Hyperbolic techniques for scalar degenerate parabolic problems

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a general survey of approaches,
ongoing works on IBVP with Mohamed K. Gazibo [1], Guy Vallet [2]

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HYPERBOLIC TECHNIQUES FOR PHASE TRANSITIONS
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Plan of the talk

Problems, general form.

Degenerate hyperbolic-parabolic-(elliptic) problems:

$$b(u)_t + \mathrm{div}_x f(u) - \Delta_{(p)}\phi(u) = 0$$

+ initial conditions
+ boundary conditions, interface coupling...

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- the diffusion operator is Leray-Lions (laplacian, p-laplacian,...)

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- $b(\cdot)$ and $\phi(\cdot)$ are non-decreasing
- ... and they may have flat regions.

Case $b = Id$, $\phi \equiv 0 \implies$ hyperbolic conservation law!

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Examples: Stefan, Hele-Shaw, sedimentation,...
In some examples "flat" and "non-flat" regions of $b(\cdot)$ and $\phi(\cdot)$ correspond to phases.

Techniques of the hyperbolic case

Goal: highlight hyperbolic techniques that remain useful (or even crucial) for the degenerate hyperbolic-parabolic case.

In the sequel: assume $b = Id$ (no elliptic degeneracy) .

NB: elliptic degeneracy is "easy" to treat using in addition techniques of [Alt, Luckhaus '83], [Otto '96] , [Ammar, Wittbold '03] . Example (somewhat extreme): "triply degenerate" problems [A., Bendahmane, Karlsen, Ouaro'09] : $\exists$, !, stability wrt all data

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Some selected techniques and facts from the hyperbolic case:

- entropy formulations, doubling of variables
- L^p stability ($1 \leq p \leq \infty$), BV stability, L^1 contraction
- kinetic, renormalized, mild (nonlinear semigroup) solutions

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- measure valued solutions / strong precompactness properties
- initial traces (time continuity), strong boundary traces
- delicate formulations of boundary conditions (BLN, Otto...)

Defining entropy solutions

- Classical solutions, free boundary formulations... difficulties ! Key property at the formal level (say, classical solutions or viscosity approximation or monotone numerical scheme): one can multiply by $\mathrm{sign}\,(u-v)$ and get L^1 contraction
- 1D formulations based on mild solutions [Bénilan, Touré '84,'95]
- Early attempt of global treatment with entropy inequalities: [Vol'pert, Hudjaev '69] in BV setting. Wrong jump conditions and uniqueness proof, corrected in [Wu, Yin '89], [Evje, Karlsen '00]

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- Entropy formulation of [Carrillo '99]: $u \in L^\infty$, $\phi(u) \in L^2(0,T;H^1_{loc})$
  Idea: require "Kruzhkov" entropy inequalities

$$\forall k \quad |u-k|_t + \mathrm{div}_x \operatorname{sign}(u-k)\big(f(u)-f(k)-\nabla\phi(u)\big) \le 0$$

and use doubling of variables + dissipation information

$$|u-k|_t + \mathrm{div}_x \operatorname{sign}(u-k)\big(f(u)-f(k)-\nabla\phi(u)-D\big) \le \lim_{\alpha\downarrow 0} \frac{1}{\alpha} 1\!\!1_{0<|\phi(u)-\phi(k)|<\alpha} \nabla\phi\cdot(D-\nabla\phi) \quad \forall k \notin \mathrm{Flat}(\phi), \quad \forall D.$$

Well-posedness: [Carrillo '99],[Carrillo, Wittbold '99],...

Alternative and related notions of solution

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Related problems: degenerate fractional parabolic, $u_t+\operatorname{div} f(u)+(-\Delta)^{\lambda/2}[\phi(u)]=0$. One can keep the dissipation into a non-singular term even in the Kruzhkov-like formulation! [Alibaud '07], [Karlsen, Ulusoy '12], [Alibaud, Cifani, Jakobsen '12] . Idea: see the local problem as limit ($\lambda\uparrow 2$) of the fractional one!

“Hyperbolic” technical elements that remain operational

- Doubling of variables works if one keeps record of parabolic dissipation (useful even for nondegenerate pb: [A., Igbida ’06,’12])
- Kinetic approach (for quasilinear problems) works
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- Well-posedness in the whole space, for Lipschitz/nice Hölder flux [Maliki, Touré ’03], [Maliki, Ouédraogo ’09], [A., Maliki ’10]
- Stability wrt nonlinearities [Karlsen, Risebro ’03], [Chen, Karlsen ’06]

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Conclusion: hyperbolic techniques work, at least away from boundary.
Limitations: e.g. $u_t + \mathrm{div}_x f(u) - \Delta_{p(x)}\phi(u) = 0$, doubling doesn’t work

Techniques for hyperbolic boundary/interface problems

Principle: BC should generate boundary terms that are dissipative

Problem: boundary terms are, most of the time, formal

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The mainstream nowadays are "strong trace" approaches:

- [Bardos, LeRoux, Nédélec '79],[Vasseur '01]
  If $u = u^D$ is "imposed" at the boundary , then there exists a unique solution verifying $\gamma u \in BLN(u^D)$: trace belongs to the domain of a maximal monotone graph , [Dubois, LeFloch '88].
- [Bürger, Frid, Karlsen '07] If f is compactly supported, there exists a unique entropy solution to $u_t + \mathrm{div}_x f(u) = 0$ with $f(u) \cdot \nu = 0$, and the zero-flux BC is satisfied literally.
- [A., Sbihi '07,'14] general "dissipative" BC $f(u) \cdot \nu \in \beta(u)$ for a maximal monotone graph β: results similar to BLN case

