Formulation and analysis of Discrete Duality Finite Volume schemes. Part I. Co-volume schemes for the $p(x)$-laplacian.

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based on joint works with
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Plan of the talk

1. **Structural stability for a nonlinear diffusion PDE**
   - A model PDE problem
   - Structural stability
2. **Generalities on FV schemes and Discrete Duality**
   - Principles of construction of Finite Volume schemes
   - Gradient approximation on diamonds
   - Discrete Duality
3. **The classical co-volume scheme (=lumped P1 FE)**
   - Construction of a 2D co-volume scheme from a triangulation
   - Discrete Duality property and its consequences
   - Convergence of the co-volume scheme for the $p(x)$-laplacian
4. **Co-volume scheme on general Donald (median dual) mesh**
   - A gradient reconstruction formula on general polygons
   - Generalized co-volume scheme, consistency and Discrete Duality

Structural stability for a nonlinear diffusion PDE

A nonlinear diffusion PDE

Consider the problem (Prob):

$$\begin{cases} u - \Delta_{p(x)} u = f \ \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

- Ω: a bounded Lipschitz domain of $\mathbb{R}^d$, $d = 1, 2, 3$
- f: an L^∞ (for simplicity) source term
- the $p(x)$-laplacian operator $\Delta_{p(x)} u := \operatorname{div} (\vec{\nabla} u)^{p(x)-1}$, where $(\vec{\xi})^{p(x)-1} := |\vec{\xi}|^{p(x)-2} \vec{\xi}$ is the “french power” of $\vec{\xi} \in \mathbb{R}^d$.
  Notation: $\vec{\mathcal{F}}[u] := A(x, \vec{\nabla} u)$ is the flux.
  The equation writes $u - \operatorname{div} \vec{\mathcal{F}}[u] = f$.
- p: a regular function on $\overline{\Omega}$ with values in $[p_-, p_+] \subset (1, +\infty)$

Applications: e.g., in image processing.
Part of Navier-Stokes models for complex fluids.

Notion of solution:
$u \in W_0^{1,1}(\Omega)$ with $|\nabla u(\cdot)|^{p(\cdot)} \in L^1(\Omega)$.
Weak formulation (multiply by a test function, integrate by parts).

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Structural stability

Basic numerical analysis ? Try to mimic structural stability analysis !

Structural stability: consider $(f_n)_n$, $(p_n)_n$ converging to f, p resp. Does the associated solutions' sequence $(u_n)_n$ of $(Prob_n)$ converge to a solution u of the limit problem?

A way to prove structural stability:

1. Prove uniform estimates on the sequence of solutions $(u_n)_n$
2. Create an accumulation point u for the sequence (compactness arguments). Usually, one starts with a weak convergence.
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in the formulation, including nonlinearities. I.e. for all $v \in C_0^\infty(\Omega)$, can we let $n \to \infty$ in

$$\int_\Omega u_n v + (\vec{\nabla} u_n)^{p_n(x)-1} \cdot \vec{\nabla} v = \int_\Omega f_n v \quad ?$$

The question boils down to:
Is the weak limit of $(\vec{\nabla} u_n)^{p_n(\cdot)-1}$ equal to $(\vec{\nabla} u)^{p(\cdot)-1}$?
Answer : exploit weak PDE formulations both for u_n and for u...

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Sketch of structural stability proof for (*Prob*)

Assume $f_n \to f$ say, in L^2. Assume $p_n \to p$ a.e. on Ω. Follow the steps:

1. Take test function u_n in weak formulation of (*Prob*) + Young inequality $\Rightarrow$

$$\|u_n\|_{L^2}^2 + \left\||\vec\nabla u_n|^{p_n(\cdot)}\right\|_{L^1} = \int_\Omega |u_n|^2 + |\vec\nabla u_n|^{p_n(x)} \le C$$

2. Weak compactness + equi-integrability arguments $\Rightarrow$

$$u_n \rightharpoonup u \;\text{ in } L^2, \quad \vec\nabla u_n \rightharpoonup \vec{\mathcal{G}} \;\text{ in } L^1, \quad (\vec\nabla u_n)^{p_n(\cdot)-1} \rightharpoonup \vec{\mathcal{F}} \;\text{ in } L^1.$$

