Formulation and analysis of Discrete Duality Finite Volume schemes. Part II. DDFV schemes in 2D and 3D.

B. Andreianov[1]

based on joint works with
Franck Boyer and Florence Hubert (Marseille), Stella Krell (Nice),
Mostafa Bendahmane (Bordeaux), Kenneth H. Karlsen (Oslo).

Original ideas: F. Hermeline – K. Domelevo and P. Omnès (2D)
Ch.Pierre – F. Hermeline – Y. Coudière and F. Hubert (3D)
J. Droniou – R. Eymard – R. Herbin (DDFV $\subset$ gradient schemes)

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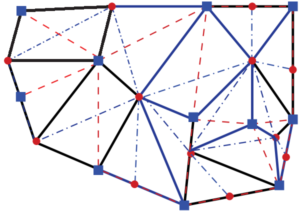
CEA-EDF-INRIA School **New Trends in Compatible Discretizations**

Plan of the talk

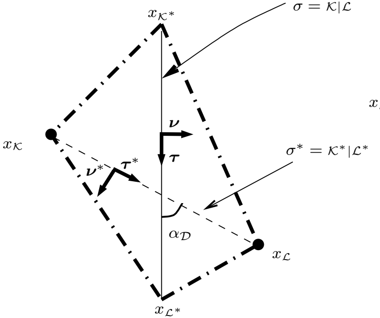
2D DIAMOND PARTITION AND DISCRETE GRADIENT

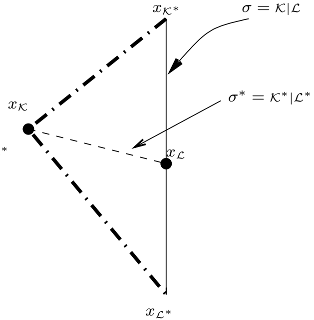
Partition into quadrilateral diamonds

Partition Ω into quadrilateral (possibly degenerate) diamonds.

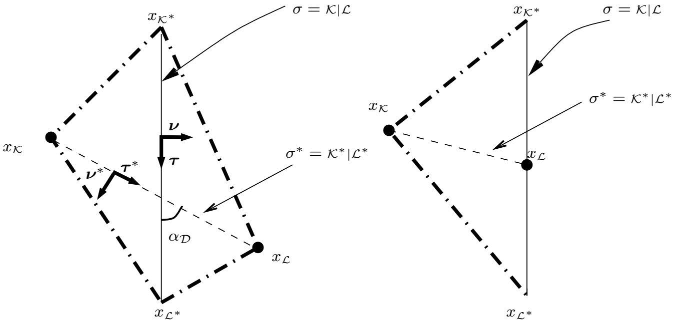


Quadrilateral has 4 vertices $=$ 2 pairs of "opposite" vertices. Call them K, L and K^*, L^*.





2D DDFV gradient reconstruction



Attach DOFs u_K, u_L and u_{K^*}, u_{L^*} to the couples of opposite vertices.

DDFV gradient reconstruction: $\vec{\nabla} u^{\mathfrak{T}} := \dfrac{1}{\sin \alpha_{\mathcal{D}}} \left(\dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \boldsymbol{\nu} + \dfrac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \boldsymbol{\nu}^* \right)$

Lemma (consistency of the 2D DDFV gradient)

This reconstruction is exact on affine functions.

Proof: Take scalar products by $\vec{\tau}, \vec{\tau}^*$.

Centers of diamonds, elements and DDFV volumes.

In order to create a FV scheme, one should attach volumes to DOFs.
Specificity of DDFV: overlapping volumes.
Two partitions: $\mathfrak{M}$ ("primal" volumes κ) and $\mathfrak{M}^*$ ("dual" volumes κ^*).

