Formulation and well-posedness of a nonlinear parabolic problem with inhomogeneous absorption

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Plan of the talk

1. Elliptic and parabolic absorption problems
2. Natural (and less natural) estimates for solutions
3. Diffuse measures, obstacles, notion(s) of solution
4. Uniqueness, contraction, comparison... Existence ?
5. Subsequent approximations and existence

Absorption problems

Goal: study diffusion problems with absorption terms:

$$u_t + \operatorname{div} a(u, Du) + \mu = f, \quad \mu(\cdot) \in \beta(\cdot, u(\cdot))$$

with initial condition $u(0, \cdot) = u_0$
and boundary conditions (e.g., homogeneous Dirichlet).

Assumptions:

- diffusion $\operatorname{div} a(u, Du)$ of Leray-Lions type (laplacian, p-laplacian,...)
- $\beta(x, r) = \partial j(x, r)$ with $j(\cdot, r)$ proper convex l.s.c. functional on $\mathbb{R}$ (and $\beta(\cdot, 0) \ni 0$) $\Rightarrow$ the term μ of the eqn is an absorption term.
- data u_0, f: first in L^∞ or in energy space; then in L^1 .

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Notions of solution:

- strong solution (M. Crandall and Ph. Bénilan): the absorption term μ is an L^1 function
- generalized solution (P. Wittbold): the absorption term μ is realized as a measure.
- both versions (strong/generalized) can be considered within variational, or renormalized, or entropy formulation .

Literature on the diffusion-absorption problems

• [Bénilan, Crandall'91] , [Bénilan, Wittbold'93] :
abstract approach, notion of completely accretive operator,
framework of strong solutions: one has $\mu \ll f$,...
but the approach is limited to x-independent absorptions $\beta(u)$.

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- [Petitta, Ponce, Porretta'11] :
a way to use "parabolic diffuse measures"
based upon parabolic p-capacities of [Droniou, Porretta, Prignet'03]

- [Karami, Igbida] :
an elegant way to avoid looking at the singular part of the measure μ
(only test functions obeying the obstacle conditions are allowed)

Natural estimates and comparison of solutions

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- $u_0 \leq \widehat{u}_0,\ f \leq \widehat{f} \;\Rightarrow$ solutions verify $u \leq \widehat{u}$
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One finer estimate :

- u_0, f in L^∞ and $\psi'(r) \ge \kappa > 0 \Rightarrow \beta \circ u \in L^\infty(0, T; L^1(\Omega))$

Characterization of measure μ: elliptic case

Let $\mathcal{J} : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \mapsto [0, +\infty]$,

$$\mathcal{J}[w] = \int_\Omega j(x, w(x))\, dx \leq +\infty.$$

Then ([Bouchitté'86]) there exist $\gamma_-(x) \leq 0 \leq \gamma_+(x)$ such that

$$\overline{\mathrm{Dom}(\mathcal{J})}^{\|\cdot\|_{W^{1,p}}} = \{w \in W_0^{1,p} \,|\, \gamma_- \leq w \leq \gamma_+\}.$$

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The space of “Lebesque in t, diffuse in x” measures

In the parabolic case, the PDE in $\mathcal{D}'$ ensures that

$$u_t + \mu \in L^{p'}(0,T; W^{-1,p'}) + L^1((0,T)\times\Omega).$$

Difficulty: separate the two terms in order to give sense to the equation using standard tools (J.-L. Lions, Alt and Luckhaus,...).

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Key ingredient:

the space $L^1(0,T;\, w-\mathcal{M}_0(\Omega))$ and "maximal regularity" for $u_t + \mu$

Lemma (regularity of "Lebesgue in t, diffuse in x" measures)

The space $L^1(0,T;\, w-\mathcal{M}_0(\Omega)$ is continuously embedded into $L^{p'}(0,T;\, W^{-1,p'}) + (L^\infty)^*((0,T)\times\Omega)) = \left(L^p(0,T;\, W_0^{1,p}) \cap L^\infty((0,T)\times\Omega)\right)^*$

Proof: define $<\mu,\phi>$ by approximation of ϕ with $\phi_n \in \mathbf{C}^0$. Prove that $<\mu,\phi_n>$ is a Cauchy sequence combining absolute continuity of the integral in t for μ with capacitary estimates in x for $(\phi-\phi_n)$.

