Formulation and well-posedness of a nonlinear parabolic problem with inhomogeneous absorption

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\textbf{NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS - MAZÓN ’60}

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Plan of the talk

1. Elliptic and parabolic absorption problems
2. Natural (and less natural) estimates for solutions
3. Diffuse measures, obstacles, notion(s) of solution
4. Uniqueness, contraction, comparison... Existence ?
5. Subsequent approximations and existence
Absorption problems

Goal: study diffusion problems with absorption terms:

\[ u_t + \text{div} \ a(u, Du) + \mu = f, \quad \mu(\cdot) \in \beta(\cdot, u(\cdot)) \]

with initial condition \( u(0, \cdot) = u_0 \)
and boundary conditions (e.g., homogeneous Dirichlet).

Assumptions:

- diffusion \( \text{div} \ a(u, Du) \) of Leray-Lions type (laplacian, \( p \)-laplacian,...)
- \( \beta(x, r) = \partial j(x, r) \) with \( j(\cdot, r) \) proper convex l.s.c. functional on \( \mathbb{R} \)
  (and \( \beta(\cdot, 0) \ni 0 \)) \( \Rightarrow \) the term \( \mu \) of the eqn is an absorption term.
- data \( u_0, f \): first in \( L^\infty \) or in energy space; then in \( L^1 \).
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Notions of solution:
- strong solution (M. Crandall and Ph. Bénilan):
  the absorption term \( \mu \) is an \( L^1 \) function
- generalized solution (P. Wittbold):
  the absorption term \( \mu \) is realized as a measure.
- both versions (strong/generalized) can be considered within variational, or renormalized, or entropy formulation.
Literature on the diffusion-absorption problems

- [Bénilan, Crandall’91], [Bénilan, Wittbold’93]: abstract approach, notion of completely accretive operator, framework of strong solutions: one has $\mu \ll f$, ... but the approach is limited to $x$-independent absorptions $\beta(u)$.

- [Wittbold’97]: elliptic case, framework of generalized solutions. Key ingredient: characterization of $\mu$ as a diffuse measure with singular part concentrated on the obstacle ([Bouchitté’86])
Absorption problems

Natural estimates

Solution notion

Uniqueness

Subsequent approximations, existence

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• [A., Sbihi, Wittbold ’08]: will be discussed in this talk.
  Attempt to extend the approach of Wittbold to parabolic case.
  Idea: we manage to prove that $\mu$ is diffuse in $x$, for a.e. $t$. 
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- [Petitta, Ponce, Porretta’11]: a way to use “parabolic diffuse measures” based upon parabolic $p$-capacities of [Droniou, Porretta, Prignet’03]

- [Karami, Igbida]: an elegant way to avoid looking at the singular part of the measure $\mu$ (only test functions obeying the obstacle conditions are allowed)
Consider absorption problems (with penalization term $\psi(u)$):

$$u_t + \text{div} a(u, Du) + \psi(u) + \beta \circ u = f, \quad u(0, \cdot) = u_0.$$
Natural estimates and comparison of solutions

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Order-preservation, contraction properties: (test fct. $\text{sign}^+(u - \hat{u}), \ldots$)

- $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^1(\Omega)}(t) \leq \|u_0 - \hat{u}_0\|_{L^1(\Omega)} + \int_0^t \|f(\tau, \cdot) - \hat{f}(\tau, \cdot)\|_{L^1(\Omega)} d\tau$
- $u_0 \leq \hat{u}_0$, $f \leq \hat{f} \Rightarrow$ solutions verify $u \leq \hat{u}$
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Main estimates (formal; can be made rigorous):

- $f \in L^{p'}(0, T; W^{-1,p'})$ and $u_0 \in L^2 \Rightarrow$ solution $u$ is in the energy space $L^p(0, T; W^{1,p}_0)$ (use $u$ itself as test function)
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- $f, u_0$ in $L^1 \Rightarrow$ truncations $T_k(u)$ are in the energy space and in addition, $\int_{|k<|u|<k+1|} |Du|^p$ vanishes as $k \to \infty$. 
Natural estimates and comparison of solutions