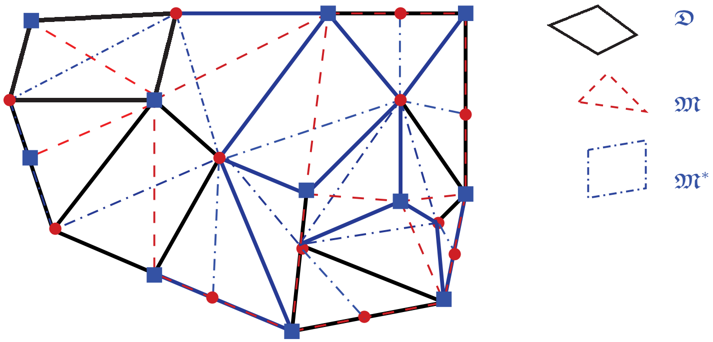
Given a choice of x_D "centers" of diamonds D,
the volumes are constructed from "elements".

2D DDFV Element:
triangle with vertices x_D, one $\in \{x_K, x_L\}$, and one $\in \{x_{K^*}, x_{L^*}\}$.

"Primal" and "dual" 2D DDFV volumes:
Assemble κ from all elements having κ for vertice.
Idem for κ^* = union of elements having κ^* for vertice.
"Primal" mesh $\mathfrak{M} := \{\text{all } \kappa\}$; "dual" mesh $\mathfrak{M}^* := \{\text{all } \kappa^*\}$.
Boundary primal and dual volumes (possibly degenerate) appear.

For implementation:
Only the diamond mesh and the measures of elements are relevant !
The matrix of the DDFV method is assembled "per diamond".

Simplest choice of diamond centers x_D: the diagonals' intersection



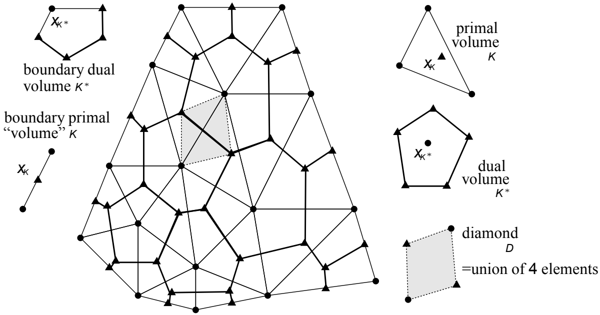
(Main) example of diamond, primal and dual meshes:
x_D taken at diagonals' intersection.

NB: another natural choice: x_D the barycenter of vertices of D.
(easy to calculate the measures of elements from given vertices of D).

Primal mesh-oriented DDFV

The 2D DDFV scheme with x_D at diagonals' intersection
is the original one: Hermeline '98,'00 , Domelevo, Omnès '05 .

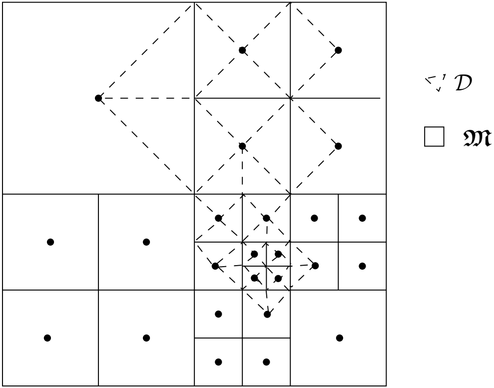
Construction from a given primal mesh $\mathfrak{M}$:
natural if a mesh is given ($\Rightarrow$ DDFV = cell+vertex - centered scheme)



Construction from diamonds:
most natural from the GS viewpoint Droniou, Eymard, Herbin '15

"Almost arbitrary" primal meshes allowed

Example of a primal mesh allowing for the 2D DDFV construction:



Some conditions on families $\mathfrak{T}_h$ of primal meshes are needed such as uniform lower bound on angles $(\alpha_D)_{D\in\mathfrak{D}_h}$
(condition satisfied for the meshes like those pictured above).

