On hyperbolic-parabolic equations with discontinuous flux,

or

How much information should be contained in (adapted) entropy inequalities ?

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ongoing work with Clément Cancès [2] and Shyam S. Ghoshal [3,1,2]

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Plan of the talk

Problem that we want to study

Degenerate hyperbolic-parabolic equation with discontinuous flux:

$$u_t + (f(x,u) - \varphi(x,u)_x)_x = 0$$

Applications: in sedimentation models [S. Diehl], [R. Bürger et al.] ,...)

Main feature addressed in this talk:

heterogeneous models with discontinuous dependence of f, φ in x, physically motivated ways to define (entropy) solutions.

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with $f_{L,R}$ (Lipschitz) continuous,

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Strategy: use well-known characterization of solution away from the interface (“local” entropy solution); make explicit the interface coupling (or interface transmission conditions).

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Physical feature: $\varphi_{L,R}(u)_x$ is in fact $\lambda_{L,R}(u)\pi_{L,R}(u)_x$. $\Rightarrow$ Coupling:

both quantities $\pi(x, u(x))$ and $\mathcal{F}[u](x) := f(x, u(x)) - \varphi(x, u(x))_x$ should be transmitted across the interface $\{x = 0\}$.

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– The interface coupling should also be “dissipative” , in some sense.
– There are infinitely many choices to prescribe such coupling ; accordingly, there are infinitely many “nice” solution semigroups.

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Our goals:
– describe and classify all possible transmission conditions, make a link with modeling assumptions
– Design an approximation procedure for every kind of coupling.

The homogeneous hyperbolic and degen. parabolic problems

Homogeneous medium (=no dependence on x):
notion of solution and well-posedness are well established :

– hyperbolic case, [Kruzhkov'69]
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- approximate solutions naturally satisfy the L^p bounds for all p,
  the L^1 contraction property $\frac{d}{dt}\|u^\varepsilon - \widehat{u}^\varepsilon\|_{L^1} \leq 0$ and
  the Kato inequality $|u^\varepsilon - \widehat{u}^\varepsilon|_t + Q(u^\varepsilon, \widehat{u}^\varepsilon)_x \leq \varepsilon |u^\varepsilon - \widehat{u}^\varepsilon|_{xx}$ in $\mathcal{D}'$

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- approximations converge strongly (a.e.) as $\varepsilon \to 0$, in particular, L^p bounds, L^1 contraction, Kato ineq. are inherited at the limit.
  The viscosity limits are considered as admissible weak solutions
  ($\Rightarrow$ solution $\equiv$ a singular limit).

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  The viscosity limits are considered as admissible weak solutions
  ($\Rightarrow$ solution $\equiv$ a singular limit).
- the limits are characterized intrinsically by a family of inequalities
  ($\Rightarrow$ entropy / integral / kinetic / renormalized / ... solutions).
  Uniqueness for such notion(s) of solution can be proved.

Characterization and uniqueness of admissible solutions

Intrinsic characterization of solutions of $u_t + (f(u))_x =$ "$0^+ u_{xx}$" :
most usually, entropy inequalities of [Kruzhkov] are used
(other possibilities: kinetic formulation [Lions, Perthame, Tadmor] ,
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Entropy solutions and integral solutions are based on close ideas.

- For entropy solutions (L^∞ data):
  - exhibit very few explicit solutions, namely,
    say that every constant function $\widehat{u} \equiv k$ is an evident solution
  - postulate the Kato inequality $|u - k|_t + Q(u, k)_x \leq 0$ in $\mathcal{D}'$
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Degen. parabolic case $u_t + (f(u) - \varphi(u)_x)_x =$ "$0^+ u_{xx}$": fully analogous

Discontinuous-flux problem: the hyperbolic case

Strange feature: there are infinitely many L^1-contractive semigroups for $u_t + (f_L(u)\mathbb{1}_{x<0} + f_R(u)\mathbb{1}_{x>0})_x = 0$.

A physically motivated class of solutions [A.,Cancès'13] :
start with "discontinuous-viscosity" $\varepsilon(\lambda(x,u)\pi(x,u)_x)_x$,
here with $\varepsilon\pi'_{L,R} > 0$ (uniformly parabolic).
Actually $\pi_{L,R}$ may come from a more precise model (in the case of [A.,Cancès] they account for negligible capillary pressure effects for the Buckley-Leverett equation in porous media models).

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For $\varepsilon > 0$, the solution u^ε satisfies two transmission conditions:

$$f_L(u^\varepsilon) - \varepsilon\lambda_L(u^\varepsilon)\pi_L(u^\varepsilon)_x|_{x=0^-} = f_R(u^\varepsilon) - \varepsilon\lambda_R(u^\varepsilon)\pi_R(u^\varepsilon)_x|_{x=0^+}$$

and also $\pi_L(u^\varepsilon)|_{x=0^-} = \pi_R(u^\varepsilon)|_{x=0^+}$.