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- $f, u_0$ in $L^1 \Rightarrow$ solution $u$ is in $L^1$ and $\beta \circ u$ is in $L^1$ (test fct. sign $(u)$)
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One finer estimate:
- $u_0, f$ in $L^\infty$ and $\psi'(r) \geq \kappa > 0 \Rightarrow \beta \circ u \in L^\infty(0, T; L^1(\Omega))$
Characterization of measure $\mu$: elliptic case

Let $\mathcal{J} : W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \mapsto [0, +\infty]$, $\mathcal{J}[w] = \int\limits_{\Omega} j(x, w(x)) \, dx \leq +\infty$.

Then ([Bouchitté'86]) there exist $\gamma_-(x) \leq 0 \leq \gamma_+(x)$ such that $\overline{\text{Dom}(\mathcal{J})}^{\|\cdot\|_W^{1,p}} = \{ w \in W^{1,p}_0 | \gamma_- \leq w \leq \gamma_+ \}$.

All the relations (equalities,...) are understood quasi-everywhere.
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$\mathcal{M}_0(\Omega)$ is space of diffuse measures on $\Omega$ (zero on sets of null $p$-capacity). [Boccardo, Gallouët, Orsina’96]: $\mathcal{M}_0 \subset L^1 + W^{-1,p}$
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Assume $f \in L^1 \cap L^\infty$. Then $u \in W^{1, p}(\Omega) \cap L^1(\Omega)$ is a solution of 

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$\mu_r \in \beta \circ u + \partial \Pi_{[\gamma^-, \gamma^+]} ...$
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The space of “Lebesque in $t$, diffuse in $x$” measures

In the parabolic case, the PDE in $\mathcal{D}'$ ensures that

$$u_t + \mu \in L^{p'}(0,T; W^{-1,p'}) + L^1((0,T) \times \Omega)).$$

Difficulty: separate the two terms in order to give sense to the equation using standard tools (J.-L. Lions, Alt and Luckhaus,...).
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Difficulty: separate the two terms in order to give sense to the equation using standard tools (J.-L. Lions, Alt and Luckhaus,...). Key ingredient:

the space $L^1(0, T; w - \mathcal{M}_0(\Omega))$ and “maximal regularity” for $u_t + \mu$

Lemma (regularity of “Lebesgue in $t$, diffuse in $x$” measures)

The space $L^1(0, T; w - \mathcal{M}_0(\Omega))$ is continuously embedded into $L^{p'}(0, T; W^{-1,p'}) + (L^\infty)^*((0, T) \times \Omega)) = (L^p(0, T; W^{1,p}_0) \cap L^\infty((0, T) \times \Omega)))^*$

Proof: define $<\mu, \phi>$ by approximation of $\phi$ with $\phi_n \in C^0$. Prove that $<\mu, \phi_n>$ is a Cauchy sequence combining absolute continuity of the integral in $t$ for $\mu$ with capacitary estimates in $x$ for $(\phi - \phi_n)$.

Corollary (“Maximal regularity”)

$\mu \in L^1(0, T; w - \mathcal{M}_0(\Omega)) \Rightarrow \mu, u_t \in (L^p(0, T; W^{1,p}_0) \cap L^\infty((0, T) \times \Omega)))^*$
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if $\mu \in L^1(0,T; w - \mathcal{M}_0(\Omega))$.
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if $\mu \in L^1(0,T; w - \mathcal{M}_0(\Omega))$ and for a.e. $t$, $\mu(t) = \mu_r(t) + \mu_s(t)$ fulfills

- $\mu_r(t, \cdot) \in \beta(\cdot, u(t, \cdot)) + \partial \mathcal{I}_{[\gamma^-(\cdot), \gamma^+(\cdot)]}$
- and $u(t, \cdot) = \gamma^\pm(\cdot)$ on the support of $\mu^{\pm}_s(t)$. 