2D DDFV SCHEME

Discrete functions and fields. Discrete gradient and divergence operators

2D DDFV triple $\mathfrak{T}$ = diamond mesh $\mathfrak{D}$ + volumes meshes $\mathfrak{M}, \mathfrak{M}^*$

- Consider two kinds of objects:
  discrete functions $u^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{M}+\#\mathfrak{M}^*}$, discrete fields $\vec{\mathcal{F}}^{\mathfrak{T}} \in \left(\mathbb{R}^{\#\mathfrak{D}}\right)^2$:

$$u^{\mathfrak{T}} = \left(\left(u_K\right)_{K\in\mathfrak{M}}, \left(u_{K^*}\right)_{K^*\in\mathfrak{M}^*} \right), \quad \vec{\mathcal{F}}^{\mathfrak{T}} = \left(\vec{\mathcal{F}}_D\right)_{D\in\mathfrak{D}}.$$

- Discrete gradient operator:
  the DDFV per diamond reconstruction from opposite vertices

$$\vec{\nabla}^{\mathfrak{T}} : \mathbb{R}^{\#\mathfrak{M}+\#\mathfrak{M}^*} \longrightarrow \left(\mathbb{R}^{\#\mathfrak{D}}\right)^d,$$

$$\vec{\nabla}_D u^{\mathfrak{T}} := \frac{1}{\sin\alpha_D}\left(\frac{u_L - u_K}{d_{K,L}}\vec{n}_{K^*,L^*} + \frac{u_{L^*} - u_{K^*}}{d_{K^*,L^*}}\vec{n}_{K,L}\right)$$

- Discrete divergence operator: standard FV discretization
  per primal/dual volume integration + Green-Gauss

$$\operatorname{div}^{\mathfrak{T}} : \left(\mathbb{R}^{\#\mathfrak{D}}\right)^d \longrightarrow \mathbb{R}^{\#\mathfrak{M}+\#\mathfrak{M}^*},$$

$$\operatorname{div}_K \vec{\mathcal{F}}^{\mathfrak{T}} := \frac{1}{|K|}\sum_D |\partial K\cap D|\vec{\mathcal{F}}_D \cdot \vec{n}_K \qquad \operatorname{div}_{K^*} \vec{\mathcal{F}}^{\mathfrak{T}} := \frac{1}{|K^*|}\sum_D |\partial K^*\cap D|\vec{\mathcal{F}}_D \cdot \vec{n}_{K^*}$$

Scalar products and Discrete Duality

- Inner product on the space of discrete functions:

$$\left[\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] := \frac{1}{2}\Big(\sum_{K\in\mathfrak{M}} |K| u_K v_K + \sum_{K^*\in\mathfrak{M}^*} |K^*| u_{K^*} v_{K^*}\Big)$$

NB: The GS viewpoint will be:

$$\left[\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right]_{GS} := \sum_{K\in\mathfrak{M},\, K\in\mathfrak{M}^*} |K\cap K^*| \frac{u_K + u_{K^*}}{2} \frac{v_K + v_{K^*}}{2}$$

- inner product on the space of discrete fields:

$$\left\{\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\right\} := \sum_{D\in\mathfrak{D}} |D| \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D$$

- Discrete Duality (DD) property :

$$\forall\, u^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{M}+\#\mathfrak{M}^*} \quad \forall\, \vec{\mathcal{F}}^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{D}}$$

$$\left[\!\left[u^{\mathfrak{T}}, \mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} \right]\!\right] + \left\{\!\left\{ \vec{\nabla} u^{\mathfrak{T}}, \vec{\mathcal{F}}^{\mathfrak{T}} \right\}\!\right\} = 0 \ \ (\text{ or } = \left\langle\!\left\langle \ldots, \ldots \right\rangle\!\right\rangle_{\partial\Omega}).$$

Calculation of discrete gradient and proof of the Discrete Duality.

