Stability of solutions of the $p(x)$-laplacian and numerical approximation.

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Plan of the talk

PROBLEMS CONSIDERED

Problems considered...

- Prototype Problem 1: $p(x)$-laplacian

$$u - \overbrace{\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)}^{\Delta_{p(x)} u} = f(x), \quad u|_{\partial\Omega} = 0 :$$

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- Prototype Problem 2: $p(u)$-laplacian

$$u - \overbrace{\mathrm{div}\left(|\nabla u|^{p(u)-2}\nabla u\right)}^{\Delta_{p(u)} u} = f(x), \quad u|_{\partial\Omega} = 0 :$$

This is a hypothetic “local auto-rheological model”.

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- Prototype Problem 3: $p[u]$-laplacian

$$\begin{cases} u - \Delta_{p(x,v)} u = f(x,u,v), & u|_{\partial\Omega} = 0, \\ v - \Delta v = g(x,u,v), & v|_{\partial\Omega} = 0 : \end{cases}$$

This is a sample “nonlocal auto-rheological model” (similar coupled systems can model, e.g., thermo-rheological fluids).

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- General local problem:

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Here $\mathfrak{a}(x,z,\xi)$ is a Leray-Lions kind nonlinearity with variable coercivity/growth exponent $p(x,z)$, i.e.

- $x \in \Omega \subset \mathbb{R}^N$, $z \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\mathfrak{a}(x,z,\xi)$ is measurable in x, cont. in z,ξ;
- $\mathfrak{a}$ is monotone in ξ : $(\mathfrak{a}(x,z,\xi) - \mathfrak{a}(x,z,\eta)) \cdot (\xi - \eta) > 0$ if $\xi \neq \eta$;
- $\mathfrak{a}(x,z,\cdot)$ is $p(x,z)$-coercive : $\mathfrak{a}(x,z,\xi) \cdot \xi \geq c\,|\xi|^{p(x,z)}$;
- $\mathfrak{a}(x,z,\cdot)$ "maps $L^{p(x,z)}$ to $L^{p'(x,z)}$": $|\mathfrak{a}(x,z,\xi)|^{p'(x,z)} \leq C\,(1 + |\xi|^{p(x,z)})$;
- the variable exponent $p(x,z)$ is measurable in x, continuous in z, with values in $[p_-, p_+] \subset (1,+\infty)$, and $\frac{1}{p(x,z)} + \frac{1}{p'(x,z)} = 1$.

MOTIVATION

Motivation of the stability analysis wrt $p(\cdot)$

- A flexible and comprehensive monotonicity argument
  (Young measures' approach versus Minty trick)
- Numerical analysis for the $p(x)$-laplacian:
  choice of approximation, proof of convergence
- Existence for "auto-rheological" problems
  (e.g., $p(u)$- and $p[u]$-laplacian)

MCC (MINTY) ARGUMENT

The monotonicity "compensated compactness" argument...

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- In addition, $\int \chi \cdot v \geq \liminf_{n\to\infty} \int \chi_n \cdot v_n$ or, in an abstract setting,

  $$< \chi\,,\, v >_{E',E} \geq \liminf_{n\to\infty} < \chi_n\,,\, v_n >_{E',E} .$$

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In the "$\overline{}$ terminology", we thus have $\overline{\mathfrak{a}(v)} = \mathfrak{a}(\overline{v})$. This can be seen as a (very ancient!) kind of "compensated compactness" property.

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The use of the above monotonicity compensated compactness (MCC) property:

- Existence for a wide class of PDEs (the Leray-Lions class of problems)
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A modification is needed:

in some problems, e.g. the auto-rheological ones, we cannot hope for a fixed duality framework $E - E'$.

- in these cases, we must make the space E depend on n:

$$v_n \in E_n,\, \chi_n \in E'_n \quad \text{instead of} \quad v_n \in E,\, \chi_n \in E'.$$

- certainly, we need some "convergence" of E_n to E. Example and details will appear in the sequel...

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  - The proof allows for making E_n vary
  - The proof permits to include easily technical details such as dependency of $\mathfrak{a}$ on x and n (case $\chi_n = \mathfrak{a}_n(x, v_n)$)
  - The proof looks "optimal" !

