Preface

One of the major open problems in the representation theory of finite groups is the determination of the irreducible representations of the symmetric group $\mathfrak{S}_n$ over a field of characteristic $p > 0$. Thanks to the work of James [179] in the 1970s, we do have a natural parametrisation of the irreducible representations in the framework of the theory of Specht modules, but explicit combinatorial formulae for their dimensions are not known in general! Note that the analogous problem in characteristic 0 has been solved for a long time, by the work of Frobenius around 1900.

In a wider context, this problem is a special case of the problem of determining the irreducible representations of Iwahori–Hecke algebras. These algebras arise naturally in the representation theory of finite groups of Lie type, but they can also be defined abstractly as certain deformations of group algebras of finite Coxeter groups, where the deformation depends on one or several parameters. For the purposes of this introduction, let us assume that all the parameters are integral powers of a fixed element in the base field. If this base parameter has infinite order and the base field has characteristic 0, then we are in the "generic case" where the algebras are semisimple; this case is quite well understood [132], [231]. Also note that, both for historical reasons and as far as applications are concerned, the case where all parameters are equal is particularly important.

The main focus in this text will be on the "modular case" where the algebras are non-semisimple. This situation typically occurs over fields of positive characteristic (a familiar phenomenon from the representation theory of finite groups), but it also occurs over fields of characteristic 0 when the base parameter is a root of unity. While leading to a highly interesting and rich theory in its own right, it turns out that the study of the characteristic 0 situation also provides a crucial step for understanding the positive characteristic case, which is most important for applications to finite groups of Lie type.

Over the last two decades, there has been considerable progress on the characteristic 0 situation. One of the most spectacular advances is the "LLT conjecture" [208] (where "LLT" stands for Lascoux, Leclerc, Thibon) and its proof by Ariki [7], [10]. This brings deep geometric methods and the combinatorics of crystal/canonical bases of quantum groups into the picture, opening the way for a variety of new theoretical connections and practical applications. Combined with sophisticated computational methods, the theory has now reached the following state:

- The classical theory of "Specht modules" has been generalised to Iwahori–Hecke algebras associated to arbitrary finite Coxeter groups, giving rise to natural parametrisations of the irreducible representations.
- Explicit descriptions of these parametrisations are now known in terms of so-called "canonical basic sets". Also, the dimensions of the irreducible representations are known, either by purely combinatorial algorithms (for the classical types) or in the form of explicit tables (for the exceptional types).

These results remain valid over fields of characteristic $p > 0$, as soon as p is larger than some bound depending on the type of the algebra. As far as the parametrisation of the irreducible representations is concerned, the bound is very mild. For example, in the equal-parameter case, it will turn out that it is sufficient to assume that the characteristic is "good" in the sense of the theory of algebraic groups. However, as far as the dimensions of the irreducible representations are concerned, no explicit bound on p is known at the present state of knowledge.

But there is a general conjecture – first formulated by James [181] in type A – specifying such a bound. This conjecture has been verified in a number of cases, including algebras of type A_n for $n \leqslant 9$ (see [181]) and all algebras of exceptional type (see Geck, Lux, and Müller [94], [126], [129]). If true, this conjecture would also yield explicit results about the dimensions of the irreducible representations of the symmetric group $\mathfrak{S}_n$ in characteristic $p > 0$ where p is such that $p^2 > n$.

The purpose of this book is to develop the general theory along the above lines and to show how it is transformed into explicit results. In a sense, this book tries to do for representations of Iwahori–Hecke algebras at roots of unity what the book by Geck and Pfeiffer [132] did for the "generic case". However, while [132] was essentially self-contained, the situation is more complex here. In fact, in order to obtain our main results, we rely on the following sources:

- Ariki's proof [7] of the LLT conjecture.
- Certain deep properties of Kazhdan–Lusztig cells [222], [231] which do not seem to be accessible by elementary methods.
- The existence and basic properties of "canonical bases" and "crystals" for the Fock space representations of certain quantised enveloping algebras.

