ON THE SZLENK INDEX AND THE WEAK*-DENTABILITY INDEX

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ABSTRACT. We prove that if the Szlenk index $Sz(X)$ and the weak*-dentability index $\delta^*(X)$ of a Banach space X are countable, then they are determined by the closed separable linear subspaces of X. From this we deduce the existence of an absolute function ψ from ω_1 to ω_1 (first uncountable ordinal) such that $\delta^*(X)$ is bounded above by $\psi(Sz(X))$, and that the condition $Sz(X) < \omega_1$ yields the existence of an equivalent norm on X whose dual norm is locally uniformly convex. As an other application, we compute $Sz(C(K))$, where K is a scattered compact space with $K^{(\omega_1)} = \emptyset$. Finally we solve the three space problem for the condition $Sz(X) < \omega_1$.

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1. INTRODUCTION.

Let X be a Banach space. We will first define the two ordinal indices $\delta^*(X)$ and $Sz(X)$.

Weak*-dentability index, $\delta^*(X)$:

Let F be a closed bounded subset of X^*. For $\varepsilon > 0$, $F'_\varepsilon = \{x^* \in F$ such that any weak*-slice of F containing x^* is of diameter $> \varepsilon\}$.

For α ordinal we construct F_ε^α inductively :

$F_\varepsilon^0 = F$

$F_\varepsilon^{\alpha+1} = (F_\varepsilon^\alpha)'_\varepsilon$

$F_\varepsilon^\alpha = \bigcap\limits_{\beta<\alpha} F_\varepsilon^\beta$, if α is a limit ordinal.

Then

$$\Delta_\varepsilon(F) = \begin{cases} \inf\{\alpha : F_\varepsilon^\alpha = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

And $\Delta(F) = \sup\limits_{\varepsilon>0} \Delta_\varepsilon(F)$.

Finally, we denote $\delta^*(X,\varepsilon) = \Delta_\varepsilon(B_{X^*})$ and $\delta^*(X) = \Delta(B_{X^*})$, where B_{X^*} is the unit ball of X^*.

Szlenk index, $Sz(X)$:

Let F be a closed bounded subset of X^*. For $\varepsilon > 0$, $F_\varepsilon^{[']} = \{x^* \in F$ such that for any weak*-neighborhood V of x^*, diam $(V \cap F) > \varepsilon\}$.

We denote :

$F_\varepsilon^{[0]} = F$

$F_\varepsilon^{[\alpha+1]} = (F_\varepsilon^{[\alpha]})_\varepsilon^{[']}$

$F_\varepsilon^{[\alpha]} = \bigcap\limits_{\beta<\alpha} F_\varepsilon^{[\beta]}$, if α is a limit ordinal.

$$S_\varepsilon(F) = \begin{cases} \inf\{\alpha : F_\varepsilon^{[\alpha]} = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

$S(F) = \sup\limits_{\varepsilon>0} S_\varepsilon(F)$

$Sz(X,\varepsilon) = S_\varepsilon(B_{X^*})$ and $Sz(X) = S(B_{X^*})$.

Clearly $Sz(X) \le \delta^*(X)$.

Our main objective is to prove that, for a Banach space X, $Sz(X) < \omega_1$ if and only if $\delta^*(X) < \omega_1$, where ω_1 is the first uncountable ordinal. Then, answering a question suggested by R. Haydon, we will be able to deduce that if $Sz(X) < \omega_1$, then X admits an equivalent norm whose dual norm is locally uniformly convex. An important step in the proof of this result is that, if $Sz(X)$ is countable then $Sz(X) = \sup\{Sz(Y), Y$ closed separable subspace of $X\}$ and if $\delta^*(X)$ is countable then $\delta^*(X) = \sup\{\delta^*(Y), Y$ closed separable subspace of $X\}$. In section 5 we use this fact to compute $Sz(C(K))$, for K scattered compact space such that its ω_1^{th} derived set $K^{(\omega_1)}$ is empty. In the last section of this paper we give a quantitative answer to the three space problem for the condition $Sz(X) < \omega_1$.

2. SEPARABLE CASE.

It is well known that if X is a separable Banach space, then the following are equivalent :

i) $Sz(X) < \omega_1$

ii) $\delta^*(X) < \omega_1$

iii) X^* is separable.

In this section we will explain how to obtain the following improvement.

PROPOSITION 2.1. *There exists a function $\psi : \omega_1 \to \omega_1$ so that, for any separable Banach space X and for any $\alpha < \omega_1, Sz(X) \leq \alpha$ implies $\delta^*(X) \leq \psi(\alpha)$.*

This is the consequence of ideas developed by B. Bossard in a slightly different and also more general setting [B].

Before proceeding to the proof of this proposition, we will introduce a few notations :

Let $K = (B_{l^\infty}, \sigma(l^\infty, l^1))$. K is a compact metrizable space. We denote by $\mathcal{F}(K)$ the collection of all closed subsets of K and we equip $\mathcal{F}(K)$ with the Hausdorff topology $\mathcal{T}$ generated by the sets of the form $\{F \in \mathcal{F}(K) : F \cap V \neq \emptyset\}$ and $\{F \in \mathcal{F}(K) : F \subset V\}$, where V is an open subset of K. $(\mathcal{F}(K), \mathcal{T})$ is a compact metrizable space.