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Represent $\vec\nabla u = L^1weak\text{-lim}\, \vec\nabla u_n$ by a Young measure $(\nu_x(\vec\cdot))_x$ ($\Rightarrow$ this also permits to represent $\vec{\mathcal{F}}$ and $\vec{\mathcal{F}} \cdot \vec\nabla u$). "Anti-weak" ineq. $\Rightarrow$

$$\int_\Omega \int_{(R^d)^2} \left((\vec\lambda)^{p_n(\cdot)-1} - (\vec\mu)^{p_n(\cdot)-1}\right) \cdot (\vec\lambda - \vec\mu)\, d\nu_x(\vec\lambda) d\nu_x(\vec\mu) \le 0.$$

From the monotonicity, deduce that the Young measure is concentrated ("multi-valued limit is single-valued") and thus $\vec\nabla u_n \to \vec\nabla u$ pointwise .

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GENERALITIES ON FINITE VOLUME SCHEMES. DISCRETE DUALITY.

Generalities on FV schemes

We use Finite Volumes (FV, for short) for space discretization of (*Prob*) .
Generally, let us think of discretizing an elliptic equation of the kind

$$u - \operatorname{div} \vec{\mathcal{F}} = f, \qquad \mathcal{F} = A(x, \vec{\nabla} u).$$

The principles for FV approximation of such equations are the following:

- A partition of the space domain Ω into "volumes" K is given; the partition is called "mesh" and denoted by $\mathfrak{T}$.
- An unknown u_K is associated to each volume (usually regarded as the value at a "center" point of the volume, denoted x_K); the whole set of the unknowns is called "discrete solution" and denoted by $u^{\mathfrak{T}}$.
- If K, L are "neighbours" (adjacent volumes), the divided differences $\frac{u_L - u_K}{d_{KL}}$ are used to "reconstruct" the "discrete gradient" $\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}$ of $u^{\mathfrak{T}}$.

  NB: In the schemes we think of, the reconstruction is done "by hands" . We do not solve any equations to compute the values of $\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}$ from those of $u^{\mathfrak{T}}$, but fix *ad hoc* formulas for $\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}$ in terms of $u^{\mathfrak{T}}$.

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Generalities on FV schemes (cont[d])

- Each couple of neighbour volumes possesses a part of the common border , called "interface" and denoted by $K|L$. Combining the values u_K, u_L and the values of the reconstructed discrete gradient $\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}$ near $K|L$, we produce the discretization $\vec{\mathcal{F}}_{K|L}$ of the flux $\vec{\mathcal{F}} = A(x, \vec{\nabla} u)$ on $K|L$.
- In two steps, the PDE is replaced by a set of algebraic equations:
  - Firstly, the PDE is "projected on the mesh" . One integrates the continuous equation on K; the Green-Gauss formula is used to reduce $\int_K \operatorname{div} \vec{\mathcal{F}}$ to $\int_{\partial K} \vec{\mathcal{F}} \cdot \vec{n}_K$.

    This yields a discrete system of equalities, one per volume.
  - And secondly, these equalities are approximated by replacing $u|_K$ with the unknown u_K and replacing $\vec{\mathcal{F}}|_{K|L}$ with the expression of $\vec{\mathcal{F}}_{K|L} = A(x_{K|L}, \vec{\nabla}_{K|L} u^{\mathfrak{T}})$.

  NB: e.g. in TPFA schemes all components of $\vec{\nabla}_{K|L} u^{\mathfrak{T}}$ need not be reconstructed, since only the normal fluxes $\vec{\mathcal{F}}_{K|L} \cdot \vec{n}_K$ appear.
  However, for $(Prob)$ or for linear anisotropic diffusion $\mathbb{K}\nabla u$ (cf. R.Eymard's lectures) : one should(?) define (per interface) $\vec{\nabla}_{K|L} u^{\mathfrak{T}}$.
- As a result, we obtain a closed system of algebraic equations which (hopefully...) can be solved , often in an approximate way.
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Volumes and diamonds. Discrete gradient and divergence operators.

