Formulation and analysis of Discrete Duality Finite Volume schemes. Part I. Co-volume schemes for the $p(x)$-laplacian.

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based on joint works with
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Plan of the talk

1. **Structural stability for a nonlinear diffusion PDE**
   - A model PDE problem
   - Structural stability

2. **Generalities on FV schemes and Discrete Duality**
   - Principles of construction of Finite Volume schemes
   - Gradient approximation on diamonds
   - Discrete Duality

3. **The classical co-volume scheme (=lumped P1 FE)**
   - Construction of a 2D co-volume scheme from a triangulation
   - Discrete Duality property and its consequences
   - Convergence of the co-volume scheme for the $p(x)$-laplacian

4. **Co-volume scheme on general Donald (median dual) mesh**
   - A gradient reconstruction formula on general polygons
   - Generalized co-volume scheme, consistency and Discrete Duality
STRUCTURAL STABILITY FOR A NONLINEAR DIFFUSION PDE
A model PDE problem

A nonlinear diffusion PDE

Consider the problem (Prob):

\[
\begin{aligned}
    u - \Delta_{p(x)}u &= f \text{ in } \Omega \\
    u|_{\partial \Omega} &= 0
\end{aligned}
\]

- \(\Omega\): a bounded Lipschitz domain of \(\mathbb{R}^d\), \(d = 1, 2, 3\)
- \(f\): an \(L^\infty\) (for simplicity) source term
- the \(p(x)\)-laplacian operator \(\Delta_{p(x)}u := \text{div} (\vec{\nabla} u)^{p(x)-1}\),
  where \((\xi)^{p(x)-1} := |\xi|^{p(x)-2}\xi\) is the “french power” of \(\xi \in \mathbb{R}^d\).
  Notation: \(\vec{F}[u] := A(x, \vec{\nabla} u)\) is the flux.
  The equation writes \(u - \text{div} \vec{F}[u] = f\).
- \(p\): a regular function on \(\overline{\Omega}\) with values in \([p_-, p_+] \subset (1, +\infty)\)

Applications: e.g., in image processing.
Part of Navier-Stokes models for complex fluids.

Notion of solution:
\(u \in W^{1,1}_0(\Omega)\) with \(|\nabla u(\cdot)|^{p(\cdot)} \in L^1(\Omega)\).
Weak formulation (multiply by a test function, integrate by parts).
A nonlinear diffusion PDE

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where $(\vec{\xi})^{p(x)-1} := |\vec{\xi}|^{p(x)-2} \vec{\xi}$ is the “french power” of $\vec{\xi} \in \mathbb{R}^d$.

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The equation writes $\quad u - \text{div} \vec{F}[u] = f$.

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Weak formulation (multiply by a test function, integrate by parts).
**Structural stability**

Basic numerical analysis? Try to mimic structural stability analysis!

**Structural stability**: consider \((f_n)\), \((p_n)\) converging to \(f\), \(p\) resp. Does the associated solutions’ sequence \((u_n)\) of \((\text{Prob}_n)\) converge to a solution \(u\) of the limit problem?

A way to prove structural stability:

1. Prove uniform estimates on the sequence of solutions \((u_n)\)
2. Create an accumulation point \(u\) for the sequence (compactness arguments). Usually, one starts with a weak convergence.
3. Prove that the accumulation point is a solution of the equation
   
   \[
   \int_{\Omega} u_n v + (\nabla u_n)^{p_n(x) - 1} \cdot \nabla v = \int_{\Omega} f_n v
   \]

The question boils down to:
Is the weak limit of \((\nabla u_n)^{p_n(\cdot) - 1}\) equal to \((\nabla u)^{p(\cdot) - 1}\) ?
Answer: exploit weak PDE formulations both for \(u_n\) and for \(u\)...
Basic numerical analysis? Try to mimic structural stability analysis!

**Structural stability**: consider \((f_n)_n, (p_n)_n\) converging to \(f, p\) resp. Does the associated solutions’ sequence \((u_n)_n\) of \((Prob_n)\) converge to a solution \(u\) of the limit problem?

A way to prove structural stability:

1. Prove **uniform estimates** on the sequence of solutions \((u_n)_n\)
2. Create an **accumulation point** \(u\) for the sequence (compactness arguments). Usually, one starts with a **weak convergence**.
3. Prove that the accumulation point is a solution of the equation \(\equiv\) pass to the limit in the formulation, including nonlinearities.

I.e. for all \(v \in C_0^\infty(\Omega)\), can we let \(n \to \infty\) in

\[
\int_\Omega u_n v + (\nabla u_n)^{p_n(x)-1} \cdot \nabla v = \int_\Omega f_n v
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The question boils down to:
Is the weak limit of \((\nabla u_n)^{p_n(\cdot)-1}\) equal to \((\nabla u)^{p(\cdot)-1}\)?

