Convergence of finite volume schemes for the macroscopic bidomain model of the heart electric activity.

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Plan of the talk

Model and main results

Problem considered...

We provide a convergence analysis of certain Finite Volume schemes (namely, different variants of DDFV schemes) for the problem (Prob):

$$\partial_t v - \operatorname{div} \mathbf{M}_i(x)\nabla u_i + h(v) = I_{\text{app}}, \qquad (t,x) \in Q_T,$$
$$-\partial_t v - \operatorname{div} \mathbf{M}_e(x)\nabla u_e - h(v) = -I_{\text{app}}, \qquad (t,x) \in Q_T,$$

with Neumann boundary conditions

$$(\mathbf{M}_{i,e}(x)\nabla u_{i,e}) \cdot n = s_{i,e} \quad \text{on } (0,T) \times \partial\Omega,$$

and with initial datum: $v(0,x) = v_0(x),\ x \in \Omega.$ Here :

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and with initial datum: $v(0,x) = v_0(x),\ x \in \Omega.$ Here :

- Ω is a time-independent Lipschitz domain in $\mathbb{R}^3$ (the heart does not move...); the space-time domain is $Q_T = (0,T) \times \Omega$;
- u_i, u_e is the intra- (respectively, extra-) cellular electric potential;
- $v := u_i - u_e$ is the transmembrane potential;
- $I_{app}(t,x)$ is a given stimulation current;
- $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$ are the intra- and extra-cellular conductivity tensors (assumed symmetric, positive definite, but anisotropic).
- $h(v)$ is an *ad hoc* model for the transmembrane ionic current. We focus on the case where h is close to a cubic polynomial.

Assumptions on the ionic current

We assume that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist $r \in (2, +\infty)$ and constants $C, L, l > 0$ such that

$$\frac{1}{C}\,|v|^r \leq |h(v)v| \leq C\left(|v|^r + 1\right), \tag{1}$$

$$\tilde{h} : z \mapsto h(z) + Lz + l \quad \text{is strictly increasing on } \mathbb{R}, \text{ with } \tilde{h}(0) = 0. \tag{2}$$

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Our results will cover the cases

— of a general r, for the convergence of the fully implicit scheme ;

— of $r < 4$ (strictly !), for the linearized implicit scheme .

There is some hope for attaining the critical case $r = 4$
(V.V. Zhikov's lemmas...)

Main results

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  - either as $[\,h(v)\,]^{n+1} = h(v^n)$ (time-explicit);
  - or as $[\,h(v)\,]^{n+1} = h(v^{n+1})$ (time-implicit);
  - or as $[\,h(v)\,]^{n+1} = \frac{h(v^n)+Lv^n+I}{v^n}\, v^{n+1} - Lv^{n+1} - I$ (linearized implicit).

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- We have existence for each of the schemes .
- For the fully implicit scheme , we show convergence . Drawback : the scheme is nonlinear !
- For the linearized implicit scheme, we show convergence if $r < 4$ (whereas $r = 4$ is regarded as the practical case).

NB : In practice, the explicit discretization of the ionic current works (cf. Ch. Pierre), but we fail to prove *a priori* estimates.

Mathematical Setting

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A weak solution to (Prob) is a triple of functions $(u_i, u_e, v) : \Omega \to \mathbb{R}^3$ such that $u_{i,e} \in L^2(0, T; V)$, $v = u_i - u_e$, $v \in L^r(Q)$, and (Prob) is satisfied in $\mathcal{D}'([0, T) \times \overline{\Omega})$.

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We normalize u_i by requiring that $\int_\Omega u_i(t,\cdot) = 0$ for a.e. $t \in (0,T)$.

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It is convenient to give a "variational" formulation of (*Prob*).

The following spaces are (almost) relevant for the analysis of (*Prob*):

$$L^2(0,T;V) \cap L^r(Q) \text{ and its dual space } L^2(0,T;V') + L^{r'}(Q);$$

the notation $< \cdot, \cdot >$ refers to their duality pairing .

Variational (re)formulation.

Remark

If (u_i, u_e, v) *is a weak solution of* $(Prob)$*, the system and initial condition are satisfied in the space* $L^2(0,T;V') + L^{r'}(Q)$*, i.e. the distributional derivative* $\partial_t v$ *can be identified with an element of* $L^2(0,T;V') + L^{r'}(Q)$ *.*

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With this identification, one has for all $\varphi \in L^2(0,T;V) \cap L^r(Q)$

$$\int_0^T < \partial_t v, \varphi > + \iint_Q \Big(\mathbf{M}_i(x, \nabla u_i) \cdot \nabla \varphi + h(v)\varphi \Big) - \int_0^T \int_{\partial\Omega} s_i \, \varphi = \iint_Q I_{\text{app}} \varphi,$$

$$\int_0^T < \partial_t v, \varphi > - \iint_Q \Big(\mathbf{M}_e(x, \nabla u_e) \cdot \nabla \varphi + h(v)\varphi \Big) + \int_0^T \int_{\partial\Omega} s_e \, \varphi = \iint_Q I_{\text{app}} \varphi;$$

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and for all $\varphi \in L^2(0,T;V)$ *such that* $\partial_t \varphi \in L^\infty(Q)$ *and* $\varphi(T,\cdot) = 0$,

$$\int_0^T < \partial_t v, \varphi > = - \iint_Q v \, \partial_t \varphi - \int_\Omega v_0(\cdot) \, \varphi(0,\cdot).$$

A regularization lemma. Well-posedness.

The following tool helps to work with the variational formulation.

Lemma

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. There exists a family of linear operators

$$(\mathcal{R}_\varepsilon)_\varepsilon \quad \text{acting from } L^2(0,T;V) \text{ into } \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$$

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- *for all $w \in L^2(0,T;V)$, $\mathcal{R}_\varepsilon(w)$ converges to w in $L^2(0,T;V)$;*
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This lemma is used in order to regularize $u_{i,e}$, so that one could take $\mathcal{R}_\varepsilon(u_{i,e})$ as test functions in the equations (*Prob*). E.g., *a priori* estimates on the weak solution are obtained in this way. Another application is the uniqueness theorem below.

