Solutions of Buckley-Leverett equation in heterogeneous medium and their approximation

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Microscopic motivations

Consider an immiscible incompressible two-phase flow (e.g. oil and water) within a homogeneous porous medium $\Omega$:

- The interface between the wetting phase (blue) and the non-wetting phase (yellow) is not flat, because of surface tension.
- The capillary force is related to the radius of the pore channels: the smaller is the radius, the greater is the capillary pressure.

$\Rightarrow$ In a nearly homogeneous porous medium, capillarity has only a small influence on the relative motion of the phases.
Microscopic motivations

Consider now the same fluid in a heterogeneous porous medium:

- The resulting capillary force cannot be neglected at the interface between the two different subdomains.

→ In a heterogeneous porous medium, capillarity plays a major role in the phase displacements.
Consider a heterogeneous porous medium \( \Omega = \mathbb{R} \), made of two homogeneous subdomains \( \Omega_1 = (\infty, 0) \) and \( \Omega_2 = (0, +\infty) \), separated by an interface \( \Gamma = \{ x = 0 \} \).

\[
\phi_i \partial_t s + \partial_x (f_i(s) - \partial_x \varphi_i(s)) = 0,
\]

where \( s \) denotes the saturation of one phase, \( q \) is the total flow rate and

\[
f_i(s) = q \frac{\eta_{o,i}(s)}{\eta_{o,i}(s) + \eta_{w,i}(s)} + (\rho_w - \rho_o) K_i \frac{\eta_{o,i}(s)\eta_{w,i}(s)}{\eta_{o,i}(s) + \eta_{w,i}(s)}
\]

\text{global convection}

\[
\varphi_i(s) = \int_0^s \frac{\eta_{o,i}(r)\eta_{w,i}(r)}{\eta_{o,i}(r) + \eta_{w,i}(r)} \pi_i'(r) \, dr.
\]

\text{buoyancy}
Typical behavior of the flux functions $f_i$

**A1** The functions $f_i$ are Lipschitz continuous and compatible, i.e.

$$f_1(0) = f_2(0) = 0, \quad f_1(1) = f_2(1) = q.$$  

**A2** The functions $f_i$ are bell-shaped, i.e. there exist $\bar{s}_i \in [0, 1]$ s.t.

$$f_i'(s)(\bar{s}_i - s) > 0 \text{ a.e. in } (0, 1).$$
Transmission conditions at the interface

At the interface $\Gamma = \{x = 0\}$, one requires:

- **continuity of the oil flux**:
  \[ [f_1(s) - \partial_x \varphi_1(s)]_{|x=0} = [f_2(s) - \partial_x \varphi_2(s)]_{|x=0} ; \]

- **continuity of the capillary pressure**$^3$
  \[ \pi_1(s_1) = \pi_2(s_2) . \]

**Initial condition**: One sets
\[ s_{|t=0} = s_0 \in L^\infty(\mathbb{R}), \quad 0 \leq s_0 \leq 1. \]

---

3. generalization in [Buzzi-Lenziger-Schweizer'09], [Cancès-Gallouët-Porretta'09], [Cancès-Pierre'12]
Existence/Uniqueness of the solution

**Theorem ([Cancès 09’])**

- Given \( s_0 \in L^\infty(\mathbb{R}; [0, 1]) \), there exists a unique mild solution \( s \) with \( 0 \leq s \leq 1 \) to the one-dimensional problem.

- Let \( s_0, \tilde{s}_0 \) two initial data, and let \( s, \tilde{s} \) their corresponding mild solution, then

\[
\|s(\cdot, t) - \tilde{s}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|s_0 - \tilde{s}_0\|_{L^1(\mathbb{R})}, \quad \forall t > 0.
\]

- The unique mild solution is computable by the mean of an implicit finite volume scheme, which is proved to be convergent.

**Idea:** The flux between two cells is discretized with

- the Godunov scheme for the convection and a usual three point approximation for the diffusion if the edge between the two cells is in \( \Omega_i \);

- the introduction of additional variables at the interface \(^4\), so that the continuity of the flux and of the capillary pressure can be prescribed.

\(^4\) At each time step, one has to solve a nonlinear problem by an iterative method. Each iteration requires the resolution of a (small) non-linear problem. The solution is thus quite expensive to compute.
Existence/Uniqueness of the solution

**Theorem ([Cancès 09’])**

- Given $s_0 \in L^\infty(\mathbb{R}; [0, 1])$, there exists a unique *mild solution* $s$ with $0 \leq s \leq 1$ to the one-dimensional problem.

