Formulation and analysis of Discrete Duality Finite Volume schemes. Part II. DDFV schemes in 2D and 3D.

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based on joint works with
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Original ideas: F. Hermeline – K. Domelevo and P. Omnès (2D)
Ch.Pierre – F. Hermeline – Y. Coudière and F. Hubert (3D)
J. Droniou – R. Eymard – R. Herbin (DDFV ⊂ gradient schemes)

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CEA-EDF-INRIA School New Trends in Compatible Discretizations
Plan of the talk

1. 2D diamond partition and DDFV discrete gradient
   - Quadrilateral diamonds and a gradient reconstruction formula
   - Centers of diamonds, elements and DDFV volumes

2. 2D DDFV scheme
   - Scalar products and Discrete Duality
   - Some properties of DDFV operators and schemes

3. The 3D CeVe-DDFV scheme
   - A primal mesh-oriented construction
   - 3D CeVe-DDFV: scalar products, discrete duality and reconstruction.

4. The 3D CeVeFE-DDFV scheme

5. Successful applications of DDFV schemes
2D DIAMOND PARTITION AND DISCRETE GRADIENT
Partition into quadrilateral diamonds

Partition $\Omega$ into quadrilateral (possibly degenerate) diamonds.

Quadrilateral has 4 vertices
$= 2$ pairs of “opposite” vertices.
Call them $\kappa, \ell$ and $\kappa^*, \ell^*$.
2D DDFV gradient reconstruction

Attach DOFs $u_K, u_L$ and $u_{K^*}, u_{L^*}$ to the couples of opposite vertices.

**DDFV gradient reconstruction:**

$$\nabla \tilde{u} := \frac{1}{\sin \alpha_D} \left( \frac{u_L - u_K}{m_{\sigma^*}} \nu + \frac{u_{L^*} - u_{K^*}}{m_\sigma} \nu^* \right)$$

**Lemma (consistency of the 2D DDFV gradient)**

*This reconstruction is exact on affine functions.*

**Proof:** Take scalar products by $\vec{\tau}, \vec{\tau}^*$. 
In order to create a FV scheme, one should attach volumes to DOFs.

**Specificity of DDFV:** overlapping volumes.

Two partitions: \( \mathcal{M} \) (“primal” volumes \( \kappa \)) and \( \mathcal{M}^* \) (“dual” volumes \( \kappa^* \)).

Given a choice of \( x_D \) “centers” of diamonds \( D \), the volumes are constructed from “elements”.

**2D DDFV Element:**
triangle with vertices \( x_D \), one \( \in \{ x_K, x_L \} \), and one \( \in \{ x_{K^*}, x_{L^*} \} \).

“Primal” and “dual” 2D DDFV volumes:
Assemble \( \kappa \) from all elements having \( \kappa \) for vertice.
Idem for \( \kappa^* = \) union of elements having \( \kappa^* \) for vertice.

“Primal” mesh \( \mathcal{M} := \{ \text{all } \kappa \} \); “dual” mesh \( \mathcal{M}^* := \{ \text{all } \kappa^* \} \).
Boundary primal and dual volumes (possibly degenerate) appear.

**For implementation:**
Only the diamond mesh and the measures of elements are relevant!
The matrix of the DDFV method is assembled “per diamond”.

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**Centers of diamonds, elements and DDFV volumes**

**2D diamond partition and gradient**

**2D DDFV scheme**

**The 3D CeVe-DDFV scheme**

**The 3D CeVeFE-DDFV scheme**

**Applications of DDFV schemes**
Simplest choice of diamond centers $x_D$: the diagonals’ intersection

(Main) example of diamond, primal and dual meshes: $x_D$ taken at diagonals’ intersection.

NB: another natural choice: $x_D$ the barycenter of vertices of $D$. (easy to calculate the measures of elements from given vertices of $D$).
Primal mesh-oriented DDFV

The 2D DDFV scheme with $x_D$ at diagonals’ intersection is the original one: Hermeline ’98,’00, Domelevo, Omnès ’05.

