Weak and entropy solutions to fractional conservation laws.

Nathaël Alibaud[1,2] Boris Andreianov[1]

[1]Université de Franche-Comté, Besançon, France
[2]École Nationale Supérieure de Mécanique et Microtechniques

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Plan of the talk

PROBLEM CONSIDERED

Problem considered

We look at the “fractal conservation laws”

$$\partial_t u + \mathrm{div}_x f(u) + \mathcal{L}_\lambda[u] = 0, \quad \text{in } (0,\infty)\times\mathbb{R}^N, \qquad (1)$$

$$u(0,\cdot) = u_0, \quad \text{on } \mathbb{R}^N, \qquad (2)$$

$\mathcal{L}_\lambda$ denotes the fractional power $(-\Delta)^\lambda/2$ of $-\Delta$.
This is a non-local, pseudodifferential operator of order $\lambda, 0<\lambda<2$.

Motivations :

- gaz detonation (Clavin-Denet-He 01), phenomenological; rather $1\leq\lambda<2$
- “abnormal diffusion” phenomena (Woyczynski 01, Biler et al. 1998), probabilistic connection; also $\lambda<1$

Generalization of $\mathcal{L}_\lambda$: Lévi processes.

The two reference (limit) case, throuhly studied :

- $\lambda=2$: the parabolic case, similar to the heat equation
- $\lambda=0$: the pure hyperbolic case (scalar conservation law)

The behaviour of generic solutions is very different in the two cases.

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What is the fractional laplacian ?

If φ is regular (e.g., for a function φ from the Schwartz class $\mathcal{S}(\mathbb{R})$), $\mathcal{L}_\lambda[\varphi]$ can be defined through the Fourier transform :

$$\mathcal{F}(\mathcal{L}_\lambda[\varphi])(\xi) := |\xi|^\lambda \mathcal{F}(\varphi)(\xi). \tag{3}$$

In absence of regularity, a more general definition is provided by the *Lévi-Khinchine formula* : (case $0 < \lambda < 1$: the integral is convergent)

$$const\, \mathcal{L}_\lambda[\varphi](x) := \; v.p. \int_{\mathbb{R}} \frac{\varphi(x+z) - \varphi(x)}{|z|^{N+\lambda}}\, dz. \tag{4}$$

Hint : The kernel $\frac{1}{|z|^{N+\lambda}}$ being singular at the origin, Droniou-Imbert 05 split (4) into regular ("order zero") and singular ("order λ") parts:

$$\mathcal{L}_\lambda[\varphi] = -G_\lambda\Big(\int_{\{|z|>r\}} \frac{\varphi(\cdot+z)-\varphi(\cdot)}{|z|^{N+\lambda}}\, dz + \int_{\{|z|<r\}} \frac{\varphi(\cdot+z)-\varphi(\cdot)}{|z|^{N+\lambda}}\, dz \Big)$$

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Heuristics: competition of regularizing and de-regularizing effects

Heuristically, the knowledge of the purely hyperbolic ($\lambda = 0$) and of the parabolic ($\lambda = 2$) cases permits to set up a few conjectures :

- in the case $0 < \lambda < 1$, the fractional diffusion term $\mathcal{L}[u]$ is dominated by the term $\mathrm{div}_x f(u)$. In particular:
  - one expects that even for very smooth initial data, there is no globally defined in time classical solution
  - the notion of a weak (distributional) solution permits to get existence for rather general data
  - the notion of a weak solution leads to non-uniqueness
  - a "good" notion of solution should be inspired by the kinetic solutions/entropy solutions coming from conservation laws.
- in the case $1 < \lambda < 2$, the fractional diffusion operator $\mathcal{L}[u]$ is the leading term. In particular:
  - smooth data give rise to globally defined smooth solutions
  - non-smooth data undergo an instanteneous regularizing effect
  - there is well-posedness in the framework of weak solutions.
- Case $\lambda = 1$. Hmmmm... no *a priori* conjectures !
  For some applications, one needs techniques that allow for a wide range of values of λ, including $\lambda = 1$...