At the limit, the first one is inherited (Rankine-Hugoniot)
but the second one is relaxed ("Bardos-LeRoux-Nédélec") .
Moreover, at the limit the first condition is fully determined by $f_{L,R}$.
But the second condition at the limit keeps depending on profiles $\pi_{L,R}$ that are not given beforehand!

Generalized pressure-and-flux transmission condition

Indeed, assume that $\pi_L(u)(t,0^-) \equiv \pi_R(u^\varepsilon)(t,0^+) \longrightarrow p(t)$ as $\varepsilon \to 0$. Then,

the limit u solves the pair of Dirichlet problems doubly coupled at $\{x=0\}$:

$u_t + f_L(u)_x = 0$ in $\{x<0\}$ with BC $u(t,0^-) = \pi_L^{-1}(p(t))$

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Equivalently, using the notation $(a,b) \mapsto F_{L,R}(a,b)$ for the Godunov fluxes and writing $u_L = u(0^-), u_R = u(0^+)$, one requires

$$f_L(u_L) = F_L(u_L, \pi_L^{-1}(p)) = F_R(\pi_R^{-1}(p), u_R) = f_R(u_R).$$

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$\Rightarrow$ an intrinsic description:
the vanishing capillarity limit is a local entropy solution satisfying the generalized transmission conditions.
How to prove uniqueness of solution characterized in this way?

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NB. The transmission condition depends on the choice of $\pi_{L,R}$!
(reflecting, e.g., the - formally neglected - capillary pressure forces)

Admissibility germs and well-posedness theory

Following many preceding works (since early 1990ies) on interface conditions and uniqueness, in [A., Karlsen, Risebro'11] we have classified all dissipative interface coupling conditions as follows.

- We fix a set $\mathcal{G} \subset \mathbb{R}^2$.
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  Flux transmission $f_L(u_L) = f_R(u_R)$ is mandatory for $(u_L, u_R) \in \mathcal{G}$.
- Such a set is called L^1-dissipative germ if $\forall (u_L, u_R), (c_L, c_R) \in \mathcal{G}$

$$\operatorname{sign}(u_L - c_L)(f_L(u_L) - f_L(c_L)) \geq \operatorname{sign}(u_R - c_R)(f_R(u_R) - f_R(c_R))$$

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Example. The germ $\mathcal{G}_{\pi_{L,R}}$ defined by: $(u_L, u_R) \in \mathcal{G}_{\pi_{L,R}} \Leftrightarrow$

$$\exists p \quad f_L(u_L) = F_L(u_L, \pi_L^{-1}(p)) = F_R(\pi_R^{-1}(p), u_R) = f_R(u_R).$$

This example is rather generic (cf. [Adimurthi,Mishra,Gowda]).

Elementary solutions and adapted entropy inequalities

There is a very simple and natural way to embed the resulting theory into the Kruzhkov framework.

- Elementary $\mathcal{G}$-solutions: for $(c_L, c_R) \in \mathcal{G}$, the function $k(x) := c_L \mathbf{1}_{x<0} + c_R \mathbf{1}_{x>0}$ is a local entropy solution and it fulfills the interface coupling condition imposed by the choice of a complete maximal L^1-dissipative germ $\mathcal{G}$.

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  adapted entropy ineq.[Audisse,Perthame],[Bürger,Karlsen,Towers]
- The maximality of $\mathcal{G}$ implies that traces $(u(t,0^-), u(t,0^+))$ belong to $\mathcal{G}$ (if such traces exist !.. [Vasseur],[Panov])
- Then the L^1-dissipativity of $\mathcal{G}$ implies uniqueness
- The completeness of $\mathcal{G}$ implies existence (WFT, Glimm,...)

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- As Kruzhkov has done, we postulate the Kato inequality between u and $\hat{u}(t,x) = k(x)$: $|u - k(x)|_t + (Q(x, u, k(x))_x \leq 0$:

  adapted entropy ineq.[Audisse,Perthame],[Bürger,Karlsen,Towers]
- The maximality of $\mathcal{G}$ implies that traces $(u(t,0^-), u(t,0^+))$ belong to $\mathcal{G}$ (if such traces exist !.. [Vasseur],[Panov])
- Then the L^1-dissipativity of $\mathcal{G}$ implies uniqueness
- The completeness of $\mathcal{G}$ implies existence (WFT, Glimm,...)

NB. The notion of integral solution can also be useful, e.g., when it is difficult to get traces ([A.'13]). In the situation where $f_{L,R}$ depend (smoothly) on x, existence of traces $u(t,0^\pm)$ is not known. But it is much simpler to get traces for the stationary problem $u - f(x,u)_x = g$

Germs (extended) and a wider set of elementary solutions

There are few references on the subject. Relevant for us:
[Karlsen,Risebro,Towers'03],[Diehl'09] , [Cancès,Gallouët,Porretta'09]

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Known formulations use:

– Traces $u(t,0^{\pm})$ of the solution on the interface
– Traces $\mathcal{F}[u](t,0^{\pm})$ of the flux $\mathcal{F}(u) = f(x,u) - \varphi(x,u)_x$.