Construction from a given primal mesh $\mathcal{M}$:
natural if a mesh is given ($\Rightarrow$ DDFV = cell+vertex - centered scheme)

Construction from diamonds:
most natural from the GS viewpoint Droniou, Eymard, Herbin ’15
“Almost arbitrary” primal meshes allowed

Example of a primal mesh allowing for the 2D DDFV construction:

Some conditions on families $\mathcal{T}_h$ of primal meshes are needed such as uniform lower bound on angles $(\alpha_D)_{D \in \mathcal{D}_h}$ (condition satisfied for the meshes like those pictured above).
2D DDFV SCHEME
Discrete functions and fields. Discrete gradient and divergence operators

2D DDFV triple $\mathcal{I} = \text{diamond mesh } \mathcal{D} + \text{volumes meshes } \mathcal{M}, \mathcal{M}^*$

- Consider two kinds of objects:
  - discrete functions $u^\mathcal{I} \in \mathbb{R}^{\#\mathcal{M} + \#\mathcal{M}^*}$, discrete fields $\vec{F}^\mathcal{I} \in (\mathbb{R}^\mathcal{D})^2$:
    \[
    u^\mathcal{I} = \left( (u_K)_{K \in \mathcal{M}}, (u_{K^*})_{K^* \in \mathcal{M}^*} \right), \quad \vec{F}^\mathcal{I} = \left( \vec{F}_D \right)_{D \in \mathcal{D}}.
    \]

- Discrete gradient operator:
  the DDFV per diamond reconstruction from opposite vertices
    \[
    \vec{\nabla}^\mathcal{I} : \mathbb{R}^{\#\mathcal{M} + \#\mathcal{M}^*} \rightarrow (\mathbb{R}^\mathcal{D})^d,
    \vec{\nabla}^\mathcal{D}_D u^\mathcal{I} := \frac{1}{\sin \alpha_D} \left( \frac{u_L - u_K}{d_{K,L}} n_{K^*,L^*} + \frac{u_{L^*} - u_{K^*}}{d_{K^*,L^*}} n_{K,L} \right)
    \]

- Discrete divergence operator: standard FV discretization per primal/dual volume integration + Green-Gauss
    \[
    \text{div}^\mathcal{I} : (\mathbb{R}^\mathcal{D})^d \rightarrow \mathbb{R}^{\#\mathcal{M} + \#\mathcal{M}^*},
    \text{div}_K \vec{F}^\mathcal{I} := \frac{1}{|K|} \sum_D |\partial K \cap D| \vec{F}_D \cdot \vec{n}_K, \quad \text{div}_{K^*} \vec{F}^\mathcal{I} := \frac{1}{|K^*|} \sum_D |\partial K^* \cap D| \vec{F}_D \cdot \vec{n}_{K^*}
    \]
Scalar products and Discrete Duality

Inner product on the space of discrete functions:

\[
\left[ u^\varnothing, v^\varnothing \right] := \frac{1}{2} \left( \sum_{K \in \mathcal{m}} |K| u_K v_K + \sum_{K^* \in \mathcal{m}^*} |K^*| u_{K^*} v_{K^*} \right)
\]

NB: The GS viewpoint will be:

\[
\left[ u^\varnothing, v^\varnothing \right]_{GS} := \sum_{K \in \mathcal{m}, K \in \mathcal{m}^*} |K \cap K^*| \frac{u_K + u_{K^*}}{2} \frac{v_K + v_{K^*}}{2}
\]

Inner product on the space of discrete fields:

\[
\left\{ \vec{F}^\varnothing, \vec{G}^\varnothing \right\} := \sum_{D \in \mathcal{D}} |D| \vec{F}_D \cdot \vec{G}_D
\]

Discrete Duality (DD) property:

\[
\forall u^\varnothing \in \mathbb{R}^{\# \mathcal{m} + \# \mathcal{m}^*} \quad \forall \vec{F}^\varnothing \in \mathbb{R}^{\# \mathcal{D}}
\]

\[
\left[ u^\varnothing, \text{div}^\varnothing \vec{F}^\varnothing \right] + \left\{ \nabla u^\varnothing, \vec{F}^\varnothing \right\} = 0 \quad \text{(or} \quad \left\langle \ldots, \ldots \right\rangle_{\partial \Omega} \text{)}.
\]
Calculation of discrete gradient and proof of the Discrete Duality.