  The proof will be described at the end of the talk, for the case of the "non-local auto-rheological" Prototype problem 3.

VARIABLE EXPONENT SPACES

Variable exponent Lebesgue and Sobolev spaces...

Definition

Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function.

- $L^{\pi(\cdot)}(\Omega)$ is the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the modular is finite:
$$\rho_{\pi(\cdot)}(f) := \int_\Omega |f(x)|^{\pi(x)}\, dx < +\infty.$$
It is equipped with the so-called Luxembourg norm.
- $W^{1,\pi(\cdot)}(\Omega)$ is the space of all functions $f \in L^{\pi(\cdot)}(\Omega)$ such that the distributional gradient ∇f belongs to $L^{\pi(\cdot)}(\Omega)$. It is equipped with the norm
$$\|f\|_{W^{1,\pi(\cdot)}} := \|f\|_{L^{\pi(\cdot)}} + \|\nabla f\|_{L^{\pi(\cdot)}}.$$
Then $\boxed{W_0^{1,\pi(\cdot)}(\Omega)}$ is the closure of $C_0^\infty(\Omega)$ wrt $W^{1,\pi(\cdot)}(\Omega)$ norm.
- In addition, $$\boxed{\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)} := \{\, f \in W_0^{1,1}(\Omega) \mid \nabla f \in L^{\pi(\cdot)}(\Omega) \,\}.$$
This space is equipped with the norm $\|f\|_{\dot{\mathrm{E}}^{\pi(\cdot)}} := \|\nabla f\|_{L^{\pi(\cdot)}}$.

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  Then $L^{\pi(\cdot)}$ possesses the most useful properties of usual (constant-exponent) reflexive Lebesgue spaces,
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  Log-Hölder continuity means that $\pi(\cdot)$ on $\overline{\Omega}$ is continuous with a modulus of continuity $\frac{const}{\log|x-y|}$ (weaker than $\pi \in C^{0,\alpha}(\overline{\Omega})$, $\alpha > 0$).

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In the absence of the log-Hölder continuity assumptions, there are counterexamples (very few, and all are somewhat similar !) showing that C^∞ is not dense in $W^{1,\pi(\cdot)}$ (Zhikov). Thus

- in general, we may have two (in fact, more !) "interpretations" of the zero Dirichlet BC: we have $W_0^{1,\pi(\cdot)}(\Omega) \subsetneq \dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.
  Stating that u belongs to one of these spaces means $u|_{\partial\Omega} = 0$, in a stronger (case $W_0^{1,\pi(\cdot)}(\Omega)$) or weaker(case $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$) sense.

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  When the spaces differ, it is clear that the problems of the variational kind such as the $\pi(x)$-laplacian, which solutions are minimizers of the energy

$$\mathcal{J}_{\pi(\cdot)} \,:\, u \mapsto \int_\Omega \frac{1}{\pi(x)} |\nabla u(x)|^{\pi(x)}\, dx,$$

  can possess two different solutions:

  – the narrow one, i.e. argmin $\mathcal{J}_{\pi(\cdot)}$ in "narrow space" $W_0^{1,\pi(\cdot)}(\Omega)$;

  – the broad one, i.e. argmin $\mathcal{J}_{\pi(\cdot)}$ in "broad space" $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.

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  – the broad one, i.e. argmin $\mathcal{J}_{\pi(\cdot)}$ in “broad space” $\dot{\mathrm{E}}^{\pi(\cdot)}(\Omega)$.
- if $\pi(\cdot)$ is log-Hölder continuous (e.g. if $\pi(\cdot) \in C^{0,\alpha}$), then broad and narrow solutions coincide because the spaces coincide.

THE $p(x)$-LAPLACIAN

Notions of solution and known well-posedness results...

Let us investigate the general local problem

$$(P) \quad u - \operatorname{div} \mathfrak{a}(x, u, \nabla u) = f(x), \quad u|_{\partial\Omega} = 0.$$

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and distinguish the following two notions of weak solutions.

Definition

(i) A function $u \in W_0^{1,p(\cdot,u(\cdot))}(\Omega)$ is called a narrow weak solution of problem (P) if $u \in L^1(\Omega)$ and $u - \operatorname{div} \mathfrak{a}(x,u,\nabla u) = f$ in $\mathcal{D}'(\Omega)$.

