C. R. Acad. Sci. Paris, Ser. I 348 (2010) 759–762

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Renormalized solutions of the fractional Laplace equation

Solutions renormalisées de l'équation de Laplace fractionnaire

Nathaël Alibaud [a,b], Boris Andreianov [a], Mostafa Bendahmane [c]

[a] *Laboratoire de mathématiques, UMR CNRS 6623, 16, route de Gray, 25030 Besançon cedex, France*
[b] *École nationale supérieure de mécanique et des microtechniques, 26 chemin de l'Épitaphe, 25030 Besançon cedex, France*
[c] *Institut de mathématiques de Bordeaux, université Bordeaux 2, 3ter, place de la Victoire, 33076 Bordeaux, France*

ARTICLE INFO

Article history:
Received 11 May 2010
Accepted 31 May 2010
Available online 23 June 2010

Presented by Jean-Michel Bony

ABSTRACT

We define renormalized solutions for the problems of the kind $\beta(u) + (-\Delta)^{s/2}u \ni f$ in $\mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$. Here β is a maximal monotone graph in $\mathbb{R}$, and $(-\Delta)^{s/2}$, $s \in (0, 2)$, is the fractional Laplace operator which is a particular case of Lévy diffusions. We prove well-posedness in the framework of renormalized solutions. Then the Cauchy problem for the associated evolution equations can be solved using the Crandall–Liggett semigroup technique.

RÉSUMÉ

Nous introduisons une notion de solution renormalisée pour les problèmes du genre $\beta(u) + (-\Delta)^{s/2}u \ni f$ in $\mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$. Ici β est un graphe maximal monotone dans $\mathbb{R}$, et $(-\Delta)^{s/2}$, $s \in (0, 2)$, est l'opérateur de Laplace fractionnaire qui est un représentant type des diffusions de Lévy. Nous montrons que le problème est bien posé dans le cadre des solutions renormalisées. Le problème de Cauchy pour l'équation d'évolution associée peut alors se traiter par les techniques de semigroupes.

1. Introduction

Beyond the variational framework, the idea of truncation test functions led to well-posedness theories for L^1 data (and even for well-chosen measure data) for many classical elliptic and parabolic equations (Laplace equation, heat equation, porous medium and fast diffusion equations, p-Laplacian and general Leray–Lions problems, viscous conservation laws, etc.). The adequate notions of solutions are the renormalized solutions, introduced in an unpublished work of Lions and Murat (cf. [6,4]), and the entropy solutions introduced by Bénilan et al. in [1]. A related notion is Stampaccia's duality solutions (see [5] and references therein). In [5], Karlsen et al. develop a duality solutions theory for equations involving fractional diffusion elliptic operators, of which the fractional Laplace operator $(-\Delta)^{s/2}$, $s \in (0, 2)$, is the prototype. Here we give the basics of a theory of renormalized solutions for such equations. While the function $(-\Delta)^{s/2}u$ is easily defined in terms of multiplication of the Fourier transform $(\mathcal{F}u)(\xi)$ by $|\xi|^s$, we look at the more general definition in terms of integral Lévy operators, namely, $(-\Delta)^{s/2} = \mathcal{L}$ with

E-mail addresses: nathael.alibaud@ens2m.fr (N. Alibaud), bandreia@univ-fcomte.fr (B. Andreianov), mostafa_bendahmane@yahoo.fr (M. Bendahmane).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.05.006

$$(\mathcal{L}u)(x) = -\int_{\mathbb{R}^n} \big[u(x+z) - u(x) - z \cdot \nabla u(x) \mathbb{1}_{\{|z|<1\}}\big]\, \mathrm{d}\mu(z) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1}$$

where $\mathrm{d}\mu(z)$ is the measure with the density $G_s|z|^{-(n+s)}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, G_s being a normalization constant. For $0<s<1$, the term $\frac{z}{|z|^{n+s}} \cdot \nabla u(x)$ in (1) can be omitted, since in this case it does not influence the integrability of the expression in brackets, and its integral equals zero.