It is useful (both for proofs and for implementation) to rewrite $\nabla_D u^\xi$:

$$\nabla_D u^\xi = \frac{1}{2|D|} \left( (u_L - u_K) \tilde{N}_{KL} + (u_L^* - u_K^*) \tilde{N}_{K*L}^* \right),$$

where

$$\tilde{N}_{KL} := \int_{K|L \cap D} \tilde{n}_K, \quad \tilde{N}_{K*L}^* := \int_{K|L \cap D} \tilde{n}_K^*$$

being the quantities that also appear in discrete divergence: e.g.,

$$\text{div}_K \vec{F}^\xi := \frac{1}{|K|} \sum_{D: D \cap K \neq \emptyset} \vec{F}_D \cdot \tilde{N}_{KL}$$

Lemma (2D DDFV has Discrete Duality property, Dirichlet BC)

Let $u^\xi$ be a discrete function with zero entries in boundary volumes, and $\vec{F}^\xi$ de a discrete field. Then

$$\left[ u^\xi, \text{div}^\xi \vec{F}^\xi \right] + \left\{ \nabla u^\xi, \vec{F}^\xi \right\} = 0.$$

Proof: Write $\left[ \cdot, \cdot \right]$ using the $\tilde{N}_{KL}$ writing, gather terms with $\vec{F}_D$ per diamond, use conservativity of fluxes (summation-by-parts idea).
Lift of the discrete solution to $\Omega$. Relation to gradient schemes.

- The discrete fields are lifted to $\Omega$ by $\left(\Pi, \tilde{\mathcal{F}}\right)(x) := \sum_D \mathcal{F}_D \mathbf{1}_D(x)$.

- For the reasons that will become clear while studying asymptotic compactness, we have to lift $u^\Xi$ on $\Omega$ by
  \[
  (\Pi u^\Xi)(x) := \frac{1}{2} \left( \sum_K u_K \mathbf{1}_K(x) + \sum_{K^*} u_{K^*} \mathbf{1}_{K^*}(x) \right) \equiv \sum_{K,K^*} \frac{u_K + u_{K^*}}{2} \mathbf{1}_{K \cap K^*}(x).
  \]

Relation to gradient schemes: in the GS setting, one uses
\[
\left[u^\Xi, v^\Xi\right]_{GS} := \int_\Omega (\Pi u^\Xi)(x) (\Pi v^\Xi)(x) \, dx \neq \text{the DDFV scalar product}.
\]

Yet if $v^\Xi$ is the discretization of $v \in C^1(\Omega)$,
\[
\forall x \in K \cap K^* \quad |v_K - \frac{v_K + v_{K^*}}{2}|, \quad |v_{K^*} - \frac{v_K + v_{K^*}}{2}| \leq \text{diam}(K) \quad \Rightarrow
\]
\[
\left[u^\Xi, v^\Xi\right]_{GS} = \sum_{K \in \mathcal{W}, K^* \in \mathcal{W}^*} |K \cap K^*| \left| \frac{u_K + u_{K^*}}{2} \frac{v_K + v_{K^*}}{2} \right|
\]
\[
= \frac{1}{2} \left( \sum_{K \in \mathcal{W}, K^* \in \mathcal{W}^*} |K \cap K^*| u_K \frac{v_K + v_{K^*}}{2} + \sum_{K \in \mathcal{W}, K^* \in \mathcal{W}^*} |K \cap K^*| u_{K^*} \frac{v_K + v_{K^*}}{2} \right)
\]
\[
\approx \frac{1}{2} \left( \sum_{K \in \mathcal{W}} |K| u_K v_K + \sum_{K^* \in \mathcal{W}^*} |K^*| u_{K^*} v_{K^*} \right) = \left[u^\Xi, v^\Xi\right]
\]
up to a term of order size($\Xi$)$\|u^\Xi\|_{L^1}$.
Lift of the discrete solution to $\Omega$. Relation to gradient schemes.

- The discrete fields are lifted to $\Omega$ by $(\vec{\Pi}_D \vec{\xi}) (x) := \sum_D \vec{\xi}_D \mathbf{1}_D (x)$.
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$$= \frac{1}{2} \left( \sum_{K \in \mathcal{M}, K^* \in \mathcal{M}^*} |K \cap K^*| u_K \frac{v_K + v_{K^*}}{2} + \sum_{K \in \mathcal{M}, K^* \in \mathcal{M}^*} |K \cap K^*| u_{K^*} \frac{v_K + v_{K^*}}{2} \right)$$

$$\approx \frac{1}{2} \left( \sum_{K \in \mathcal{M}} |K| u_K v_K + \sum_{K^* \in \mathcal{M}^*} |K^*| u_{K^*} v_{K^*} \right) = \left[ u^\Xi, v^\Xi \right]$$

up to a term of order size($\Xi$)$\|u^\Xi\|_{L^1}$. 