PROPOSITION 2.2. *There exists a function $\psi : \omega_1 \to \omega_1$ so that, for any closed subset F of K and for any $\alpha < \omega_1, S(F) \leq \alpha$ implies $\Delta(F) \leq \psi(\alpha)$.*

Proof. We will need the following result of B. Bossard [B] :

$$\text{for } \varepsilon > 0 \quad \begin{array}{c} d_\varepsilon : \mathcal{F}(K) \to \mathcal{F}(K) \\ F \mapsto F'_\varepsilon \end{array} \quad \text{and} \quad \begin{array}{c} D_\varepsilon : \mathcal{F}(K) \to \mathcal{F}(K) \\ F \mapsto F^{[']}_\varepsilon \end{array}$$

are Borel derivations.

Therefore, for any $\alpha < \omega_1$, $\mathcal{B}_\alpha = \{F \in \mathcal{F}(K) : S(F) \le \alpha\} = \bigcap_{n=1}^{\infty} \{F \in \mathcal{F}(K) : S_{1/n}(F) \le \alpha\}$ is a Borel set in $(\mathcal{F}(K), \mathcal{T})$.

Moreover, for any $n \ge 1$, $\mathcal{B}_\alpha \subseteq \{F \in \mathcal{F}(K) : \Delta_{1/n}(F) < \omega_1\}$. Indeed, if $S(F) < \omega_1$, then F is norm separable and therefore every weak*-closed subset of F is weak*-dentable. So $(F^\alpha_{1/n})_\alpha$ is strictly decreasing and must stabilize at $\emptyset$ before ω_1.

Now, by a result of C. Dellacherie [Del] about the applications of the Kunen-Martin theorem to the study of the analytic derivations, there exists $\psi_n(\alpha) < \omega_1$ such that :

$$\mathcal{B}_\alpha \subseteq \{F \in \mathcal{F}(K) : \Delta_{1/n}(F) \le \psi_n(\alpha)\}.$$

We can conclude the proof by taking $\psi(\alpha) = \sup_{n \ge 1} \psi_n(\alpha)$. $\square$

Proof of Proposition 2.1. Let X be a separable Banach space and $\alpha < \omega_1$.
There is a closed linear subspace Y of $\ell_1(\mathbb{N})$ such that X is isomorphic to $\dfrac{\ell_1(\mathbb{N})}{Y}$.
So $Sz(X) = Sz(\dfrac{\ell_1(\mathbb{N})}{Y}) = S(B_{Y^\perp})$
and $\delta^*(X) = \delta^*(\dfrac{\ell_1(\mathbb{N})}{Y}) = \Delta(B_{Y^\perp})$.
Thus by proposition 2.2, if $Sz(X) \le \alpha$, then $\delta^*(X) \le \psi(\alpha)$. $\square$

3. WHEN COUNTABLE, $Sz(X)$ AND $\delta^*(X)$ ARE SEPARABLY DETERMINED.

Our goal in this section is to prove the two following statements :

PROPOSITION 3.1. *Let X be a Banach space and let $\alpha < \omega_1$.*

If $Sz(X) > \alpha$, then there exists a separable closed subspace Y of X such that $Sz(Y) > \alpha$.

PROPOSITION 3.2. *Let X be a Banach space and let $\alpha < \omega_1$.*

If $\delta^*(X) > \alpha$, then there exists a separable closed subspace Y of X such that $\delta^*(Y) > \alpha$.

Proof of Proposition 3.1 : We will give our original proof in which we construct "by hand" the space Y. In order to do this we will use a family $(T_\alpha)_{\alpha<\omega_1}$ of trees on ω (first infinite ordinal) constructed inductively in the following way :

$T_0 = \{\emptyset\}$

$T_{\alpha+1} = \{\emptyset\} \cup \bigcup\limits_{n=0}^{\infty} n^\frown T_\alpha$, where $n^\frown T_\alpha = \{n^\frown s, s \in T_\alpha\}$.

$T_\alpha = \{\emptyset\} \cup \bigcup\limits_{n=0}^{\infty} n^\frown T_{\alpha_n}$, if α is a limit ordinal, $(\alpha_n)_{n=0}^{\infty}$ being an enumeration of the ordinals less than α.

Remarks :

1) The height of T_α is $ht(T_\alpha) = \alpha$.

2) For s in T_α we denote $T_\alpha(s) = \{t \in \omega^{<\omega} : s^\frown t \in T_\alpha\}$, where $\omega^{<\omega}$ is the set of all finite sequences of elements of ω. If we call $h_\alpha(s) = ht(T_\alpha(s))$, we have that $T_\alpha(s) = T_{h_\alpha(s)}$.

We will need the following :

LEMMA 3.3. *For any $1 \leq \alpha < \omega_1$, there exists a bijection $\varphi_\alpha : \omega \to T_\alpha$ such that :*

For any s, s' in T_α, $s < s'$ implies $\varphi_\alpha^{-1}(s) < \varphi_\alpha^{-1}(s')$.