Formula (1) with the measure $\mathrm{d}\mu$ specified above is known as the Lévy–Khintchine formula. In view of (1), the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions, more exactly, we take the assumptions

$$\mathrm{d}\mu(z) = g(z)\,\mathrm{d}z \quad \text{with } g \geqslant 0, \qquad g(z) = g(-z) \quad \text{for all } z \in \mathbb{R}^n, \quad \text{and} \quad \int_{\mathbb{R}^n} \min\big\{1, |z|^2\big\} g(z)\,\mathrm{d}z < +\infty. \tag{2}$$

Fix a Lévy operator given by (1) with a measure $\mathrm{d}\mu$ as in (2). Fix a maximal monotone graph β on $\mathbb{R}$ with $0 \in \beta(0)$; we make the simplifying assumptions $\mathrm{Dom}\,\beta = \mathbb{R}$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$b + \mathcal{L}u = f, \quad b \in \beta(u), \tag{3}$$

with data $f \in L^1(\mathbb{R}^n)$. Our results are: existence and uniqueness of a renormalized solution, and the L^1-contraction and comparison property for solutions u and $\hat{u}$ of (3) associated with data f and $\hat{f}$, resp.:

$$\int_{\mathbb{R}^n} (b-\hat{b})^+ \leqslant \int_{[b\neq\hat{b}]} \mathrm{sign}^+(b-\hat{b})(f-\hat{f}) + \int_{[b=\hat{b}]} (f-\hat{f})^+ =: [b-\hat{b}, f-\hat{f}]^+_{L^1(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^n} (f-\hat{f})^+. \tag{4}$$

Precise definition and result are stated in Section 2, along with some basic comments explaining the definitions. In Section 3, the proof is sketched. We refer to a forthcoming paper for technical details and generalizations.

Notice that, on the basis of property (4) and of the existence result for (3), one defines an $m-T$-accretive operator $A_{\beta,\mathcal{L}}$ in $L^1(\mathbb{R}^n)$ associated with the formal expression $b \mapsto \mathcal{L}\beta^{-1}(b)$. It is easy to see that $A_{\beta,\mathcal{L}}$ is densely defined; by the standard nonlinear semigroup techniques (see e.g. [2]), it follows that there exists a unique mild and integral solution $b(\cdot) \in C([0,+\infty); L^1(\mathbb{R}^n))$ of the associated abstract evolution problem

$$\frac{\mathrm{d}}{\mathrm{d}t} b + A_{\beta,\mathcal{L}} b \ni f, \quad b(0) = b_0 \quad \big(\text{with data } b_0 \in L^1\big(\mathbb{R}^n\big),\ f \in L^1_{loc}\big([0,+\infty); L^1\big(\mathbb{R}^n\big)\big)\big).$$

E.g. in the case where β is the identity graph and $\mathcal{L}$ is the fractional Laplacian $(-\Delta)^{s/2}$, the function $u(t,x) \equiv b(t)(x)$ is a formal solution of the fractional heat equation $\partial_t u + (-\Delta)^{s/2} u = f$, $u|_{t=0} = b_0$. One can show that under the natural integrability assumptions on f and u_0, the function $u(t,x) \equiv b(t)(x)$ is also the $L^2(0,T; H^{s/2}(\mathbb{R}^n))$ solution of the fractional heat equation. In general, the semigroup solution $b(\cdot)$ can be characterized in terms of renormalized solutions of parabolic equations driven by Lévy diffusions; this question will be addressed elsewhere, along with a study of entropy solutions in the spirit of [1].

2. Renormalized solutions for the nonlocal elliptic problem (3)

Before turning to definitions, we need some notation. We denote by $\mathrm{d}\pi(x,y)$ the measure $\frac{1}{2} g(x-y)\,\mathrm{d}x\,\mathrm{d}y$ on $\mathbb{R}^{2n}$ (recall that $g(\cdot)$ is described in (2); we assume $g \not\equiv 0$). For $k > 0$, set

$$T_k : r \mapsto \mathrm{sign}\, r \min\big\{k, |r|\big\} \quad \text{and} \quad \Phi_k : r \mapsto T_k(r+1) - T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k > 0$. We will write $T_k u(x)$, $Hu(x)$ for $T_k(u(x))$, $H(u(x))$, etc.