- Let $s_0, \tilde{s}_0$ two initial data, and let $s, \tilde{s}$ their corresponding *mild solution*, then

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- The unique *mild solution* is computable by the mean of an *implicit finite volume scheme*, which is proved to be *convergent*.

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\(^4\) [Enchery-Eymard-Michel’06]
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Numerical results

For a reasonable choice of parameters:

- The capillary diffusion has a small influence on the solution in each $\Omega_i$.
- The discontinuity of the medium yields a discontinuity of the saturation at the interface.
- Two waves are leaving the interface.
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**Goal:** neglect the capillary diffusion in the numerical procedure.
Neglecting the capillary diffusion

As stressed before, the capillary forces have a small influence within $\Omega_i$, but a large influence at the interface $\Gamma = \{x = 0\}$. By a convenient scaling $(t := \frac{t}{\varepsilon}, x := \frac{x}{\varepsilon})$, one gets

$$\phi_i \partial_t s^\varepsilon + \partial_x f_i(s^\varepsilon) - \varepsilon \partial_{xx} \varphi_i(s^\varepsilon) = 0,$$

in $\Omega_i \times (0, T)$,

$$\left[ f_1(s^\varepsilon) - \varepsilon \partial_x \varphi_1(s^\varepsilon) \right]_{x=0} = \left[ f_2(s^\varepsilon) - \varepsilon \partial_x \varphi_2(s^\varepsilon) \right]_{x=0},$$

on $\Gamma \times (0, T)$,

$$\tilde{\pi}_1(s_1^\varepsilon) \cap \tilde{\pi}_2(s_2^\varepsilon) \neq \emptyset,$$

on $\Gamma \times (0, T)$.

Letting $\varepsilon \to 0$, and assume that $s^\varepsilon \to s$ a.e. in $\mathbb{R} \times (0, T)$, then

$$\left\{ \begin{array}{ll}
\phi_i \partial_t s + \partial_x f_i(s) = 0, & \text{in } \Omega_i \times (0, T), \\
 f_1(s_1) = f_2(s_2) & \text{on } \Gamma \times (0, T).
\end{array} \right.$$

More precisely, denoting by $\Phi_i(a, b) = \text{sign}(a - b)(f_i(a) - f_i(b))$, the solution $s$ satisfies, for all $\kappa \in [0, 1]$, the Kruzhkov entropy inequalities away from the interface $^5$

$$\phi_i \partial_t |s - \kappa| + \Phi_i(s, \kappa) \leq 0$$

in $D'(\Omega_i \times (0, T))$ (2)

$$f_1(s_1) = f_2(s_2)$$

on $\Gamma \times (0, T)$.

But: the connection of capillary pressures may not be inherited at the limit!

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5. This means that constants are “evident solutions” of homogeneous scalar conservation law and that all other solutions verify a (localized) $L^1$ contraction property with respect to these constant solutions.
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Overcompressive and undercompressive discontinuities

Let $u$ be the Kruzhkov entropy solution of

$$
\partial_t u + \partial_x f(u) = 0, \quad u|_{t=0} = u_0.
$$

- Even for smooth initial data $u_0$, the solution $u$ can become discontinuous after a finite time.
- If the solution is discontinuous, then the Rankine-Hugoniot condition is satisfied:

$$
\sigma = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.
$$

- The shocks are overcompressive, i.e. they only destroy information, due to the Lax condition:

$$
f'(u^-) \geq \sigma \geq f'(u^+).
$$
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Compressive and undercompressive discontinuities

For physical reasons, it can be relevant to allow the generation of information at some \( x_0 \in \mathbb{R} \):

- **traffic flow**: lights, change of the road size, ... [Andreianov, Goatin, Seguin ’10]
- **fluid mechanics**: small particle in an inviscid fluid
  [Andreianov, Seguin ’12], [Andreianov, Lagoutière, Takahashi, Seguin ’10 and ’13+]ε
- **porous media flow**: change of the rock type, [Kaasschieter ’99],
  [Adimurthi, Veerappa Gowda with Jaffré, Mishra, ’04-’05...], [Andreianov-Cancès ’13]

We will have to consider undercompressive shocks !!!