Answer: exploit weak PDE formulations both for \(u_n\) and for \(u\)...
Assume $f_n \to f$ say, in $L^2$. Assume $p_n \to p$ a.e. on $\Omega$. Follow the steps:

1. Take test function $u_n$ in weak formulation of $(Prob)$ + Young inequality $\Rightarrow$
   $$\|u_n\|_{L^2}^2 + \|\nabla u_n|^{p_n(x)}\|_{L^1} = \int_\Omega |u_n|^2 + |\nabla u_n|^{p_n(x)} \leq C$$

2. Weak compactness + equi-integrability arguments $\Rightarrow$
   $$u_n \rightharpoonup u \text{ in } L^2, \quad \nabla u_n \rightharpoonup \vec{G} \text{ in } L^1, \quad (\nabla u_n)^{p_n(x)-1} \rightharpoonup \vec{F} \text{ in } L^1.$$ 
   Identification of the weak limit $\vec{G}$ of $\nabla u_n$ with $\nabla u$: let $n \to \infty$ in
   $$\forall \vec{\psi} \in C_0^\infty(\Omega) \int_\Omega u_n \text{div } \vec{\psi} = -\int_\Omega \nabla u_n \cdot \vec{\psi}.$$ 

3. Identification of the weak limit $\vec{F}$ of $(\nabla u_n)^{p_n(x)-1}$ with $(\nabla u)^{p(x)-1}$:
   the starting point is the “anti-weak-convergence” inequality
   $$\int_\Omega \vec{F} \cdot \nabla u = \int_\Omega (f - u)u \geq \liminf \int_\Omega (f_n - u_n)u_n = \lim \int_\Omega (\nabla u_n)^{p_n(x)-1} \cdot \nabla u_n.$$ 
   Represent $\nabla u = L^1$ weak-lim $\nabla u_n$ by a Young measure $(\nu_x(\vec{\cdot}))_x$ 
   ($\Rightarrow$ this also permits to represent $\vec{F}$ and $\vec{F} \cdot \nabla u$). “Anti-weak” ineq. $\Rightarrow$
   $$\int_\Omega \int_{\mathbb{R}^d} \left( (\vec{\lambda})^{p_n(x)-1} - (\vec{\mu})^{p_n(x)-1} \right) \cdot (\vec{\lambda} - \vec{\mu}) \, d\nu_x(\vec{\lambda}) d\nu_x(\vec{\mu}) \leq 0.$$ 
   From the monotonicity, deduce that the Young measure is concentrated 
   (“multi-valued limit is single-valued”) and thus $\nabla u_n \rightharpoonup \nabla u$ pointwise.
Structural stability

Sketch of structural stability proof for \((Prob)\)

Assume \(f_n \to f\) say, in \(L^2\). Assume \(p_n \to p\) a.e. on \(\Omega\). Follow the steps:

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\|u_n\|_{L^2}^2 + \|\nabla u_n|^{p_n(x)}\|_{L^1} = \int_{\Omega} |u_n|^2 + |\nabla u_n|^{p_n(x)} \leq C
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2. Weak compactness + equi-integrability arguments \(\Rightarrow\)

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u_n \rightharpoonup u \text{ in } L^2, \quad \nabla u_n \rightharpoonup \overrightarrow{G} \text{ in } L^1, \quad (\nabla u_n)^{p_n(x)-1} \rightharpoonup \overrightarrow{F} \text{ in } L^1.
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Identification of the weak limit \(\overrightarrow{G}\) of \(\nabla u_n\) with \(\nabla u\): let \(n \to \infty\) in

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\forall \psi \in C^\infty_0(\Omega) \quad \int_{\Omega} u_n \text{div} \psi = - \int_{\Omega} \nabla u_n \cdot \psi.
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3. Identification of the weak limit \(\overrightarrow{F}\) of \((\nabla u_n)^{p_n(x)-1}\) with \((\nabla u)^{p(x)-1}\):

the starting point is the “anti-weak-convergence” inequality

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\int_{\Omega} \overrightarrow{F} \cdot \nabla u = \int_{\Omega} (f - u)u \geq \lim \int_{\Omega} (f_n - u_n)u_n = \lim \int_{\Omega} (\nabla u_n)^{p(x)-1} \cdot \nabla u_n.
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Represent \(\nabla u = L^1\text{-weak-lim } \nabla u_n\) by a Young measure \((\nu_x(\cdot))_x\)

\(\Rightarrow\) this also permits to represent \(\overrightarrow{F}\) and \(\overrightarrow{F} \cdot \nabla u\). “Anti-weak” ineq. \(\Rightarrow\)

\[
\int_{\Omega} \int_{\mathbb{R}^d} ((\lambda)^{p(x)-1} - (\mu)^{p(x)-1}) \cdot (\lambda - \mu) \, d\nu_x(\lambda) d\nu_x(\mu) \leq 0.
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From the monotonicity, deduce that the Young measure is concentrated (“multi-valued limit is single-valued") and thus \(\nabla u_n \rightharpoonup \nabla u\) pointwise.
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**Structural stability for** $p(x)$-laplacian

**Generalities on FV schemes and Discrete Duality**

**Classical co-volume scheme**

**Co-volume scheme on Donald mesh**

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GENERALITIES ONFINITE VOLUMESCHMES.