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Throughout this section, we only consider $\mathfrak{a}(x,z,\xi)$ with the coercivity/growth exponent $p(x,z)$ independent of z.
The model case is the $p(x)$-laplacian: $\mathfrak{a}(x,\xi) = |\xi|^{p(x)-2}\xi$.

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$$\mathcal{J} : u \mapsto \int_\Omega \Big(\frac{u^2}{2} + A(x,\nabla u) - fu \Big)$$

over the broad or the over the narrow spaces introduced above.

NB: The source f should be from the dual space; $p(\cdot)$ often assumed regular.

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– To treat general L^1 source terms, one replaces the (broad or narrow) weak solutions with (broad or narrow) entropy solutions/ renormalized solutions, following Bénilan&al./Murat&al.. Existence, uniqueness, L^1 contraction.

Idea: one looks at the truncates of u, $T_\gamma(u) := \max\{-k\,,\,\min\{\,u\,,\,k\,\}\,\}$...

References: Sanchon & Urbano; Bendahmane & Wittbold; Ouaro & Traoré.

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References: Zhikov, then many other authors.

– The standard Leray-Lions techniques (existence by the Galerkine method) apply for general monotone $\mathfrak{a}$. Existence, uniqueness, continuous dep.

– To treat general L^1 source terms, one replaces the (broad or narrow) weak solutions with (broad or narrow) entropy solutions/ renormalized solutions, following Bénilan&al./Murat&al.. Existence, uniqueness, L^1 contraction.

Idea: one looks at the truncates of u, $T_\gamma(u) := \max\{-k\,,\,\min\{\,u\,,\,k\,\}\}$...

References: Sanchon & Urbano; Bendahmane & Wittbold; Ouaro & Traoré.

– The renormalized setting also allows to go beyond the L^1 source terms (take measure source terms). Existence is true.

References: Lukkari; Ouaro & Soma.

Stability wrt perturbations of $p(\cdot)$...

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- f_n converges to f weakly in $L^1(\Omega)$.

Remark: Speaking of weak solutions, we make an implicit assumption of the kind $f_n \in L^{p'_n(\cdot)}$, $f \in L^{p'(\cdot)}$ (or $(p')^*$ instead of $p'(\cdot)$...).

Formulating convergence of f_n in "moving spaces" is a nightmare... Fortunately, the weak L^1 convergence of $(f_n)_n$ is enough, if we use the tools of renormalized solutions while treating the weak solutions !

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Remark: same result OK for renormalized broad/narrow solutions.

FINITE VOLUMES FOR $p(x)$-LAPLACIAN

Numerics for $p(x)$-laplacian, etc.

Applications of the result/of the method:

- One elementary corollary is existence (and stability) for the problem where the diffusion takes the form $\mathfrak{a}(x, u, \nabla u)$ but the coercivity/growth exponent $p(\cdot)$ does not depend on u, only on x.

  Argument: just consider $\mathfrak{a}(\cdot, u_n(\cdot), \xi)$ as $\mathfrak{a}_n(\cdot, \xi)$ and use the above Theorem. Use e.g. the Galerkine approximations.

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- Application to convergence of FV schemes for the $p(x)$-laplacian.

  DDFV≡"discrete duality finite volume" scheme is a tricky family of (originally, 2D) FV schemes designed for anisotropic linear elliptic problems (or even for $-\Delta u = f$ on non-structured meshes !).

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  DDFV schemes were recognized as a convenient and robust discretization tool (general geometries including distorted and non-conformal meshes, local mesh refinement, discontinuous coefficients, domain decomposition, adaptations to 3D... and also for nonlinear problems !). Drawback: no maximum principle.

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  Refs.: Hermeline; Domelevo&Omnès; BA&Boyer&Hubert; Coudière& Pierre; Delcourte&Omnès; BA&MB&KHK...

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  Writing such DDFV scheme for a $p(x)$-laplacian kind problem is straightforward. Naturally, piecewise constant variable exponents $p_n(\cdot)$ appear. Convergence proof follows, step by step, the proof of our stability Theorem. NB: no hope for getting broad solutions?

  Work in progress: BA&MB&Ruiz Baier

...Numerics for $p(x)$-laplacian...