One might have (actually, only for one well-chosen couple \((u_-, u_+)\))

\[
f'(u^-) < \sigma (= 0) < f'(u^+).
\]
Scalar conservation laws with discontinuous flux

There exists infinitely many solutions to the problem (2)–(3). We have to select the relevant one. To do so, we use the frame of [Bürger-Karlsen-Towers ’09], [Andreianov-Karlsen-Risebro ’11] based on adapted entropy inequalities. The idea is to recognize admissible solutions of the form $A_1_{\Omega_1} + B_1_{\Omega_2}$ that would replace Kruzhkov constants $\kappa$, and use them to define relative (adapted) entropies.
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There exists infinitely many solutions\(^6\) to the problem (2)–(3). We have to select the relevant one. To do so, we use the frame of [Bürger-Karlsen-Towers ’09], [Andreianov-Karlsen-Risebro ’11] based on adapted entropy inequalities\(^7\).

The idea is to recognize admissible solutions of the form \(A_1 \Omega_1 + B_1 \Omega_2\) that would replace Kruzhkov constants \(\kappa\), and use them to define relative (adapted) entropies.

Define the set \(\mathcal{U}\) of the under-compressive shocks and by \(\mathcal{O}\) the set of the over-compressive (or “regular” shocks) satisfying the conservativity condition:

\[
\mathcal{U} = \left\{(A, B) \in [0, 1]^2 \mid f_1(A) = f_2(B), A \geq \bar{s}_1 \text{ and } B \leq \bar{s}_2\right\},
\]

\[
\mathcal{O} = \left\{(A, B) \in [0, 1]^2 \mid f_1(A) = f_2(B), A < \bar{s}_1 \text{ or } B > \bar{s}_2\right\}.
\]

\(^6\) [Adimurthi-Mishra-Gowda ’05]
\(^7\) [Audusse, Perthame ’05]
Characterization by the connection

Let us state the claims of [Andreianov-Karlsen-Risebro ’11] that are useful here.

- The problem (2)–(3) admits infinitely many solutions, corresponding to infinitely many contraction semi-groups, i.e. such that

\[ \|s(t) - \tilde{s}(t)\|_{L^1_{\phi}} \leq \|s_0 - \tilde{s}_0\|_{L^1_{\phi}}, \quad \forall t > 0; \]

- There is a bijection between the set $S$ of the contraction semi-groups and the set $U$ of the under-compressive shocks

\[ \forall (A, B) \in U, \text{ a unique solution semigroup admits } A1_{\Omega_1} + B1_{\Omega_2} \text{ as a stationary solution.} \]

\[ \forall S \in S, \text{ a unique couple } (A, B) \in U \text{ is such that } A1_{\Omega_1} + B1_{\Omega_2} \text{ is a stationary solution.} \]

- Let $(A, B) \in U$, and let $s$ be some trajectory of the associated semigroup $S_{(A,B)}$, then its traces $(s_1, s_2)$ on $\{x = 0\}$ belong to

\[ G^*_{(A,B)} := \{(s_1, s_2) \mid f_1(s_1) = f_2(s_2) \text{ and } \Phi_2(s_2, B) - \Phi_1(s_1, A) \leq 0\} \]

In particular, the only solutions $A^*1_{\Omega_1} + B^*1_{\Omega_2}$ are such that $(A^*, B^*) \in G^*_{(A,B)}$.

Heuristically : the “maximal germ” $G^*_{(A,B)}$ contains all information on the chosen semigroup ; and the “definite germ” $G_{(A,B)} := \{(A, B)\}$ is the minimal information.
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- Let \((A, B) \in U\), and let \( s \) be some trajectory of the associated semigroup \( S_{(A,B)} \), then its traces \((s_1, s_2)\) on \( \{x = 0\} \) belong to

\[ G^*_{(A,B)} := \{(s_1, s_2) \mid f_1(s_1) = f_2(s_2) \text{ and } \Phi_2(s_2, B) - \Phi_1(s_1, A) \leq 0\} \]

In particular, the only solutions \( A^*1_{\Omega_1} + B^*1_{\Omega_2} \) are such that \((A^*, B^*) \in G^*_{(A,B)}\).

Heuristically: the “maximal germ” \( G^*_{(A,B)} \) contains all information on the chosen semigroup; and the “definite germ” \( G_{(A,B)} := \{(A, B)\} \) is the minimal information.
Characterization by the connection

Let us state the claims of [Andreianov-Karlsen-Risebro '11] that are useful here.