Theorem

There exists at most one weak solution (u_i, u_e, v) to (Prob).

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Proof.

Take $\zeta \in \mathcal{D}([0,T))$. We take $\zeta(t)\mathcal{R}_\varepsilon(u_i - \hat{u}_i)(t,x)$ as the test function in the first equation , and $\zeta(t)\mathcal{R}_\varepsilon(u_e - \hat{u}_e)(t,x)$, in the second equation of $(Prob)$. We subtract the equations obtained and let $\varepsilon \to 0$. By the linearity of $\mathcal{R}_\varepsilon(\cdot)$ and the convergences of $\mathcal{R}_\varepsilon$, using the chain rule we get as $\varepsilon \to 0$,

$$\iint_Q -\frac{(v-\hat{v})^2}{2}\,\partial_t\zeta - \int_\Omega \frac{(v_0-\hat{v}_0)^2}{2}\,\zeta(0) + \iint_Q \big(\,h(v)-h(\hat{v})\,\big)\,(v-\hat{v})$$
$$+\sum\nolimits_{j=i,e}\iint_Q \Big((\mathbf{M}_j(x)\nabla u_j - \mathbf{M}_j(x)\nabla\hat{u}_j))\cdot(\nabla u_j - \nabla\hat{u}_j) \le 0.$$

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For a.e. $t>0$, letting ζ converge to the characteristic function of $[0,t]$, thanks to the monotonicity assumption on $\tilde{h}$ we deduce

$$\int_\Omega (v-\hat{v})^2(t) \le \int_\Omega (v_0-\hat{v}_0)^2 + 2L\int_0^t\int_\Omega (v-\hat{v})^2.$$

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Then also $\hat{u}_i = u_i$ because of the monotonicity assumption on $\mathbf{M}_i(x)$ and the normalization condition on u_i; finally, $\hat{u}_e = u_e$. ◇

Time-implicit discretization : a minimization problem !

Rq: if one replaces $\partial_t v$ by the backward time differencing

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then... at each time step, we have to solve the optimization pb.:

find $\min_{V\times V} J^{n+1}$, where (given v^n)

$$J^{n+1}(u_i,u_e) := \int_\Omega \frac{1}{2}(u_i-u_e)^2 - \int_\Omega v^n(u_i-u_e)$$
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This feature will be preserved in our fully implicit scheme ;
thus the scheme (although non linear) can be solved by descent methods , since it derives from minimization of a coercive functional (NB: for time steps $(t^{n+1}-t^n)$ small enough, the functional J^{n+1} will be strictly convex).

DDFV Schemes

Generalities on FV schemes

We use Finite Volumes (FV, for short) for space discretization of $(Prob)$; for the time discretization, we'll pick the (essentially implicit) Euler scheme.

Thus we describe the scheme at one time step .

Generally, let us think of discretizing an elliptic equation of the kind

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The principles for FV approximation of such equations are the following:

- A partition of the space domain Ω into "volumes" κ is given; the partition is called "mesh" and denoted by $\mathfrak{T}$.
- An unknown u_κ is associated to each volume (usually regarded as the value at a "center" point of the volume, denoted x_κ); the whole set of the unknowns is called "discrete solution" and denoted by $u^{\mathfrak{T}}$.

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- If K, L are "neighbours" (adjacent volumes), the divided differences $\frac{u_L - u_K}{d_{KL}}$ are used to "reconstruct" the "discrete gradient" $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ of $u^{\mathfrak{T}}$.

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  NB: In the schemes we think of, the reconstruction is done "by hands" . We do not solve any equations to compute the values of $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ from those of $u^{\mathfrak{T}}$, but fix *ad hoc* formulas for $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ in terms of $u^{\mathfrak{T}}$.

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  Here we may have too much liberty ! "DDFV" schemes we'll use is one way to reconstruct, among many others .

Generalities on FV schemes (cont^{d})

- Each couple of neighbour volumes possesses a part of the common border , called “interface” and denoted by $_{K\!|\!L}$. Combining the values u_K, u_L and the values of the reconstructed discrete gradient $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ near $_{K\!|\!L}$, we produce a discretization $\mathcal{F}_{K\!|\!L}$ of the flux $\mathcal{F} = A(x, u, \nabla u)$ on $_{K\!|\!L}$.

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- Each couple of neighbour volumes possesses a part of the common border , called "interface" and denoted by $\kappa|\!\!|L$. Combining the values u_κ, u_L and the values of the reconstructed discrete gradient $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ near $\kappa|\!\!|L$, we produce a discretization $\mathcal{F}_{\kappa|\!\!|L}$ of the flux $\mathcal{F} = A(x, u, \nabla u)$ on $\kappa|\!\!|L$.
- In two steps, the PDE is replaced by a set of algebraic equations :
  - Firstly, the PDE is "projected on the mesh" . One integrates the continuous equation on κ; the Green-Gauss formula is used to reduce $\int_\kappa \operatorname{div} \mathcal{F}$ to $\int_{\partial\kappa} \mathcal{F} \cdot n$.

    This yields a discrete system of equalities, one per volume.
  - And secondly, these equalities are approximated by replacing $u|_\kappa$ with the unknown u_κ and replacing $\mathcal{F}|_{\kappa|\!\!|L}$ with the expression of $\vec{\mathcal{F}}_{\kappa|\!\!|L}$ (in our setting, $\vec{\mathcal{F}}_{\kappa|\!\!|L}$ is a closed-form formula involving u_κ and the neighbour unknowns u_L).
- As a result, we obtain a closed system of algebraic equations which (hopefully...) can be solved , often in an approximate way. This yields $u^{\mathfrak{T}}$ (and also $\nabla u^{\mathfrak{T}}$) that may be viewed as approximations of u and ∇u, respectively...

Generalities on FV schemes (cont^{d})

- Each couple of neighbour volumes possesses a part of the common border , called "interface" and denoted by $K|L$. Combining the values u_K, u_L and the values of the reconstructed discrete gradient $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ near $K|L$, we produce a discretization $\mathcal{F}_{K|L}$ of the flux $\mathcal{F} = A(x, u, \nabla u)$ on $K|L$.
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We do not discuss the efficiency of different approximations, but only look at mathematical properties of the schemes that allow to justify the convergence.