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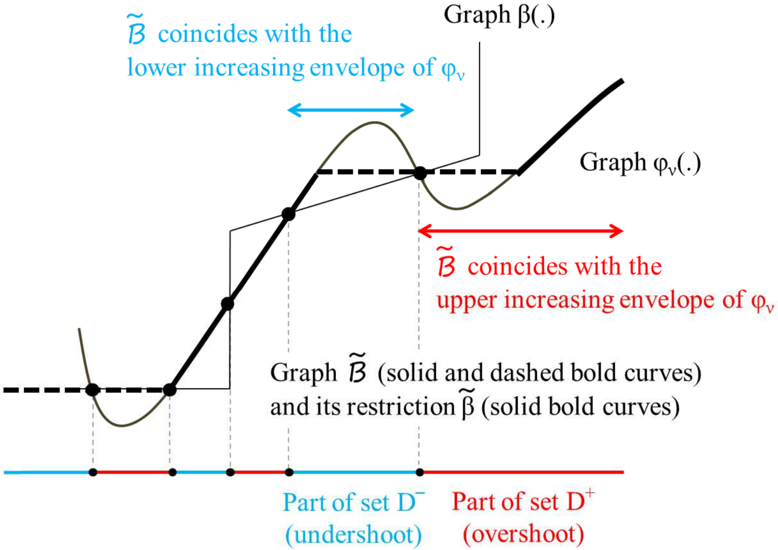
Further, there were much more delicate "weak trace" approaches:

- [Otto '96] / [Malek, Nečas, Rokyta, Ruzička '96] an approach to the general Dirichet BC pb. using Otto's boundary entropies and the weak trace framework of [Chen, Frid '99]
- [Carrillo '99],[Vovelle '02],[Ammar, Carrillo, Wittbold '06] using semi-Kruzhkov (Serre) entropies and... "no traces at all".

Example of general BC problem: projection procedure [A., Sbihi '14]



BC: techniques that still apply, and limitations

What does work:

- Otto weak-trace technique for Dirichlet boundary conditions, with some regularity limitations: [Mascia, Porretta, Terracina '02], [Michel, Vovelle '02]; [Vallet '05] , (t, x)-dependence ok
- under strong BV assumptions on the data, Dirichlet conditions can be given an explicit non-formal interpretation [Rouvre, Gagneux '99], [Evje, Karlsen '00],...
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What does not work well:

- for zero-flux BC: arguments exploiting weak traces...
- ...and justification of existence of strong boundary traces of the normal flux $(f(u) - \nabla\phi(u)) \cdot \nu$ is very difficult !

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Fortunately, one particular case yields strong flux traces: the 1D stationary problem $u+\big(f(u)-\phi(u)_x\big)_x=g$ degenerate elliptic
Density + semigroup arguments help to treat 1D evolution problem.

Entropy formulation and technical difficulty

Zero-flux problem $(f(u) - \nabla\phi(u)) \cdot \nu = 0$ on $\Sigma = [0, T] \times \partial\Omega$.
Assumption: $f(0) = 0 = f(u_{max})$, i.e., enforcing L^∞ bound.
Definition of solution [A., Gazibo '12] , cf. [Bürger, Frid, Karlsen '07] :

$$\forall k \quad |u-k|_t + \big(\mathrm{sign}\,(u-k)\big(f(u) - f(k) - \phi(u)_x\big)_x \leq |f(k)\cdot\nu|\, d\mathcal{H}^1_\Sigma$$

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Existence: vanishing viscosity, standard.
Uniqueness: make doubling of variables, boundary terms appear :

$$|u-v|_{(t+s)} + \cdots \leq \mathrm{sign}\,(u-v)(f(u) - \phi(u)_x)(t,x) \cdot \nu(y) \, d\mathcal{H}^1_\Sigma(s,y) + \text{symm.}$$

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Way out: use stationary problem and nonlinear semigroup tools [Bénilan '72] : integral solutions.

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Hint: 1D stationary solutions are trace-regular $\Longrightarrow$ well-posedness, [A., Gazibo '12] . Ongoing [Gazibo '??] : convergence of FV scheme?
Tool: introduction of auxiliary integral-process solutions :-)

Strong sense of BC for the Dirichlet problem

In the light of recent advances of hyperbolic techniques
+ the above example (dealing with zero-flux problem):
what can be done for degenerate parabolic problem with other BC?

Strong trace approach: [A., Gazibo, Vallet '??]
reformulate the formal Dirichlet BC as a general "BLN-kind" BC:

- standard Dirichlet on $\{x : \ u^D(x) \notin \text{Flat}(\phi)\}$
- on $\{x : \ u^D(x) \in \text{Flat}(\phi)\}$, parab./hyperbolic strongly interfere:
  the BLN graph is cut by obstacles at "corner points" of ϕ.

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Program (ongoing) for Dirichlet BC and more:

- trace-regularity ok in very few situations; try to drop 1D limitation !??
- formulate notion of solution without trace-regularity
- uniqueness using the approach of "general-to-regular" comparision
  + density of regular solutions
- justify the singular limit projection procedure:
  Dirichlet for approximate solutions $\Rightarrow$ BLN-like BC at the limit
- extend to other BC (e.g., zero-flux without $f(0) = 0 = f(u_{max})$).

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- trace-regularity ok in very few situations; try to drop 1D limitation !??
- formulate notion of solution without trace-regularity
- uniqueness using the approach of "general-to-regular" comparision
  + density of regular solutions
- justify the singular limit projection procedure:
  Dirichlet for approximate solutions $\Rightarrow$ BLN-like BC at the limit
- extend to other BC (e.g., zero-flux without $f(0) = 0 = f(u_{max})$).

THANK YOU FOR YOUR ATTENTION !