For representing discrete gradients , many FV schemes use "diamond mesh".
A diamond $\widehat{\kappa\mathcal{L}}$ is a neighbourhood of interface $\kappa|\mathcal{L}$.
Notation ($\sim$ Gradient Schemes) useful to formulate "Discrete Duality":

- Consider $\mathfrak{T} = (\mathcal{D}, \mathcal{M})$ consisting of two partitions of Ω:
  volumes κ (mesh $\mathcal{M}$) and diamonds D (diamond mesh $\mathcal{D}$) .
- Consider two kinds of objects:
  discrete functions $u^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{M}}$ and discrete fields $\vec{\mathcal{F}}^{\mathfrak{T}} \in \left(\mathbb{R}^{\#\mathfrak{D}}\right)^d$:

  $$u^{\mathfrak{T}} = \left(u_K\right)_{K\in\mathfrak{M}}, \quad \vec{\mathcal{F}}^{\mathfrak{T}} = \left(\vec{\mathcal{F}}_D\right)_{D\in\mathfrak{D}}.$$

  They are lifted to Ω as piecewise constant (per volume / per diamond) functions $\Pi u^{\mathfrak{T}}$ and fields $\vec{\Pi}\vec{\mathcal{F}}$ (cf. Gradient Schemes).
- Discrete gradient operator: some "per diamond reconstruction"

  $$\vec{\nabla}^{\mathfrak{T}} : \mathbb{R}^{\#\mathfrak{M}} \longrightarrow \left(\mathbb{R}^{\#\mathfrak{D}}\right)^d, \quad \vec{\nabla}_D u^{\mathfrak{T}} := \text{Formula}\big(u_K \text{ for } \kappa \text{ near } D\big)$$

- Discrete divergence operator: per volume integration + Green-Gauss:

  $$\mathrm{div}^{\mathfrak{T}} : \left(\mathbb{R}^{\#\mathfrak{D}}\right)^d \longrightarrow \mathbb{R}^{\#\mathfrak{M}}, \quad \mathrm{div}_K \vec{\mathcal{F}}^{\mathfrak{T}} := \frac{1}{|\kappa|} \sum_D |\partial\kappa \cap D| \vec{\mathcal{F}}_D \cdot \vec{n}_K .$$

Scalar products of discrete functions/fields and Discrete Duality

- inner product on the space of discrete functions:

$$\left[\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] := \sum_{K\in\mathfrak{M}} |K| u_K v_K$$

- inner product on the space of discrete fields:

$$\left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\!\right\} := \sum_{D\in\mathfrak{D}} |D| \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D$$

- Discrete Duality (DD) property holds if $\mathrm{div}^{\mathfrak{T}}$ and $-\vec{\nabla}^{\mathfrak{T}}$ fulfill :

$$\forall u^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{M}} \quad \forall \vec{\mathcal{F}}^{\mathfrak{T}} \in \mathbb{R}^{\#\mathfrak{D}}$$

$$\left[\!\left[u^{\mathfrak{T}}, \mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} \right]\!\right] + \left\{\!\!\left\{ \vec{\nabla} u^{\mathfrak{T}}, \vec{\mathcal{F}}^{\mathfrak{T}} \right\}\!\!\right\} = 0 \;\; \left(\text{or } = \left\langle\!\!\left\langle \ldots, \ldots \right\rangle\!\!\right\rangle_{\partial\Omega} \right).$$

Use of DD property: it permits to mimic the structural stability arguments for proving convergence of the scheme

Additional ingredient: (natural) consistency properties

- for the discrete gradient applied to projections of test functions
- for the discrete divergence applied to projection of test fields.

Classical co-volume scheme aka lumped P1 Finite Element scheme

The co-volume scheme in 2D

- Consider a Delaunay triangulation $\mathcal{D}$ ("diamonds") of Ω
- Attach the DOF to the nodes of the diamond mesh $\mathcal{D}$.
  Interpret the DOF as values of the const/volume discrete solution $u^{\mathfrak{T}} = (u_K)_K$ on the dual (Voronoï) mesh $\mathcal{M}$
- Given $u^{\mathfrak{T}}$, the const/diamond discrete gradient $\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}} :=$ the gradient of the P1 EF reconstruction from node DOFs .

Lemma (explicit representation of $\vec{\nabla}_D u^{\mathfrak{T}}$)

This "co-volume discrete gradient" is exact on affine functions and is given by

$$\vec{\nabla}_D u^{\mathfrak{T}} = \frac{|K\!L \cap D|(u_L - u_K)\vec{n}_{K,L} + |L\!M \cap D|(u_M - u_L)\vec{n}_{L,M} + |M\!K \cap D|(u_K - u_M)\vec{n}_{M,K}}{|D|}$$

The co-volume scheme for (*Prob*) is formulated "pointwise" as :

$$\forall K \in \mathcal{M} \quad u_K - \frac{1}{|K|}\sum_D |\partial K \cap D| \, (\vec{\nabla}_D u^{\mathfrak{T}})^{p_D - 1} \cdot \vec{n}_K = f_K$$

with $(p_D)_D$, $(f_K)_K$ piecewise constant discretizations of $p(\cdot)$ and $f(\cdot)$.
Dirichlet BC: attach the values 0 to the nodes $x_K \in \partial\Omega$.