DDFV : pro&contra

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- Unless the mesh is "orthogonal" and the problem is isotropic, the discrete maximum principle for linear elliptic problems fails
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+ in FV, local conservativity is built in explicitly (which does not mean that, e.g., finite elements schemes are not locally conservative)
+ in the DDFV schemes, we have a naturally consistent discretization of the gradient , and in many cases, the discrete gradient converges strongly to the exact solution gradient

DDFV : pro&contra (cont^d)

- + DDFV schemes support quite unstructured meshes (non-orthogonal, non conforming, locally refined)
- + they work on heterogeneous, anisotropic linear problems (like our (*Prob*) !) and on nonlinear problems (e.g., non-Newtonian fluids)
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The consequences are :

- the key mathematical features that underlie the existence theory for (*Prob*) are preserved : coercivity, monotonicity, variational structure . This ensures that the scheme is unconditionally stable.
- The variational structure being preserved, the scheme (although not necessarily linear) derives from a convex minimization problem; therefore the scheme can be efficiently solved .
- Calculations can be written in a concise and easy-to-interprete way, thanks to the far-reaching analogy between the continuous and the discrete frameworks.

DDFV in dimension two

The DDFV schemes were devised in 2D, first by F. Hermeline '98 , then rediscovered and popularized by K. Domelevo & P. Omnès '05 .

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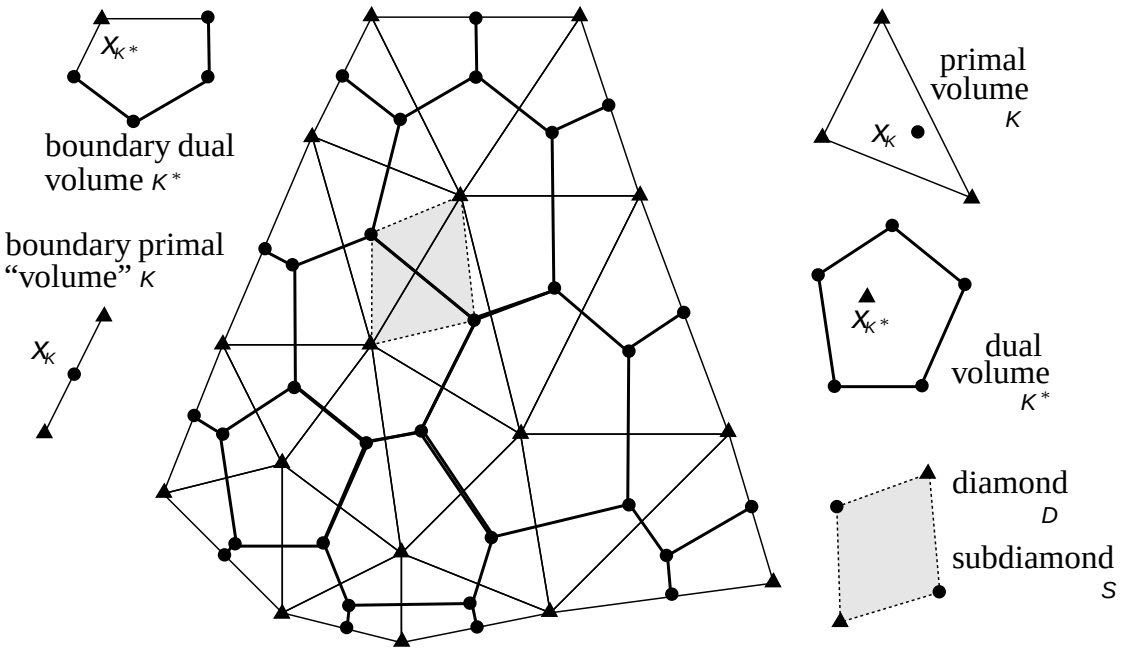
Different FV schemes solving this difficulty were proposed by Coudière & Vila & Villedieu, Afif & Amaziane, Handlovicova & Mikula & Sgallari, Aavatsmark et al., A. & Gutnic & Wittbold, A. & Boyer & Hubert, Eymard & Droniou, Le Potier, ... and more recently, a number of schemes by Eymard & Gallouët & Herbin .

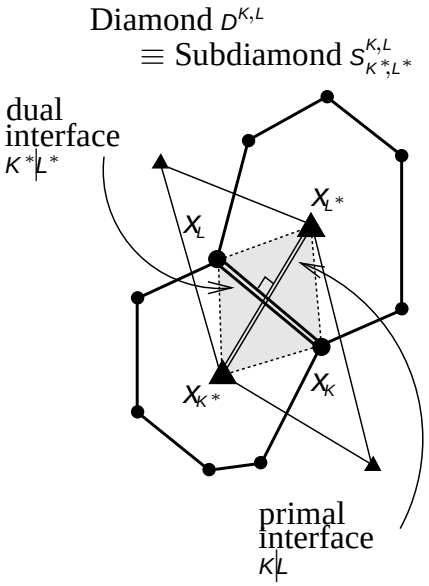
DDFV in dimension two



The idea of the "DDFV" schemes (which should rather be called "double" schemes) is to use the unknowns on both "primal" and "dual" meshes.

This induces a partition of Ω into quadrilaterals called "diamonds" D ;
the discrete gradient is reconstructed diamond-wise ,
using the divided differences along the diagonals of D.