Proof. Let $\{\mathcal{B}_n\}_{n=0}^{\infty}$ be an enumeration of the branches of T_α. In order to define φ_α we enumerate successively $\mathcal{B}_1$, $\mathcal{B}_2 \setminus \mathcal{B}_1, ...,$ $\mathcal{B}_{n+1} \setminus \bigcup\limits_{k=1}^{\infty} \mathcal{B}_k, ...$ (each enumeration of $\mathcal{B}_{n+1} \setminus \bigcup\limits_{k=1}^{\infty} \mathcal{B}_k$ following the natural partial order on T_α). $\square$

LEMMA 3.4. *Let $1 \leq \alpha < \omega_1$ and $\varepsilon > 0$ and let X be a Banach space. If $x^* \in (B_{X^*})_\varepsilon^{[\alpha]}$, then there exist a separable subspace Y of X and a family $(x_s^*)_{s \in T_\alpha} \subseteq B_{X^*}$ such that*

i) $x_\emptyset^* = x^*$

ii) $\forall\ s \in (T_\alpha)', \forall\ n \in \omega : \|(x_{s^\frown n}^* - x_s^*)_{|Y}\| > \dfrac{\varepsilon}{2}$.

iii) $\forall\ s \in (T_\alpha)' : (x_{s^\frown n}^* - x_s^*)_{|Y} \overset{\sigma(Y^*,Y)}{\longrightarrow} 0$.

(Note : $(T_\alpha)' = \{s \in T_\alpha, \exists\ n \in \omega : s^\frown n \in T_\alpha\}$*).*

Proof. We will construct, by induction on n, $(x^*_{\varphi_\alpha(n)})_{n=0}^\infty$ in B_{X^*} and $(x_n)_{n=1}^\infty$ in B_X so that :

a) $x^*_{\varphi_\alpha(0)} = x^*_\emptyset = x^*$.

b) $\forall\, n \in \omega,\ x^*_{\varphi_\alpha(n)} \in (B_{X^*})_\varepsilon^{[h_\alpha(\varphi_\alpha(n))]}$

c) $\forall\, n \geq 1, (x^*_{\varphi_\alpha(n)} - x^*_{s_n})(x_n) > \dfrac{\varepsilon}{2}$, where $\varphi_\alpha(n) = s_n^\frown k_n$ with $k_n \in \omega$.

d) $\forall\, n \geq 2, \forall\, 1 \leq k \leq n-1, |(x^*_{\varphi_\alpha(n)} - x^*_{s_n})(x_k)| \leq \dfrac{1}{2^n}$.

Assume $x^*_{\varphi_\alpha(k)}$ for $0 \leq k \leq n-1$ and x_k for $1 \leq k \leq n-1$ have been constructed and satisty a)...d). By Lemma 3.3, there exists $i_n < n$ such that $\varphi_\alpha(n) = \varphi_\alpha(i_n)^\frown k_n$, with $k_n \in \omega$. By induction hypothesis $x^*_{\varphi_\alpha(i_n)} \in (B_{X^*})_\varepsilon^{[h_\alpha(\varphi_\alpha(i_n))]}$. Since $h_\alpha(\varphi_\alpha(i_n)) \geq h_\alpha(\varphi_\alpha(n)) + 1$, we have that $x^*_{\varphi_\alpha(i_n)} \in (B_{X^*})_\varepsilon^{[h_\alpha(\varphi_\alpha(n))+1]}$. So for any weak*-neighborhood V of $x^*_{\varphi_\alpha(i_n)}$, diam $(V \cap (B_{X^*})_\varepsilon^{[h_\alpha(\varphi_\alpha(n))]}) > \varepsilon$. In particular there exists $x^*_{\varphi_\alpha(n)} \in (B_{X^*})_\varepsilon^{[h_\alpha(\varphi_\alpha(n))]}$ such that : $\left\|x^*_{\varphi_\alpha(n)} - x^*_{\varphi_\alpha(i_n)}\right\| > \dfrac{\varepsilon}{2}$ and
$\left|(x^*_{\varphi_\alpha(n)} - x^*_{\varphi_\alpha(i_n)})(x_k)\right| \leq \dfrac{1}{2^n}, \forall\, 1 \leq k \leq n-1$.
We conclude the induction by choosing x_n in B_X such that $(x^*_{\varphi_\alpha(n)} - x^*_{\varphi_\alpha(i_n)})(x_n) > \dfrac{\varepsilon}{2}$.

Let Y be the closed linear span of $\{x_n\}_{n=1}^\infty$. Y and the family $(x^*_s)_{s \in T_\alpha}$ constructed by induction satisfy the properties claimed in Lemma 3.4. □

It is now easy to show that $x^*_{\restriction Y} \in (B_{Y^*})_{\varepsilon/2}^{[\alpha]}$. This completes the proof of Proposition 3.1. □

Proof of Proposition 3.2 : It is possible, by using convex combinations, to adapt the proof of Proposition 3.1. But we will use instead a simpler and more global technique that has been indicated to us.

We will show by transfinite induction that for any countable ordinal α, there is a separable subspace Z_α of X such that for any $\gamma \leq \alpha : x^* \in (B_{X^*})_\varepsilon^\gamma$ implies $x^*_{\restriction Z_\alpha} \in (B_{Z_\alpha^*})_\varepsilon^\gamma$.
First we pick x in $X \notin \{0\}$ and call $Z_0 = \mathbb{R}x$.
Assume that the previous statement is true for any $\beta < \alpha$.

If α is a limit ordinal, we choose Z_α to be the closed linear span of $\bigcup_{\beta<\alpha} Z_\beta$.