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2D co-volume scheme is a Discrete Duality scheme

Lemma (the DD property)

The 2D co-volume scheme has the Discrete Duality property.

Proof:

$$\sum_{K\in\mathfrak{M}} |K| \Big(\frac{1}{|K|} \sum_{D} |\partial K \cap D| \vec{\mathcal{F}}_D \cdot \vec{n}_K \Big) v_K =$$

$$\Big(\text{gather by } D\text{; notice } \vec{\mathcal{F}}_D \cdot \vec{n}_K \equiv -\vec{\mathcal{F}}_D \cdot \vec{n}_L \Big)$$

$$= \sum_{D\in\mathfrak{D}} |D| \vec{\mathcal{F}}_D \cdot \frac{|K|L \cap D|(v_L - v_K)\vec{n}_{K,L} + \cdots + |M|K \cap D|(v_K - v_M)\vec{n}_{M,K}}{|D|}$$

$$= \sum_{D\in\mathfrak{D}} |D| \vec{\mathcal{F}}_D \cdot \vec{\nabla}_D v^{\mathfrak{T}}.$$

NB: this is a “miracle property” (unlike in mimetic schemes) !
Two “naturally defined” operators **turn out** to be dual.

Consistency and "Gradient scheme" point of view

Pointwise form of the discrete duality:

Let us write the DD in a formalism reminiscent of Gradient Schemes.

Associate to $u^{\mathfrak{T}}$, the function $(\Pi u^{\mathfrak{T}})(x) := \sum_K u_K \mathbb{1}_K(x)$ on Ω;

and to $\vec{\mathcal{F}}^{\mathfrak{T}}$, the field $(\vec{\Pi}\vec{\mathcal{F}}^{\mathfrak{T}})(x) := \sum_D \vec{\mathcal{F}}_D \mathbb{1}_D(x)$.

Then the DD property writes $\sim$ to the limit-conformity property of GS:

$$\int_\Omega (\Pi \mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}})(x)(\Pi v^{\mathfrak{T}})(x) + (\vec{\Pi}\vec{\mathcal{F}}^{\mathfrak{T}})(x)\cdot(\vec{\Pi}\vec{\nabla} v^{\mathfrak{T}})(x) = 0.$$

The limit-conformity can be deduced using consistency lemmas. E.g.

Lemma (consistency of discretization of divergence)

Let $\vec{\psi} \in C_c^\infty(\Omega)^d$. Given a sequence of meshes $\mathfrak{T}$ with size$(\mathfrak{T}) \to 0$, define $\vec{\psi}^{\mathfrak{T}}$ by $\vec{\psi}_D := \frac{1}{|D|}\int_D \vec{\psi}(x)\,dx$ for each diamond D. Then

$$\Pi \mathrm{div}^{\mathfrak{T}} \vec{\psi}^{\mathfrak{T}} \to \mathrm{div}\,\vec{\psi} \ \text{ in } L^\infty(\Omega).$$

Proof: straightforward using smoothness of $\vec{\psi}$.

Finer versions support Sobolev regularity of fields/functions.

NB Other consistency lemmas: for discrete functions, for gradients.

Consequences of the discrete duality

Corollary (the weak formulation of the scheme)

The scheme is equivalent to: find $u^{\mathfrak{T}} \in \mathbb{R}_0^{\#\mathfrak{M}}$ *such that*

$$\forall v^{\mathfrak{T}} \in \mathbb{R}_0^{\#\mathfrak{M}} \quad \left[\!\left[u^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] + \left\{\!\left\{ (\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}})^{p^{\mathfrak{T}}-1}, \nabla^{\mathfrak{T}} v^{\mathfrak{T}} \right\}\!\right\} = \left[\!\left[f^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right].$$

Corollary (discrete variational form of the scheme)