Scalar products and Discrete Duality

- 2D DDFV scheme
- The 3D CeVe-DDFV scheme
- The 3D CeVeFE-DDFV scheme
- Applications of DDFV schemes
Discrete compactness and lift of discrete functions.

Introduce primal and dual lift of discrete functions:

\[ \Pi^0 u^\Xi := \sum_K u_K 1_K(x), \quad \Pi^* u^\Xi := \sum_{K^*} u_{K^*} 1_{K^*}(x) \]

\[ \implies \Pi u^\Xi = \frac{\Pi^0 u^\Xi + \Pi^* u^\Xi}{2} \]

Using the DD property + ad hoc consistency properties, one proves

**Proposition (Discrete asymptotic compactness)**

Consider a sequence of meshes \( \Xi_h \) with \( h = \text{size}(\Xi_h) \to 0 \). Assume \( (\Pi^0 u^\Xi_h)_h, (\Pi^* u^\Xi_h)_h \) and \( (\Pi \nabla^\Xi_h u^\Xi_h)_h \) are bounded in \( L^{1+s\text{thg}}(\Omega) \).

Then there exist \( u^0, u^* \) such that

\[ \Pi^0 u^\Xi_h \to u^0, \quad \Pi^* u^\Xi_h \to u^* \quad \text{in} \ L^1(\Omega) \]

Moreover, \( \Pi \nabla^\Xi_h u^\Xi_h \rightharpoonup \frac{u^0 + u^*}{2} \) weakly in \( L^1(\Omega) \).

\[ \implies \] this is the reason to fix reconstruction \( \Pi \)!

(cf. also GS analysis of Droniou, Eymard, Herbin ’15)
Handling of nonlinearities (reaction terms,...). Penalization.

**Nonlinearities I:** for $b(u^\tau)$, use the GS reconstruction “per $K \cap K^*$”. I.e., do not use the lumped approximations “$b(u^\tau) \sim b(u_K)$ on $K$” but use “$b(u^\tau) \sim b(\frac{u_K+u_K^*}{2})$ on $K \cap K^*$” (natural for GS ideology).

**Nonlinearities II:** alternatively, add penalization and keep the natural DDFV (lumped) approximations for $b(u^\tau)$.

**Penalization operator:** add to $-\text{div}_K \vec{\nabla}^\tau u^\tau$ the extra diffusion

$$(\mathcal{P}^\tau u^\tau)_K := \frac{1}{|K|} \sum_{K^*} |K \cap K^*| \frac{u_K-u_K^*}{\text{size}(\tau)}.$$  

(cf. S. Krell’s stabilization in DDFV for Stokes pb.)

**Lemma (contributions of the penalization operator)**

(i) Assume $v^\tau$ has zero DOF at boundary nodes. Then

$$\left[\mathcal{P}^\tau u^\tau, v^\tau\right] = \sum_{K,K^*} \frac{(u_K-u_K^*)(v_K-v_K^*)}{\text{size}(\tau)}.$$  

(ii) If $\left[\mathcal{P}^\tau u^\tau, u^\tau\right] \leq C$ uniformly w.r.t $h = \text{size}(\mathcal{I}_h) \to 0$, then $u^o = \lim_h \Pi^o u^\tau_h$ and $u^* = \lim_h \Pi^* u^\tau_h$ coincide.
Handling of nonlinearities (reaction terms,...). Penalization.

**Nonlinearities I:** for \( b(u^\overline{x}) \), use the GS reconstruction “per \( K \cap K^* \)”. I.e., do not use the lumped approximations \( b(u^\overline{x}) \sim b(u_K) \) on \( K \) but use \( b(u^\overline{x}) \sim b\left(\frac{u_K + u_{K^*}}{2}\right) \) on \( K \cap K^* \) (natural for GS ideology).

**Nonlinearities II:** alternatively, add penalization and keep the natural DDFV (lumped) approximations for \( b(u^\overline{x}) \).

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\[
(P^\overline{x} u^\overline{x})_K := \frac{1}{|K|} \sum_{K^*} |K \cap K^*| \frac{u_K - u_{K^*}}{\text{size}(\overline{x})}.
\]

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**Lemma (contributions of the penalization operator)**

(i) Assume \( v^\overline{x} \) has zero DOF at boundary nodes. Then

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(ii) If \( \left[ P^\overline{x} u^\overline{x}, u^\overline{x} \right] \leq C \) uniformly w.r.t \( h = \text{size}(\overline{x}_h) \to 0 \), then \( u^0 = \lim_h \Pi^0 u^\overline{x}_h \) and \( u^* = \lim_h \Pi^* u^\overline{x}_h \) coincide.
Many classical functional inequalities, etc. are transposed to the discrete DDFV setting.