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Notions of solution: weak solutions

Definition (Weak solution)

Let $u_0 \in L^\infty(\mathbb{R}^N)$. A function $u \in L^\infty((0,\infty)\times\mathbb{R}^N)$ is said to be a weak solution to (1),(2) if for all $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)$,

$$\int_0^\infty \int_{\mathbb{R}^N} (u\partial_t\varphi + f(u)\cdot\nabla_x\varphi - u\mathcal{L}_\lambda[\varphi]) + \int_{\mathbb{R}^N} u_0\varphi(0) = 0.$$

Remark: for regular u and v there holds the integration-by-parts formula

$$\int \mathcal{L}_\lambda[u]v = \int u\mathcal{L}_\lambda[v] = const \iint (u(x)-u(y))(v(x)-v(y))\frac{dxdy}{|x-y|^{N+\lambda}}.$$

Therefore the definition just says,

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Here η is an "entropy", q is the associated "entropy flux"; these notions are inherited from the Kruzhkov theory of conservation laws.

The definition of Alibaud is based upon the fractional Kato inequality :

$$\forall r > 0 \quad \eta'(u)\, \mathcal{R}^r_\lambda[u] + \mathcal{S}^r_\lambda[\eta(u)] \le \eta'(u)\, \mathcal{L}_\lambda[u].$$

To be compared with the Kato inequality used in the Kruzhkov theory:

$$-\varepsilon\Delta\eta(u) \le -\varepsilon\Delta\eta(u) + \varepsilon\eta''(u)|\nabla u|^2 = \eta'(u)\,(-\varepsilon\Delta u)\,.$$

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WELL- AND ILL-POSEDNESS RESULTS

Well and ill-posedness results

- for the case $1/2 < \lambda < 2$, an H^1 solution exists globally and is unique for small H^1 data (Biler-Funaki-Woyczynski 98)
- for $1 < \lambda < 2$, there exists a unique weak solution for L^∞ data, and $u(t,\cdot)$ falls within C^∞ for $t > 0$ (Droniou-Gallouët-Vovelle 02)
- for $0 < \lambda < 2$, there exists a unique entropy solution (Alibaud 06); tools : doubling of var. with $\mathcal{R}^r_\lambda, \mathcal{S}^r_\lambda$ in the entropy formulation; kernel $K_t(x) = \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda})$ associated with $\mathcal{L}_\lambda$; time splitting

Further, or $0 < \lambda < 1$ and the Burgers flux $f(u) = \frac{u^2}{2}$ in dim. one:

- assume the initial datum u_0 presents an initial discontinuity (say, at zero) with $u_0(0-) > u_0(0+)$ and belongs to a class $\mathcal{C}$ $\Longrightarrow$ the discontinuity is persistent, at least for small times
- specially selected smooth initial data in $\mathcal{C}$ $\Longrightarrow$ the unique entropy solution, which becomes discontinuous (but not instantly)
- small Lipschitz data in the class $\mathcal{C}$ $\Longrightarrow$ global Lipschitz solutions

(Alibaud-Droniou-Vovelle 07; the main tool : characteristics)

For the same fractional Burgers equation in the "hyperbolic regime",

- Non-entropy weak solutions can be constructed
- Consequently, uniqueness of a weak solution may fail .

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CONSTRUCTION OF A "WRONG" WEAK SOLUTION

Ideas of the construction

- We try to mimic the simplest “wrong” weak solution of the Burgers conservation law $\partial_t u + \left(\frac{u^2}{2}\right) = 0$. This is the discontinuous stationary solution
$$u(t,x) := \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$
- One simple reason why this is not an entropy solution is that it fails to satisfy the Oleĭnik inequality $\partial_x u(t,x) \leq \frac{const}{t}$ in $\mathcal{D}'$.
- We prove that the Oleĭnik inequality still holds for entropy solutions of the fractional Burgers equation
- We work in the space of odd in x functions discontinuous at zero. If we ensure that $u(0+) = -u(0-) > 0$, then the Oleĭnik condition is violated.
- a comparison principle for odd “sub-super-solutions” holds; to prove it, we use adapted entropies $\eta(x; u, k) = (u-k)^+ 1\!\!1_{\{x>0\}} + (u-k)^- 1\!\!1_{\{x<0\}}$.

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Fractional laplacian on the space of odd functions...

Let H^1_* be the space of functions u on $\mathbb{R}_* = \mathbb{R}^- \cup \mathbb{R}^+$ such that $u1\!\!1_{\{x>0\}} \in H^1(\mathbb{R}^+)$, $u1\!\!1_{\{x<0\}} \in H^1(\mathbb{R}^-)$.