DISCRETE DUALITY.
We use Finite Volumes (FV, for short) for space discretization of (Prob).

Generally, let us think of discretizing an elliptic equation of the kind

$$u - \text{div } \vec{F} = f, \quad \vec{F} = A(x, \nabla u).$$

The principles for FV approximation of such equations are the following:

- A partition of the space domain $\Omega$ into “volumes” $\kappa$ is given; the partition is called “mesh” and denoted by $\mathcal{K}$.

- An unknown $u_\kappa$ is associated to each volume (usually regarded as the value at a “center” point of the volume, denoted $x_\kappa$); the whole set of the unknowns is called “discrete solution” and denoted by $u^\mathcal{K}$.

- If $\kappa, \ell$ are “neighbours” (adjacent volumes), the divided differences $\frac{u_\ell - u_\kappa}{d_{KL}}$ are used to “reconstruct” the “discrete gradient” $\nabla^\mathcal{K} u^\mathcal{K}$ of $u^\mathcal{K}$.

NB: In the schemes we think of, the reconstruction is done “by hands”. We do not solve any equations to compute the values of $\nabla^\mathcal{K} u^\mathcal{K}$ from those of $u^\mathcal{K}$, but fix ad hoc formulas for $\nabla^\mathcal{K} u^\mathcal{K}$ in terms of $u^\mathcal{K}$. 
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The principles for FV approximation of such equations are the following:

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- If \(\mathcal{K}, \mathcal{L}\) are “neighbours” (adjacent volumes), the divided differences \(\frac{u_\mathcal{L} - u_\mathcal{K}}{d_{\mathcal{KL}}}\) are used to “reconstruct” the “discrete gradient” \(\vec{\nabla}^\mathcal{T} u^\mathcal{T}\) of \(u^\mathcal{T}\).

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Generalities on FV schemes

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Principles of construction of Finite Volume schemes

Generalities on FV schemes (cont’d)

- Each couple of neighbour volumes possesses a part of the common border, called “interface” and denoted by $K_L$. Combining the values $u_K$, $u_L$ and the values of the reconstructed discrete gradient $\vec{\nabla} u^x$ near $K_L$, we produce the discretization $\vec{F}_{K_L}$ of the flux $\vec{F} = A(x, \vec{\nabla} u)$ on $K_L$.

- In two steps, the PDE is replaced by a set of algebraic equations:
  - Firstly, the PDE is “projected on the mesh”. One integrates the continuous equation on $K$; the Green-Gauss formula is used to reduce $\int_K \text{div} \vec{F}$ to $\int_{\partial K} \vec{F} \cdot \vec{n}_K$.
    This yields a discrete system of equalities, one per volume.
  - And secondly, these equalities are approximated by replacing $u|_K$ with the unknown $u_K$ and replacing $\vec{F}|_{K_L}$ with the expression of $\vec{F}_{K_L} = A(x_{K_L}, \vec{\nabla}_{K_L} u^x)$.

NB: e.g. in TPFA schemes all components of $\vec{\nabla}_{K_L} u^x$ need not be reconstructed, since only the normal fluxes $\vec{F}_{K_L} \cdot \vec{n}_K$ appear.

However, for (Prob) or for linear anisotropic diffusion $\mathbb{K} \nabla u$ (cf. R.Eymard’s lectures) : one should(?) define (per interface) $\vec{\nabla}_{K_L} u^x$.

- As a result, we obtain a closed system of algebraic equations which (hopefully...) can be solved, often in an approximate way.
  This yields $u^x$ and $\vec{\nabla} u^x$, approximations of $u$ and $\vec{\nabla} u$, respectively.
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Generalities on FV schemes (cont’d)

- Each couple of neighbour volumes possesses a part of the common border, called “interface” and denoted by $K_L$. Combining the values $u_K, u_L$ and the values of the reconstructed discrete gradient $\nabla^\mathcal{I} u^\mathcal{I}$ near $K_L$, we produce the discretization $\mathcal{F}_{K_L}$ of the flux $\mathcal{F} = A(x, \nabla u)$ on $K_L$.

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NB: e.g. in TPFA schemes all components of $\nabla_{K_L} u^\mathcal{I}$ need not be reconstructed, since only the normal fluxes $\mathcal{F}_{K_L} \cdot \mathbf{n}_K$ appear. However, for (Prob) or for linear anisotropic diffusion $\mathbf{K} \nabla u$ (cf. R.Eymard’s lectures): one should(?) define (per interface) $\nabla_{K_L} u^\mathcal{I}$.

- As a result, we obtain a closed system of algebraic equations which (hopefully...) can be solved, often in an approximate way. This yields $u^\mathcal{I}$ and $\nabla u^\mathcal{I}$, approximations of $u$ and $\nabla u$, respectively.
For representing discrete gradients, many FV schemes use “diamond mesh”. A diamond $\hat{K}_L$ is a neighbourhood of interface $K_L$.