Definition 2.1. Let $f \in L^1(\mathbb{R}^n)$. A measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is called renormalized solution of problem (3) if there exists a function $b \in L^1(\mathbb{R}^n)$ such that $b(x) \in \beta(u(x))$ for a.e. $x \in \mathbb{R}^n$, and

(i)

$$\text{for all } k > 0, \quad \iint_{\mathbb{R}^{2n}} \big(u(x) - u(y)\big)\big(T_k u(x) - T_k u(y)\big)\,\mathrm{d}\pi(x,y) < +\infty; \tag{5}$$

$$\lim_{k\to+\infty} \iint_{\mathbb{R}^{2n}} \big(u(x) - u(y)\big)\big(\Phi_k u(x) - \Phi_k u(y)\big)\,\mathrm{d}\pi(x,y) = 0; \tag{6}$$

(ii) for all compactly supported renormalization functions $H \in W^{1,\infty}(\mathbb{R})$, for all test functions $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} bHu\phi + \iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(Hu(x)-Hu(y)\big)\frac{\phi(x)+\phi(y)}{2}\,\mathrm{d}\pi(x,y) + \iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\phi(x)-\phi(y)\big)\frac{Hu(x)+Hu(y)}{2}\,\mathrm{d}\pi(x,y) = \int_{\mathbb{R}^n} fHu\phi. \tag{7}$$

A careful analysis shows that thanks to (5), (6) and to the choice of $H(\cdot)$, all terms in (7) make sense (cf. (8) below). Let us explain the different points of the definition, as compared to the well-known case $\mathcal{L} = -\Delta$ (cf. [6,4]). Bound (5) replaces the regularity property $T_k u \in H^1(\mathbb{R}^n)$ of the truncates of a renormalized solution u, while the constraint $\lim_{k\to\infty}\int_{[k<|u|<k+1]} |\nabla u|^2 = 0$ is replaced by relation (6) (see also the reformulation (8)). Next, the set of integral identities (7) expresses the weak formulation of Eq. (3) formally multiplied by $H(u)$. At this stage, we give sense to the nonlocal term $(\mathcal{L}u)Hu\phi$ through the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^n)}$ (cf. [3, Lemma A.2]):

Proposition 2.2. *For all* $u, v \in \mathcal{D}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}(\mathcal{L}u)v = \iint_{\mathbb{R}^{2n}}(u(x)-u(y))(v(x)-v(y))\,\mathrm{d}\pi(x,y)$.

Then symmetrization of the difference $(Hu\phi)(x) - (Hu\phi)(y)$ yields the two middle terms in (7). Proposition 2.2 makes the link between sufficiently regular renormalized solutions of (7) and classical solutions.

Theorem 2.3. *For all* $f \in L^1(\mathbb{R}^n)$, *there exists a renormalized solution* u *of* (3)*. The contraction and comparison inequality* (4) *holds for renormalized solutions* u, $\hat{u}$ *associated with data* f, $\hat{f}$*; in particular, the function* b *in Definition* 2.1 *is unique.*

3. Techniques and arguments in use

For v measurable, we write $(\delta_{x,y}v)$ for $(v(x)-v(y))$. Set $H_\mu := \{v \mid \delta_{x,y}v \in L^2(\mathbb{R}^{2n}, \mathrm{d}\pi)\}$; the quotient space $H_\mu/\{v \equiv \mathit{const}\}$ is a Hilbert space under the scalar product $(\phi,\psi) \mapsto \iint_{R^{2n}}(\delta_{x,y}\phi)(\delta_{x,y}\psi)\,\mathrm{d}\pi(x,y)$.

Integrability constraints: A close examination shows that Definition 2.1(i) is equivalent to the properties

$$T_k u \in H_\mu \quad\text{and}\quad \lim_{k\to+\infty}\int\int_{[(u(x),u(y))\in A_k]} \big|u(x)-u(y)\big|\,\mathrm{d}\pi(x,y) = 0, \tag{8}$$

where $A_k := \{(u,v) \in \mathbb{R}^2 \mid k+1 \leqslant \max\{|u|,|v|\} \text{ and } (\min\{|u|,|v|\} \leqslant k \text{ or } uv < 0)\}$.