The scheme is equivalent to: find $u^{\mathfrak{T}} \in \mathbb{R}_0^{\#\mathfrak{M}}$ *that minimizes*

$$\mathbb{E} : u^{\mathfrak{T}} \mapsto \sum_{K\in\mathfrak{M}} |K| \frac{|u_K - f_K|^2}{2} + \sum_{D\in\mathfrak{D}} |D| \frac{|\vec{\nabla}_D u^{\mathfrak{T}}|^{p^{\mathfrak{T}}}}{p^{\mathfrak{T}}}.$$

Proposition (asymptotic compactness)

Assume that for a sequence of meshes $\mathfrak{T}_h$ *with* $h = size(\mathfrak{T}_h) \to 0$, $u^{\mathfrak{T}_h} \in \mathbb{R}_0^{\#\mathfrak{T}_h}$ *and* $(\Pi u^{\mathfrak{T}_h})_h$, $(\vec{\Pi}\vec{\nabla}^{\mathfrak{T}_h} u^{\mathfrak{T}_h})_h$ *are bounded in* $L^{1+stgh}(\Omega)$.
Then there exists $u \in W_0^{1,1}(\Omega)$ *such that*

$$\Pi u^{\mathfrak{T}_h} \to u \text{ in } L^1(\Omega) \text{ and } \vec{\Pi}\vec{\nabla}^{\mathfrak{T}_h} u^{\mathfrak{T}_h} \rightharpoonup \vec{\nabla} u \text{ weakly in } L^1(\Omega).$$

Proof: Use DD + consistency for test fields $\vec{\psi}$ and div$\vec{\psi}$.

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Proof of convergence of the co-volume scheme for $(Prob)$:

Follow the lines of the Steps of the structural stability proof.

1. Take test function $u^{\mathfrak{T}}$ in the discrete weak formulation of $(Prob^{\mathfrak{T}})$. Young inequality $\Rightarrow$

$$\|\Pi u^{\mathfrak{T}}\|_{L^2}^2 + \left\| |\vec{\Pi}\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}}|^{p^{\mathfrak{T}}(\cdot)} \right\|_{L^1}$$
$$= \left[\!\left[u^{\mathfrak{T}}, u^{\mathfrak{T}} \right]\!\right] + \left\{\!\!\left\{ (\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}})^{p^{\mathfrak{T}}-1}, \vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}} \right\}\!\!\right\} \leq C$$

2. "Asymptotic compactness" property + equi-integrability $\Rightarrow$

$$\Pi u^{\mathfrak{T}} \rightharpoonup u \text{ in } L^2, \quad \vec{\Pi}\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}} \rightharpoonup \vec{\nabla} u \text{ in } L^1, \quad (\vec{\Pi}\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}})^{p_n(\cdot)-1} \rightharpoonup \vec{\mathcal{F}} \text{ in } L^1.$$

3. Identification of the weak limit $\vec{\mathcal{F}}$ of $(\vec{\Pi}\vec{\nabla}^{\mathfrak{T}} u^{\mathfrak{T}})^{p^{\mathfrak{T}}(\cdot)-1}$ with $(\vec{\nabla} u)^{p(\cdot)-1}$ relies on the "anti-weak-convergence" inequality

$$\int_\Omega \vec{\mathcal{F}} \cdot \vec{\nabla} u = \int_\Omega (f-u)u \geq \overline{\lim} \left[\!\left[f^{\mathfrak{T}} - u^{\mathfrak{T}}, u^{\mathfrak{T}} \right]\!\right]$$
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Starting from this "anti-weak convergence" inequality, use Young measures as in the continuous case .

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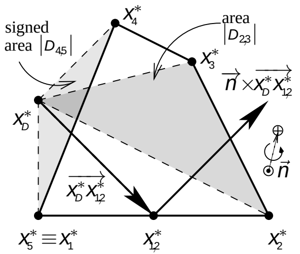
Generalized co-volume scheme on Donald (median dual) mesh

A gradient reconstruction formula

Polygon $D \subset \Theta$, oriented by $\vec{n} \perp \Theta$



Let Θ be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $D \subset \Theta$ be a polygon.

Introduce the vertices x_i^*, $i = 1, \ldots, \ell$ (numbered counter-clockwise w.r.t. the orientation of Θ induced by $\vec{n}$).

Denote the area of σ by $|D|$, we have $|D| = \sum_{i=1}^{\ell} |D_{i,i+1}|$ (sub-areas are signed).