The first two ultimately rely on deep geometric theories, an exposition of which would go far beyond the scope of this text. Fortunately, this material is now more readily accessible through a number of books; for example, Kirwan [201], Chriss and Ginzburg [50], Hotta et al. [159], Kiehl and Weissauer [197]. Also note that the geometry only plays a role in the proofs, but not in the formulation of the results! (It is not completely impossible that, some day, more direct and purely algebraic proofs will be found.) Much of what we need about crystal and canonical bases can be found in Ariki's book [10]; see also Jantzen [185], Kashiwara [191], Lusztig [230]. Our general policy regarding these topics is that we shall introduce the required notation to state the results that we need, but we will not endeavour to give the proofs. In this way, we can keep the size of this text within reasonable limits, and yet present some substantial results and applications.

The origin of the theory of Iwahori–Hecke algebras lies in the representation theory of finite groups of Lie type, where these algebras arise as endomorphism algebras of certain induced representations. Via some natural functors, a well-defined part of the representation theory of a finite group of Lie type is controlled by the representation theory of Iwahori–Hecke algebras. Thus, the theory and the results that we are going to present in this book form a contribution to the general project of determining the irreducible representations of all non-abelian finite simple groups. Note that such a group is either an alternating group of degree at least 5, or a simple group of Lie type, or one of 26 sporadic simple groups; see Gorenstein et al. [142].

A rough outline of the contents of this book now follows.

Chapters 1 and 2 provide a general introduction to the representation theory of Iwahori–Hecke algebras and, thus, may be of some independent interest. The discussion will be based on the Kazhdan–Lusztig theory of "cells" [195], [219]. In Lusztig's work [220] on characters of reductive groups over finite fields, a crucial role is played by the "**a**-function", which associates with every irreducible representation E of a finite Coxeter group a numerical invariant $\mathbf{a}_E$. One of the main themes of this book will be to show that these invariants play a similarly important role for "modular" representations. In Theorem 2.6.12, this culminates in the construction of a "cell datum" in the sense of Graham and Lehrer [144], giving rise to a general theory of "Specht modules" for Iwahori–Hecke algebras. (These results originally appeared in [111], [112].) Thus, we now see that the original Specht module theory in type A, due to Dipper and James [62] and Murphy [256], [257] (see also the exposition by Mathas [245]), is the prototype of a picture which applies to all Iwahori–Hecke algebras associated with finite Coxeter groups.

In our exposition, we pay a particular attention to treating Iwahori–Hecke algebras of type A as a model case. The required results on Kazhdan–Lusztig cells will be established in a complete and self-contained manner, where no use of geometry is required; see Section 2.8. This treatment of type A is new and entirely independent of the original approach by Dipper, James, and Murphy.

In Chapter 3, we study non-semisimple Iwahori–Hecke algebras in the spirit of Brauer's classical "modular representation theory" involving, in particular, blocks and decomposition numbers. We shall assume that the reader has some familiarity which the basic features of this theory (for a general finite-dimensional associative algebra); this is readily accessible in standard reference texts, like Curtis and Reiner [53] and Feit [83]. In this setting, we define the key concept of a "canonical basic set" in Section 3.2. This concept is independent of the existence of a Graham–Lehrer cell datum, but, in a sense, it captures precisely those features of a cell datum which can be seen by looking only at the decomposition matrix of the algebra. Again, we treat Iwahori–Hecke algebras of type A as a model case. In Section 3.5 we give a new proof of the classification of the modular irreducible representations of these algebras. For this purpose, we have found it convenient to introduce the formal concept of an "abstract Fock datum" in Section 3.4. In another direction, we present a factorisation result for decomposition matrices and formulate a general version of

James's conjecture. The exposition in Section 3.7 unifies the original formulation of James [181] with the further developments in [92], [98], [129], [133].