Existence of variational solutions: Let $j : \mathbb{R} \to [0,+\infty]$ be the convex l.s.c. function of which β is the subdifferential; we first replace β by its bi-Lipschitz approximation β_i, $\beta_i(0) = 0$, $\beta_i = \partial j_i$. For f in the space $L^\infty_c(\mathbb{R}^n)$ of compactly supported bounded functions, there exists a variational solution $u_i \in L^2(\mathbb{R}^n) \cap H_\mu$ with $j_i(u_i) \in L^1(\mathbb{R}^n)$ and $b_i := \beta_i(u_i) \in L^2(\mathbb{R}^n)$; u_i minimizes the associated coercive convex l.s.c. functional $J_i : v \mapsto \int_{\mathbb{R}^n}(j_i(v) - fv) + \iint_{\mathbb{R}^{2n}} \mid \delta_{x,y}v|^2\,\mathrm{d}\pi(x,y) =: J_i[v] \in (-\infty,+\infty]$ on $L^2(\mathbb{R}^n) \cap H_\mu$.

Estimates: For the above f, for $u = u_i$, $b = b_i$, testing the Euler–Lagrange equation with $T_k u$, $\Phi_k u$ we get

$$\int_{\mathbb{R}^n} |bT_k u| + \iint_{\mathbb{R}^{2n}} (\delta_{x,y}u)(\delta_{x,y}T_k u)\,\mathrm{d}\pi(x,y) \leqslant k\int_{\mathbb{R}^n} |f|; \qquad \iint_{\mathbb{R}^{2n}} (\delta_{x,y}u)(\delta_{x,y}\Phi_k u)\,\mathrm{d}\pi(x,y) \leqslant \int_{[|u|\geqslant k]} |f|. \tag{9}$$

A variational solution is also a renormalized one: It is enough to take $Hu\phi$ as test function in the variational formulation of (3) to get Definition 2.1(ii). Properties (i) are straightforward from estimates (9).

A partial comparison inequality: Let $u, \hat{u}$ be two renormalized solutions (we also allow for constant solutions) of (3). Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then by (8), this solution also belongs to H_μ.

Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leqslant \xi \leqslant 1$. Then $\xi \in H_\mu$ and for all $k > 0$, $\phi := \frac{1}{k}T_k(u - \hat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu$. By approximation, take ϕ for the test function in the renormalized formulations (7) for both u and $\hat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint (8) we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive

$$\forall \xi \in \mathcal{D}\big(\mathbb{R}^n\big), \quad 0 \leqslant \xi \leqslant 1, \qquad \int_{\mathbb{R}^n} \big(\mathrm{sign}^+(u-\hat{u}) + \xi \mathbb{1}_{[u=\hat{u}]}\big)(b-\hat{b}) \leqslant \int_{\mathbb{R}^n} \big(\mathrm{sign}^+(u-\hat{u}) + \xi \mathbb{1}_{[u=\hat{u}]}\big)(f-\hat{f}). \tag{10}$$

Letting ξ go to $\mathrm{sign}^+(b-\hat{b})$, we obtain (4). As a byproduct, we get $\int_{\mathbb{R}^n}|b| \leqslant \int_{\mathbb{R}^n}|f|$ (a weaker bound stems from (9)). Moreover, if $f \leqslant \hat{f}$, using the test function $T_k(u-\hat{u})^+$ with $k \to +\infty$, we get $(u-\hat{u})^+ \equiv \mathit{const}$.

The maximum principle: Clearly, a constant $c \in \mathrm{Dom}(\beta)$ is a bounded renormalized solution of (3) with datum $f \in \beta(c)$. By the above comparison principle, using the surjectivity of β, we see that the previously constructed variational (and thus, renormalized) solutions u with L^∞ data f do belong to $L^\infty(\mathbb{R}^n)$.

Existence: Keeping $f \in L^\infty_c(\mathbb{R}^n)$, with the uniform bounds on u_i in H_μ, on b_i (thus also on u_i) in L^∞, with classical convexity/monotonicity arguments we pass to the limit in $(u_i)_i$, $(b_i)_i$ and deduce existence of a variational and renormalized solution $u \in L^\infty(\mathbb{R}^n)$ with $b \in L^1(\mathbb{R}^n)$, $b \in \beta(u)$ a.e., and (9) holds.