Note the existence of traces $u(0+)$,$u(0-)$.

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Lemma (Fract. lapl. on the space of piecewise H^1 functions)

Let $\lambda \in (0,1)$ and $\mathcal{L}_\lambda$ defined by the Lévi-Khinchine formula. Then

- *The linear operators $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda/2}$ are bounded as operators:*

  a) $\mathcal{L}_\lambda : C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*) \longrightarrow C(\mathbb{R}_*)$;
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  Moreover, $\mathcal{L}_\lambda$ is sequentially continuous as an operator:

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- *If $v \in H^1_*$,the definition of $\mathcal{L}_\lambda$ by Fourier transform makes sense.*
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Lemma (continued)

- *If $v \in H^1_*$ is odd (resp., even), then $\mathcal{L}_\lambda[v]$ is odd (resp., even).*
- *Let $0 \not\equiv v \in C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*)$ be odd. Assume that $x_* > 0$ is such that*
$$v(x_*) = \sup_{\mathbb{R}^+_*} v \geq 0 \quad \big(\textit{resp.} \; = \inf_{\mathbb{R}^+_*} \leq 0\big).$$
Then $\mathcal{L}_\lambda[v](x_*) > 0$ (resp. < 0) ("strong max. principle").
- *For $k \in \mathbb{R}$, let $\eta(x; \cdot, k) = (\cdot - k)^+ 1\!\!1_{\{x>0\}} + (\cdot - k)^- 1\!\!1_{\{x<0\}}$. Then, for all odd $v \in C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*)$, for all $x > 0$*
$$\eta'(x; v(x), k)\; \mathcal{L}_\lambda[v](x) \;\geq\; \mathcal{L}_\lambda[\eta(x; v(x), k)](x).$$
(adapted entropy) ; the same holds with $\mathcal{S}^r_\lambda$ in the place of $\mathcal{L}_\lambda$.

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Proof I

We construct solutions in the space H^1_{odd} of the regularized stationary problem

$$\varepsilon(u - \Delta u) + \left(\frac{u^2}{2}\right)_x + \mathcal{L}[u] = 0, \quad u(0\pm) = \pm 1.$$

(they are $\mathcal{D}'(\mathbb{R}_*)$ solutions;
in $\mathcal{D}'(\mathbb{R})$, the singular source term $-2\varepsilon(\delta_0)_x$ appears in the rhs !)

Techniques:

- "Freeze" the convection term, truncate the nonlinearity (in u) and its support (in x):

$$\text{replace } \left(\frac{u^2}{2}\right)_x \quad \text{by} \quad \rho_n(x)\left(\frac{(\rho_n(x)\, T_n(\bar{u}))^2}{2}\right)_x.$$

- The problem obtained in this way is the Euler-Lagrange equation for a quite standard convex minimization problem.
- Obtain *a priori* estimates to ensure boundedness; the strict convexity enforces compactness.
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Proof II

Pass to the limit, as $\varepsilon \downarrow 0$. The things to be cared of:

- Compactness (in $H^{\lambda/2}(\mathbb{R}^{\pm})$-weak and for the a.e. convergence): this comes from the uniform in ε estimate in $H^{\lambda/2}(\mathbb{R}^{\pm})$
- Passage to the limit in the weak formulation: straightforward
- Guarantee that the discontinuity of u^{ε} at $x = 0$ persists at the limit

The last item is challenging. We have two proofs .

– a first one , with an explicit construction of barriers m, M such that $\pm m(x) \leq \pm u^{\varepsilon}(x) \leq \pm M(x)$ for $\pm x > 0$, and $u_m(0+) = 1 = u_M(0+)$.
The tools are: explicit sub-and-supersolutions, and the comparison principle (deduced with the help of the adapted entropies from the Kato inequality).

– a second proof , with a passage to the limit in the traces of u^{ε} .
This looks a bit hopeless starting from the sole a.e. convergence of u^{ε} to u (nothing seems to prevent the formation of a boundary layer).
Fortunately, the Green-Gauss formula and the PDE in hand permit to pass to the (weak) limit in the traces of the flux $\frac{1}{2}(u^{\varepsilon})^2(0\pm)$ and get $u(0\pm) = \pm 1$.

Conclusion: the so constructed u is stationary, discontinuous, it is a weak solution to the fractional Burgers equation ; it violates the entropy condition .

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And that's it...

GRAZIE !!!