In Chapter 4 we explain the fundamental connection between Iwahori–Hecke algebras and representations of a finite group of Lie type $G(\mathbb{F}_q)$ (where q is a power of a prime number and $\mathbb{F}_q$ denotes a finite field with q elements). We begin with a self-contained discussion of the Schur functor and its variations, where we combine the original approach of Dipper [58] with later developments by Cline et al. [51] and Schubert [279]. Following [109], we then show in Theorem 4.4.1 how our results on "cell data" and "canonical basic sets" lead to a natural parametrisation of the modular irreducible representations of $G(\mathbb{F}_q)$ which admit non-zero vectors fixed under a Borel subgroup. This generalises classical results from the characteristic 0 situation (due to Bourbaki, Iwahori, Tits, ...) to positive characteristic. We also explain how this fits into a (conjectural) classification of all irreducible representations of $G(\mathbb{F}_q)$ in the "non-defining characteristic case".

The determination of canonical basic sets for the classical types B_n and D_n has turned out to be an extremely difficult problem. At the end of Chapter 4 we shall discuss some cases that can be dealt with by elementary methods, based on the work of Dipper, James, and Murphy [66], [68]. The solution in the general case requires completely new methods; this will be achieved as a consequence of the results presented in Chapters 5 and 6.

For this purpose, it will be convenient to work in the framework of the theory of *Ariki–Koike algebras*, which are generalisations of Iwahori–Hecke algebras of type B_n. The main idea of Chapter 5 is to try to generalise as much as possible the combinatorial constructions involved in the discussion of type A in Chapter 3. This leads us to consider in Section 5.7 certain special choices of the parameters which arise from the combinatorics of "FLOTW multipartitions" (where FLOTW stands for Foda, Leclerc, Okado, Thibon, Welsh [88]); these special choices cover, in particular, the equal parameter case for Iwahori–Hecke algebras of type B_n and D_n. As a consequence, in Theorem 5.8.2, we can state the main result concerning the determination of canonical basic sets for this choice of parameters. The methods in Chapter 5 do not allow us to complete the proof of this theorem. The missing piece is a result about the number of irreducible representations of Ariki–Koike algebras which is due to Ariki and Mathas [15] and which relies on the deep work of Ariki [7] on the proof of the LLT conjecture. This will be discussed in Chapter 6.

The idea that FLOTW multipartitions are relevant in the modular representation theory of Iwahori–Hecke algebras of classical type first appeared in the work of Jacon [172], [173], [174]. Originally, the base field for the algebras was assumed to be of characteristic 0. The new approach developed in Chapter 5 shows that these results also hold for fields of positive characteristic.

In Chapter 6 we introduce the quantised enveloping algebra $\mathscr{U}_q(\widehat{\mathfrak{sl}_e})$ and study the canonical bases of certain Fock space representations. The associated "crystals" carry some rich combinatorial structure which will be discussed in detail. We can state (without proof) Ariki's theorem [7] which links the canonical bases of the Fock space representations to the irreducible representations of Ariki–Koike algebras at roots of unity. This allows us to complete the proofs of the main results of the previous chapter; see Section 6.3. Since this only covers certain choices of the parameters for Ariki–Koike algebras, we then go further and present some deep results of Uglov [291] on the canonical bases of the Fock space representations. We show how this leads to an explicit description of the "canonical basic sets" for Ariki–Koike algebras at roots of unity – and, hence, of Iwahori–Hecke algebras of classical type – for any choice of the parameters, assuming that the base field is of characteristic 0; see Theorem 6.7.2. We also derive purely combinatorial algorithms for computing decomposition numbers and the dimensions of the irreducible representations (in characterictic 0).

Finally, Chapter 7 contains explicit results concerning Iwahori–Hecke algebras of exceptional type H_3, H_4, F_4, E_6, E_7, E_8. We also explain some basic algorithmic methods, including Parker's MEATAXE. The project of computing the decomposition matrices for these algebras (over fields of characteristic 0) was started almost 20 years ago in [126] and finally completed in [129]; the matrices for type E_8 appear here for the first time in print. From these matrices, one can simply read off the corresponding "canonical basic sets".

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