Hints on discretization :

- If p is log-Hölder continuous on $\overline{\Omega}$, one can take :
  – either $p_D := p(x_D)$, where $x|_D$ is a center of D
  – or $p_D := \frac{1}{|D|}\int_D p(x)\,dx$
  — (in theory !) for discretization of $\mathfrak{a}(x,\xi)$, one can take $\mathfrak{a}_D(\xi) := \frac{1}{|D|}\int_D \mathfrak{a}(x,\xi)\,dx$.
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- To approximate the broad solution, take $p_D := \min_{x\in D} p(x)$? The question of convergence (to the broad solution) is open.

...Numerics for $p(x)$-laplacian.

Numerical analysis tools used :

- The scheme derives from a discrete energy functional ($\Longrightarrow$ existence, uniqueness, practical implementation)
- Discrete Poincaré inequality with fixed exponent p_-
- Discrete energy estimate ($L^{p^{\mathcal{T}}(\cdot)}$ estimate for the discrete gradient $\nabla^{\mathcal{T}} u^{\mathcal{T}}$)
- Discrete weak compactness in W_0^{1,p_-} (fixed exponent !)
- Consistency in W_0^{1,p_+} (fixed exponent !)
- “Summation-by-parts” (discrete duality) formula
- If renormalized solutions : in the proof of a priori estimates, we need the orthogonality of meshes.

THE $p(u)$-LAPLACIAN

Existence for $p(u)$-laplacian and the density problem.

The density problem: in the proof of the stability Theorem, we encounter the following (major ?) problem:

– the limit u is found to lie in the “broad” space;

– the limit PDE can be tested with functions in the “narrow” space;

– the key MCC argument requires testing the limit PDE with u!

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Open problem: suppress the assumption $p_- > N$ in the below thm:

Theorem (Existence for the $p(u)$-laplacian)

Assume $\mathfrak{a} = \mathfrak{a}(x, z, \xi)$ satisfies the coercivity/growth assumptions with $p(x, z)$ such that $p(\cdot, \cdot)$ is log-Hölder continuous and $p(\cdot, \cdot) \geq p_- > N$.

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Remark: the solution u is both broad and narrow. It is L^∞ (even C^α); no need of considering entropy/renormalized solutions.
Remark: the case $N = 1$ is trivial: no density problem occurs at all, and solutions are always $W^{1,\infty}$. (Bénilan&Touré,Ouaro&Touré).

Regularity and uniqueness for $p(u)$-laplacian...

In fact, this existence result is of the SOLA
($\equiv$ solutions obtained as limit of approximation) kind:
we can construct a solution map $f \mapsto u_f$ which is an order-preserving contraction in L^1.

SOLAs were an early alternative to entropy/renormalized solutions, designed to cope with the non-uniqueness (the Serrin example) for elliptic problems with L^1 data.

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What about uniqueness of a weak solution ?

In our case, uniqueness proof relies on the regularity result.

The classical regularity techniques (...,DiBenedetto, Lieberman,...) also apply to the variable exponent elliptic problems and yield (under very mild assumptions) a $C^{1,\alpha}(\overline{\Omega})$ regularity for solutions corresponding to L^∞ data.

References: Fan&Zhao, Alkhutov, Fan

...Regularity and uniqueness for $p(u)$-laplacian.

Theorem (well-posedness for $p(u)$-laplacian)

In addition to the assumptions of the existence theorem ($p(\cdot,\cdot)$ Hölder continuous, $p_- > N$) require some Hölder regularity of $\mathfrak{a}$ and Ω. Then there is uniqueness of a weak solution to (P). The solution map is an order-preserving contraction in $L^1(\Omega)$.

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In particular, in a Hölder domain Ω, for all locally Hölder $p : \mathbb{R} \to \mathbb{R}$, the $p(u)$-laplacian problem $u - \Delta_{p(u)} = f$ possesses a unique and L^1 stable weak solution for all $f \in L^1(\Omega)$.

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Idea of the proof:

- compare two solutions, of which one is arbitrary, and the other is “good” (for our purposes, only $W^{1,\infty}$ bound on solutions needed).
- Combine the L^1 contraction (between 1 “good” & 1 arbitrary solution) with the density of the “good data”.