DDFV in dimension two (cont^{d})



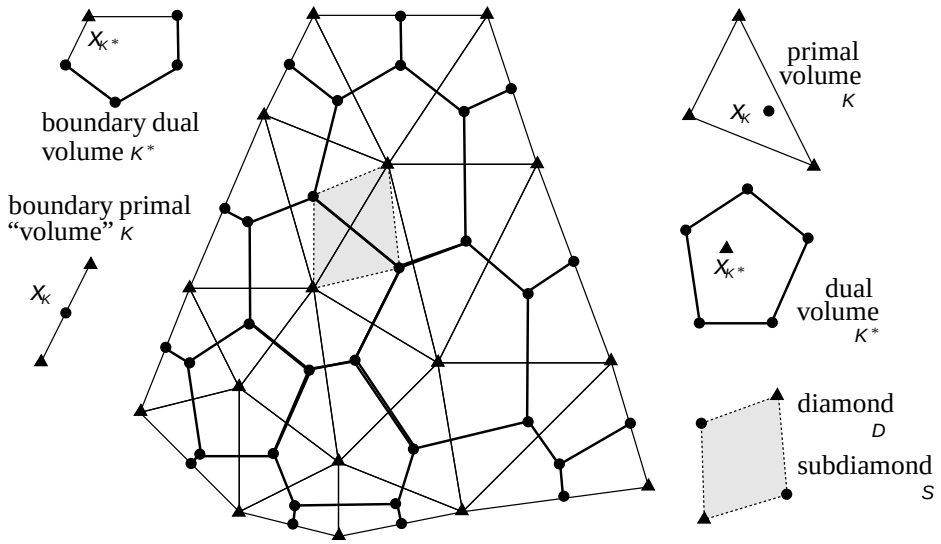
In 2D, we reconstruct the discrete gradient on D (and hence also on $K|L$ and $K^*|L^*$) from the two given projections:

$$\nabla^{\mathfrak{T}} u^{\mathfrak{T}}|_{K|L} = \nabla^{\mathfrak{T}} u^{\mathfrak{T}}|_{K^*|L^*}$$

$$= \nabla^{\mathfrak{T}} u^{\mathfrak{T}}|_{D^{K,L}_{K^*,L^*}} = \text{the vector of } \mathbb{R}^2 \text{ with the projections} \begin{cases} \dfrac{u_L - u_K}{d_{KL}}, & \text{on } \overrightarrow{x_K x_L} \\ \dfrac{u_{L^*} - u_{K^*}}{d_{K^*L^*}}, & \text{on } \overrightarrow{x_{K^*} x_{L^*}}. \end{cases}$$

DDFV in dimension two (cont^{d})

On the preceding figures, the primal mesh (denoted $\mathfrak{M}$) consists of triangles, and the dual mesh (denoted $\mathfrak{M}^*$) has the vertices of primal volumes $K \in \mathfrak{M}$ for centers x_{K^*} of dual volumes K^*, and it has the primal centers x_K for vertices of the dual volumes K^*.



Because the discrete gradient has been reconstructed on each primal interface $K|L$ and on each dual interface $K^*|L^*$, we can "integrate" the diffusion term $-\mathrm{div}\,\mathcal{F} \equiv -\mathrm{div}\,A(x, u, \nabla u)$ on each primal volume K and on each dual volume K^*. This yields a closed system of equations on

$$u^{\mathfrak{T}} := \left((u_K)_{K \in \mathfrak{M}} \, , \, (u_{K^*})_{K^* \in \mathfrak{M}^*} \right) .$$

DDFV in 2D: discrete duality features

Notice that in the DDFV approach, the field $\mathcal{F}$ is discretized with one vector value per diamond $D \in \mathfrak{D}$. In general, we will say that a vector-valued set $(\vec{\mathcal{F}}_D)_{D\in\mathfrak{D}}$ is a discrete function , denoted by $\vec{\mathcal{F}}^{\mathfrak{T}}$.

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The discrete gradient $\nabla^{\mathfrak{T}}$ can be seen as an operator acting from the space $\mathbb{R}^{\mathfrak{T}}$ of discrete functions to the space $(\mathbb{R}^2)^{\mathfrak{D}}$ of discrete fields.

Reciprocally, the standard FV discrete divergence operator $\mathrm{div}^{\mathfrak{T}}$ acts from the space $(\mathbb{R}^2)^{\mathfrak{D}}$ of discrete fields to the space $\mathbb{R}^{\mathfrak{T}}$ of discrete functions.

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In order to state the discrete duality, we introduce two scalar products:

– for discrete functions $w^{\mathfrak{T}}, v^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}$,

$$\left[\!\left[w^{\mathfrak{T}},\, v^{\mathfrak{T}} \right]\!\right] = \frac{1}{2}\sum_{K\in\mathfrak{M}} m_K\; w_K v_K \;+\; \frac{1}{2}\sum_{K^*\in\mathfrak{M}^*} m_{K^*}\; w_{K^*} v_{K^*};$$

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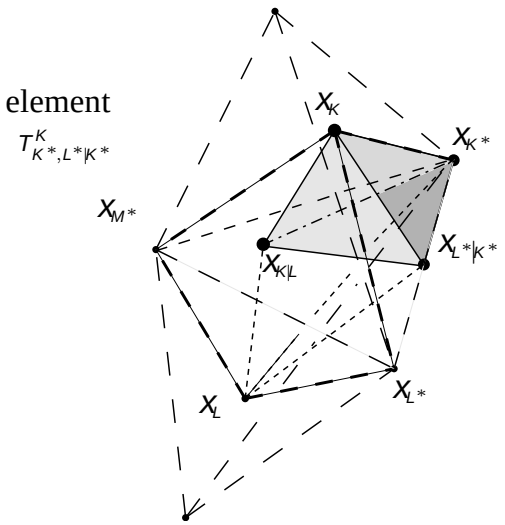
Proposition (The Discrete Duality property)

The discrete divergence and gradient operators $-\mathrm{div}^{\mathfrak{T}}$, $\nabla^{\mathfrak{T}}$ *are linked by*

$$\forall\, w^{\overline{\mathfrak{T}}} \in \mathbb{R}_0^{\overline{\mathfrak{T}}} \quad \forall\, \vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}} \qquad \left[\!\left[-\,\mathrm{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\, w^{\mathfrak{T}} \right]\!\right] = \left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}},\; \nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}} \right\}\!\!\right\}.$$

DDFV in dimension three

In 3D, the primal mesh $\mathfrak{M}$ consists of tetrahedra; the description of the dual mesh $\mathfrak{M}^*$ is a bit tedious. A generic dual volume K^* consists of all "elements" having x_{K^*} for a vertice; a generic element is depicted on the figure:



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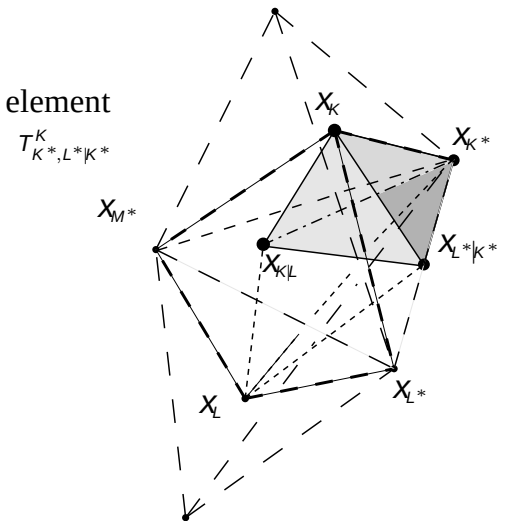

Similarly to the 2D case, a 3D diamond $D = D^{K,L}_{K^*,L^*,M^*}$

– isolates the interaction of neighbours K, L and dual neighbours K^*, L^*, M^*;

– serves to reconstruct the discrete gradient $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ of a discrete function $u^{\mathfrak{T}}$.

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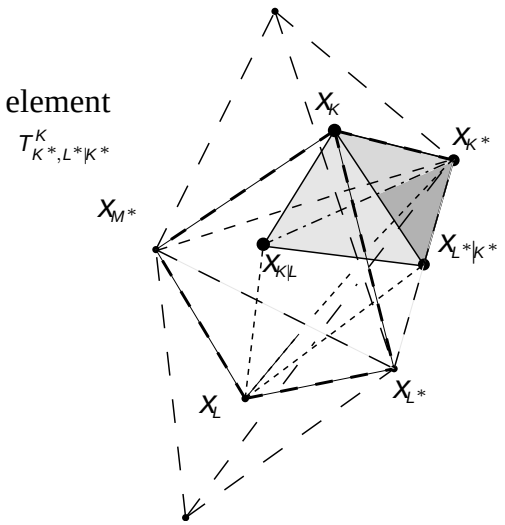

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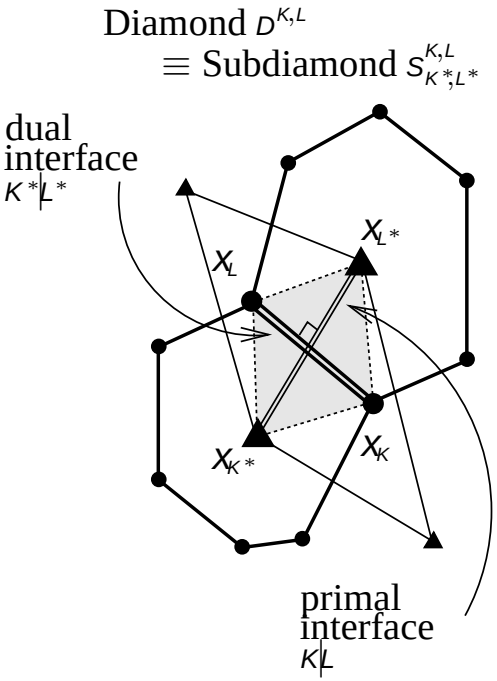
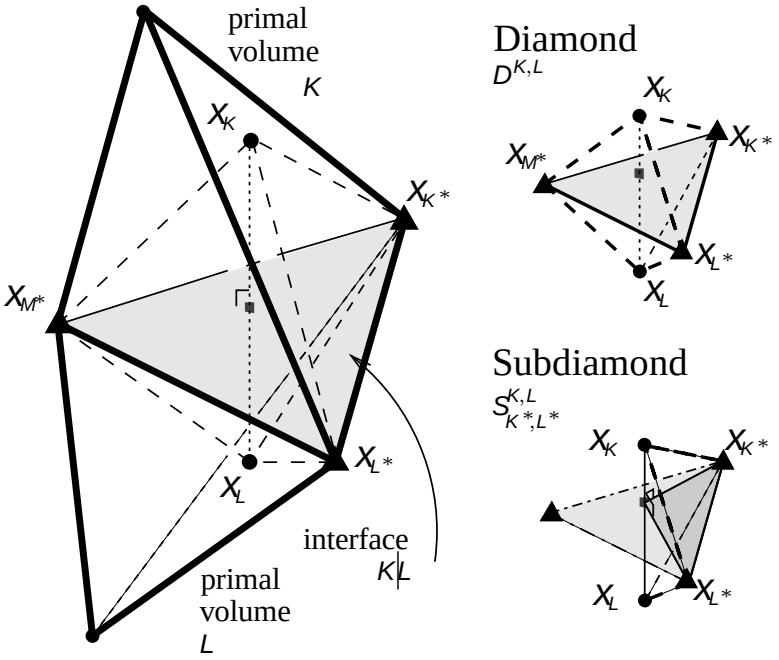
Similarly, a 3D subdiamond $s = s^{K,L}_{K^*,L^*}$

– isolates the interaction of neighbours K, L and dual neigbours K^*, L^*;
– serves to reconstruct the discrete divergence $\mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}}$ of a discrete field $\vec{\mathcal{F}}^{\mathfrak{T}}$.

DDFV in dimension three (cont^d)

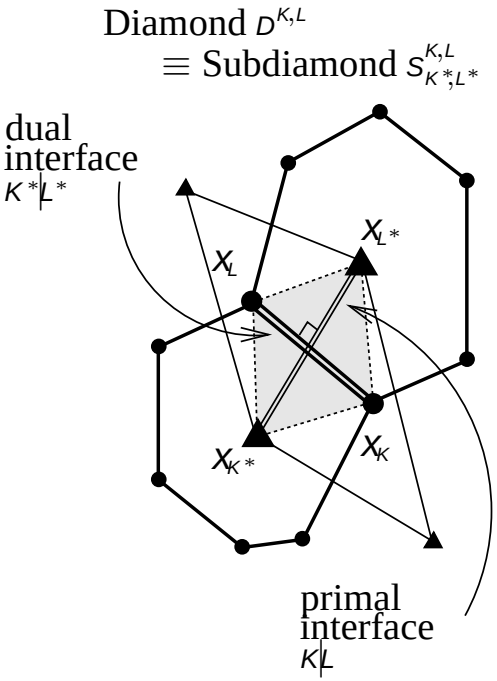


(a) 2D (sub)diamond.