- **Poincaré-Friedrichs:**
  A., Boyer, Hubert ’07 (weakest assumptions on $\Omega$)

- **Poincaré-Wirtinger, Sobolev:**
  A., Bendahmane, Karlsen, Pierre ’11, Omnès, Le ’14, Bessemoulin-Chatard, Chainais, Filbet ’14

NB: The two meshes are treated separately (like TPFA)

- **Korn inequality:**
  Delcourte, Omnès – Krell, strongly uses DD property

- **tools for Stokes and elasticity:**
  duality formulas involving $\tilde{\mathbf{\nabla}} \tilde{u}$ for vector-valued $\tilde{u}$
  Delcourte, Domelevo, Omnès – Krell

- **inf – sup stability:** Boyer, Krell, Nabet ’15
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Maximum principle and entropy compatibility. 2D $m$-DDFV

- **Maximum principle**: true on orthogonal meshes.
  E.g., primal Delaunay triangulation and dual Voronoï mesh; cartesian meshes (relevant for image processing)
- (hyperbolic problems) compatibility with entropy inequalities: true on orthogonal meshes A., Bendahmane, Karlsen ’11.

Theoretical convergence orders:
- order $h$, for linear problems with smooth $K$, Domelevo, Omnès ’05
- possibility of keeping order $h$ for $-\text{div } K \nabla u$ and piecewise constant $K$: $m$-DDFV extension of DDFV Boyer, Hubert ’08
- handling domain decomposition Boyer, Hubert, Krell, Gander
- order $h^{\min\{p^{-1}, \frac{1}{p-1}\}}$ for $p$-laplacian A., Boyer, Hubert ’07
- $h^2$ on uniform cartesian meshes? Cf. A., Boyer, Hubert ’06

Numerical convergence:
- in practice, particularly good approximation of gradients
- orders between $h$ and $h^2$ for the solutions, depending on meshes

Details: Herbin, Hubert FVCA5 benchmark in 2D.
Maximum principle and entropy compatibility. 2D $m$-DDFV

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3D CeVe-DDFV Scheme
A primal mesh-oriented construction

Construction starting from a primal mesh

A first 3D construction (Pierre – Hermeline — A. et al.) uses primal+dual mesh

3D CeVe-DDFV gradient:

Diamond

Gradient reconstruction: one direction from $x_K, x_L$ and two directions from the vertices of $K\cap L$. 

$D^{K,L}$
If the face $\mathcal{K}\mathcal{L}$ is a triangle $D = D_{K^*,L^*,M^*}^{K,L}$, the reconstruction of all the components of the gradient is obvious:

$$\nabla^\mathcal{K} u^\mathcal{K} \big|_D = \text{the vector of } \mathbb{R}^3 \text{ with the projections}$$

$$\left\{ \begin{array}{l}
\frac{u_L - u_K}{d_{KL}}, \quad \text{on } \overrightarrow{X_KX_L} \\
\frac{u_{L^*} - u_{K^*}}{d_{K^*L^*}}, \quad \text{on } \overrightarrow{X_{K^*}X_{L^*}} \\
\frac{u_{K^*} - u_{M^*}}{d_{M^*K^*}}, \quad \text{on } \overrightarrow{X_{M^*}X_{K^*}} \\
\end{array} \right.$$

And what if it is a general polygon? (e.g., a quadrilateral, as for cartesian primal mesh)?
Let $\Theta$ be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $D \subset \Theta$ be a polygon.

Introduce the vertices $x_i^*$, $i = 1, \ldots, \ell$ (numbered counter-clockwise w.r.t. the orientation of $\Theta$ induced by $\vec{n}$).

Denote the area of $\sigma$ by $|D|$, we have $|D| = \sum_{i=1}^{\ell} |D_{i,i+1}|$ (sub-areas are signed).

Let $x_0^* \in \Theta$ be a distinguished point.

Take $x_{i,i+1}^*$ the midpoints of the edges.