Bi-monotone sequence of variational solutions, compactness: For given $f \in L^1(\mathbb{R}^n)$, consider the bounded compactly supported data $f^{l,m} = \min\{f^+, l\}\mathbb{1}_{|x|<l} - \min\{f^-, m\}\mathbb{1}_{|x|<m}$; the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leqslant |f|$. Hence associated variational solutions $u^{l,m}$ obey uniform estimates of the form (9); recall that (10) yields a uniform L^1 bound. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus $(b^{l,m})_{l,m}$ is monotone in l, m. Going back to $u_i^{l,m}$, we see that $(u^{l,m})_{l,m}$ is also bi-monotone. We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b. By (9), there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e. The subdifferential relation "$\forall v,\ j(u^{l,m}) - j(v) \leqslant b^{l,m}(u^{l,m} - v)$" holds a.e.; by passage to the limit, it follows that $b \in \beta(u)$. Since $b(\mathbb{R}) = \mathbb{R}$, u is finite a.e.

Passage to the limit in (5)–(7): We already have a renormalized formulation for $u^{l,m}$. Since $|f^{l,m}| \leqslant |f|$ and $u^{l,m}$ converge pointwise to an a.e. finite limit u, one shows that the bounds (9) for $u^{l,m}$ are uniform. Hence by the Fatou lemma, u fulfills (5) and (6). From the identities $(7)_{l,m}$ (let us denote this way the identities (7) written for $u^{l,m}$), for the limit u we first get a "rough" version of identity (7) with the second term replaced by $\bar{\bar{o}}_H(1)$ as $H(\cdot)$ goes to 1. Within, we take $\phi = T_k u$ for the test function; and we take $\phi = T_k u^{l,m}$ for the test function in $(7)_{l,m}$. Letting $H(\cdot)$ go to 1 on $\mathbb{R}$ (here (9), (8) are used), we find that

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u)(\delta_{x,y} T_k u)\, \mathrm{d}\pi(x,y) \geqslant \limsup_{l\to\infty} \limsup_{m\to\infty} \iint_{\mathbb{R}^{2n}} \big(\delta_{x,y} u^{l,m}\big)\big(\delta_{x,y} T_k u^{l,m}\big)\, \mathrm{d}\pi(x,y).$$

Hence the sequence $((\delta_{x,y} u^{l,m})(\delta_{x,y} T_k u^{l,m}))_{l,m}$ converges in $L^1(\mathbb{R}^{2n}, \mathrm{d}\pi)$, thus it is equi-integrable. Now using the a.e. convergence and (8), we can pass to the limit in all the terms of $(7)_{l,m}$ and get (7) for u.

Extension of the comparison property: Let $f \in L^1(\mathbb{R}^n)$; let $\hat{u}$ be a renormalized solution with datum f. A renormalized solution u was constructed as the limit of bounded solutions $u^{l,m}$. Passing to the limit in the previously obtained comparison inequality for $u^{l,m}, \hat{u}$, we find that $b = \hat{b}$ (further analysis shows that $\iint_{\mathbb{R}^{2n}} |\delta_{x,y}(u - \hat{u})|^2 = 0$, and in many cases we get $u = \hat{u}$). Hence by the density argument, using the upper semi-continuity of the bracket $[\cdot,\cdot]^+_{L^1(\mathbb{R}^n)}$ w.r.t. the L^1 convergence, one justifies (4) in full generality.

References

[1] Ph. Bénilan, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 22 (2) (1995) 241–273.

[2] Ph. Bénilan, M.G. Crandall, A. Pazy, Nonlinear evolution equations governed by accretive operators, Preprint.

[3] S. Cifani, E.R. Jacobsen, K.H. Karlsen, The discontinuous Galerkin method for fractal conservation laws, IMA J. Numer. Anal. (2010), doi:10.1093/imanum/drq006.

[4] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 28 (4) (1999) 741–808.

[5] K.H. Karlsen, F. Petitta, S. Ulusoy, A duality approach to the fractional Laplacian with measure data, Preprint, 2010.

[6] F. Murat, Équations elliptiques non linéaires avec second membre L^1 ou mesure, Actes du 26ème Congrès d'Analyse Numérique, Les Karellis, France, 1994.