If $\alpha = \beta + 1$: let us call $V_0 = Z_\beta$. Let D_0 be a countable dense subset of V_0 and $\mathcal{S}_0$ be the collection of half spaces $S = \{x^* \in X^* :\ x^*(z) > q\}$ with z in D_0 and q in $\mathbb{Q}$.
If $S \cap (B_{X^*})_\varepsilon^{\gamma+1} \neq \emptyset$ for some $\gamma \leq \beta$, then diam$(S \cap (B_{X^*})_\varepsilon^\gamma > \varepsilon$ and therefore we can find $u^*,\ v^*$ in $S \cap (B_{X^*})_\varepsilon^\gamma$ and $x = x(\gamma, S)$ in B_X such that $(u^* - v^*)(x) > \varepsilon$.

Let us denote by V_1 the closed linear span of $Z_\beta \cup \bigcup_{\gamma \leq \beta} \bigcup_{S \in \mathcal{S}_0} x(\gamma, S)$.
Then we consider D_1 a countable dense subset of V_1 and we construct V_2 similarly.
Finally $Z_{\alpha+1}$ is the closed linear span of $\bigcup_{n=0}^{\infty} V_n$.

We now need to prove by induction that for any $\gamma \leq \alpha : x^* \in (B_{X^*})^\gamma_\varepsilon$ implies $x^*_{\restriction Z_\alpha} \in (B_{Z^*_\alpha})^\gamma_\varepsilon$. The case $\gamma = 0$ and the limit case are trivial, so let us assume that this is true for γ.
Let $x^* \in (B_{X^*})^{\gamma+1}_\varepsilon$ and let S be a slice of $(B_{Z^*_\alpha})^\gamma_\varepsilon$ containing x^*. We may assume that S is defined by a z in some D_n and by a q in $\mathbb{Q}$. Let u^* and v^* in $S \cap (B_{X^*})^\gamma_\varepsilon$ such that $(u^* - v^*)(x(\gamma, S)) > \varepsilon$.
By induction hypothesis $u^*_{\restriction Z_\alpha}$ and $v^*_{\restriction Z_\alpha}$ belong to $(B_{Z^*_\alpha})^\gamma_\varepsilon$.
Thus $\mathrm{diam}(S \cap (B_{Z^*_\alpha})^\gamma_\varepsilon) > \varepsilon$ and $x^*_{\restriction Z_\alpha} \in (B_{Z^*_\alpha})^{\gamma+1}_\varepsilon$. $\square$

Remark : This method gives similar results about ordinals with a different cardinality and the subspaces of X with corresponding density character.

However a refinement of the technique used in the proof of Proposition 3.1. allows us to obtain the following extension :

PROPOSITION 3.5. *Let X be a Banach space with a separable dual and let $\alpha < \omega_1$. If $Sz(X) > \alpha$, then there is a subspace Z of X such that $\frac{X}{Z}$ has a shrinking basis and $Sz(\frac{X}{Z}) > \alpha$.*

Proof : It will follow from a slight modification of W.B. Johnson and H.P. Rosenthal's proof of the existence of a quotient with a shrinking basis for any Banach space with separable dual ([J-R]).

Since X^* is separable, we may assume that the norm of X is such that the weak* and the norm topologies coincide on the unit sphere of X^*.

Let $\varepsilon > 0$ such that $0 \in (B_{X^*})^{[\alpha]}_{2\varepsilon}$, $(\varepsilon_n)_{n\geq 1} \subseteq (0,1)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $(x_n)_{n\geq 1}$ be a dense subset of B_X. We will construct by induction $(x^*_{\varphi_\alpha(n)})_{n=0}^\infty \subseteq B_{X^*}$ and $(F_n)_{n=1}^\infty$ an increasing sequence of finite subsets B_X verifying :

a) $x^*_{\varphi_\alpha(0)} = 0$.

b) $\forall n \geq 0,\ x^*_{\varphi_\alpha(n)} \in (B_{X^*})^{[h_\alpha(\varphi_\alpha(n))]}_{2\varepsilon}$.

c) $\forall n \geq 1,\ ||x^*_{\varphi_\alpha(n)} - x^*_{s_n}|| > \varepsilon$, (let us denote $y^*_n = \dfrac{x^*_{\varphi_\alpha(n)} - x^*_{s_n}}{||x^*_{\varphi_\alpha(n)} - x^*_{s_n}||}$).

d) For any f in $([y^*_k]^n_{k=1})^*$ with $||f|| \leq 1$, there is $x \in F_n$ such that :

$$\forall y^* \in [y^*_k]^n_{k=1},\ |f(y^*) - y^*(x)| < \frac{\varepsilon_n}{3}||y^*||.$$

e) $\forall x \in F_n\ \ |y^*_{n+1}(x)| \leq \dfrac{\varepsilon_n}{3}$.

f) $\forall n \geq 1\ \ (x_k)^n_{k=1} \subseteq F_n$.

Suppose $(x^*_{\varphi_\alpha(k)})^n_{k=0}$ and F_{n-1} have been constructed. Take F_n satisfying d) and f). As in the proof of Proposition 3.1, we now choose $x^*_{\varphi_\alpha(n+1)}$ in $(B_{X^*})^{[h_\alpha(\varphi_\alpha(n+1))]}_{2\varepsilon}$ such that $||x^*_{\varphi_\alpha(n)} - x^*_{s_n}|| > \varepsilon$ and $|y^*_{n+1}(x)| \leq \dfrac{\varepsilon_n}{3}$ for all x in F_n.