**Lemma**

$$\bar{r} = \frac{1}{|D|} \sum_{i=1}^{\ell} (\bar{r} \cdot x_i^* x_{i+1}^*) \left[ \vec{n} \times x_i^* x_{i+1}^* \right] \equiv \frac{2}{|D|} \sum_{i=1}^{\ell} |D_{i,i+1}| (\bar{r} \cdot \vec{e}_{i,i+1}) \vec{e}_{i,i+1}',$$

where $\vec{e}_{i,i+1} := x_i^* x_{i+1}^* / \|x_i^* x_{i+1}^*\|$ and $\vec{e}_{i,i+1}' := \left[ \vec{n} \times x_i^* x_{i+1}^* \right] / \| \vec{n} \times x_i^* x_{i+1}^* \|$. 

The formula can be derived from the “magical formula” of Droniou, Eymard.

NB: if $\ell > 3$, this is one of infinitely many affine reconstruction formulas!
Corollary (Consistency of the gradient reconstruction)

Take \((w_i^*)_{i=1}^\ell \subset \mathbb{R}\), \(w_{\ell+1} := w_1^*\). If \(w_i^*\) are the values of an affine function \(w\) at the vertices \(x_i^*\) of the polygon \(\sigma\), then

\[
\nabla w = \frac{1}{|D|} \sum_{i=1}^\ell (w_{i+1}^* - w_i^*) \left[ \vec{n} \times \overrightarrow{x_i^* x_{i+1}^*} \right] = \frac{2}{|D|} \sum_{i=1}^\ell |D_{i,i+1}| \frac{w_{i+1}^* - w_i^*}{d_{i,i+1}} \vec{e}_{i,i+1}',
\]

where \(d_{i,i+1} := \|\overrightarrow{x_i^* x_{i+1}^*}\|\) and \(\vec{e}_{i,i+1}' := \left[ \vec{n} \times \overrightarrow{x_i^* x_{i+1}^*} \right] / \|\vec{n} \times \overrightarrow{x_i^* x_{i+1}^*}\|\)

\(\vec{n}\) signed area

\(x_{1,2} \equiv x_1^*\)

\(x_{4,5} \equiv x_5^*\)
There are infinitely many possibilities of reconstructing tangential components of $\vec{\nabla}_D$. One leads to Discrete Duality property.

**The 3D CeVe-DDFV gradient reconstruction:**
use the 2D co-volume reconstruction (cf. Part I of these lectures) in the polygon $KL$ which vertices are dual centers $x_K^*, x_L^*, x_M^*, ...$

**Coercivity concern:**
is the kernel of $\vec{\nabla}$ reduced to constants, i.e.,

$$\vec{\nabla} = 0 \quad \Rightarrow \quad \Pi^0 u^{\nabla} \equiv \text{const}, \quad \Pi^* u^{\nabla} \equiv \text{const} ?$$

Yes, if $\ell = 3$, e.g. for tetrahedral primal meshes.
Yes, on cartesian meshes ($\ell = 4$) or topologically equivalent to them.
No coercivity, in general.

**NB:** 3D CeVeFE-DDFV scheme fixes the coercivity issue
Coercivity or lack of coercivity of CeVe-DDFV

There are infinitely many possibilities of reconstructing tangential components of $\vec{\nabla}_D$. One leads to Discrete Duality property.

**The 3D CeVe-DDFV gradient reconstruction:**
use the 2D co-volume reconstruction (cf. Part I of these lectures) in the polygon $\kappa L$ which vertices are dual centers $x_{K^*}, x_{L^*}, x_{M^*}, \ldots$

**Coercivity concern:**
is the kernel of $\vec{\nabla}_x$ reduced to constants, i.e.,

$$\vec{\nabla}_x \equiv \vec{0} \quad \Rightarrow \quad \Pi^0 u^x \equiv const, \quad \Pi^* u^x \equiv const ?$$

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No coercivity, in general.

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Scalar products, discrete duality, reconstruction.