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Idea: define a notion of germ $\mathcal{H} \subset \mathbb{R}^4$ which has the sense of

$$(u_L, \mathcal{F}_L; u_R, \mathcal{F}_R) \in \mathcal{H} \quad \Leftrightarrow \quad \begin{array}{l} u_{L,R} \text{ are traces of solution } u \\ \mathcal{F}_{L,R} \text{ are traces of flux } \mathcal{F}[u] \end{array}$$

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Elements of the hyperbolic theory revisited: structure of $\mathcal{H}$

- some restrictions on $\mathcal{H}$ coming from degeneracy .
  E.g., $\varphi_{L,R} \equiv 0$ enforces $\mathcal{F}_{L,R} = f_{L,R}(u_{L,R})$

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  E.g., $\varphi_{L,R} \equiv 0$ enforces $\mathcal{F}_{L,R} = f_{L,R}(u_{L,R})$
- natural generalization of Rankine-Hugoniot condition : $\mathcal{F}_L = \mathcal{F}_R$
- natural generalization of L^1-dissipativity condition :

$$\operatorname{sign}(u_L - c_L)(\mathcal{F}_L - \mathcal{E}_L) \geq \operatorname{sign}(u_R - c_R)(\mathcal{F}_R - \mathcal{E}_R)$$

- maximality $\equiv$ absence of nontrivial extension
- completeness ??? define it as solvability of the stationary pb.!

How many adapted entropy inequalities we need ?

Constants have zero derivative :-) ...
$\Rightarrow$ To write adapted entropy inequalities,
one needs a wider class of elementary solutions !

Recall:
– [Kruzhkov],[Bulíček,Gwiazda,Málek,Świerczewska] : k constant
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Our choice: for each $(c_L, \mathcal{F}_L; c_R, \mathcal{F}_R) \in \mathcal{H}$,
construct **one** solution to $(f(x,u) - \text{“}\varphi(x,u)_x\text{”})_x = g$
with some source g (whatever) , with $u(0^-) = u_L$, $u(0^+) = u_R$
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A conclusion:
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NB. $\Rightarrow$ another application: alternative approach to boundary-value Dirichlet problem ([Carrillo],[Mascia,Porretta,Terracina],...) restricted to the 1D case, work in progress [A,Gazibo Karimou,Vallet]

Results coming from the germ approach to degen. parabolic pb.

- Notion of solution determined by a maximal $L^1 D$ germ $\mathcal{H}$
- Definition of $\mathcal{H}$-entropy solution for the stationary problem: local entropy solution in the sense of [Carrillo] with left/right solution/flux traces on interface belonging to $\mathcal{H}$ (all traces exist, gratis!). Easy: Uniqueness of such solution.

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- Assuming completeness of $\mathcal{H}$, we get an m-accretive operator $\Rightarrow$ existence and uniqueness of integral solutions to evol. pb.
- For the example with "transmission conditions" enforced by

$$\text{“ } (\pi_L + \varepsilon\Pi_L)(u_L^\varepsilon) = (\pi_R + \varepsilon\Pi_R)(u_R^\varepsilon) \text{ ”}$$

  we find the associated L^1-dissipative germ $\mathcal{H}_{\Pi_{L,R}}$
  and we prove completeness ($\Rightarrow$ and maximality) of this germ using viscosity / using finite volume scheme ([A.,Cancès'13]) constructed directly from the generalized transmission condition

Results coming from the germ approach to degen. parabolic pb.

- Solutions to stat. pb. are also characterized by a small family of entropy (Kato) inequalities (one inequality per element of $\mathcal{H}$).
NB: traces disappear from the definition !
- Analogous notion of entropy solution for the evol. pb. ,
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- If the entropy solution is *trace-regular* , i.e., there exist

  – strong L^1 traces $u(t, 0^{\pm})$ of the solution [Kwon'09],[Gazibo'13]

  – **weak $\mathbf{L^1}$ (more than** [Chen,Frid] **$\mathbf{H^{-1/2}}$!)** traces $\mathcal{F}[u](t, 0^{\pm})$

  (some sufficient conditions known:
  BV flux estimates [Karlsen,Evje],[Karlsen,Risebro,Towers],... ,
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  then trace-regular entropy solution is an integral solution,
  and therefore it is unique.
- Alternative description: in some cases, $\mathcal{H}$-description of solution can be replaced by a generalization of "Γ-condition" ([Diehl])
  (Diehl's case corresponds to the choice $\pi_{L,R} = \Pi_{L,R} = Id$)

Thank you !

Děkuji !