Consequences of this construction : By d) and e), $(y^*_n)^\infty_{n=1}$ is a basic sequence in X^* and, if we denote by P_n the natural projections from $[y^*_k]^\infty_{k=1}$ onto $[y^*_k]^n_{k=1}$, we have that $||P_n|| \to 1$. Let $(y_k)^\infty_{k=1} \subseteq ([y^*_k]^\infty_{k=1})^*$ be the biorthogonal functionnals associated to the basis $(y^*_k)^\infty_{k=1}$. Following the paper of W.B. Johnson and H.P. Rosenthal ([J-R]) it is now possible to check that the operator :

$$T : X \to ([y^*_k]^\infty_{k=1})^*$$
$$x \mapsto Tx, \quad \text{where } Tx(y^*) = y^*(x)$$

maps X onto Y the closed linear span of $(y_k)^\infty_{k=1}$. From this we can deduce, as in [J-R], that $(y^*_k)^\infty_{k=1}$ is a weak*-basic sequence. Finally, since the norm and the weak* topologies coincide on the unit sphere of X^* we can see, still following [J-R], that $(y^*_k)^\infty_{k=1}$ is boundedly complete. Therefore, $(y_k)^\infty_{k=1}$ is shrinking.

Moreover our construction insures that $Sz(Y) > \alpha$. Thus we can conclude the proof by taking $Z = \text{Ker } T$. □

4. MAIN RESULTS.

THEOREM 4.1. *There exists a function $\psi : \omega_1 \to \omega_1$ so that, for any Banach space X and for any countable ordinal α : $Sz(X) \leq \alpha$ implies $\delta^*(X) \leq \psi(\alpha)$.*

Proof. This is an immediate consequence of Proposition 3.2 and Proposition 2.1. The function ψ is the same as the function given by Proposition 2.1. □

Remarks.

1) For a Banach space X, it is possible to define a dentability index $\delta(X)$ and a "weak-Szlenk" index $Sz_\omega(X)$ by peeling the unit ball of X with slices of small diameter or with weakly open sets of small diameter. But the two conditions "$\delta(X) < \omega_1$" and "$Sz_\omega(X) < \omega_1$" are not equivalent, even in the separable case. Indeed the predual B of the James tree space has the Point of Continuity Property and is separable, so $Sz_\omega(B) < \omega_1$; but B does not have the Radon-Nikodym Property, so $\delta(X) = \infty$ (see R.C. James [J], J. Lindenstrauss and C. Stegall [L-S], C.A. Edgar and R.F. Wheeler [E-W]).

2) In general $\psi(\alpha) > \alpha$. For instance, if X is finite dimensional, $Sz(X) = 1$, while $\delta^*(X) = \omega$. Moreover, the condition $\delta^*(X) = \omega$ is equivalent to X super reflexive. But this is not true for the Szlenk index. For example it is easy to check that $Sz((\sum_{n=1}^{\infty} l_1^n)_{l_2}) = \omega$.

On the other hand, the descriptive set theory approach used in section 2 implies that

$$\{\alpha < \omega_1 : \{X \text{ Banach space } : Sz(X) < \alpha\} = \{X \text{ Banach space } : \delta^*(X) < \alpha\}\}$$

contains a closed cofinal subset of ω_1.

THEOREM 4.2. *Let X be a Banach space. If $Sz(X) < \omega_1$, then X admits an equivalent norm whose dual norm is locally uniformly convex. In particular, there is an equivalent Fréchet-differentiable norm on X.*

Proof. This result is proven in [L] under the "a priori" stronger hypothesis : $\delta^*(X) < \omega_1$. □

5. $Sz(\mathcal{C}(K))$ for K SCATTERED COMPACT SPACE.

For a topological space K, the derived space K' is defined to be $K \setminus \{x : x \text{ isolated point of } K\}$; for ordinals α we define $K^{(\alpha)}$ inductively by $K^{(0)} = K, K^{(\alpha+1)} = (K^{(\alpha)})', K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$ for α limit ordinal. Then the space K is said to be scattered if $K^{(\alpha)} = \emptyset$ for some α.

THEOREM 5.1. *Let K be a scattered compact space such that $K^{(\omega_1)} = \emptyset$. Let $\alpha < \omega_1$ be the ordinal such that $K^{(\omega^\alpha)} \neq \emptyset$ and $K^{(\omega^{\alpha+1})} = \emptyset$. Then $Sz(\mathcal{C}(K)) = \omega^{\alpha+1}$.*

As a corollary we obtain the following result of R. Deville [Dev] : if K is a compact space such that $K^{(\omega_1)} = \emptyset$, then there is an equivalent norm on $\mathcal{C}(K)$ whose dual norm is locally uniformly convex.

We will need two classical lemmas.

LEMMA 5.2. *Let K be a compact space and X be a separable subspace of $\mathcal{C}(K)$. Then there exists a compact space L such that :*

i) $\mathcal{C}(L)$ is separable.

ii) X embeds isometrically into $\mathcal{C}(L)$.

iii) there is a map $s : K \to L$ which is continuous and onto.

