Weak and entropy solutions to fractional conservation laws.

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Plan of the talk

1. Problem and notions of solution

2. Some of the known results

3. Construction of a non-entropy weak solution
   - Ideas of the construction
   - Fractional laplacian on the space of odd functions
   - Proof sketched
PROBLEM CONSIDERED
Problem considered

We look at the “fractal conservation laws”

\[
\partial_t u + \text{div}_x f(u) + \mathcal{L}_\lambda[u] = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^N, \tag{1}
\]
\[
u(0, \cdot) = u_0, \quad \text{on } \mathbb{R}^N, \tag{2}
\]

\(\mathcal{L}_\lambda\) denotes the fractional power \((-\Delta)^{\lambda/2}\) of \(-\Delta\).

This is a non-local, pseudodifferential operator of order \(\lambda, 0 < \lambda < 2\).

Motivations:

- gaz detonation (Clavin-Denet-He 01), phenomenological; rather \(1 \leq \lambda < 2\)
- “abnormal diffusion” phenomena (Woyczynski 01, Biler et al. 1998), probabilistic connection; also \(\lambda < 1\)

Generalization of \(\mathcal{L}_\lambda\) : Lévi processes.

The two reference (limit) case, thoroughly studied:

- \(\lambda = 2\) : the parabolic case, similar to the heat equation
- \(\lambda = 0\) : the pure hyperbolic case (scalar conservation law)

The behaviour of generic solutions is very different in the two cases.
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The behaviour of generic solutions is very different in the two cases.
What is the fractional laplacian?

If \( \varphi \) is regular (e.g., for a function \( \varphi \) from the Schwartz class \( \mathcal{S} (\mathbb{R}) \)), \( \mathcal{L}_\lambda [\varphi] \) can be defined through the Fourier transform:

\[
\mathcal{F}(\mathcal{L}_\lambda [\varphi])(\xi) := |\xi|^{\lambda} \mathcal{F}(\varphi)(\xi).
\]  

(3)

In absence of regularity, a more general definition is provided by the Lévi-Khinchine formula: (case \( 0 < \lambda < 1 \): the integral is converent)

\[
\text{const } \mathcal{L}_\lambda [\varphi](x) := \text{v.p.} \int_{\mathbb{R}} \frac{\varphi(x + z) - \varphi(x)}{|z|^{N+\lambda}} \, dz.
\]  

(4)

Hint: The kernel \( \frac{1}{|z|^{N+\lambda}} \) being singular at the origin, Droniou-Imbert 05 split (4) into regular (“order zero”) and singular (“order \( \lambda \)”) parts:

\[
\mathcal{L}_\lambda [\varphi] = -G_\lambda \left( \int_{\{|z|>r\}} \frac{\varphi(z) - \varphi(0)}{|z|^{N+\lambda}} \, dz + \int_{\{|z|<r\}} \frac{\varphi(z) - \varphi(0)}{|z|^{N+\lambda}} \, dz \right)
\]  

=: \( \mathcal{R}_\lambda^r [\varphi] + \mathcal{S}_\lambda^r [\varphi] \).
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\]
Heuristics: competition of regularizing and de-regularizing effects

Heuristically, the knowledge of the purely hyperbolic ($\lambda = 0$) and of the parabolic ($\lambda = 2$) cases permits to set up a few conjectures:

- In the case $0 < \lambda < 1$, the fractional diffusion term $\mathcal{L}[u]$ is dominated by the term $\text{div}_x f(u)$. In particular:
  - one expects that even for very smooth initial data, there is no globally defined in time classical solution
  - the notion of a weak (distributional) solution permits to get existence for rather general data
  - the notion of a weak solution leads to non-uniquness
  - a “good” notion of solution should be inspired by the kinetic solutions/entropy solutions coming from conservation laws.

- In the case $1 < \lambda < 2$, the fractional diffusion operator $\mathcal{L}[u]$ is the leading term. In particular:
  - smooth data give rise to globally defined smooth solutions
  - non-smooth data undergo an instantaneous regularizing effect
  - there is well-posedness in the framework of weak solutions.