- The problem (2)–(3) admits infinitely many solutions, corresponding to infinitely many contraction semi-groups, i.e. such that

\[ \| s(t) - \tilde{s}(t) \|_{L^1_\phi} \leq \| s_0 - \tilde{s}_0 \|_{L^1_\phi}, \quad \forall t > 0; \]

- There is a bijection between the set \( S \) of the contraction semi-groups and the set \( \mathcal{U} \) of the under-compressive shocks

\[ \forall (A, B) \in \mathcal{U}, \text{a unique solution semigroup admits } A1_{\Omega_1} + B1_{\Omega_2} \text{ as a stationary solution.} \]

\[ \forall S \in S, \text{a unique couple } (A, B) \in \mathcal{U} \text{ is such that } A1_{\Omega_1} + B1_{\Omega_2} \text{ is a stationary solution.} \]

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\[ \mathcal{G}_{(A,B)}^* := \{ (s_1, s_2) \mid f_1(s_1) = f_2(s_2) \text{ and } \Phi_2(s_2, B) - \Phi_1(s_1, A) \leq 0 \} \]

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The optimal entropy solution

A particular solution is a natural extension of the usual entropy solution when \( f_1 \equiv f_2 \): the optimal entropy solution. The work of [Kaasschieter ‘99] was interpreted to say that this is the right notion of solution that arises from the vanishing capillarity approach...

Define \( \bar{F}^{\text{opt}} \) by

\[
\bar{F}^{\text{opt}} = \min_{i \in \{1, 2\}} \left( \max_{s \in [0, 1]} f_i(s) \right),
\]

then \((s_1^{\text{opt}}, s_2^{\text{opt}}) \in \mathcal{U}\) such that

\[
f_1(s_1^{\text{opt}}) = f_2(s_2^{\text{opt}}) = \bar{F}^{\text{opt}}
\]

\(\rightsquigarrow (s_L^{\text{opt}}, s_R^{\text{opt}}) \in \mathcal{U} \cap \mathcal{O}\).

- One has \(G^*_{(s_1^{\text{opt}}, s_2^{\text{opt}})} = \emptyset\) : there is no strictly under-compressive discontinuity.
- In the case where \( f_1 \equiv f_2 \), the optimal entropy solution coincides with the usual entropy solution...
- Importance of optimal solution was realized at an early stage ([Kaasschieter ‘99]), and simple numerical schemes for their approximation were designed\(^8\)
- But in general, the optimal solution IS NOT the vanishing capillarity limit.

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then $(s_1^{opt}, s_2^{opt}) \in \mathcal{U}$ such that

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8. [Adimurthi, Jaffré, Veerappa Gowda '04]
Determination of the good connection

Due to the bijection between couples \((A, B) \in U\) and \(L^1\)-contractive semigroups of solutions, in the place of the painstaking construction of vanishing capillarity profiles for all possible jumps ([Kaasschieter ’99]) we can look only at the jumps with states \((u(0^-), u(0^+)) \in U\). And there is only one such jump!

**Theorem ([Andreianov-Karlsen-Risebro’11], [Andreianov-Cancès ’13])**

Given capillary pressure curves \(\pi_1, \pi_2\), \((A, B) \in U\) is the relevant connection if there exists a steady state \(\tilde{s}^\varepsilon(x)\) to the problem with small capillary diffusion such that

\[ \tilde{s}^\varepsilon(x) \to A\Omega_1(x) + B\Omega_2(x) \quad \text{for a.e. } x \in \mathbb{R} \text{ as } \varepsilon \to 0. \]

In this case, \(S_{(A,B)}\) is the semigroup corresponding to the vanishing capillarity limit, that is, for all \(s_0\) the corresponding solution \(s^\varepsilon\) tends to the unique \((A,B)\)-entropy solution to the SCL with discontinuous flux function.

A simple way to determine \((A, B)\):

Denote by \(U\) the set of the under-compressive shocks

\[ U = \{(A, B) \in [0, 1]^2 \mid f_1(A) = f_2(B), A \geq \bar{s}_1 \text{ and } B \leq \bar{s}_2\} \]

and by \(P\) the maximal monotone graph of \([0, 1]^2\) corresponding to the states where the capillary pressure continuity is satisfied:

\[ P = \{(s_1, s_2) \mid \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset\}. \]
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Determination of the good connection

**Theorem ([Andreianov-Cancès ’13])**

1. Assume that $\mathcal{P} \cap \mathcal{U} = (s_1^\pi, s_2^\pi) \neq \emptyset$, then the vanishing capillarity solution is the unique $(s_1^\pi, s_2^\pi)$-entropy solution.

2. Assume that $\mathcal{P} \cap \mathcal{U} = \emptyset$, the vanishing capillarity solution is the unique optimal entropy solution.