(b) 3D primal volumes, diamond, subdiamond.

DDFV in dimension three (cont^d)



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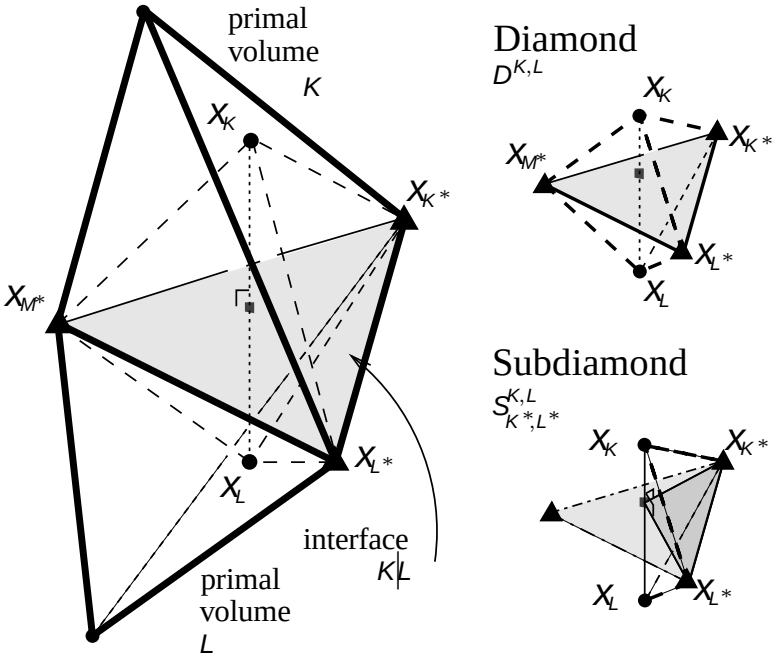

(d) 3D primal volumes, diamond, subdiamond.

Analogous gradient reconstruction applies in 3D , namely for $D = D^{K,L}_{K^*,L^*,M^*}$,

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DDFV in 3D: discrete duality features

In order to state the discrete duality and take into account the Neumann boundary condition, we need three scalar products:

– for $3D$ discrete functions $w^{\mathfrak{T}}, v^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}$,

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Here the function $w^{\mathfrak{T}}$ on Ω appears naturally:

$$w^{\mathfrak{T}} = \frac{1}{3} \sum_{K \in \mathfrak{M}} w_K \mathbb{1}_K + \frac{2}{3} \sum_{K^* \in \mathfrak{M}^*} w_{K^*} \mathbb{1}_{K^*} = \frac{1}{3} w^{\mathfrak{M}} + \frac{2}{3} w^{\mathfrak{M}^*}$$

and $w^{\partial \mathfrak{T}}$ is its restriction on $\partial\Omega$ (or, more precisely, its trace).

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$$w^{\mathfrak{T}} = \frac{1}{3} \sum_{K \in \mathfrak{M}} w_K \mathbb{1}_K + \frac{2}{3} \sum_{K^* \in \mathfrak{M}^*} w_{K^*} \mathbb{1}_{K^*} = \frac{1}{3} w^{\mathfrak{M}} + \frac{2}{3} w^{\mathfrak{M}^*}$$

and $w^{\partial \mathfrak{T}}$ is its restriction on $\partial\Omega$ (or, more precisely, its trace).

Proposition (The Discrete Duality property)

For all $w^{\overline{\mathfrak{T}}} \in \mathbb{R}^{\overline{\mathfrak{T}}}$ and all $\vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^3)^{\mathfrak{D}}$ such that $\vec{\mathcal{F}}^{\mathfrak{T}} \cdot \vec{n}|_{\partial\Omega} = s^{\mathfrak{T}}$,

$$\left[\!\left[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}], w^{\mathfrak{T}} \right]\!\right] = \left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}} \right\}\!\!\right\} - \left\langle\!\!\left\langle s^{\mathfrak{T}}, w^{\partial \mathfrak{T}} \right\rangle\!\!\right\rangle.$$

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Proposition (Consistency of the discrete gradient)

Assume $w \in L^p(\Omega)$, $p < \infty$.

Denote $$w_K := \frac{1}{m_K}\int_{m_K} w, \qquad w_{K^*} := \frac{1}{m_{K^*}}\int_{m_{K^*}} w.$$

Set

$$w^{\mathfrak{M}_h} := \left(w_K\right)_{K\in\mathfrak{M}_h} \quad \text{(assimilated to } \sum_{K\in\mathfrak{M}_h} w_K \mathbb{1}_K\text{)},$$

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– *Then* $w^{\mathfrak{M}_h} \to w$ *and* $w^{\mathfrak{M}^*_h} \to w$ *in* $L^p(\Omega)$.

– *Moreover, if* $\nabla w \in L^q(\Omega)$, *then* $\nabla^{\mathfrak{T}_h} w^{\mathfrak{T}_h} \to \nabla w$ *in* $L^q(\Omega)$, $q < \infty$.