Ref.: Andr&Bouhsiss.

THE TECHNIQUES

Fundamental theorem for Young measures...

Theorem (Young, Ball, Pedregal,...; Norbert Hungerbühler)

(i) *Let $\Omega \subset \mathbb{R}^N$ (bounded) domain. Consider a sequence $(v_n)_n$ of $\mathbb{R}^d$-valued functions such that $(v_n)_n$ is equi-integrable on Ω.*

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Then there exists a subsequence $(n_k)_k$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$, weakly measurable in x wrt the Lebesgue measure on Ω,

such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^t$,

$$\lim_{k\to+\infty} \int_\Omega F(x, v_{n_k}(x))\, dx = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)\, d\nu_x(\lambda) dx$$

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whenever the sequence $\big(F(\cdot, v_n(\cdot)) \big)_n$ is equi-integrable on Ω.

In particular, $v(x) := \int_{\mathbb{R}^d} \lambda\, d\nu_x(\lambda)$ is the weak limit of the sequence $(v_{n_k})_k$ in $L^1(\Omega)$, as $k \to +\infty$.

The family $(\nu_x)_x$ is called the Young measure generated by the subsequence $(v_{n_k})_k$.

...Fundamental theorem for Young measures.

Theorem (continued)

(ii) *If Ω is of finite measure, and $(\nu_x)_x$ is the Young measure generated by a sequence $(v_n)_n$, then*

$$\nu_x = \delta_{v(x)} \text{ for a.e. } x \in \Omega \quad \Longleftrightarrow \quad v_n \Longrightarrow v \text{ as } n \to +\infty.$$ [12]

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$$\nu_x = \delta_{v(x)} \text{ for a.e. } x \in \Omega \iff v_n \Longrightarrow v \text{ as } n \to +\infty.$$ [1,2]

(iii) *If $(u_n)_n$ generates a Dirac Young measure $(\delta_{u(x)})_x$ on $\mathbb{R}^{d_1}$, and $(v_n)_n$ generates a Young measure $(\nu_x)_x$ on $\mathbb{R}^{d_2}$,*

then the sequence $\big((u_n, v_n)\big)_n$ generates the Young measure $\big(\delta_{u(x)} \otimes \nu_x\big)_x$ on $\mathbb{R}^{d_1+d_2}$.

---

[1] By "δ_a" we denote the probability measure supported at the point $a \in \mathbb{R}^d$ (i.e., the Dirac measure at the point a)

[2] By "$\Longrightarrow$" we denote the convergence in measure on Ω

MCC property from Young measures' approach...

Strategy to deduce the monotonicity "comp. comp." property:

- use *a priori* estimates and Theorem (i) to represent

  the weak limit $\nabla u(x) = \int_{R^N} \lambda \, d\nu_x(\lambda)$ of $v_n := \nabla u_n$

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  (this comes from the PDE) to infer the "div-curl inequality"

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- use the monotonicity of $\mathfrak{a}$ to reduce the measure ν_x to the Dirac measure $\delta_{\nabla u(x)}$.

  This permits to identify χ with $\mathfrak{a}(\cdot, \nabla u)$ (and also to conclude, with Theorem (ii), that the convergences are in fact strong).

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Remarks:

- At no point we use a fixed duality framework $E - E'$.

  We only need that $\mathfrak{a}_n(v_n) \cdot v_n$ be integrable

  (i.e., for each n, $v_n \in E_n$ and $\chi_n = \mathfrak{a}(v_n) \in E'_n$)...

- It's fairly easy to take into account e.g.
  the dependency of $\mathfrak{a}_n(\cdot, u_n, \nabla u_n)$ on u_n:

  in case u_n converges in measure to u, Theorem (iii) does the job.

Existence for a non-local $p[u]$-laplacian problem...

As an example of use of the Young measures MCC technique, let us give one proof.

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Theorem (Existence for a non-local $p[u]$- laplacian)

Let Ω be a domain of $\mathbb{R}^N$ with $C^{0,\alpha}$ boundary, with some $\alpha > 0$.

Let $g, h : \Omega\times(\mathbb{R}\times\mathbb{R}) \longrightarrow \mathbb{R}$ be globally bounded Carathéodory functions. Assume p is a locally $C^{0,\alpha}$ continuous function on $\overline{\Omega}\times\mathbb{R}$ taking values in $[p_-, p_+] \subset (1, +\infty)$.