**Remark:** The vanishing capillarity profile depends (in a continuous way) on the shape of the capillary pressure functions !!!
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General principles of finite volume approximation for conservation laws

Finite volumes provide piecewise constant approximation of solution, which is advantageous because shocks may appear sharply. For homogeneous scalar conservation law

$$\partial_t u + \partial_x f(u) = 0$$

the main ingredient of a stable, convergent FV scheme is the numerical flux $u_K, u_L \mapsto F(u_K, u_L)$ needed to approximate $f(u)$ on the boundary $K|L$ between two volumes $K, L$ with values $u_K, u_L$ of the approximated solution $u$.

The following is required on the map $F(\cdot, \cdot)$:

- consistency: $F(a, a) = f(a)$
- Lipschitz dependence on its arguments
  (allows for an explicit scheme under a CFL condition)
- monotonicity: $a \mapsto F(a, b)$ is ↗ and $b \mapsto F(a, b)$ is ↘
  (this adds “numerical diffusion”, and thus ensures entropy inequalities)

**Example:**
if $f(\cdot) = f_\uparrow(\cdot) + f_\downarrow(\cdot)$ then $F(a, b) = f_\uparrow(a) + f_\downarrow(b)$ is a good numerical flux.

The only modification to standard schemes we have to apply for our case is:

find a suitable numerical flux $F_{int}(\cdot, \cdot)$ at the interface $\{x = 0\}$.

For the optimal connection $(A^{opt}, B^{opt})$ the answer is well known [Adimurthi, Jaffré, Veerappa Gowda ’04] for the most precise numerical flux (Godunov flux):

$$F^{opt}(a, b) = \min \left\{ f_1(\min\{\bar{s}_1, a\}), f_2(\max\{\bar{s}_2, b\}) \right\}.$$ 

What can be done in the general case?
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Observation: Flux limitation at the interface

Given \((A, B) \in \mathcal{U}\), we calculate \(G^*_{(A,B)}\) and find that

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\forall (A^*, B^*) \in G^*_{(A,B)} \quad f_1(A^*) = f_2(B^*) \leq \bar{F}_{(A,B)} := f_1(A) = f_2(B);
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thus \((A, B)\) prescribes the maximal level of flux at the interface.

NB : for this reason, the Buckley-Leverett equation in two-rocks’ medium is able to model the oil trapping phenomenon, i.e., the situation where the interface acts as a barrier for the exchange of phases.

**IDEA:** ([Andreianov, Goatin, Seguin '10], in the context of constrained road traffic)
For numerical approximation, simply apply the flux limitation on the top of the optimal flux of [Adimurthi, Jaffré, Veerappa Gowda '04]:

\[
F_{int}(a, b) = \min \left\{ \bar{F}_{(A,B)}, F_{opt}^{opt}(a, b) \right\}.
\]

And this works!

**ALGORITHM:**
- calculate the flux constraint value \(\bar{F}_{(A,B)}\) from \(\pi_1, \pi_2\);
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Alternative: introduction of “pressure transmission variables” at the interface

We mimic, at the numerical level, the properties of vanishing capillarity approximation: flux conservation and connection of capillary pressures. To do so, we introduce additional “pressure unknown” $p$ at the interface used to connect everything:

$$F_{int}(a, b) = \text{the common value } F_1(a, \pi_1^{-1}(p)) = F_2(\pi_2^{-1}(p), b),$$

which is actually a nonlinear equation on $p \in \mathbb{R}$. This equation has an (almost) unique solution thanks to monotonicity of $F_1(\cdot, \cdot), F_2(\cdot, \cdot)$.

This means that:
- we connect $a$ to some $s_1$ by the standard scheme on the left of the interface;
- we connect pressures at the interface: $\pi_1(s_1) = \pi_2(s_2)$;
- then we connect the resulting $s_2$ to $b$ by the standard scheme on the right from the interface.

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**Algorithm:**
- introduce the additional interface unknown $p$ and solve the equation for $p$;
- use the obtained quantity $F_{int}(a, b)$ as the interface numerical flux.

Disadvantage: it works slower than flux limitation;
Advantage: this applies in the multi-d case where we have additional coupling...
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The phase-by-phase upstream scheme... does converge to the right solution!

The phase-by-phase upstream scheme (see, e.g., [Brenier, Jaffré '91]) is frequently used in practice of petroleum engineering. Is it compatible with medium heterogeneity? Interpretations of [Jaffré, Mishra '10],[Tveit, Aavatsmark '12] of the vanishing capillarity analysis of [Kaasschieter '99] led to the following important conclusion:

? ...the phase-by-phase upstream scheme yields a wrong solution... ?

Firstly, we stress that these results rely on the erroneous conclusion that the right vanishing capillarity limit is always the optimal solution.