Proof. Let X be a separable subspace of $\mathcal{C}(K)$. We denote by L_0 the metrizable compact space $(B_{X^*}, \sigma(X^*, X))$. For x in K we call δ_x the element of $(\mathcal{C}(K))^*$ defined by : for any f in $\mathcal{C}(K)$, $\delta_x(f) = f(x)$.
We have that the application $s : K \to L_0$

$$x \mapsto \delta_x\restriction_X \quad \text{is continuous.}$$

Let $L = s(K)$. X embeds isometrically and in a canonical way into $\mathcal{C}(L)$. □

LEMMA 5.3. *Let K and L be two compact spaces and let $s : K \to L$ be continous and onto. Then, for any ordinal α, $L^{(\alpha)} \subseteq s(K^{(\alpha)})$.*

The proof of this lemma is an easy transfinite induction.

Proof of theorem 5.1. Let K be a compact space such that $K^{(\omega^{\alpha+1})} = \emptyset$, with $\alpha < \omega_1$. Let X be a separable subspace of $\mathcal{C}(K)$. By Lemma 5.2, there is a compact space L such that $\mathcal{C}(L)$ is separable, X embeds isometrically in $\mathcal{C}(L)$ and there is a continuous map s from K onto L. Then by Lemma 5.3, $L^{(\omega^{\alpha+1})} = \emptyset$. Since $\mathcal{C}(L)$ is separable and L is scattered we have that L is countable. Now, it is known that for countable compact spaces, $L^{(\omega^{\alpha+1})} = \emptyset$ implies $Sz(\mathcal{C}(L)) \leq \omega^{\alpha+1}$ (see C. Samuel [Sa]). Therefore, for any separable subspace X of $\mathcal{C}(K), Sz(X) \leq \omega^{\alpha+1}$. Thus, by Proposition 3.1, $Sz(\mathcal{C}(K)) \leq \omega^{\alpha+1}$.

Let us mention that the definition of the Szlenk index that we use is not the definition introduced by W. Szlenk [Sz] and used by C. Samuel [Sa]. But the two definitions coincide for X separable Banach space not containing any isomorphic copy of $l_1(\mathbb{N})$ (see [L]) and therefore for $\mathcal{C}(L)$ with L countable compact space.

On the other hand, if $K^{(\omega^\alpha)} \neq \emptyset$, then $Sz(\mathcal{C}(K)) > \omega^\alpha$. More precisely we have that, for any ordinal $\alpha : x \in K^{(\alpha)} \Rightarrow \delta_x \in (B_{\mathcal{C}(K)^*})_1^{[\alpha]}$, where δ_x is the point evaluation at x.

Therefore, under the assumptions of theorem 5.1 we have that $\omega^\alpha < Sz(\mathcal{C}(K)) \leq \omega^{\alpha+1}$. The conclusion of this proof follows immediately from the next proposition.

PROPOSITION 5.4. *Let X be a Banach space such that $Sz(X) < \omega_1$. Then there exists a countable ordinal α so that $Sz(X) = \omega^\alpha$.*

Proof. We will use the following fact : for any Banach space X and any ordinal α

$$(*) \qquad \frac{1}{2}(B_{X^*})_\varepsilon^{[\alpha]} + \frac{1}{2}B_{X^*} \subseteq (B_{X^*})_{\varepsilon/2}^{[\alpha]}.$$

The proof of this is a straightforward transfinite induction.

Claim : $Sz(X) > \omega^\alpha \Rightarrow Sz(X) \geq \omega^{\alpha+1}$.

If $Sz(X) > \omega^\alpha$ then we can find $\varepsilon > 0$ and $x^* \in B_{X^*}$ such that $x^* \in (B_{X^*})_{2\varepsilon}^{[\omega^\alpha]}$. Then, by (*), $0 \in (B_{X^*})_\varepsilon^{[\omega^\alpha]}$.

Thus $\frac{1}{2}B_{X^*} \subseteq (B_{X^*})_{\varepsilon/2}^{[\omega^\alpha]}$. So $(\frac{1}{2}B_{X^*})_{\varepsilon/2}^{[\omega^\alpha]} \subseteq (B_{X^*})_{\varepsilon/2}^{[\omega^\alpha.2]}$. But $0 \in (B_{X^*})_\varepsilon^{[\omega^\alpha]} \Rightarrow 0 \in (\frac{1}{2}B_{X^*})_{\varepsilon/2}^{[\omega^\alpha]}$. Hence $0 \in (B_{X^*})_{\varepsilon/2}^{[\omega^\alpha.2]}$.

Proceeding inductively, we show that for any n in $\omega, 0 \in (B_{X^*})_{\varepsilon/2^n}^{[\omega^{\alpha}.2^n]}$.

Therefore $Sz(X) \geq \omega^{\alpha+1}$. This completes the proof of the Claim.

Now let $\alpha = \text{Inf}\{\gamma : Sz(X) \leq \omega^\gamma\}$. If α is a limit ordinal, $Sz(X) \geq \sup_{\beta<\alpha} \omega^\beta = \omega^\alpha$. So $Sz(X) = \omega^\alpha$. If $\alpha = \beta + 1$, our claim implies that $Sz(X) = \omega^\alpha$. □

Remarks.