It is useful (both for proofs and for implementation) to rewrite $\vec{\nabla}_D u^{\mathfrak{T}}$:

$$\vec{\nabla}_D u^{\mathfrak{T}} = \frac{1}{2|D|}\Big((u_L - u_K)\vec{N}_{KL} + (u_{L^*} - u_{K^*})\vec{N}_{K^*L^*}\Big),$$

$$\vec{N}_{KL} := \int_{K|L\cap D} \vec{n}_K, \quad \vec{N}_{K^*L^*} := \int_{K|L\cap D} \vec{n}_K$$

being the quantities that also appear in discrete divergence: e.g.,

$$\mathrm{div}_K \vec{\mathcal{F}}^{\mathfrak{T}} := \frac{1}{|K|}\sum_{D:\ D\cap\partial K\neq\emptyset} \vec{\mathcal{F}}_D \cdot \vec{N}_{KL}$$

Lemma (2D DDFV has Discrete Duality property, Dirichlet BC)

Let $u^{\mathfrak{T}}$ be a discrete function with zero entries in boundary volumes, and $\vec{\mathcal{F}}^{\mathfrak{T}}$ de a discrete field. Then

$$\Big[\!\Big[u^{\mathfrak{T}}, div^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} \Big]\!\Big] + \Big\{\!\Big\{ \vec{\nabla} u^{\mathfrak{T}}, \vec{\mathcal{F}}^{\mathfrak{T}} \Big\}\!\Big\} = 0.$$

Proof: Write $[\![\cdot,\cdot]\!]$ using the $\vec{N}_{KL}$ writing, gather terms with $\vec{\mathcal{F}}_D$ per diamond, use conservativity of fluxes (summation-by-parts idea).

Lift of the discrete solution to Ω. Relation to gradient schemes.

- The discrete fields are lifted to Ω by $(\vec{\Pi}\vec{\mathcal{F}}^{\mathfrak{T}})(x) := \sum_D \vec{\mathcal{F}}_D 1\!\!1_D(x)$
- For the reasons that will become clear while studying asymptotic compactness, we have to lift $u^{\mathfrak{T}}$ on Ω by

$$(\Pi u^{\mathfrak{T}})(x) := \frac{1}{2}\Big(\sum_K u_K 1\!\!1_K(x) + \sum_{K^*} u_{K^*} 1\!\!1_{K^*}(x)\Big) \equiv \sum_{K,K^*} \frac{u_K+u_{K^*}}{2} 1\!\!1_{K\cap K^*}(x).$$

Relation to gradient schemes: in the GS setting, one uses

$$[\![u^{\mathfrak{T}}, v^{\mathfrak{T}}]\!]_{GS} := \int_\Omega (\Pi u^{\mathfrak{T}})(x)(\Pi u^{\mathfrak{T}})(x)\,dx \neq \text{the DDFV scalar product.}$$

Yet if $v^{\mathfrak{T}}$ is the discretization of $v \in C^1(\overline{\Omega})$,

$$\forall x \in K\cap K^* \quad \Big|v_K - \frac{v_K+v_{K^*}}{2}\Big|, \Big|v_{K^*} - \frac{v_K+v_{K^*}}{2}\Big| \le \operatorname{diam}(K) \quad \Rightarrow$$

$$[\![u^{\mathfrak{T}}, v^{\mathfrak{T}}]\!]_{GS} = \sum_{K\in\mathfrak{M},\,K\in\mathfrak{M}^*} |K\cap K^*| \frac{u_K+u_{K^*}}{2}\frac{v_K+v_{K^*}}{2}$$
$$= \frac{1}{2}\Big(\sum_{K\in\mathfrak{M},\,K\in\mathfrak{M}^*} |K\cap K^*| u_K \frac{v_K+v_{K^*}}{2} + \sum_{K\in\mathfrak{M},\,K\in\mathfrak{M}^*} |K\cap K^*| u_{K^*} \frac{v_K+v_{K^*}}{2}\Big)$$
$$\approx \frac{1}{2}\Big(\sum_{K\in\mathfrak{M}} |K| u_K v_K + \sum_{K^*\in\mathfrak{M}^*} |K^*| u_{K^*} v_{K^*}\Big) = [\![u^{\mathfrak{T}}, v^{\mathfrak{T}}]\!]$$

up to a term of order $\operatorname{size}(\mathfrak{T})\|u^{\mathfrak{T}}\|_{L^1}$.