Notation ($\sim$ Gradient Schemes) useful to formulate “Discrete Duality”:

- Consider $\mathcal{X} = (\mathcal{D}, \mathcal{M})$ consisting of two partitions of $\Omega$:
  - volumes $K$ (mesh $\mathcal{M}$) and diamonds $D$ (diamond mesh $\mathcal{D}$).
- Consider two kinds of objects:
  - discrete functions $u^\mathcal{X} \in \mathbb{R}^\#\mathcal{M}$ and discrete fields $\vec{F}^\mathcal{X} \in (\mathbb{R}^\#\mathcal{D})^d$:
    $$u^\mathcal{X} = (u_K)_{K \in \mathcal{M}}, \quad \vec{F}^\mathcal{X} = (\vec{F}_D)_{D \in \mathcal{D}}.$$ 
- They are lifted to $\Omega$ as piecewise constant (per volume / per diamond) functions $\Pi u^\mathcal{X}$ and fields $\vec{\Pi} \vec{F}$ (cf. Gradient Schemes).
- Discrete gradient operator: some “per diamond reconstruction”
  $$\nabla^\mathcal{X} : \mathbb{R}^\#\mathcal{M} \rightarrow (\mathbb{R}^\#\mathcal{D})^d, \quad \nabla^\mathcal{X} u^\mathcal{X} := \text{Formula}(u_K \text{ for } K \text{ near } D)$$
- Discrete divergence operator: per volume integration + Green-Gauss:
  $$\text{div}^\mathcal{X} : (\mathbb{R}^\#\mathcal{D})^d \rightarrow \mathbb{R}^\#\mathcal{M}, \quad \text{div}_K \vec{F}^\mathcal{X} := \frac{1}{|K|} \sum_D |\partial K \cap D| \vec{F}_D \cdot \vec{n}_K.$$
Scalar products of discrete functions/fields and Discrete Duality

- inner product on the space of discrete functions:
  \[
  \left[ u^x, v^x \right] := \sum_{K \in \mathcal{M}} |K| u_K v_K
  \]

- inner product on the space of discrete fields:
  \[
  \left\{ \vec{F}^x, \vec{G}^x \right\} := \sum_{D \in \mathcal{D}} |D| \vec{F}_D \cdot \vec{G}_D
  \]

- Discrete Duality (DD) property holds if \( \text{div}^x \) and \( -\vec{\nabla}^x \) fulfill:
  \[
  \forall u^x \in \mathbb{R}^{\# \mathcal{M}} \quad \forall \vec{F}^x \in \mathbb{R}^{\# \mathcal{D}}
  \left[ u^x, \text{div}^x \vec{F}^x \right] + \left\{ \vec{\nabla} u^x, \vec{F}^x \right\} = 0 \quad (\text{or} \quad \langle ..., ... \rangle_{\partial \Omega}).
  \]

**Use of DD property:** It permits to mimic the structural stability arguments for proving convergence of the scheme.

**Additional ingredient:** (natural) consistency properties

- for the discrete gradient applied to projections of test functions
- for the discrete divergence applied to projection of test fields.
CLASSICAL CO-VOLUME SCHEME
AKA
LUMPED P1 FINITE ELEMENT SCHEME
The co-volume scheme in 2D

Consider a Delaunay triangulation $\mathcal{D}$ (“diamonds”) of $\Omega$.

Attach the DOF to the nodes of the diamond mesh $\mathcal{D}$.

Interpret the DOF as values of the const/volume discrete solution $u^\mathcal{V} = (u_K)_K$ on the dual (Voronoï) mesh $\mathcal{M}$.

Given $u^\mathcal{V}$, the const/diamond discrete gradient $\vec{\nabla}^\mathcal{V} u^\mathcal{V} :=$ the gradient of the P1 EF reconstruction from node DOFs.

**Lemma (explicit representation of $\vec{\nabla}^\mathcal{D}_D u^\mathcal{V}$)**

This “co-volume discrete gradient” is exact on affine functions and is given by

$$\vec{\nabla}^\mathcal{D}_D u^\mathcal{V} = \frac{|K_L \cap D| (u_L - u_K) \vec{n}_{K,L} + |M_M \cap D| (u_M - u_L) \vec{n}_{L,M} + |M_K \cap D| (u_K - u_M) \vec{n}_{M,K}}{|D|}$$

The co-volume scheme for $(\text{Prob})$ is formulated “pointwise” as:

$$\forall K \in \mathcal{M} \quad u_K = \frac{1}{|K|} \sum_D |\partial K \cap D| (\vec{\nabla}^\mathcal{D}_D u^\mathcal{V})_{D}^{p_D^{-1}} \cdot \vec{n}_K = f_K$$

with $(p_D)_D$, $(f_K)_K$ piecewise constant discretizations of $p(\cdot)$ and $f(\cdot)$.

Dirichlet BC: attach the values 0 to the nodes $x_K \in \partial \Omega$. 

Construction of a 2D co-volume scheme from a triangulation

The co-volume scheme in 2D

- Consider a Delaunay triangulation $\mathcal{D}$ (“diamonds”) of $\Omega$
- Attach the DOF to the nodes of the diamond mesh $\mathcal{D}$.
  Interpret the DOF as values of the const/volume discrete solution $u^\Sigma = (u_K)_K$ on the dual (Voronoï) mesh $\mathcal{M}$
- Given $u^\Sigma$, the const/diamond discrete gradient $\vec{\nabla}^\Sigma u^\Sigma :=$ the gradient of the P1 EF reconstruction from node DOFs.