Corollary ("Maximal regularity")

$\mu \in L^1(0,T;\, w-\mathcal{M}_0(\Omega) \Rightarrow \mu, u_t \in \left(L^p(0,T;\, W_0^{1,p}) \cap L^\infty((0,T)\times\Omega)\right)^*$

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For L^1 data, one defines entropy solutions following [Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vázquez'95] by requiring variational inequalities on $T_k(u - \phi)$ where

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Theorem (Well-posedness, [A., Sbihi, Wittbold'08])

Parabolic problem with absorption is well-posed for entropy solutions

Remark: looks nice, but... limited to t-independent absorptions $\beta(x,\cdot)$

Uniqueness, contraction and comparison principle

Contraction results: standard for this type of equations

The technique is also rather standard by now:

- use of the definition of entropy solution
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- comparing a solution u to a solution $u_{m,n}$ of the penalized problem with penalization $\psi_{m,n} : r \mapsto \frac{1}{m} r^+ - \frac{1}{n} r^-$
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Why the $L^1(0,T; w-\mathcal{M}_0(\Omega))$ space is appropriate?

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The proof is a multi-step approximation. Ingredients:
Yosida regularization, nonlinear semigroups,
penalization, control of concentrations in $\mu = \beta \circ u$,
bi-monotone penalization & data approximation [Ammar, Wittbold'03]

Scheme of the existence proof

- Start by bi-parameter Yosida approximation of β by $\beta_\lambda^+ - \beta_\nu^-$. Then there exist strong solutions $u_{\lambda,\nu(\lambda)}$ to the elliptic problem
- Use Crandall-Liggett theorem
  to construct a mild solution of the evolution problem; show it is variational strong solution ($\mu_\lambda \in L^1((0,T)\times\Omega)$) to the PDE.

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- Difficulty: control of concentrations of μ_λ.
  In space: desperate (concentration "on the obstacles" $\gamma_\pm$).
  And in time? Add penalization $\psi_{m,n}(u) = \frac{1}{m}u^+ - \frac{1}{n}u^-$.
  Its role: dominate $u_\lambda^\pm$ by constructed *ad hoc* solutions $v_\lambda^\pm$
  of auxiliary stationary problem ($\Rightarrow$ the "fine estimate")

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  to construct a mild solution of the evolution problem; show it is variational strong solution ($\mu_\lambda \in L^1((0,T)\times\Omega)$) to the PDE.
- Difficulty: control of concentrations of μ_λ.
  In space: desperate (concentration "on the obstacles" $\gamma_\pm$).
  And in time? Add penalization $\psi_{m,n}(u) = \frac{1}{m}u^+ - \frac{1}{n}u^-$.
  Its role: dominate $u^\pm_\lambda$ by constructed *ad hoc* solutions $v^\pm_\lambda$
  of auxiliary stationary problem ($\Rightarrow$ the "fine estimate")
- Fix m, n; pass to the limit with parameter λ.
  Passage in u_λ: $L^p(0,T;W^{1,p}_0)$ bound, Minty, Landes...
  Passage in μ_λ: the "fine estimate" on μ_λ in $L^\infty(0,T;\mathcal{M}_b(\Omega))$
  obtained using sub/super-solutions [Barthélemy,Bénilan'92] ;
  subdifferential relation $j_\lambda(v) \geq j_\lambda(u_\lambda) + \mu_\lambda(v - u_\lambda)$

Scheme of the existence proof

- Start by bi-parameter Yosida approximation of β by $\beta^+_\lambda - \beta^-_\nu$. Then there exist strong solutions $u_{\lambda,\nu(\lambda)}$ to the elliptic problem
- Use Crandall-Liggett theorem to construct a mild solution of the evolution problem; show it is variational strong solution ($\mu_\lambda \in L^1((0,T)\times\Omega)$) to the PDE.
- Difficulty: control of concentrations of μ_λ. In space: desperate (concentration "on the obstacles" $\gamma_\pm$). And in time? Add penalization $\psi_{m,n}(u) = \frac{1}{m}u^+ - \frac{1}{n}u^-$. Its role: dominate $u^\pm_\lambda$ by constructed *ad hoc* solutions $v^\pm_\lambda$ of auxiliary stationary problem ($\Rightarrow$ the "fine estimate")
- Fix m,n; pass to the limit with parameter λ. Passage in u_λ: $L^p(0,T;W^{1,p}_0)$ bound, Minty, Landes... Passage in μ_λ: the "fine estimate" on μ_λ in $L^\infty(0,T;\mathcal{M}_b(\Omega))$ obtained using sub/super-solutions [Barthélemy,Bénilan'92] ; subdifferential relation $j_\lambda(v) \geq j_\lambda(u_\lambda) + \mu_\lambda(v - u_\lambda)$
- Finally, use monotonicity in $m,n \Rightarrow$ strong convergence of $u_{m,n}$ and also strong convergence of $\mu_{m,n}$ because the space $L^1(0,T;w\!-\!\mathcal{M}_0(\Omega))$ is closed in $\mathcal{M}_b((0,T)\times\Omega)$.

Gracías — Thank you for your attention

GRACÍAS — THANK YOU FOR YOUR ATTENTION

FELIZ CUMPLEAÑOS, MAZÓN !