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up to a term of order size$(\mathfrak{T})\|u^{\mathfrak{T}}\|_{L^1}$.

Discrete compactness and lift of discrete functions.

Introduce primal and dual lift of discrete functions:

$$\Pi^o u^{\mathfrak{T}} := \sum_K u_K 1\!\!1_K(x), \quad \Pi^* u^{\mathfrak{T}} := \sum_{K^*} u_{K^*} 1\!\!1_{K^*}(x)$$

$$\Rightarrow \quad \Pi u^{\mathfrak{T}} = \frac{\Pi^o u^{\mathfrak{T}} + \Pi^* u^{\mathfrak{T}}}{2}$$

Using the DD property + ad hoc consistency properties, one proves

Proposition (Discrete asymptotic compactness)

Consider a sequence of meshes $\mathfrak{T}_h$ with $h = size(\mathfrak{T}_h) \to 0$. Assume $(\Pi^o u^{\mathfrak{T}_h})_h$, $(\Pi^* u^{\mathfrak{T}_h})_h$ and $(\vec{\Pi}\vec{\nabla}^{\mathfrak{T}_h} u^{\mathfrak{T}_h})_h$ are bounded in $L^{1+sthg}(\Omega)$. Then there exist u^o, u^* such that

$$\Pi^o u^{\mathfrak{T}_h} \to u^o, \quad \Pi^* u^{\mathfrak{T}_h} \to u^* \quad \text{in } L^1(\Omega)$$

$$\text{moreover,} \quad \vec{\Pi}\vec{\nabla}^{\mathfrak{T}_h} u^{\mathfrak{T}_h} \rightharpoonup \vec{\nabla}\frac{u^o + u^*}{2} \quad \text{weakly in } L^1(\Omega).$$

$\Rightarrow$ this is the reason to fix reconstruction Π !
(cf. also GS analysis of Droniou, Eymard, Herbin '15)

Handling of nonlinearities (reaction terms,...). Penalization.

Nonlinearities I: for $b(u^{\mathfrak{T}})$, use the GS reconstruction "per $\kappa \cap \kappa^*$".
I.e., do not use the lumped approximations "$b(u^{\mathfrak{T}}) \sim b(u_\kappa)$ on κ" but use "$b(u^{\mathfrak{T}}) \sim b(\frac{u_K+u_{K^*}}{2})$ on $\kappa \cap \kappa^*$" (natural for GS ideology).

Nonlinearities II: alternatively, add penalization
and keep the natural DDFV (lumped) approximations for $b(u^{\mathfrak{T}})$.

Penalization operator: add to $-\text{div}_\kappa \mathbb{K}\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}$ the extra diffusion

$$(\mathcal{P}^{\mathfrak{T}} u^{\mathfrak{T}})_\kappa := \frac{1}{|K|} \sum_{K^*} |\kappa \cap \kappa^*| \frac{u_K - u_{K^*}}{\text{size}(\mathfrak{T})}.$$

(cf. S. Krell 's stabilization in DDFV for Stokes pb.)

Lemma (contributions of the penalization operator)

(i) *Assume $v^{\mathfrak{T}}$ has zero DOF at boundary nodes. Then*

$$\left[\!\left[\mathcal{P}^{\mathfrak{T}} u^{\mathfrak{T}}, v^{\mathfrak{T}}\right]\!\right] = \sum_{K,K^*} \frac{(u_K - u_{K^*})(v_K - v_{K^*})}{size(\mathfrak{T})}$$

(ii) *If $\left[\!\left[\mathcal{P}^{\mathfrak{T}} u^{\mathfrak{T}}, u^{\mathfrak{T}}\right]\!\right] \leq C$ uniformly w.r.t $h = size(\mathfrak{T}_h) \to 0$, then $u^o = \lim_h \Pi^o u^{\mathfrak{T}_h}$ and $u^* = \lim_h \Pi^* u^{\mathfrak{T}_h}$ coincide.*

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Discrete Poincare, Sobolev inequalities. Discrete Korn inequality, etc.