**Lemma (explicit representation of $\vec{\nabla}^\Sigma_D u^\Sigma$)**

This “co-volume discrete gradient” is exact on affine functions and is given by

$$
\vec{\nabla}^\Sigma_D u^\Sigma = \frac{1}{|D|} \left[ |K_L \cap D| (u_L - u_K) \vec{n}_{K,L} + |M_M \cap D| (u_M - u_L) \vec{n}_{L,M} + |M_K \cap D| (u_K - u_M) \vec{n}_{M,K} \right]
$$

The co-volume scheme for $(Prob)$ is formulated “pointwise” as:

$$
\forall K \in \mathcal{M} \quad u_K = \frac{1}{|K|} \sum_D |\partial K \cap D| (\vec{\nabla}^\Sigma_D u^\Sigma)^{\rho_D - 1} \cdot \vec{n}_K = f_K
$$

with $(\rho_D)_D$, $(f_K)_K$ piecewise constant discretizations of $p(\cdot)$ and $f(\cdot)$. Dirichlet BC: attach the values 0 to the nodes $x_K \in \partial \Omega$. 
Consider a Delaunay triangulation \( D \) ("diamonds") of \( \Omega \).

Attach the DOF to the nodes of the diamond mesh \( D \).

Interpret the DOF as values of the const/volume discrete solution \( u^{\Xi} = (u_K)_K \) on the dual (Voronoï) mesh \( M \).

Given \( u^{\Xi} \), the const/diamond discrete gradient \( \vec{\nabla}^{\Xi} u^{\Xi} := \) the gradient of the P1 EF reconstruction from node DOFs.

### Lemma (explicit representation of \( \vec{\nabla}^{\Xi} u^{\Xi} \))

This "co-volume discrete gradient" is exact on affine functions and is given by

\[
\vec{\nabla}^{\Xi} u^{\Xi} = \frac{|K_L \cap D|(u_L - u_K)\vec{n}_{K,L} + |M_M \cap D|(u_M - u_L)\vec{n}_{L,M} + |M_K \cap D|(u_K - u_M)\vec{n}_{M,K}}{|D|}
\]

The co-volume scheme for (Prob) is formulated "pointwise" as:

\[
\forall K \in M \quad u_K - \frac{1}{|K|} \sum_D |\partial K \cap D| (\vec{\nabla}^{\Xi} u^{\Xi})^{p_D-1} \cdot \vec{n}_K = f_K
\]

with \((p_D)_D, (f_K)_K\) piecewise constant discretizations of \( p(\cdot) \) and \( f(\cdot) \).

Dirichlet BC: attach the values 0 to the nodes \( x_K \in \partial \Omega \).
2D co-volume scheme is a Discrete Duality scheme

Lemma (the DD property)

The 2D co-volume scheme has the Discrete Duality property.

Proof:

\[ \sum_{K \in \mathcal{M}} |K| \left( \frac{1}{|K|} \sum_{D} |\partial K \cap D| \vec{F}_D \cdot \vec{n}_K \right) v_K = \]

\[ \left( \text{gather by } D; \text{ notice } \vec{F}_D \cdot \vec{n}_K \equiv -\vec{F}_D \cdot \vec{n}_L \right) \]

\[ = \sum_{D \in \mathcal{D}} |D| \vec{F}_D \cdot \frac{|K_L \cap D|(v_L - v_K)\vec{n}_{K,L} + \cdots + |M_K \cap D|(v_K - v_M)\vec{n}_{M,K}}{|D|} \]

\[ = \sum_{D \in \mathcal{D}} |D| \vec{F}_D \cdot \vec{n}_D v^\perp. \]

NB: this is a “miracle property” (unlike in mimetic schemes)!
Two “naturally defined” operators turn out to be dual.
Consistency and "Gradient scheme" point of view

Pointwise form of the discrete duality:
Let us write the DD in a formalism reminiscent of Gradient Schemes. Associate to $u^\Xi$, the function $(\prod u^\Xi)(x) := \sum_K u_K \mathbb{1}_K(x)$ on $\Omega$;
and to $\vec{F}^\Xi$, the field $(\prod \vec{F}^\Xi)(x) := \sum_D \vec{F}_D \mathbb{1}_D(x)$.
Then the DD property writes $\sim$ to the limit-conformity property of GS:
\[
\int_\Omega (\prod \text{div}^\Xi \vec{F}^\Xi)(x)(\prod \nu^\Xi)(x) + (\prod \vec{F}^\Xi)(x) \cdot (\prod \vec{\nabla} \nu^\Xi)(x) = 0.
\]
The limit-conformity can be deduced using consistency lemmas. E.g.