- Case $\lambda = 1$. Hmmmm... no *a priori* conjectures!
  For some applications, one needs techniques that allow for a wide range of values of $\lambda$, including $\lambda = 1$...
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Notions of solution: weak solutions

**Definition (Weak solution)**

Let $u_0 \in L^\infty(\mathbb{R}^N)$. A function $u \in L^\infty((0, \infty) \times \mathbb{R}^N)$ is said to be a weak solution to (1),(2) if for all $\varphi \in \mathcal{D}([0, \infty) \times \mathbb{R}^N)$,

$$
\int_0^\infty \int_{\mathbb{R}^N} (u \partial_t \varphi + f(u) \cdot \nabla_x \varphi - u \mathcal{L}_\lambda[\varphi]) + \int_{\mathbb{R}^N} u_0 \varphi(0) = 0.
$$

Remark: for regular $u$ and $v$ there holds the integration-by-parts formula

$$
\int \mathcal{L}_\lambda[u] v = \int u \mathcal{L}_\lambda[v] = \text{const} \int \int (u(x) - u(y)) (v(x) - v(y)) \frac{dxdy}{|x - y|^{N+\lambda}}.
$$

Therefore the definition just says,

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\partial_t u + \text{div}_x f(u) + \mathcal{L}_\lambda[u] = 0, \quad u|_{t=0} = u_0 \quad \text{in } \mathcal{D}'.
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The following definition (case $0 < \lambda < 1$) is due to Alibaud 06:

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\partial_t \eta(u) + \text{div}_x q(u) + \eta'(u) R^r_\lambda [u] + S^r_\lambda [\eta(u)] \leq 0 \quad \text{in } \mathcal{D}'.
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Here $\eta$ is an “entropy”, $q$ is the associated “entropy flux”; these notions are inherited from the Kruzhkov theory of conservation laws.

The definition of Alibaud is based upon the fractional Kato inequality:

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\forall r > 0 \quad \eta'(u) R^r_\lambda [u] + S^r_\lambda [\eta(u)] \leq \eta'(u) L_\lambda [u].
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To be compared with the Kato inequality used in the Kruzhkov theory:

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-\varepsilon \Delta \eta(u) \leq -\varepsilon \Delta \eta(u) + \varepsilon \eta''(u)|\nabla u|^2 = \eta'(u) (-\varepsilon \Delta u).
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Remark: smaller is the parameter $r$, less information is lost.
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WELL- AND ILL-POSEDNESS RESULTS
Well and ill-posedness results

- for the case $1/2 < \lambda < 2$, an $H^1$ solution exists globally and is unique for small $H^1$ data (Biler-Funaki-Woyczynski 98)
- for $1 < \lambda < 2$, there exists a unique weak solution for $L^\infty$ data, and $u(t, \cdot)$ falls within $C^\infty$ for $t > 0$ (Droniou-Gallouët-Vovelle 02)
- for $0 < \lambda < 2$, there exists a unique entropy solution (Alibaud 06);
  tools: doubling of var. with $R^\lambda, S^\lambda$ in the entropy formulation;
  kernel $K_t(x) = \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda})$ associated with $\mathcal{L}_\lambda$; time splitting

Further, or $0 < \lambda < 1$ and the Burgers flux $f(u) = \frac{u^2}{2}$ in dim. one:
- assume the initial datum $u_0$ presents an initial discontinuity (say, at zero) with $u_0(0-) > u_0(0+)$ and belongs to a class $C$ $\implies$ the discontinuity is persistent, at least for small times
- specially selected smooth initial data in $C$ $\implies$ the unique entropy solution, which becomes discontinuous (but not instantly)
- small Lipschitz data in the class $C$ $\implies$ global Lipschitz solutions (Alibaud-Droniou-Vovelle 07; the main tool: characteristics)

For the same fractional Burgers equation in the “hyperbolic regime”,
- Non-entropy weak solutions can be constructed
- Consequently, uniqueness of a weak solution may fail.
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CONSTRUCTION OF A “WRONG” WEAK SOLUTION
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We try to mimic the simplest “wrong” weak solution of the Burgers conservation law \( \partial_t u + \left( \frac{u^2}{2} \right) = 0 \). This is the discontinuous stationary solution

\[
u(t, x) := \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}
\]

One simple reason why this is not an entropy solution is that it fails to satisfy the Ole\v{n}ik inequality \( \partial_x u(t, x) \leq \frac{\text{const}}{t} \) in \( \mathcal{D}' \).