Many classical functional inequalities, etc.
are transposed to the discrete DDFV setting.

- Poincaré-Friedrichs:
  A., Boyer, Hubert '07 (weakest assumptions on $\mathfrak{T}$)
- Poincaré-Wirtinger, Sobolev:
  A., Bendahmane, Karlsen, Pierre '11 , Omnès, Le '14 ,
  Bessemoulin-Chatard, Chainais, Filbet '14

  NB: The two meshes are treated separately (like TPFA)
- Korn inequality:
  Delcourte, Omnès – Krell , strongly uses DD property
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Maximum principle and entropy compatibility. 2D m-DDFV

- **Maximum principle:** true on orthogonal meshes.
  E.g., primal Delaunay triangulation and dual Voronoï mesh; cartesian meshes (relevant for image processing)
- (hyperbolic problems) compatibility with entropy inequalities: true on orthogonal meshes A., Bendahmane, Karlsen '11 .

Theoretical convergence orders:

- order h, for linear problems with smooth $\mathbb{K}$ Domelevo, Omnès '05
- possibility of keeping order h for $-\mathrm{div}\,\mathbb{K}\vec{\nabla}u$ and piecewise constant $\mathbb{K}$: m-DDFV extension of DDFV Boyer, Hubert '08
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Numerical convergence:

- in practice, particularly good approximation of gradients
- orders between h and h^2 for the solutions, depending on meshes

Details: Herbin, Hubert FVCA5 benchmark in 2D .

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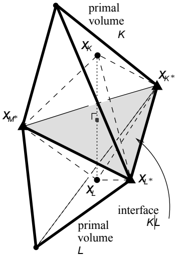
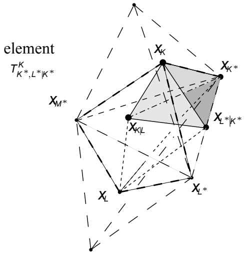
3D CeVe-DDFV scheme

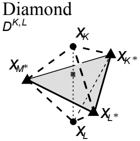
Construction starting from a primal mesh

A first 3D construction (Pierre – Hermeline — A. et al.) uses primal+dual mesh





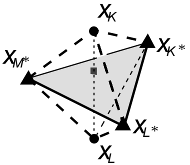
3D CeVe-DDFV gradient:



Gradient reconstruction: one direction from x_K, x_L and two directions from the vertices of $K|L$.

Gradient approximation on CeVe diamonds

If the face $K|L$ is a triangle $D = D^{K,L}_{K^*,L^*,M^*}$,
the reconstruction of all the components
of the gradient is obvious:

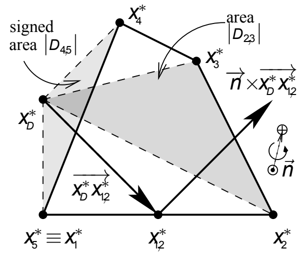


$$\nabla^{\mathfrak{T}} u^{\mathfrak{T}}|_D = \text{the vector of } \mathbb{R}^3 \text{ with the projections } \begin{cases} \frac{u_L - u_K}{d_{KL}}, & \text{on } \overrightarrow{x_K x_L} \\ \frac{u_{L^*} - u_{K^*}}{d_{K^*L^*}}, & \text{on } \overrightarrow{x_{K^*} x_{L^*}} \\ \frac{u_{K^*} - u_{M^*}}{d_{M^*K^*}}, & \text{on } \overrightarrow{x_{M^*} x_{K^*}} \\ & \text{etc.} \end{cases}$$

And what if it is a general polygon ?
(e.g., a quadrilateral, as for cartesian primal mesh)?