- For 3D discrete fields \( \vec{F}^x, \vec{G}^x \in (\mathbb{R}^#)^3 \),
  \[
  \left\{ \vec{F}^x, \vec{G}^x \right\} = \sum_{D \in D} m_D \vec{F}_D \cdot \vec{G}_D
  \]

- For 3D discrete functions \( u^x, v^x \in \mathbb{R}^#M + #M^* \),
  \[
  \left[ u^x, v^x \right] = \frac{1}{3} \sum_{K \in M} m_K u_K v_K + \frac{2}{3} \sum_{K^* \in M^*} m_{K^*} u_{K^*} v_{K^*};
  \]

- Lift of discrete functions in 3D CeVE-DDFV:
  \[
  \Pi u^x := \frac{1}{3} \Pi^o u^x + \frac{2}{3} \Pi^* u^x,
  \]
  \[
  \Pi^o := u^x(x) \sum_{K \in M} u_K \mathbf{1}_K(x), \quad \Pi^* u^x(x) := \sum_{K^* \in M^*} u_{K^*} \mathbf{1}_{K^*}(x)
  \]

Proposition (3D CeVe Discrete Duality property, Dirichlet BC)

\[
\left[ \text{div}^x [\vec{F}^x], u^x \right] + \left\{ \vec{F}^x, \nabla^x u^x \right\} = 0
\]
3D CeVeFE-DDFV scheme
An alternative: 3D CeVeFE-DDFV aka “new DDFV” scheme

There is an alternative to 3D CeVe-DDFV which is always coercive. Moreover, the construction can naturally be started from an arbitrary octahedral diamond mesh.

3D octahedron has 3 pairs of “opposite” vertices per diamond. The vertices are sorted pairwise into three classes and each pair of vertices is used to reconstruct a direction of the gradient.

Each of the three types of the vertices determines a partition of $\Omega$

$\Rightarrow$ CeVeFE-DDFV involves integration on three meshes

**Differences w.r.t. CeVe-DDFV:**

meshes weights $\frac{1}{3}$ (primal), $\frac{2}{3}$ (dual) are replaced by $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.

- **CeVeFE scalar product:**

  $\left[u^\times, v^\times\right] = \frac{1}{3} \sum_{K \in \mathcal{M}} m_K u_K v_K + \frac{1}{3} \sum_{K^* \in \mathcal{M}^*} m_{K^*} u_{K^*} v_{K^*} + \frac{1}{3} \mathcal{M}^\#$ mesh term

- **CeVeFE reconstruction of solution:**

  $\Pi u^\times := \frac{1}{3} \Pi^0 u^\times + \frac{1}{3} \Pi^* u^\times + \frac{1}{3} \Pi^\# u^\times$

**Details:** the “Part-III.pdf” (by the courtesy of F. Hubert)
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  - CeVeFE scalar product:
    \[
    \left[u^\Sigma, v^\Sigma\right] = \frac{1}{3} \sum_{K \in \mathcal{M}} m_K u_K v_K + \frac{1}{3} \sum_{K^* \in \mathcal{M}^*} m_{K^*} u_{K^*} v_{K^*} + \frac{1}{3} \mathcal{M}^\# \text{ mesh term}
    \]

  - CeVeFE reconstruction of solution:
    \[
    \Pi u^\Sigma := \frac{1}{3} \Pi^0 u^\Sigma + \frac{1}{3} \Pi^* u^\Sigma + \frac{1}{3} \Pi^\# u^\Sigma
    \]

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    \]

  - CeVeFE reconstruction of solution:
    \[
    \Pi u^\text{x} := \frac{1}{3} \Pi^0 u^\text{x} + \frac{1}{3} \Pi^* u^\text{x} + \frac{1}{3} \Pi^\# u^\text{x}
    \]

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APPLICATIONS OF DDFV SCHEMES
Problems discretized with DDFV

- Linear anisotropic and heterogeneous elliptic equations:
  Hermeline – Domelevo, Omnès – ...

- Flows in porous media:
  Boyer, Hubert – Chainais, Krell, Mouton – ...

- Nonlinear elliptic and elliptic-parabolic equations:
  A., Boyer, Hubert – Coudière, Hubert – A., Bendahmane, Hubert

- Nonlinear convection-diffusion equations:
  Coudière, Manzini – A., Bendahmane, Karlsen

- Electrocardiology (elliptic-parabolic reaction-diffusion system):
  Pierre – Coudière, Pierre, Turpault – A., Bendahmane, Karlsen, Pierre

- Stokes and Navier-Stokes problems:
  Delcourte, Domelevo, Omnès – Krell – Krell, Manzini – Goudon, Krell –
  Le, Omnès – Delcourte, Omnès – Boyer, Krell, Nabet – ...

- Linear elasticity:
  F. Pascal, B. Martin

- Image restoration, level-set curvature driven eqn.:
  Handlovičová, Kotorová – Handlovičová, Frolkovič – Hartung, Hubert
That’s all… thank you — merci !