1) A similar argument shows that if $\delta^*(X) < \omega_1$, then $\delta^*(X)$ is of the form ω^α.

2) The property described by Proposition 5.4 has been suggested by a paper of A. Sersouri [Se] about the Lavrientiev indices.

6. THREE-SPACE PROBLEM FOR THE CONDITION $Sz(X)$ COUNTABLE.

The general question we are now interested in is the following : let X be a Banach space and Y be a closed subspace of X. Assume that $Sz(Y) < \omega_1$ and $Sz(X/Y) < \omega_1$. Can we conclude that $Sz(X) < \omega_1$?

In this section we answer positively this question by proving the following result :

THEOREM 6.1. *Let X be a Banach space and Y be a closed subspace of X such that $Sz(Y) < \omega_1$ and $Sz(X/Y) < \omega_1$. Then $Sz(X) \leq Sz(X/Y).Sz(Y)$*

Remark. If we can prove this inequality when $Y^\perp$ is separable, then we can use the results of Section 3 to deduce the general case. Indeed, for any separable subspace Z of X, if we call E the closed linear space spanned by Z and Y, since $(E/Y)^*$ is separable we have

$$Sz(E) \leq Sz(E/Y).Sz(Y) \leq Sz(X/Y).Sz(Y)$$

So

$$Sz(Z) \leq Sz(X/Y).Sz(Y)$$

Hence, by Proposition 3.1

$$Sz(X) \leq Sz(X/Y).Sz(Y)$$

Therefore, from now on, we will assume that $Y^\perp$ is separable and we will denote by $\mathcal{V} = (V_n)_{n=1}^\infty$ a basis of open sets for $(B_{Y^\perp}, \sigma(Y^\perp, X/Y))$.

LEMMA 6.2. *Let $\varepsilon > 0$, $F = 3B_{Y^\perp}$ and $B = F + \frac{\varepsilon}{3}B_{X^*}$. For any ordinal α :*

$$B_\varepsilon^{[\omega.\alpha]} \subseteq F_{\varepsilon/3}^{[\alpha]} + \frac{\varepsilon}{3}B_{X^*}.$$

Proof. We will prove this by transfinite induction on α.

By definition of B, it is true for $\alpha = 0$.

Assume this property is true for any $\beta < \alpha$.

If α is a limit ordinal, we have that

$$B_\varepsilon^{[\omega.\alpha]} = \bigcap_{\beta<\alpha} B_\varepsilon^{[\omega.\beta]} \subseteq \bigcap_{\beta<\alpha} (F_{\varepsilon/3}^{[\beta]} + \frac{\varepsilon}{3} B_{X^*}) = F_{\varepsilon/3}^{[\alpha]} + \frac{\varepsilon}{3} B_{X^*},$$

because $(F_{\varepsilon/3}^{[\beta]})_{\beta<\alpha}$ is a decreasing family of $\sigma(Y^\perp, X/Y)$-compact sets.

If $\alpha = \beta+1$: let $(V_{n_i(\alpha)})_{i=1}^\infty = \{V \in \mathcal{V}$ such that $V \cap F_{\varepsilon/3}^{[\beta]} \neq \emptyset$ and $\operatorname{diam}(V \cap F_{\varepsilon/3}^{[\beta]}) \leq \frac{\varepsilon}{3}\}$. We will show by induction that for any $k \geq 1$:

$$B_\varepsilon^{[\omega.\beta+k]} \subseteq (F_{\varepsilon/3}^{[\beta]} \setminus \bigcup_{i=1}^{k} V_{n_i(\alpha)}) + \frac{\varepsilon}{3} B_{X^*}.$$

If we assume that this is true for k, we have that $B_\varepsilon^{[\omega.\beta+k]} \setminus [(F_{\varepsilon/3}^{[\beta]} \setminus \bigcup_{i=1}^{k+1} V_{n_i(\alpha)}) + \frac{\varepsilon}{3} B_{X^*}]$ is a $\sigma(X^*, X)$-open subset of $B_\varepsilon^{[\omega.\beta+k]}$ and is included in $(V_{n_{k+1}(\alpha)} \cap F_{\varepsilon/3}^{[\beta]}) + \frac{\varepsilon}{3} B_{X^*}$. So its diameter is $\leq \varepsilon$. Therefore $B_\varepsilon^{[\omega.\beta+k+1]} \subseteq (F_{\varepsilon/3}^{[\beta]} \setminus \bigcup_{i=1}^{k+1} V_{n_i(\alpha)}) + \frac{\varepsilon}{3} B_{X^*}$.

It follows from these inclusions that

$$B_\varepsilon^{[\omega.\alpha]} \subseteq (F_{\varepsilon/3}^{[\beta]} \setminus \bigcup_{i=1}^{\infty} V_{n_i(\alpha)}) + \frac{\varepsilon}{3} B_{X^*} = F_{\varepsilon/3}^{[\alpha]} + \frac{\varepsilon}{3} B_{X^*}.$$

Let q be the quotient map from X^* to $X^*/Y^\perp$. We have the following lemma.

LEMMA 6.3. *For any ordinal α, $q((B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\alpha]}) \subseteq (B_{X^*/Y^\perp})_{\varepsilon/4}^{[\alpha]}$,*

where $\gamma_\varepsilon = \omega.S_{\varepsilon/3}(F) = \omega.Sz(X/Y, \frac{\varepsilon}{9})$.