Then there exists a couple of functions $u, v : \Omega \longrightarrow \mathbb{R}$ and $\beta \in (0, 1)$ such that $u, v \in C^{0,\beta}(\overline{\Omega})$, $u \in W_0^{1,p(v)}(\Omega)$, $v \in H_0^1(\Omega)$ and

$$\begin{cases} u - \Delta_{p(x,v)} u = f(x, u, v) \\ v - \Delta v = g(x, u, v) \end{cases}$$

is fulfilled in $\mathcal{D}'(\Omega)$.

...Existence for a non-local $p[u]$-laplacian problem...

PROOF OF THE THEOREM:

- STEP 1. Construct a sequence of approximate solutions.

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Pick a countable set $(w_i)_i \subset \mathcal{D}(\Omega)$ which is weakly dense in $W^{1,\infty}(\Omega)$.
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u_n, v_n in Span$(w_1, \ldots, w_n)$ (existence: Brouwer fixed-point).

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• STEP 2. The functions u_n, v_n verify the uniform estimate

$$\int_\Omega \left(u_n^2 + |\nabla u_n|^{p(x,v_n)} + v_n^2 + |\nabla v_n|^2 \right) \leq C(\|f\|_{L^\infty}, \|g\|_{L^\infty}, \Omega).$$

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This estimate is standard; it permits to assert that, upon extracting a (not relabelled) subsequence, $v_n \to v$ in $H_0^1(\Omega)$ weakly and a.e. on Ω, that $u_n \to u$ in $W_0^{1,p_-}(\Omega) \cap L^2(\Omega)$ weakly and also a.e. on Ω, and, moreover, that

$$\nabla u(x) = \int_{\mathbb{R}^N} \lambda \, d\nu_x(\lambda),$$

where $(\nu_x)_x$ is the family of Young measures associated with the weakly convergent in $L^1(\Omega)$ sequence $(\nabla u_n)_n$.

...Existence for a non-local $p[u]$-laplacian problem...

- STEP 3. The estimate of Step 2 implies that $\nabla u \in L^{p(x,v(x))}$.

This is made by showing that $|\lambda|^{p(x,v(x))}$ is summable wrt measure $d\nu_x(\lambda)\,dx$ on $\mathbb{R}^N \times \Omega$.

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To this end, note that $p_n(\cdot) := p(\cdot, v_n(\cdot))$ converge to $p_\infty(\cdot) := p(\cdot, v(\cdot))$ a.e. on Ω, because p is uniformly continuous on $\Omega \times \mathbb{R}$ and v_n converge pointwise. In particular, the Young measure $(\mu_x)_x$ on $\mathbb{R} \times \mathbb{R}^N$ associated with the sequence $\big(\, v_n, \nabla u_n \big)_n$ is equal to $\delta_{v(x)} \otimes \nu_x$.

Then we apply the "nonlinear weak-* convergence property" to the function

$$F \,:\, \big(\, x, (\lambda_0, \lambda) \big) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N)) \;\mapsto\; |h_m(\lambda)|^{p(x,\lambda_0)},$$

where $(h_m)_m$ is a sequence of truncations. Hence

$$\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x,v(x))}\; d\nu_x(\lambda)\,dx = \int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x,y)}\; d\mu_x(y,\lambda)\,dx$$
$$= \lim_{n\to\infty} \int_\Omega |h_m(\nabla u_n)|^{p(x,v_n(x))}\,dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n(x)}\,dx \le C.$$

As $m \uparrow \infty$, we conclude using the monotone convergence theorem.

...Existence for a non-local $p[u]$-laplacian problem...

- STEP 4. The sequence

$$\chi_n := \mathfrak{a}(x, v_n(x), \nabla u_n) \equiv |\nabla u_n|^{p(x,v_n)-2}\,\nabla u_n$$

is relatively weakly compact in $L^1(\Omega)$, the weak L^1 limit χ of $(\chi_n)_n$ belongs to $L^{p'_\infty(\cdot)}(\Omega)$, and we have for a.e $x \in \Omega$,

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We only have to show that $(\chi_n)_n$ is equi-integrable on Ω. The proof of the equi-integrability is based upon the *a priori estimate* of Step 2 and on the use of the Hölder inequality (this is slightly technical).