Secondly, there is no unique extension of this scheme to two-rocks’ medium; we suggest to use the “interface pressure unknown” $p$ in order to obtain the interface flux corresponding to capillary pressure curves $\pi_1, \pi_2$.

**Theorem ([Andreianov-Cancès '13+])**

The phase-by-phase upstream scheme corresponding to this choice of interface flux converges under a CFL condition towards the vanishing capillarity solution.

**Sketch of the proof:** The proof consists in three observations:

- The scheme is $L^1$ contractive at the discrete level ($\Rightarrow$ semigroups $S^h, h > 0$)
- For each datum (at least, $BV$), it converges up to a subsequence ($\Rightarrow$ limit semigroup $S$ which inherits $L^1$ contraction)
- The limits of the scheme are Kruzhkov entropy solutions away from the interface...
- ...and the scheme does approximate the correct (with respect to $\pi_1, \pi_2$) stationary solution $A_1\Omega_1 + B_1\Omega_2$, which permits to identify the semigroup $S$. 
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**Theorem ([Andreianov-Cancès '13])**

*The phase-by-phase upstream scheme corresponding to this choice of interface flux converges under a CFL condition towards the vanishing capillarity solution.*

**Sketch of the proof:** The proof consists in three observations:

- the scheme is $L^1$ contractive at the discrete level ($\Rightarrow$ semigroups $S^h$, $h > 0$)
- for each datum (at least, $BV$), it converges up to a subsequence ($\Rightarrow$ limit semigroup $S$ which inherits $L^1$ contraction)
- the limits of the scheme are Kruzhkov entropy solutions away from the interface...
- ...and the scheme does approximate the correct (with respect to $\pi_1$, $\pi_2$) stationary solution $A_1\Omega_1 + B_1\Omega_2$, which permits to identify the semigroup $S$. 
The phase-by-phase upstream scheme... does converge to the right solution!

The phase-by-phase upstream scheme (see, e.g., [Brenier, Jaffré ’91]) is frequently used in practice of petroleum engineering. Is it compatible with medium heterogeneity? Interpretations of [Jaffré, Mishra ’10],[Tveit, Aavatsmark ’12] of the vanishing capillarity analysis of [Kaasschieter ’99] led to the following important conclusion:

? ...the phase-by-phase upstream scheme yields a wrong solution... ?

Firstly, we stress that these results rely on the erroneous conclusion that the right vanishing capillarity limit is always the optimal solution.

Secondly, there is no unique extension of this scheme to two-rocks’ medium; we suggest to use the “interface pressure unknown” $p$ in order to obtain the interface flux corresponding to capillary pressure curves $\pi_1, \pi_2$.

**Theorem ([Andreianov-Cancès ’13+])**

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Two-phase flows in two rocks' media
Vanishing capillarity solutions in 1D
Finite volume approximation
Multi-dimensional case : numerics

Numerical illustrations and comparison to small capillarity model

**Numerical Test Case:**

\[ \pi_i(s) = P_i - \ln(1 - s), \]

(a) Capillary pressures

(b) Flux functions \( f_i \)

For the initial datum, we will take the constant state \( u_0 = 1/2 \)
(we will see a non-constant solution : this underlines the active role of the interface between the two rocks !)
Numerical illustrations and comparison to small capillarity model

**Numerical test case:**

\[ \pi_i(s) = P_i - \ln(1 - s), \]

Test case 1: \( P_1 = 0, \ P_2 = 0.5 \)

The sets \( \mathcal{P} \) and \( \mathcal{U} \)

Here we should obtain the optimal entropy solution
Numerical illustrations and comparison to small capillarity model

**NUMERICAL TEST CASE:**

\[ \pi_i(s) = P_i - \ln(1 - s), \]

Test case 2: \( P_1 = 0, \ P_2 = 2 \)

(c)  
(d) Connection \((s_1^\pi, s_2^\pi)\)

Here we should obtain an entropy solution with flux limitation
Numerical results in the optimal case

We compare with the numerical solution corresponding to $s^\varepsilon$ for $\varepsilon = 10^{-3}$.

- $s$ is approximated by the explicit Godunov scheme
- $s^\varepsilon$ approximated by a convergent asymptotic preserving implicit scheme

![Diagram](image)

(e) Solution $s_h$ to the hyperbolic problem

(f) Solution $s_h^\varepsilon$ to the parabolic problem ($\varepsilon = 10^{-3}$)

9. [Cancès '09], [Andreianov-Cancès '13]
Numerical results in the non-optimal case

We compare with the numerical solution corresponding to \( s^\varepsilon \) for \( \varepsilon = 10^{-3} \).