Let $x_D^* \in \Theta$ be a distinguished point .

Take $x_{i,i+1}^*$ the midpoints of the edges .

Lemma

For all $\vec{r} \parallel \Theta$, $\vec{r} = \frac{1}{|D|} \sum_{i=1}^{\ell} (\vec{r} \cdot \overrightarrow{x_i^* x_{i+1}^*}) \, [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] \equiv \frac{2}{|D|} \sum_{i=1}^{\ell} |D_{i,i+1}| (\vec{r} \cdot \vec{e}_{i,i+1}) \vec{e}\,'_{i,i+1}$,

where $\vec{e}_{i,i+1} := \overrightarrow{x_i^* x_{i+1}^*} / \| \overrightarrow{x_i^* x_{i+1}^*} \|$ *and* $\vec{e}\,'_{i,i+1} := [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] / \| [\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}] \|$.

The formula can be derived from the “magical formula” of Droniou, Eymard .
NB: if $\ell > 3$, this is one of infinitely many affine reconstruction formulas !

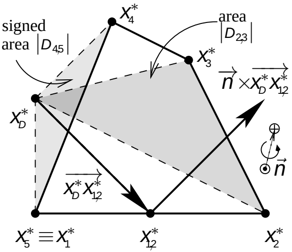
The formula (cont^{d})

Corollary (Consistency of the gradient reconstruction)

Take $(w_i^*)_{i=1}^{\ell} \subset \mathbb{R}$, $w_{\ell+1}^* := w_1^*$. *If* w_i^* *are the values of an affine function* w *at the vertices* x_i^* *of the polygon* σ, *then*

$$\vec{\nabla} w = \frac{1}{|D|} \sum_{i=1}^{\ell} (w_{i+1}^* - w_i^*) \left[\vec{n} \times \overrightarrow{x_D^* x_{i,i+1}^*}\right] \equiv \frac{2}{|D|} \sum_{i=1}^{\ell} |D_{i,i+1}| \frac{w_{i+1}^* - w_i^*}{d_{i,i+1}} \vec{e}\,'_{i,i+1},$$

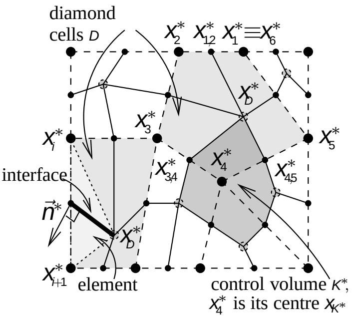
where $d_{i,i+1} := \|\overrightarrow{x_i^* x_{i+1}^*}\|$ *and* $\vec{e}\,'_{i,i+1} := \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right] / \left\| \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right] \right\|$



2D co-volume scheme: generalization to Donald (median dual) mesh.

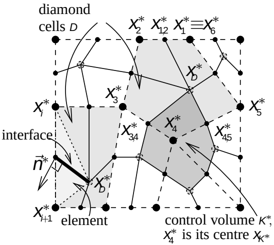
The idea of the 2D co-volume scheme on the Voronoï mesh dual to a triangulation was:
reconstruct the discrete gradient on a given triangulation (affine per triangle) and then write the FV scheme on the dual mesh .

Thus the triangles were “diamonds” of the scheme.
We can replace the triangulation $\mathcal{D}$ by an arbitrary polygonal partition.



We use the “median dual mesh” (≡“Donald dual mesh”) for $\mathcal{M}$.

Generalized co-volume scheme in 2D (cont^{d})



Here, we do not necessarily need σ's (the "diamonds") to be triangles; any polygon would do (in particular, quadrilaterals are welcome).

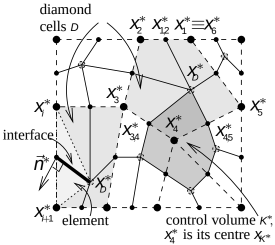
We associate to each "diamond" a value of the discrete gradient (reconstructed from the formula of the Corollary) .

We associate to the mesh the standard FV discrete divergence operator.

Theorem (DD property for $2D$ co-volume scheme on Donald mesh)

These discrete gradient and divergence operators are linked by the discrete duality (integration-by-parts) formula.

Generalized co-volume scheme in 2D (cont^d)



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To be continued... towards 2D and 3D DDFV schemes

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2D and 3D "DDFV schemes"

are awaiting us tomorrow !