We prove that the Ole\v{n}ik inequality still holds for entropy solutions of the fractional Burgers equation.

We work in the space of odd in \( x \) functions discontinuous at zero. If we ensure that \( u(0+) = -u(0-) > 0 \), then the Ole\v{n}ik condition is violated.

A comparison principle for odd “sub-super-solutions” holds; to prove it, we use adapted entropies

\[
\eta(x; u, k) = (u - k)^+ 1_{\{x > 0\}} + (u - k)^- 1_{\{x < 0\}}.
\]
Ideas of the construction

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Ideas of the construction

- We try to mimic the simplest “wrong” weak solution of the Burgers conservation law $\partial_t u + \left( \frac{u^2}{2} \right) = 0$. This is the discontinuous stationary solution
  $$u(t, x) := \begin{cases} 
-1 & x < 0 \\
1 & x > 0 
\end{cases}$$

- One simple reason why this is not an entropy solution is that it fails to satisfy the Olešnik inequality $\partial_x u(t, x) \leq \frac{\text{const}}{t}$ in $\mathcal{D}'$.

- We prove that the Olešnik inequality still holds for entropy solutions of the fractional Burgers equation.

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Let $H^1_{odd}$ be the subspace of odd functions in $H^1_*$; we have in particular $u(0+) = -u(0-)$ if $u \in H^1_{odd}$.

### Lemma (Fract. lapl. on the space of piecewise $H^1$ functions)

Let $\lambda \in (0, 1)$ and $\mathcal{L}_\lambda$ defined by the Lévi-Khinchine formula. Then

- **The linear operators** $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda/2}$ **are bounded as operators:**
  
  - a) $\mathcal{L}_\lambda : C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*) \rightarrow C(\mathbb{R}_*)$;
  - b) $\mathcal{L}_\lambda : H^1_* \rightarrow L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R} \setminus \{0\})$;
  - c) $\mathcal{L}_{\lambda/2} : H^1_* \rightarrow L^2(\mathbb{R})$.

- Moreover, $\mathcal{L}_\lambda$ is sequentially continuous as an operator:
  
  - d) $\mathcal{L}_\lambda : L^1(\mathbb{R}) \cap (BV(\mathbb{R}))_{w-} \rightarrow L^1(\mathbb{R})$.

- If $v \in H^1_*$, the definition of $\mathcal{L}_\lambda$ by Fourier transform makes sense.

- For $v, w \in H^1_*$, $\int_{\mathbb{R}} \mathcal{L}_\lambda[v] w = \int_{\mathbb{R}} v \mathcal{L}_\lambda[w] = \int_{\mathbb{R}} \mathcal{L}_{\lambda/2}[v] \mathcal{L}_{\lambda/2}[w]$. 
Fractional laplacian on the space of odd functions...

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Lemma (continued)

- If $v \in H^1_\ast$ is odd (resp., even), then $\mathcal{L}_\lambda[v]$ is odd (resp., even).

- Let $0 \neq v \in C_b(\mathbb{R}_\ast) \cap C^1(\mathbb{R}_\ast)$ be odd. Assume that $x_\ast > 0$ is such that
  \[ v(x_\ast) = \sup_{\mathbb{R}_\ast^+} v \geq 0 \quad (\text{resp. } = \inf_{\mathbb{R}_\ast^+} \leq 0). \]

  Then $\mathcal{L}_\lambda[v](x_\ast) > 0$ (resp. $< 0$) (“strong max. principle”).