Proof. Again the proof is a transfinite induction.

Since $q(B_{X^*}) = B_{X^*/Y^\perp}$, the case $\alpha = 0$ is clear.

Assume this is true for any ordinal $\alpha < \beta$.

If α is a limit ordinal, it is easy to check that the property considered is therefore true for α.

If $\alpha = \beta + 1$: let $x^* \in B_{X^*}$ so that $qx^* \notin (B_{X^*/Y^\perp})_{\varepsilon/4}^{[\alpha]}$. We need to prove that $x^* \notin (B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\alpha]}$, so we may assume that $x^* \in (B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\beta]}$ and then, by induction hypothesis,

$qx^* \in (B_{X^*/Y^\perp})_{\varepsilon/4}^{[\beta]}$. Therefore there is a $\sigma(X^*/Y^\perp, Y)$-neighborhood $\tilde{V}$ of qx^* such that $\text{diam}(\tilde{V} \cap (B_{X^*/Y^\perp})_{\varepsilon/4}^{[\beta]}) \leq \frac{\varepsilon}{4}$. $\tilde{V}$ defines a $\sigma(X^*, X)$-neighborhood V of x^*, and

$$V \cap (B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\beta]} \subseteq x^* + (3B_{Y^\perp} + \frac{\varepsilon}{3} B_{X^*}).$$

By Lemma 6.2, $(3B_{Y^\perp} + \frac{\varepsilon}{3} B_{X^*})_\varepsilon^{[\gamma_\varepsilon]} = \emptyset$.
Therefore $x^* \notin (B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\beta+\gamma_\varepsilon]} = (B_{X^*})_\varepsilon^{[\gamma_\varepsilon.\alpha]}$. □

Proof of Theorem 6.1. We deduce directly from Lemma 6.3 that, for any $\varepsilon > 0$:

$$Sz(X, \varepsilon) \leq \omega.Sz(X/Y, \frac{\varepsilon}{9}).Sz(Y, \frac{\varepsilon}{4})$$

We will use the following easy and technical fact :

Claim : let α and β be two ordinals ≥ 1. If $\gamma < \omega^\alpha$ then $\omega.\gamma.\omega^\beta \leq \omega^\alpha.\omega^\beta$

We want now to prove that $Sz(X) \leq Sz(X/Y).Sz(Y)$. It is clear that if $\dim(Y)$ is finite then $Sz(X) = Sz(X/Y)$ and that if $\dim(X/Y)$ is finite then $Sz(X) = Sz(Y)$. Therefore we may assume that $Sz(Y) \geq \omega$ and that $Sz(X/Y) \geq \omega$. Then, if we combine the claim above with Proposition 5.4 we can conclude that $Sz(X) \leq Sz(X/Y).Sz(Y)$. □

We will end this section with a slight improvement of the above inequality in the case where Y is complemented in X. This will allow us to compute $Sz(X)$ in some particular cases.

LEMMA 6.4. *Let X a Banach space and Y a complemented subspace of X. If $Sz(Y) < \omega_1$ and $Sz(\frac{X}{Y}) < \omega_1$, then there exists a constant $C > 0$ such that :*

$$\text{for any } \varepsilon > 0,\ Sz(X, \varepsilon) \leq Sz(Y, \frac{\varepsilon}{C}).Sz(\frac{X}{Y}, \frac{\varepsilon}{C}).$$

Proof : It is enough to show that if $X = Y \oplus_1 Z$, then for any $\varepsilon > 0$:
$Sz(X, \varepsilon) \leq Sz(Y, \varepsilon).Sz(Z, \varepsilon)$.
This can be done by a straightforward double transfinite induction. □

Remark : Now it is not difficult to see that if $Sz(Y) \leq \omega$ and $\dim(Z) = \infty$, then $Sz(X) = Sz(Z)$.

If we combine this remark with Proposition 3.1, we get the following result :

PROPOSITION 6.5. *Let X be a Banach space and Y be an infinite codimensional subspace of X isomorphic to $c_0(\mathbb{N})$.*

If $Sz(\frac{X}{Y}) < \omega_1$, then $Sz(X) = Sz(\frac{X}{Y})$.

Proof. By Proposition 3.1, it is enough to show that for any separable subspace E of X containing Y and such that Y is of infinite codimension in E, we have $Sz(E) \leq Sz(\frac{X}{Y})$. But Sobczyk's theorem (see [So]) implies that Y is complemented in E. Moreover it is easy to check that $Sz(c_0(\mathbb{N})) = \omega$. Therefore, by the above remark, $Sz(E) = Sz(\frac{E}{Y}) \leq Sz(\frac{X}{Y})$. □

Example : Let JL be the space constructed by W.B. Johnson et J. Lindenstrauss (see [J-L] for the definition and the main properties of this space).
JL contains a subspace Y isometric to $c_0(\mathbb{N})$ and such that $\frac{JL}{Y}$ is isometric to $l_2(\Gamma)$, where Γ is a certain uncountable set.
Thus, by Proposition 6.5, $Sz(JL) = Sz(l_2(\Gamma))$.
But $l_2(\Gamma)$ is uniformly convex, so $Sz(l_2(\Gamma)) = \omega = Sz(JL)$. □

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