The next result is due to Chen and Jüngel . Consider the following class of cross-diffusion systems with quadratic cross-diffusion:

$$\partial_t u - D_1 \Delta u - \operatorname{div}\left(c_1 u \nabla u + \nabla(uv)\right) = F(u,v),$$
$$\partial_t v - D_2 \Delta v - \operatorname{div}\left(c_2 v \nabla v + \nabla(uv)\right) = G(u,v)$$

The fact that the $\nabla(uv)$ part of the cross-diffusion term has coefficient one in both equations is a normalization due to a scaling argument.

For c_1, c_2 large enough, the cross-diffusion matrix

$$\mathcal{A}(u,v) = \begin{pmatrix} c_1 u + v & u \\ v & u + c_2 v \end{pmatrix}.$$

corresponding to this example is positive , more exactly,

$$\left(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w}\right) \geq C\,(u+v)\,\|\mathbf{w}\|^2.$$

Then this is a "weak" cross-diffusion, and the previous result applies .

And if the matrix is not positive ? This does happen in modelling practice..!

A family of (stronger) cross-diffusions.

The next result is due to Chen and Jüngel . Consider the following class of cross-diffusion systems with quadratic cross-diffusion:

$$\partial_t u - D_1 \Delta u - \mathrm{div}\left(c_1 u \nabla u + \nabla(uv)\right) = F(u,v),$$
$$\partial_t v - D_2 \Delta v - \mathrm{div}\left(c_2 v \nabla v + \nabla(uv)\right) = G(u,v)$$

The fact that the $\nabla(uv)$ part of the cross-diffusion term has coefficient one in both equations is a normalization due to a scaling argument.

For c_1, c_2 large enough, the cross-diffusion matrix

$$\mathcal{A}(u,v) = \begin{pmatrix} c_1 u + v & u \\ v & u + c_2 v \end{pmatrix}.$$

corresponding to this example is positive , more exactly,

$$\Big(\mathcal{A}(u,v)\mathbf{w},\mathbf{w}\Big) \;\geq\; C\,(u+v)\,\|\mathbf{w}\|^2.$$

Then this is a “weak” cross-diffusion, and the previous result applies .

And if the matrix is not positive ? This does happen in modelling practice..!

A family of (stronger) cross-diffusions.

The next result is due to Chen and Jüngel . Consider the following class of cross-diffusion systems with quadratic cross-diffusion:

$$\partial_t u - D_1 \Delta u - \mathrm{div}\,\big(c_1 u \nabla u + \nabla(uv)\big) = F(u,v),$$
$$\partial_t v - D_2 \Delta v - \mathrm{div}\,\big(c_2 v \nabla v + \nabla(uv)\big) = G(u,v)$$

The fact that the $\nabla(uv)$ part of the cross-diffusion term has coefficient one in both equations is a normalization due to a scaling argument.

For c_1, c_2 large enough, the cross-diffusion matrix

$$\mathcal{A}(u,v) = \begin{pmatrix} c_1 u + v & u \\ v & u + c_2 v \end{pmatrix}.$$

corresponding to this example is positive , more exactly,

$$\big(\mathcal{A}(u,v)\mathbf{w}, \mathbf{w}\big) \ \geq \ C\,(u+v)\,\|\mathbf{w}\|^2.$$

Then this is a “weak” cross-diffusion, and the previous result applies .

And if the matrix is not positive ? This does happen in modelling practice..!

The "$u \log u$" entropy

A (yet somewhat formal) solution comes from the following calculation :

$$\nabla(uv)\cdot\Big(\nabla \ln u + \nabla \ln v\Big) = \nabla(uv)\cdot\Big(\frac{\nabla u}{u} + \frac{\nabla v}{v}\Big)$$
$$= \frac{1}{uv}|\nabla(uv)|^2 \equiv 4|\nabla\sqrt{uv}|^2 \;\geq\; 0.$$

Therefore we take (formally) the test function $\ln u$ in the first equation , the test function $\ln v$ in the second equation , and get the *a priori* estimate

$$\int_\Omega \Big(E_0(u)+E_0(v)\Big) + \int_0^T\int_\Omega \Big(c_1|\nabla u|^2 + c_2|\nabla v|^2\Big)$$
$$+\int_0^T\int_\Omega\Big(\frac{1}{u}|\nabla u|^2 + \frac{1}{v}|\nabla v|^2 + \frac{1}{uv}|\nabla uv|^2\Big)$$
$$\leq \int_\Omega \Big(E_0(u_0)+E_0(v_0)\Big),$$

with the entropy $E_0(z) := \int_0^z \ln s\, ds = z(\ln z - 1)$.

The "$u \log u$" entropy

A (yet somewhat formal) solution comes from the following calculation :

$$\nabla(uv) \cdot \Big(\nabla \ln u + \nabla \ln v\Big) = \nabla(uv) \cdot \Big(\frac{\nabla u}{u} + \frac{\nabla v}{v}\Big)$$
$$= \frac{1}{uv}|\nabla(uv)|^2 \equiv 4|\nabla\sqrt{uv}|^2 \;\; \geq \;\; 0.$$

Therefore we take (formally) the test function $\ln u$ in the first equation , the test function $\ln v$ in the second equation , and get the *a priori* estimate

$$\int_\Omega \Big(E_0(u) + E_0(v)\Big) + \int_0^T\int_\Omega \Big(c_1|\nabla u|^2 + c_2|\nabla v|^2\Big)$$
$$+ \int_0^T\int_\Omega \Big(\frac{1}{u}|\nabla u|^2 + \frac{1}{v}|\nabla v|^2 + \frac{1}{uv}|\nabla uv|^2\Big)$$
$$\leq \int_\Omega \Big(E_0(u_0) + E_0(v_0)\Big),$$

with the entropy $E_0(z) := \displaystyle\int_0^z \ln s\, ds = z(\ln z - 1)$.

Oufff !!!

Merci !