Back to the 2D co-volume reconstruction property

Polygon $D \subset \Theta$, oriented by $\vec{n} \perp \Theta$



Let Θ be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $D \subset \Theta$ be a polygon.

Introduce the vertices x_i^*, $i = 1, \dots, \ell$ (numbered counter-clockwise w.r.t. the orientation of Θ induced by $\vec{n}$).

Denote the area of σ by $|D|$, we have $|D| = \sum_{i=1}^{\ell} |D_{i,i+1}|$ (sub-areas are signed).

Let $x_D^* \in \Theta$ be a distinguished point .

Take $x_{i,i+1}^*$ the midpoints of the edges .

Lemma

For all $\vec{r} \parallel \Theta$, $\displaystyle \vec{r} = \frac{1}{|D|} \sum_{i=1}^{\ell} (\vec{r} \cdot \overrightarrow{x_i^* x_{i+1}^*}) \, [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] \equiv \frac{2}{|D|} \sum_{i=1}^{\ell} |D_{i,i+1}| (\vec{r} \cdot \vec{e}_{i,i+1}) \vec{e}\,'_{i,i+1}$,

where $\vec{e}_{i,i+1} := \overrightarrow{x_i^* x_{i+1}^*} / \| \overrightarrow{x_i^* x_{i+1}^*} \|$ *and* $\vec{e}\,'_{i,i+1} := [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] / \| [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] \|$.

The formula can be derived from the “magical formula” of Droniou, Eymard .
NB: if $\ell > 3$, this is one of infinitely many affine reconstruction formulas !

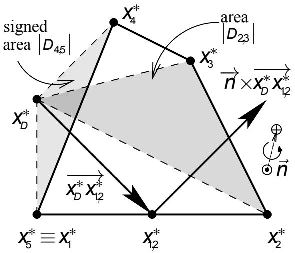
The formula (cont[d])

Corollary (Consistency of the gradient reconstruction)

Take $(w_i^*)_{i=1}^{\ell} \subset \mathbb{R}$, $w_{\ell+1}^* := w_1^*$. *If* w_i^* *are the values of an affine function* w *at the vertices* x_i^* *of the polygon* σ*, then*

$$\vec{\nabla} w = \frac{1}{|D|} \sum_{i=1}^{\ell} (w_{i+1}^* - w_i^*) \left[\vec{n} \times \overrightarrow{x_D^* x_{i,i+1}^*}\right] \equiv \frac{2}{|D|} \sum_{i=1}^{\ell} |D_{i,i+1}| \frac{w_{i+1}^* - w_i^*}{d_{i,i+1}} \vec{e}\,'_{i,i+1},$$

where $d_{i,i+1} := \|\overrightarrow{x_i^* x_{i+1}^*}\|$ *and* $\vec{e}\,'_{i,i+1} := \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right] / \left\| \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right] \right\|$



Coercivity or lack of coercivity of CeVe-DDFV

There are infinitely many possibilities of reconstructing tangential components of $\vec{\nabla}_{\mathcal{D}}$. One leads to Discrete Duality property.

The 3D CeVe-DDFV gradient reconstruction:
use the 2D co-volume reconstruction (cf. Part I of these lectures) in the polygon $K|L$ which vertices are dual centers $x_{K^*}, x_{L^*}, x_{M^*}, \ldots$

Coercivity concern:
is the kernel of $\vec{\nabla}^{\mathfrak{T}}$ reduced to constants, i.e.,

$$\vec{\nabla}^{\mathfrak{T}} \equiv \vec{0} \quad \Rightarrow \quad \Pi^o u^{\mathfrak{T}} \equiv const, \Pi^* u^{\mathfrak{T}} \equiv const \; ?$$

Yes, if $\ell = 3$, e.g. for tetrahedral primal meshes.

Yes, on cartesian meshes ($\ell = 4$) or topologically equivalent to them.
No coercivity, in general .

NB: 3D CeVeFE-DDFV scheme fixes the coercivity issue

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Scalar products, discrete duality, reconstruction.