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and $u(t, \cdot) = \gamma_{\pm}(\cdot)$ on the support of $\mu_{\pm}(t)$.

For $L^1$ data, one defines entropy solutions following [Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vázquez’95] by requiring variational inequalities on $T_k(u - \phi)$ where

- $T_k$ is truncation of levels $\pm k$
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**Theorem (Well-posedness, [A., Sbihi, Wittbold’08])**

*Parabolic problem with absorption is well-posed for entropy solutions*

Remark: looks nice, but... limited to $t$-independent absorptions $\beta(x, \cdot)$.
Uniqueness, contraction and comparison principle

Contraction results: standard for this type of equations

The technique is also rather standard by now:
- use of the definition of entropy solution
- doubling of the time variable ([Kruzhkov’69],[Otto’96])
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NB: actually, the proof of comparison between $u$ and $\hat{u}$ goes by
- comparing a solution $u$ to a solution $u_{m,n}$ of the penalized problem
  with penalization $\psi_{m,n} : r \mapsto \frac{1}{m} r^+ - \frac{1}{n} r^-$
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Difficulty: existence.
Why the $L^1(0,T; w - \mathcal{M}_0(\Omega))$ space is appropriate?
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Difficulty: existence.
Why the $L^1(0,T; w - M_0(\Omega))$ space is appropriate?
The proof is a multi-step approximation. Ingredients:
Yosida regularization, nonlinear semigroups,
penalization, control of concentrations in $\mu = \beta \circ u$,
bi-monotone penalization & data approximation [Ammar, Wittbold’03]
Scheme of the existence proof

- Start by bi-parameter Yosida approximation of $\beta$ by $\beta_\lambda^+ - \beta^-\nu$. Then there exist strong solutions $u_{\lambda,\nu}(\lambda)$ to the elliptic problem.
- Use Crandall-Liggett theorem to construct a mild solution of the evolution problem; show it is variational strong solution ($\mu_\lambda \in L^1((0,T) \times \Omega)$) to the PDE.
Scheme of the existence proof

- Start by bi-parameter Yosida approximation of $\beta$ by $\beta_\lambda^+ - \beta_\nu^-$. Then there exist strong solutions $u_{\lambda,\nu}(\lambda)$ to the elliptic problem.
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- **Difficulty:** control of concentrations of $\mu_\lambda$.
  In space: desperate (concentration “on the obstacles” $\gamma^\pm$).
  And in time? Add penalization $\psi_{m,n}(u) = \frac{1}{m} u^+ - \frac{1}{n} u^-$. Its role: dominate $u_\lambda^\pm$ by constructed *ad hoc* solutions $v_\lambda^\pm$ of auxiliary stationary problem (⇒ the “fine estimate”).
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- Fix $m, n$; pass to the limit with parameter $\lambda$.
  - Passage in $u_{\lambda}$: $L^p(0,T; W^{1,p}_0)$ bound, Minty, Landes...
  - Passage in $\mu_{\lambda}$: the “fine estimate” on $\mu_{\lambda}$ in $L^\infty(0,T; \mathcal{M}_b(\Omega))$ obtained using sub/super-solutions [Barthélemy, Bénilan’92]; subdifferential relation $j_{\lambda}(v) \geq j_{\lambda}(u_{\lambda}) + \mu_{\lambda}(v - u_{\lambda})$. 

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- Finally, use monotonicity in $m, n$ $\Rightarrow$ strong convergence of $u_{m,n}$ and also strong convergence of $\mu_{m,n}$ because the space $L^1(0, T; w - M_0(\Omega))$ is closed in $M_b((0, T) \times \Omega)$. 
GRACÍAS — THANK YOU FOR YOUR ATTENTION
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FELIZ CUMPLEAÑOS, MAZÓN!