- For $k \in \mathbb{R}$, let $\eta(x; \cdot, k) = (\cdot - k)^+ 1_{\{x > 0\}} + (\cdot - k)^- 1_{\{x < 0\}}$. Then, for all odd $v \in C_b(\mathbb{R}_\ast) \cap C^1(\mathbb{R}_\ast)$, for all $x > 0$
  \[ \eta'(x; v(x), k) \mathcal{L}_\lambda[v](x) \geq \mathcal{L}_\lambda[\eta(x; v(x), k)](x). \]
  (adapted entropy) ; the same holds with $S^\lambda_\chi$ in the place of $\mathcal{L}_\lambda$.

Proof: essentially, by looking at the Lévi-Khinchine formula.
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- For $k \in \mathbb{R}$, let $\eta(x; \cdot, k) = (\cdot - k)^+ 1_{\{x > 0\}} + (\cdot - k)^- 1_{\{x < 0\}}$.
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- If $v \in H^1_{\ast}$ is odd (resp., even), then $\mathcal{L}_\lambda[v]$ is odd (resp., even).
- Let $0 \not\equiv v \in C_b(\mathbb{R}_{\ast}) \cap C^1(\mathbb{R}_{\ast})$ be odd. Assume that $x_{\ast} > 0$ is such that

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Proof: essentially, by looking at the Lévi-Khinchine formula.
We construct solutions in the space $H^1_{\text{odd}}$ of the regularized stationary problem

$$\varepsilon (u - \Delta u) + \left( \frac{u^2}{2} \right)_x + \mathcal{L}[u] = 0, \quad u(0\pm) = \pm 1.$$ 

(they are $\mathcal{D}'(\mathbb{R}_\ast)$ solutions; in $\mathcal{D}'(\mathbb{R})$, the singular source term $-2\varepsilon (\delta_0)_x$ appears in the rhs !)

Techniques:
- “Freeze” the convection term, truncate the nonlinearity (in $u$) and its support (in $x$):
  
  replace $\left( \frac{u^2}{2} \right)_x$ by $\rho_n(x) \left( \frac{\rho_n(x) T_n(\bar{u})^2}{2} \right)_x$.

- The problem obtained in this way is the Euler-Lagrange equation for a quite standard convex minimization problem.
- Obtain a priori estimates to ensure boundedness; the strict convexity enforces compactness.
- One can use the Schauder fixed-point theorem.
- A maximum principle permits to get rid of the truncation $T_n$ in $u$; a passage to the limit removes the truncation $\rho_n$ in space.
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Proof II

Pass to the limit, as $\varepsilon \downarrow 0$. The things to be cared of:

- **Compactness** (in $H^{\lambda/2}(\mathbb{R}^\pm)$-weak and for the a.e. convergence): this comes from the uniform in $\varepsilon$ estimate in $H^{\lambda/2}(\mathbb{R}^\pm)$
- Passage to the limit in the weak formulation: straightforward
- Guarantee that the discontinuity of $u^\varepsilon$ at $x = 0$ persists at the limit

The last item is challenging. We have two proofs.

- **First proof**, with an explicit construction of barriers $m, M$ such that $\pm m(x) \leq \pm u^\varepsilon(x) \leq \pm M(x)$ for $\pm x > 0$, and $u_m(0+) = 1 = u_M(0+)$. The tools are: explicit sub-and-supersolutions, and the comparison principle (deduced with the help of the adapted entropies from the Kato inequality).

- **Second proof**, with a passage to the limit in the traces of $u^\varepsilon$. This looks a bit hopeless starting from the sole a.e. convergence of $u^\varepsilon$ to $u$ (nothing seems to prevent the formation of a boundary layer).

Fortunately, the Green-Gauss formula and the PDE in hand permit to pass to the (weak) limit in the traces of the flux $\frac{1}{2}(u^\varepsilon)^2(0\pm)$ and get $u(0\pm) = \pm 1$.

Conclusion: the so constructed $u$ is stationary, discontinuous, it is a weak solution to the fractional Burgers equation; it violates the entropy condition.
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And that’s it...

Grazie !!!