- \( s \) is approximated by the explicit Godunov scheme
- \( s^\varepsilon \) approximated by a convergent asymptotic preserving implicit scheme

\( [\text{Cances '09}, \text{Andreianov-Cances '13}] \)

-\( (g) \) Solution \( s_h \) to the hyperbolic problem
-\( (h) \) Solution \( s_h^\varepsilon \) to the parabolic problem (\( \varepsilon = 10^{-3} \))
The multi-dimensional model

Our starting point is the model where immiscible incompressible two-phase flow in the homogeneous porous medium $\Omega_i$ is governed by the coupling of the degenerate parabolic equation on the saturation $s$:

$$\phi_i \partial_t s + \nabla \cdot (u_t f_i(s) + K_i \lambda_i(s) (-\varepsilon \nabla \pi_i(s) + \rho g)) = q_o(s), \quad (4)$$

with the uniformly elliptic equation on the total fluid velocity $u_t$:

$$\nabla \cdot u_t = q_o(s) + q_w(s), \quad u_t = -K_i (M_i(s) \nabla P - \zeta_i(s) g). \quad (5)$$

The function $P$ is the so-called “global pressure” [Chavent, Jaffré ’86].

Coupling: fluxes that appear in the SCL with discontinuous flux depend on $u_t$.

Transmission conditions on the interface $\Gamma$ between $\Omega_1$ and $\Omega_2$:

- conservation of mass

$$\sum_{i=1,2} u_t \cdot n_i = 0, \quad \sum_{i=1,2} (u_t f_i(s) + K_i \lambda_i(s) (-\varepsilon \nabla \pi_i(s) + \rho g)) \cdot n_i = 0. \quad (6)$$

- “continuity” of the extended phase pressures: $\exists p : \Gamma \times (0, T) \rightarrow \mathbb{R}$ such that

$$p \in \bar{\pi}_1(s_1) \cap \bar{\pi}_2(s_2) \quad \text{and} \quad P_1 - Z_1(p) = P_2 - Z_2(p), \quad (7)$$

where $Z_i(\cdot)$ are suitably defined nonlinearities and $s_i, P_i$ denote the traces of $s, P$ on $\Gamma \times (0, T)$ from the side of $\Omega_i$. Indeed, $P$ is non-physical, it can be discontinuous at the interface; but physical pressures expressed as $P_i - Z_i(s)$ must be connected.

Goal: set $\varepsilon = 0$ in this model without losing the pressure connection at the interface.
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- “continuity” of the extended phase pressures: $\exists p : \Gamma \times (0, T) \rightarrow \mathbb{R}$ such that
  $$p \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \quad \text{and} \quad P_1 - Z_1(p) = P_2 - Z_2(p), \quad (7)$$

where $Z_i(\cdot)$ are suitably defined nonlinearities and $s_i, P_i$ denote the traces of $s, P$ on $\Gamma \times (0, T)$ from the side of $\Omega_i$. Indeed, $P$ is non-physical, it can be discontinuous at the interface; but physical pressures expressed as $P_i - Z_i(s)$ must be connected.

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A decoupled IMPES scheme

The “decoupled” IMplicit Pressure - Explicit Saturation (IMPES) scheme based on additional interface pressure variables $p^n_{\sigma}$ : [Andreianov, Brenner, Cancès '13 ?]

- **Initialization** : the saturation values $(s^0_K)_K$ are given ; the values $p^{n+1}_{\sigma}$ of initial capillary pressure are non-physical and must be interpolated :
  
  $p_{\sigma}^0 = \text{mean value}\{\pi_K(s^0_K), \pi_L(s^0_L)\}, \quad \forall \sigma = K|L \in \mathcal{E}_\Gamma.$

- Values $(s^n_K)_K$ and $(p^n_{\sigma})_{\sigma}$ being given, solve the linear system on $(P^{n+1}_K)_K, (u^{n+1}_K)_K$

  $\sum_{\sigma \in \mathcal{E}_K} |\sigma| u^{n+1}_{K,\sigma} = |K| q_o(s^n_K) + |K| q_w(s^n_K), \quad \forall K, \sigma \notin \mathcal{E}_\Gamma ;$

  $u^{n+1}_{K,\sigma} = 0$ if $\sigma \subset \partial \Omega,$

  $u^{n+1}_{K,\sigma} = \ldots (P^{n+1}_K - P^{n+1}_L) + \ldots$ if $\sigma = K|L, \sigma \notin \mathcal{E}_\Gamma,$