Lemma (consistency of discretization of divergence)

Let $\vec{\psi} \in C_\infty^\infty(\Omega)^d$. Given a sequence of meshes $\Xi$ with $\text{size}(\Xi) \to 0$, define $\vec{\psi}^\Xi$ by $\vec{\psi}_D := \frac{1}{|D|} \int_D \vec{\psi}(x) \, dx$ for each diamond $D$. Then
\[
\prod \text{div}^\Xi \vec{\psi}^\Xi \to \text{div} \vec{\psi} \text{ in } L^\infty(\Omega).
\]

Proof: straightforward using smoothness of $\vec{\psi}$.
Finer versions support Sobolev regularity of fields/functions.
NB Other consistency lemmas: for discrete functions, for gradients.
Consequences of the discrete duality

**Corollary (the weak formulation of the scheme)**

The scheme is equivalent to: find \( u^\tau \in \mathbb{R}_0^{\#\mathcal{m}} \) such that

\[
\forall v^\tau \in \mathbb{R}_0^{\#\mathcal{m}} \quad \left[ u^\tau, v^\tau \right] + \left\{ (\nabla^\tau u^\tau)^{\rho^\tau - 1}, \nabla^\tau v^\tau \right\} = \left[ f^\tau, v^\tau \right].
\]

**Corollary (discrete variational form of the scheme)**

The scheme is equivalent to: find \( u^\tau \in \mathbb{R}_0^{\#\mathcal{m}} \) that minimizes

\[
E : u^\tau \mapsto \sum_{K \in \mathcal{m}} |K| \frac{|u_K - f_K|^2}{2} + \sum_{D \in \mathcal{D}} |D| \frac{|\nabla_D u^\tau|^p}{p^\tau}.
\]

**Proposition (asymptotic compactness)**

Assume that for a sequence of meshes \( \mathcal{T}_h \) with \( h = \text{size}(\mathcal{T}_h) \to 0 \), \( u^{\mathcal{T}_h} \in \mathbb{R}_0^{\#\mathcal{T}_h} \) and \( (\nabla u^{\mathcal{T}_h})_h \), \( (\nabla^\tau u^{\mathcal{T}_h})_h \) are bounded in \( L^{1+s\text{thg}}(\Omega) \). Then there exists \( u \in W_0^{1,1}(\Omega) \) such that

\[
\nabla u^{\mathcal{T}_h} \to u \text{ in } L^1(\Omega) \text{ and } \nabla^\tau u^{\mathcal{T}_h} \to \nabla u \text{ weakly in } L^1(\Omega).
\]

**Proof:** Use DD + consistency for test fields \( \tilde{\psi} \) and \( \text{div} \tilde{\psi} \).
Consequences of the discrete duality

**Corollary (the weak formulation of the scheme)**

The scheme is equivalent to: find $u^\mathcal{F} \in \mathbb{R}_0^\# \mathcal{M}$ such that
\[
\forall v^\mathcal{F} \in \mathbb{R}_0^\# \mathcal{M} \quad \left[ u^\mathcal{F}, v^\mathcal{F} \right] + \left\{ (\nabla^\mathcal{F} u^\mathcal{F})^p^\mathcal{F} - 1, \nabla^\mathcal{F} v^\mathcal{F} \right\} = \left[ f^\mathcal{F}, v^\mathcal{F} \right].
\]

**Corollary (discrete variational form of the scheme)**

The scheme is equivalent to: find $u^\mathcal{F} \in \mathbb{R}_0^\# \mathcal{M}$ that minimizes
\[
\mathbb{E} : u^\mathcal{F} \mapsto \sum_{K \in \mathcal{M}} |K| \left| u_K - f_K \right|^2 + \sum_{D \in \mathcal{D}} |D| \left| \nabla^D u^\mathcal{F} \right|^p^\mathcal{F}.
\]

**Proposition (asymptotic compactness)**

Assume that for a sequence of meshes $\mathcal{F}_h$ with $h = \text{size}(\mathcal{F}_h) \to 0$, $u^\mathcal{F}_h \in \mathbb{R}_0^\# \mathcal{F}_h$ and $(\nabla u^\mathcal{F}_h)_h$, $(\nabla^\mathcal{F}_h u^\mathcal{F}_h)_h$ are bounded in $L^{1+sthg}(\Omega)$.

Then there exists $u \in W^{1,1}_0(\Omega)$ such that
\[
\nabla u^\mathcal{F}_h \to u \text{ in } L^1(\Omega) \quad \text{and} \quad \nabla^\mathcal{F}_h u^\mathcal{F}_h \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega).
\]

**Proof:** Use DD + consistency for test fields $\mathcal{F}$ and $\text{div} \mathcal{F}$. 
Proof of convergence of the co-volume scheme for (Prob):

Follow the lines of the Steps of the structural stability proof.