- For 3D discrete fields $\vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \in (\mathbb{R}^{\#\mathfrak{D}})^3$,

$$\left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\!\right\} = \sum_{D\in\mathfrak{D}} m_D \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D$$

- For $3D$ discrete functions $u^{\mathfrak{T}}, v^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{M}+\#\mathfrak{M}^*}$,

$$\left[\!\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\!\right] = \frac{1}{3} \sum_{K\in\mathfrak{M}} m_K \, u_K v_K \; + \; \frac{2}{3} \sum_{K^*\in\mathfrak{M}^*} m_{K^*} \, u_{K^*} v_{K^*} \,;$$

- Lift of discrete functions in 3D CeVE-DDFV:

$$\Pi u^{\mathfrak{T}} := \frac{1}{3}\Pi^o u^{\mathfrak{T}} + \frac{2}{3}\Pi^* u^{\mathfrak{T}},$$

$$\Pi^o := u^{\mathfrak{T}}(x) \sum_{K\in\mathfrak{M}} u_K 1\!\!1_K(x), \quad \Pi^* u^{\mathfrak{T}}(x) := \sum_{K^*\in\mathfrak{M}^*} u_{K^*} 1\!\!1_{K^*}(x)$$

Proposition (3D CeVe Discrete Duality property, Dirichlet BC)

$$\left[\!\!\left[\mathrm{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}], u^{\mathfrak{T}} \right]\!\!\right] + \left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \nabla^{\mathfrak{T}} u^{\mathfrak{T}} \right\}\!\!\right\} = 0$$

3D CeVeFE-DDFV scheme

An alternative: 3D CeVeFE-DDFV aka "new DDFV" scheme

There is an alternative to 3D CeVe-DDFV which is always coercive. Moreover, the construction can naturally be started from an arbitrary octahedral diamond mesh.

3D octahedron has 3 pairs of "opposite" vertices per diamond . The vertices are sorted pairwise into three classes and each pair of vertices is used to reconstruct a direction of the gradient .

Each of the three types of the vertices determines a partition of Ω
$\Rightarrow$ CeVeFE-DDFV involves integration on three meshes

Differences w.r.t. CeVe-DDFV:
meshes weights $\frac{1}{3}$ (primal), $\frac{2}{3}$ (dual) are replaced by $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.

- CeVeFE scalar product:

$$\left[\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] = \frac{1}{3} \sum_{K \in \mathfrak{M}} m_K \, u_K v_K + \frac{1}{3} \sum_{K^* \in \mathfrak{M}^*} m_{K^*} \, u_{K^*} v_{K^*} + \frac{1}{3} \mathfrak{M}^{\#} \text{ mesh term}$$

- CeVeFE reconstruction of solution:

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Details: the "Part-III.pdf" (by the courtesy of F. Hubert)

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Applications of DDFV schemes

Problems discretized with DDFV

- Linear anisotropic and heterogeneous elliptic equations:
  Hermeline – Domelevo, Omnès – ...
- Flows in porous media:
  Boyer, Hubert – Chainais, Krell, Mouton – ...
- Nonlinear elliptic and elliptic-parabolic equations:
  A., Boyer, Hubert – Coudière, Hubert – A., Bendahmane, Hubert
- Nonlinear convection-diffusion equations:
  Coudière, Manzini – A., Bendahmane, Karlsen
- Electrocardiology (elliptic-parabolic reaction-diffusion system):
  Pierre – Coudière, Pierre, Turpault – A., Bendahmane, Karlsen, Pierre
- Stokes and Navier-Stokes problems:
  Delcourte, Domelevo, Omnès – Krell – Krell, Manzini – Goudon, Krell – Le, Omnès – Delcourte, Omnès – Boyer, Krell, Nabet –...
- Linear elasticity:
  F. Pascal, B. Martin
- Image restoration, level-set curvature driven eqn.:
  Handlovičová, Kotorová – Handlovičová, Frolkovič – Hartung, Hubert

Thank you !

That’s all... thank you — merci !