The convergence-in-measure is a perfect technical tool here.

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By the density of the family $(w_i)_i$ in $\mathcal{D}(\Omega)$ supplied with the weak $W^{1,\infty}(\Omega)$ topology, we infer that

$$\begin{cases} u - \operatorname{div}\chi = f(x,u,v) \\ v - \Delta v = g(x,u,v) \end{cases} \qquad \boxed{\text{in } \mathcal{D}'(\Omega)}.$$

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Indeed, we already know that $u \in W_0^{1,p_-}(\Omega)$ and that $\nabla u \in L^{p_\infty(\cdot)}(\Omega)$. Thus u belongs to the "broad" space $\dot{\mathrm{E}}^{p_\infty(\cdot)}(\Omega)$.
By the Hölder regularity of p and v, $p_\infty(\cdot) = p(\cdot, v(\cdot))$ is log-Hölder continuous. Thus $\dot{\mathrm{E}}^{p_\infty(\cdot)}(\Omega) = W_0^{1,p_\infty(\cdot)}(\Omega)$.

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- STEP 7. The “div-curl” inequality holds:

$$\int_{\Omega\times\mathbb{R}^N} \left(|\lambda|^{p_\infty(x)-2}\,\lambda - |\nabla u|^{p_\infty(x)-2}\,\nabla u\right)\cdot\left(\lambda - \nabla u\right)\, d\nu_x(\lambda)\,dx \le 0.$$

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Take test fct. u_n in the Galerkine eq., and test fct u in the limit eq..

Let us stress that u is an admissible test function in the limit eq.

Using the equation, the Fatou lemma, and the a.e. convergences of u_n, v_n, we can pass to the limit in all terms of the first eq. except in the $-\Delta_{p(x,v_n(x))}u_n$ term (this term is nonlinear in ∇u_n). Using the representations of ∇u and χ in terms of $\nu_x(\lambda)$, we (almost) get

$$\int_\Omega \left(\int_{\mathbb{R}^N} |\lambda|^{p_\infty(x)-2}\lambda\, d\nu_x(\lambda)\right)\cdot\left(\int_{\mathbb{R}^N}\lambda\, d\nu_x(\lambda)\right) dx \;=\; \int_\Omega \chi\cdot\nabla u$$

$$\ge \liminf_{n\to\infty}\int_\Omega \chi_n\cdot\nabla u_n \;=^{???}\; \int_\Omega \left(\int_{\mathbb{R}^N} |\lambda|^{p_\infty(x)}\; d\nu_x(\lambda)\right) dx.$$

True if we could apply the "nonlin. weak-* conv." to $F(x,\lambda) := |\lambda|^{p_\infty(x)}$.

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We perform some easy algebraic manipulations using the fact that $(\nu_x)_x$ are probability measures, and deduce the "div-curl" inequality required.

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- STEP 8. We deduce that for a.e. $x \in \Omega$, the support of the measure ν_x is reduced to the singleton $\{\nabla u(x)\}$.

In other words, (ν_x) is a Dirac Young measure which can be identified with the function ∇u on Ω.

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The claim is direct from the strict monotonicity in ξ of the map $\mathfrak{a} : (x, \xi) \mapsto |\xi|^{p_\infty(\cdot)-2}\xi$ and the "div-curl" inequality of Step 7.

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By the representation formula for χ, we deduce that $\chi = |\nabla u|^{p_\infty(x)-2}\nabla u = |\nabla u|^{p(x,v(x))-2}\nabla u$ a.e. on Ω. Therefore the limit system is exactly the $\mathcal{D}'$ formulation of our original nonlinear system.

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Notice that it can be deduced from the above results that v_n converges to v strongly in $H_0^1(\Omega)$, $\int_\Omega |\nabla u_n|^{p(x,v_n(x))} \to \int_\Omega |\nabla u|^{p(x,v(x))}$, and ∇u_n converges ∇u a.e. on Ω, up to a subsequence.

Remark: some uniqueness results for this kind of coupled systems were obtained by Zhikov; Antontsev&Rodrigues.

MERCI !!!