Discrete calculus tools (cont^{d})

Proposition (Asymptotic $W^{1,p}$ compactness)

Assume $(w^{\mathfrak{T}_h})_{h\downarrow 0}$ *is a sequence of discrete functions on DDFV meshes* $\mathfrak{T}_h$ *such that*

$$\| \nabla^{\mathfrak{T}_h} w^{\mathfrak{T}_h} \|^p_{L^p(\Omega)} := \sum_{D\in\mathfrak{D}_h} m_D | \nabla_D w^{\mathfrak{T}_h} |^p \leq const,$$

and also $\|w^{\mathfrak{M}_h}\|_{L^p(\Omega)}, \|w^{\mathfrak{M}^*_h}\|_{L^p(\Omega)} \leq const$*, where* $p \in [1,\infty)$*.*

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Then there exists $w \in W^{1,p}(\Omega)$ *such that*

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Proposition (Discrete Poincaré inequality)

There exists a constant C *independent of* h *such that for all discrete function* $w^{\mathfrak{T}_h}$ *on the DDFV mesh* $\mathfrak{T}_h$, *for all* $p \in [1,\infty)$ *there holds*

$$\|w^{\mathfrak{M}_h}\|_{L^p(\Omega)} \leq const \Big(\| \nabla^{\mathfrak{T}_h} w^{\mathfrak{T}_h} \|_{L^p(\Omega)} + \frac{1}{|\Omega|} \Big| \int_\Omega w^{\mathfrak{M}_h} \Big| \Big),$$

and idem for $w^{\mathfrak{M}_h}$ *replaced with* $w^{\mathfrak{M}^*_h}$.

DDFV Scheme for the Model Problem

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$$\begin{cases} \text{find } \Big((u_i^{\mathfrak{T},n+1}, u_e^{\mathfrak{T},n+1}, v^{\mathfrak{T},n+1})\Big)_{n=0,\dots,N} \subset (R^{\mathfrak{T}})^3,\ N \approx T/\Delta t, \text{ satisfying} \\ \dfrac{v^{\mathfrak{T},n+1} - v^{\mathfrak{T},n}}{\Delta t} - \operatorname{div}^{\mathfrak{T}}[\mathbf{M}_i^{\mathfrak{T}} \nabla^{\mathfrak{T}} u_i^{\mathfrak{T},n+1}] + (h(v^{\mathfrak{T},???}))^{\mathfrak{T},n+1} = I_{\text{app}}^{\mathfrak{T},n+1}, \\ \dfrac{v^{\mathfrak{T},n+1} - v^{\mathfrak{T},n}}{\Delta t} + \operatorname{div}^{\mathfrak{T}}[\mathbf{M}_e^{\mathfrak{T}} \nabla^{\mathfrak{T}} u_e^{\mathfrak{T},n+1}] + (h(v^{\mathfrak{T},???}))^{\mathfrak{T},n+1} = I_{\text{app}}^{\mathfrak{T},n+1}, \\ v^{\mathfrak{T},n+1} = u_i^{\mathfrak{T},n+1} - u_e^{\mathfrak{T},n+1}, \qquad v^{\mathfrak{T},0} = v_0^{\mathfrak{T}}. \end{cases}$$

Here the ionic current discretization term $(h(v^{\mathfrak{T},???}))^{\mathfrak{T},n+1}$ should be precised.

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we take for the value in a volume (e.g., in a primal volume κ) the expression

$$\frac{1}{m_K}\int_K h(v^{\mathfrak{T},n+1}), \quad \text{i.e.,} \quad \frac{1}{m_K}\sum_{K^*\in\mathfrak{M}^*} m_{K\cap K^*}\, h\Big(\frac{1}{3}v_K^{n+1} + \frac{2}{3}v_{K^*}^{n+1}\Big)$$

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– for the linearized implicit scheme , we write $h(z) = b(z)z - Lz - I$ and take

$$\frac{1}{m_K}\int_K \big(b(v^{\mathfrak{T},n})v^{\mathfrak{T},n+1} - Lv^{\mathfrak{T},n+1} - I\big), \quad \text{with analogous interpretation.}$$

The variational nature of the time-implicit scheme

It is straightforward to check that the scheme derives from a convex coercive functional. At each time step, we have to solve the optimization problem:

find $\min_{\mathbb{R}^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{T}}} J^{\mathfrak{T},n+1}$, where (given $v^{\mathfrak{T},n}$)

$$J^{n+1}(u_i^{\mathfrak{T}},u_e^{\mathfrak{T}}) := \frac{1}{2}\left[\!\left[u_i^{\mathfrak{T}}-u_e^{\mathfrak{T}},u_i^{\mathfrak{T}}-u_e^{\mathfrak{T}}\right]\!\right] - \left[\!\left[u_i^{\mathfrak{T}}-u_e^{\mathfrak{T}},v^{\mathfrak{T},n}\right]\!\right]$$

$$+\frac{\Delta t}{2}\left\{\!\!\left\{\mathbf{M}_i\,\nabla^{\mathfrak{T}}u_i^{\mathfrak{T}},\,\nabla^{\mathfrak{T}}u_i^{\mathfrak{T}}\right\}\!\!\right\} + \frac{\Delta t}{2}\left\{\!\!\left\{\mathbf{M}_e\,\nabla^{\mathfrak{T}}u_e^{\mathfrak{T}},\,\nabla^{\mathfrak{T}}u_e^{\mathfrak{T}}\right\}\!\!\right\}$$

$$+\Delta t\sum_{K\in\mathfrak{M},\ K^*\in\mathfrak{M}^*} m_{K\cap K^*}\,H\left(\frac{1}{3}(u_{i,K}-u_{e,K})+\frac{2}{3}(u_{i,K^*}-u_{e,K^*})\right);$$

recall that H is a primitive of h , $H : z \mapsto \int_0^z h(s)\,ds$.

Convergence Analysis

Convergence : the time-implicit scheme

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$$\sum_n \Delta t \sum_{K,\, K^*} m_{K\cap K^*}\, h\Big(\frac{1}{3}v_K^{n+1} + \frac{2}{3}v_{K^*}^{n+1}\Big)\Big(\frac{1}{3}v_K^{n+1} + \frac{2}{3}v_{K^*}^{n+1}\Big) \equiv \iint_{Q_T} h(v^{\mathfrak{T},\Delta t})v^{\mathfrak{T},\Delta t};$$

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  - on the L^2 time translates of $v^{\mathfrak{M},\Delta t}$, $v^{\mathfrak{M}^*,\Delta t}$, thanks to the previous estimates and the "evolution nature" of the problem.