1. Take test function $u^\xi$ in the discrete weak formulation of (Prob$^\xi$).
   Young inequality $\Rightarrow$

   \[
   \|\Pi u^\xi\|^2_{L^2} + \|\Pi \nabla^\xi u^\xi \rho^\xi(\cdot)\|_{L^1}
   \]

   \[
   = \left[ u^\xi, u^\xi \right] + \left\{ (\nabla^\xi u^\xi)^{\rho^\xi - 1}, \nabla^\xi u^\xi \right\} \leq C
   \]

2. “Asymptotic compactness” property + equi-integrability $\Rightarrow$

   $\Pi u^\xi \rightharpoonup u$ in $L^2$, $\Pi \nabla^\xi u^\xi \rightharpoonup \nabla u$ in $L^1$, $(\Pi \nabla^\xi u^\xi)^{\rho_n(\cdot) - 1} \rightharpoonup \bar{F}$ in $L^1$.

3. Identification of the weak limit $\bar{F}$ of $(\Pi \nabla^\xi u^\xi)^{\rho^\xi(\cdot) - 1}$ with $(\nabla u)^{\rho(\cdot) - 1}$ relies on the “anti-weak-convergence” inequality

   \[
   \int_\Omega \bar{F} \cdot \nabla u = \int_\Omega (f - u)u \geq \lim \left[ f^\xi - u^\xi, u^\xi \right]
   \]

   \[
   = \lim \left\{ (\nabla^\xi u^\xi)^{\rho^\xi - 1}, \nabla^\xi u^\xi \right\} = \int_\Omega (\Pi \nabla^\xi u^\xi)^{\rho^\xi - 1} \cdot \Pi \nabla^\xi u^\xi.
   \]

Starting from this “anti-weak convergence” inequality, use Young measures as in the continuous case.
Proof of convergence of the co-volume scheme for \((\text{Prob})\):

Follow the lines of the Steps of the structural stability proof.

1. Take test function \(u^\xi\) in the discrete weak formulation of \((\text{Prob}^\xi)\). Young inequality \(\Rightarrow\)

\[
\|\Pi u^\xi\|^2_{L^2} + \|\bar{\nabla}^\xi u^\xi|^{p^\xi}(\cdot)\|_{L^1} \\
= \left[u^\xi, u^\xi\right] + \left\{(\bar{\nabla}^\xi u^\xi)^{p^\xi-1}, \bar{\nabla}^\xi u^\xi\right\} \leq C
\]

2. “Asymptotic compactness” property + equi-integrability \(\Rightarrow\)

\[
\Pi u^\xi \rightharpoonup u \text{ in } L^2, \quad \bar{\nabla}^\xi u^\xi \rightharpoonup \nabla u \text{ in } L^1, \quad (\bar{\nabla}^\xi u^\xi)^{p^\xi_{n\cdot}-1} \rightharpoonup \bar{F} \text{ in } L^1.
\]

3. Identification of the weak limit \(\bar{F}\) of \((\bar{\nabla}^\xi u^\xi)^{p^\xi_{n\cdot}-1}\) with \((\nabla u)^{p(\cdot)-1}\) relies on the “anti-weak-convergence” inequality

\[
\int_{\Omega} \bar{F} \cdot \nabla u = \int_{\Omega} (f - u)u \geq \lim \left[f^\xi - u^\xi, u^\xi\right] \\
= \lim \left\{(\bar{\nabla}^\xi u^\xi)^{p^\xi_{n\cdot}-1}, \bar{\nabla}^\xi u^\xi\right\} \equiv \int_{\Omega} (\bar{\Pi} \bar{\nabla}^\xi u^\xi)^{p^\xi_{\cdot}-1} \cdot \bar{\Pi} \bar{\nabla}^\xi u^\xi.
\]

Starting from this “anti-weak convergence” inequality, use Young measures as in the continuous case.
Proof of convergence of the co-volume scheme for $(\text{Prob})$:

Follow the lines of the Steps of the structural stability proof.

1. Take test function $\bar{u}^x$ in the discrete weak formulation of $(\text{Prob}^x)$. Young inequality $\Rightarrow$

   \[
   \|\bar{\Pi}u^x\|_{L^2}^2 + \|\bar{\Pi}\bar{\nabla}^x u^x\|_{L^1}^{\rho^x(\cdot)}
   = \left[\bar{u}^x,\bar{u}^x\right] + \left\{\bar{\nabla}^x u^x\right\}^{\rho^x-1} \leq C
   \]

2. “Asymptotic compactness” property + equi-integrability $\Rightarrow$

   \[
   \bar{\Pi}u^x \rightharpoonup u \text{ in } L^2, \quad \bar{\Pi}\bar{\nabla}^x u^x \rightharpoonup \bar{\nabla} u \text{ in } L^1, \quad (\bar{\Pi}\bar{\nabla}^x u^x)^{\rho^x-1} \rightharpoonup \bar{F} \text{ in } L^1.
   \]

3. Identification of the weak limit $\bar{F}$ of $(\bar{\Pi}\bar{\nabla}^x u^x)^{\rho^x(\cdot)-1}$ with $(\bar{\nabla} u)^{\rho(\cdot)-1}$ relies on the “anti-weak-convergence” inequality

   \[
   \int_{\Omega} \bar{F} \cdot \bar{\nabla} u = \int_{\Omega} (f - u)u \geq \lim \left[ f^x - u^x, u^x \right]
   \]

   \[
   = \lim \left\{ (\bar{\nabla}^x u^x)^{\rho^x-1}, \bar{\nabla}^x u^x \right\} \equiv \int_{\Omega} (\bar{\Pi}\bar{\nabla}^x u^x)^{\rho^x-1} \cdot \bar{\Pi}\bar{\nabla}^x u^x.
   \]