  $u^{n+1}_{K,\sigma} + u^{n+1}_{L,\sigma} = 0, \quad u^{n+1}_{J,\sigma} = \ldots (P^{n+1}_J - P^{n+1}_{J,\sigma}) + \ldots, \quad J = K, L, \sigma \in \mathcal{E}_\Gamma ;$

  closed by the pressure connection relation on interface variables $P_{K,\sigma}$ :

  $P^{n+1}_{K,\sigma} - Z_K(p^n_{\sigma}) = P^{n+1}_{L,\sigma} - Z_L(p^n_{\sigma})$, \quad $\forall \sigma \in \mathcal{E}_\Gamma.$

- Now we can compute the values $p_{\sigma}^{n+1}$ for each $\sigma \in \mathcal{E}_\Gamma$ by solving one nonlinear equation per edge $\sigma \in \mathcal{E}_\Gamma$, as in 1D. Indeed, the previous step also yields the values $u^{n+1}_{K,\sigma}$ of $u_t$, thus the expressions of the fluxes ($f_1, f_2$ in the 1D case) are available.

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- Values $(s_K^n)_K$ and $(p_\sigma^n)_\sigma$ being given, solve the linear system on $(P_K^{n+1})_K, (u_K^{n+1})_K$:

  $$\sum_{\sigma \in \mathcal{E}_K} |\sigma| u_{K,\sigma}^{n+1} = |K| q_o(s_K^n) + |K| q_w(s_K^n), \quad \forall K, \sigma \notin \mathcal{E}_\Gamma;$$

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- Values $(s^n_K)_K$ and $(p^n_\sigma)_\sigma$ being given, solve the linear system on $(P^{n+1}_K)_K, (u^{n+1}_K)_K$ :

  $$\sum_{\sigma \in \mathcal{E}_K} |\sigma| u^{n+1}_{K,\sigma} = |K|q_o(s^n_K) + |K|q_w(s^n_K) \quad \forall K, \sigma \notin \mathcal{E}_\Gamma ;$$  

  $$u^{n+1}_{K,\sigma} = 0 \text{ if } \sigma \subset \partial \Omega, \quad u^{n+1}_{K,\sigma} = \ldots (P^{n+1}_K - P^{n+1}_L) + \ldots \text{ if } \sigma = K|L, \sigma \notin \mathcal{E}_\Gamma ,$$  

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- Values $(s^n_K)_K$ and $(p^n_\sigma)_\sigma$ being given, solve the linear system on $(P^{n+1}_K)_K, (u^{n+1}_K)_K$:

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  $$u^{n+1}_K + u^{n+1}_L = 0, \quad u^{n+1}_J + u^{n+1}_L = \ldots(P^{n+1}_J - P^{n+1}_J) + \ldots, \quad J = K, L, \sigma \in \mathcal{E}_\Gamma;$$

  closed by the pressure connection relation on interface variables $P_{K,\sigma}$:

  $$P^{n+1}_{K,\sigma} - Z_K(p^n_\sigma) = P^{n+1}_L, - Z_L(p^n_\sigma), \quad \forall \sigma \in \mathcal{E}_\Gamma.$$

- Now we can compute the values $p^{n+1}_\sigma$ for each $\sigma \in \mathcal{E}_\Gamma$ by solving one nonlinear equation per edge $\sigma \in \mathcal{E}_\Gamma$, as in 1D. Indeed, the previous step also yields the values $u^{n+1}_{K,\sigma}$ of $u_t$, thus the expressions of the fluxes ($f_1, f_2$ in the 1D case) are available.

- Finally, we compute the saturations $(s^{n+1}_K)_K$ by the straightforward explicit scheme.
Numerical results in 2D : a constrained flow

A 2D test case inspired from [Eymard, Guichard, Herbin, Masson ’13 ?] : The 2D domain $\Omega$ (see Fig. 1, top left), mostly consists of rock $\Omega_1$, is initially saturated in water. The flow is constrained by a presence of two barriers (rock $\Omega_2$) having a higher entry pressure. The vertical boundaries are assumed to be impermeable. At the bottom and the top of $\Omega$ we prescribe a constant rate of a total flux. The constant saturation value $s = 0.5$ is imposed on $\Gamma_{in}$. Details : [Andreianov, Brenner, Cancès ’13 ?].

**Figure**: Computational domain and the oil saturation field at time $t = 0.075, 0.2, 0.4, 0.6, 0.8$. 
Thank you - Merci - Danke!

DANKE   SCHÖN!