Convergence : the time-implicit scheme (cont^d)

- The above estimates allow to use the compactness arguments and deduce that, up to extraction of a subsequence:

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- We take a smooth test function φ , project it on the mesh to get $\varphi^{\mathfrak{T},\Delta t}$, and take it for the test function in the scheme . The discrete duality and simple rearrangements yield (for the first eqn of the system)

$$-\sum_n \Delta t \left[\!\left[v^{\mathfrak{T},n}, \frac{\varphi^{\mathfrak{T},n+1} - \varphi^{\mathfrak{T},n}}{\Delta t} \right]\!\right] + \sum_n \left\{\!\!\left\{ \mathbf{M}_i^{\mathfrak{T}} \nabla^{\mathfrak{T}} u_i^{\mathfrak{T},n+1}, \nabla^{\mathfrak{T}} \varphi^{\mathfrak{T},n+1} \right\}\!\!\right\}$$

$$+ \sum_n \Delta t \sum_{K,\, K^*} m_{K\cap K^*}\, h\left(\frac{1}{3} v_K^{n+1} + \frac{2}{3} v_{K^*}^{n+1}\right)\left(\frac{1}{3}\varphi_K^{n+1} + \frac{2}{3}\varphi_{K^*}^{n+1}\right)$$

$$= \sum_n \Delta t \left[\!\left[I_{\text{app}}^{\mathfrak{T},n+1}, \varphi^{\mathfrak{T},n+1} \right]\!\right] + \left[\!\left[v^{\mathfrak{T},0}, \varphi^{\mathfrak{T},0} \right]\!\right].$$

Convergence : the time-implicit scheme (cont^{d})

- Replacing $\varphi^{\mathfrak{T},\Delta t}$ by φ, replacing $\nabla^{\mathfrak{T}}\varphi^{\mathfrak{T},\Delta t}$ with $\nabla\varphi$, etc., we can write the "continuous form of the discrete equations" :

$$\iint_{Q_T}\left(-v^{\mathfrak{T},\Delta t}\partial_t\varphi + (\mathbf{M}_i^{\mathfrak{T}}\,\nabla u_i)\cdot\nabla\varphi + h(v^{\mathfrak{T},\Delta t})\varphi^{\mathfrak{T},\Delta t} - I_{\mathrm{app}}\varphi\right) = \int_\Omega v_0\varphi(0) + Rem$$

where the remainder terms $Rem \equiv Rem(\phi, h, \Delta t)$ converge to zero as $h, \Delta t \downarrow 0$ thanks to the consistency of $\nabla^{\mathfrak{T}}$ and to the previously stated uniform estimates on discrete solutions.

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- Passing to the limit as $h, \Delta t \downarrow 0$ in the above equation (using the aforementioned compactnesses), we get

$$\partial_t v - \operatorname{div}(\mathbf{M}_i(x)\nabla u_i) + \chi = I_{\text{app}}, \quad v(0) = v_0 \quad \text{in } \mathcal{D}'([0,T)\times\Omega)$$

- It remains to identify χ to $h(v)$, which is done by the classical monotonicity argument (Minty-Browder).

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$$\iint_{Q_T}\left(-v^{\mathfrak{T},\Delta t}\partial_t\varphi + (\mathbf{M}_i^{\mathfrak{T}}\nabla u_i)\cdot\nabla\varphi + h(v^{\mathfrak{T},\Delta t})\varphi^{\mathfrak{T},\Delta t} - I_{\text{app}}\varphi\right) = \int_\Omega v_0\varphi(0) + Rem$$

where the remainder terms $Rem \equiv Rem(\phi, h, \Delta t)$ converge to zero as $h, \Delta t \downarrow 0$ thanks to the consistency of $\nabla^{\mathfrak{T}}$ and to the previously stated uniform estimates on discrete solutions.

- Passing to the limit as $h, \Delta t \downarrow 0$ in the above equation (using the aforementioned compactnesses), we get

$$\partial_t v - \operatorname{div}(\mathbf{M}_i(x)\nabla u_i) + \chi = I_{\text{app}}, \quad v(0) = v_0 \quad \text{in } \mathcal{D}'([0,T)\times\Omega)$$

- It remains to identify χ to $h(v)$, which is done by the classical monotonicity argument (Minty-Browder).
- As a byproduct, we get strong L^2 convergence of $\nabla^{\mathfrak{T}}u_{i,e}^{\mathfrak{T},\Delta t}$ to $\nabla u_{i,e}$, the strong L^r convergence of $v^{\mathfrak{T},\Delta t}$ to v, and the strong $L^{r'}$ convergence of $(h(v^{\mathfrak{T},\Delta t}))^{\mathfrak{T}}$ to $h(v)$.
- Uniqueness for $(Prob) \Longrightarrow$ all the sequence of the DDFV approximation converges (strongly !) to the unique solution of $(Prob)$.

Convergence : the linearized implicit scheme

The only supplementary difficulty for the analysis of the linearized implicit scheme lies in obtaining the uniform estimate on $v^{\mathfrak{T},\Delta t}$ in $L^r(Q_T)$.

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- The scheme is designed in such a way that it gives a uniform estimate on
$$\iint_{Q_T} b(v^{\mathfrak{T},\Delta t}|_{(t-\Delta t,x)})\, |v^{\mathfrak{T},\Delta t}|_{(t,x)}|^2\, dtdx.$$
where $b(z) := (h(z) + Lz + I)/z$ is ≥ 0 , by the assumptions on h.
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- Recall that we also have a uniform $L^2(Q_T)$ bound on $v^{\mathfrak{T},\Delta t}$.
- Notice that the assumption $r < 4$ yields $0 \leq b(z) \ll z^2$ for $|z| \gg 1$.

  This allows to deduce an equi-integrability estimate on the ionic current term (which is, essentially, $b(v^{\mathfrak{T},n})\, v^{\mathfrak{T},n+1}$) from the decomposition
$$\forall \alpha \quad |b(v^{\mathfrak{T},n})\, v^{\mathfrak{T},n+1}| \leq \alpha\, b(v^{\mathfrak{T},n}) \; + \; \frac{1}{\alpha}\, b(v^{\mathfrak{T},n})\, |v^{\mathfrak{T},n+1}|^2.$$

This trick allows to bypass the difficulty and conclude the convergence proof also for the linearized implicit scheme .

Ouffff !!!!!

Merci !