Starting from this “anti-weak convergence” inequality, use Young measures as in the continuous case.
GENERALIZED CO-VOLUME SCHEME ON DONALD (MEDIAN DUAL) MESH
A gradient reconstruction formula on general polygons

Let $\Theta$ be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $D \subset \Theta$ be a polygon. Introduce the vertices $x_i^*$, $i = 1, \ldots, \ell$ (numbered counter-clockwise w.r.t. the orientation of $\Theta$ induced by $\vec{n}$).

Denote the area of $\sigma$ by $|\sigma|$, we have $|\sigma| = \sum_{i=1}^\ell |D_{i,i+1}|$ (sub-areas are signed).

Let $x_D^* \in \Theta$ be a distinguished point.

Take $x_{i,i+1}^*$ the midpoints of the edges.

**Lemma**

For all $\vec{r} \parallel \Theta$, 
$$\vec{r} = \frac{1}{|D|} \sum_{i=1}^\ell (\vec{r} \cdot x_i^* x_{i+1}^*) \left[ \vec{n} \times x_i^* x_{i+1}^* \right] = \frac{2}{|D|} \sum_{i=1}^\ell |D_{i,i+1}| (\vec{r} \cdot \vec{e}_{i,i+1}) \vec{e}_{i,i+1}' ,$$

where $\vec{e}_{i,i+1} := \frac{x_i^* x_{i+1}^*}{\|x_i^* x_{i+1}^*\|}$ and $\vec{e}_{i,i+1}' := \left[ \vec{n} \times x_i^* x_{i+1}^* \right] / \| \vec{n} \times x_i^* x_{i+1}^* \|$. 

The formula can be derived from the “magical formula” of Droniou, Eymard. 

NB: if $\ell > 3$, this is one of infinitely many affine reconstruction formulas!
The formula (cont'd)

**Corollary (Consistency of the gradient reconstruction)**

Take \((w^*_i)_{i=1}^\ell \subset \mathbb{R}, w^*_\ell+1 := w^*_1\). If \(w_i^*\) are the values of an affine function \(w\) at the vertices \(x_i^*\) of the polygon \(\sigma\), then

\[
\vec{\nabla} w = \frac{1}{|D|} \sum_{i=1}^\ell \left(w^*_{i+1} - w^*_i\right) \left[ \vec{n} \times \overrightarrow{x^*_D x^*_{i,i+1}} \right] \equiv 2 \sum_{i=1}^\ell \frac{|D_{i,i+1}|}{|D|} \frac{w^*_{i+1} - w^*_i}{d_{i,i+1}} \vec{e}'_{i,i+1},
\]

where \(d_{i,i+1} := \|x^*_i x^*_{i+1}\|\) and \(\vec{e}'_{i,i+1} := \frac{\left[ \vec{n} \times \overrightarrow{x^*_D x^*_{i,i+1}} \right]}{\|\vec{n} \times \overrightarrow{x^*_D x^*_{i,i+1}}\|}\)
2D co-volume scheme: generalization to Donald (median dual) mesh.

The idea of the 2D co-volume scheme on the Voronoï mesh dual to a triangulation was:
reconstruct the discrete gradient on a given triangulation (affine per triangle) and then write the FV scheme on the dual mesh.
Thus the triangles were “diamonds” of the scheme.
We can replace the triangulation $\mathcal{D}$ by an arbitrary polygonal partition.

We use the “median dual mesh” ($\equiv$“Donald dual mesh”) for $\mathcal{M}$. 
Generalization of the co-volume scheme

Generalized co-volume scheme in 2D (cont\textsuperscript{d})

Here, we do not necessarily need $\sigma$’s (the “diamonds”) to be triangles; any polygon would do (in particular, quadrilaterals are welcome).

We associate to each “diamond” a value of the discrete gradient (reconstructed from the formula of the Corollary).

We associate to the mesh the standard FV discrete divergence operator.

\textbf{Theorem (DD property for 2D co-volume scheme on Donald mesh)}

These discrete gradient and divergence operators are linked by the discrete duality (integration-by-parts) formula.
Generalization of the co-volume scheme

Generalized co-volume scheme in 2D (cont^d)

Here, we do not necessarily need σ’s (the “diamonds”) to be triangles; any polygon would do (in particular, quadrilaterals are welcome).

We associate to each “diamond” a value of the discrete gradient (reconstructed from the formula of the Corollary).

We associate to the mesh the standard FV discrete divergence operator.

**Theorem (DD property for 2D co-volume scheme on Donald mesh)**

These discrete gradient and divergence operators are linked by the discrete duality (integration-by-parts) formula.
To be continued... towards 2D and 3D DDFV schemes

2D and 3D “DDFV schemes” are awaiting us tomorrow!