“Duplex” finite-volume schemes for nonlinear elliptic problems on general 2D meshes

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Outline

The problem

⇒ Obtain a FV scheme for

$$-\text{div}\,(\varphi(z,\nabla u_e(z))) = f(z), \quad \text{in } \Omega,$$
$$u_e = g, \text{ on } \partial\Omega.$$

- Ω is a bounded polygonal domain of $\mathbb{R}^2$.
- $u \mapsto -\text{div}(\varphi(\cdot,\nabla u))$ is a Leray-Lions operator from $W^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$.
- $p \in]1,\infty[$ and $p' = \frac{p}{p-1}$.
- $f \in L^{p'}(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$.

First assumptions on the flux φ

$\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheodory function such that

- $(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) > 0,\ \forall \xi \neq \eta$ $(\mathcal{H}_1)$
- $(\varphi(z,\xi), \xi) \geq C_1 |\xi|^p - b_1(z)$ $(\mathcal{H}_2)$
- $|\varphi(z,\xi)| \leq C_2 |\xi|^{p-1} + b_2(z)$ $(\mathcal{H}_3)$

with $b_1 \in L^1(\Omega)$, $b_2 \in L^{p'}(\Omega)$.

Examples

- The anisotropic laplacian : $-\mathrm{div}(K(z)\nabla u_e) = f$.
- The p-laplacian : $-\mathrm{div}\left(|\nabla u_e|^{p-2}\nabla u_e\right) = f$.
- General models of non-newtonian fluids in porous media :

$$-\mathrm{div}\left(k(z)|F(z)+\nabla u_e|^{p-2}(F(z)+\nabla u_e)\right) = f.$$

  (Diaz-De Thelin)
- Glacier models (Glowinsky-Rappaz)

Remark : the case $f \in W^{-1,p'}(\Omega)$ can be reduced to the case $f \in L^{p'}$
If $f = f_0 + \mathrm{div} f_1$, we take

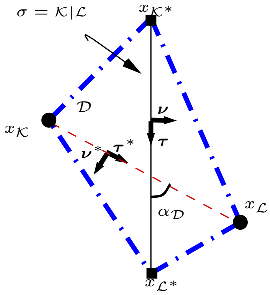
$$\begin{array}{ccc} f & \to & f_0 \\ \varphi(z,\xi) & \to & \varphi(z,\xi) + f_1(z) \end{array}$$

Warning : Non uniqueness of the decomposition $\Rightarrow$ Non uniqueness of the approximation.

The difficulty

- If ue is sufficiently enough

$$m_{\mathcal{K}|\mathcal{L}} \frac{ue(x_{\mathcal{K}}) - ue(x_{\mathcal{L}})}{d_{\mathcal{K}|\mathcal{L}}} = \int_{\mathcal{K}|\mathcal{L}} (\nabla ue(s), \boldsymbol{\tau}^*) \; ds + \mathcal{O}(h^2).$$



- Integrating the equation on a control volume $\mathcal{K}$, we have to approximate

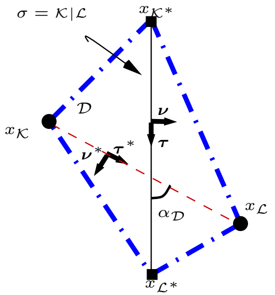
$$\int_{\mathcal{K}|\mathcal{L}} (\varphi(s, \nabla ue(s)), \boldsymbol{\nu}_{\mathcal{K}}) \; ds$$

$\Rightarrow$ As soon as $x_{\mathcal{K}} x_{\mathcal{L}} \not\perp \sigma$ or $\varphi(z, \xi) \neq \alpha \xi$, we have to reconstruct the whole gradient.

Construction of the whole gradient

If u_e is affine on $\mathcal{D}$:

$$\nabla u_e = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_e(x_{\mathcal{L}}) - u_e(x_{\mathcal{K}})}{m_{\sigma^*}} \boldsymbol{\nu} + \frac{u_e(x_{\mathcal{L}^*}) - u_e(x_{\mathcal{K}^*})}{m_{\sigma}} \boldsymbol{\nu}^* \right)$$



⇒ New unknowns on the vertices of the mesh $u_{\mathcal{K}^*}, u_{\mathcal{L}^*}$

⇒ Definition of the discrete gradient

$$\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \boldsymbol{\nu}^* \right)$$

Warning : State one additional equation per vertex $x_{\mathcal{K}^*}$.

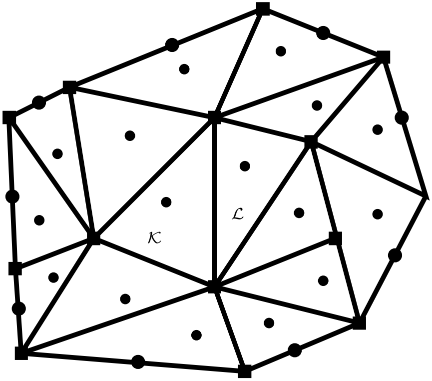
Strategies

- For the laplacian ⇒ introduce unknows $u_{\mathcal{K}^*}$ at the vertices $x_{\mathcal{K}^*}$ of each cell. The discrete gradient is taken constant on any diamond cell.
  - Coudière : $u_{\mathcal{K}^*}$ is an interpolation from the neighbouring cells.
  - Hermeline and Domelevo & Omnès : $x_{\mathcal{K}^*}$ at the center of the dual control volume .
    ⇒Integration of the Laplace equation on each dual control volumes.
- A different approach ⇒ reconstruction of a discrete gradient, constant on any control volume $\mathcal{K}$.
  - Droniou & Eymard : for the anisotropic laplacian.
  - Droniou : for general nonlinear Leray-Lions operators.

The meshes

- The primal mesh

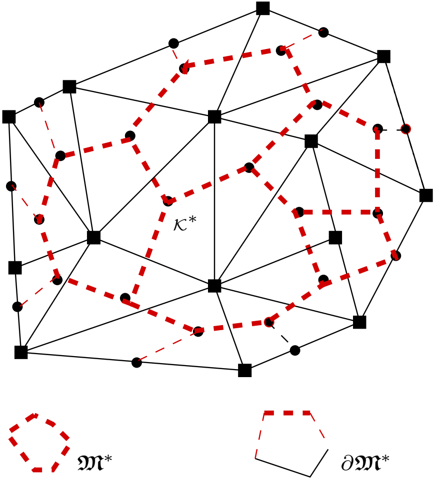
$\mathfrak{M} = \{\mathcal{K}\}$, $\partial\mathfrak{M} = \{\mathcal{K} \subset \partial\Omega\}$



The meshes

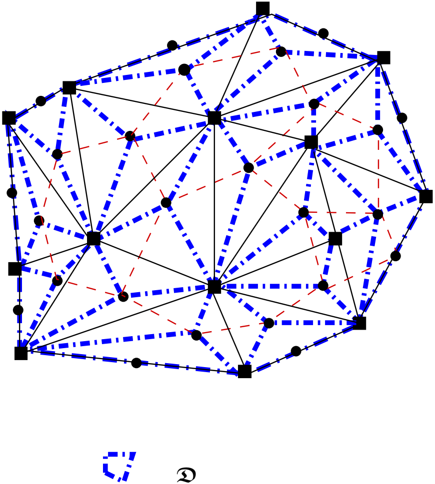
- The dual mesh

$\mathfrak{M}^* = \{\kappa^*\}$, $\partial\mathfrak{M}^* = \{\kappa^* \subset \partial\Omega\}$



The meshes

- The diamond mesh



The mesh's variables

- $\mathrm{size}(\mathcal{T})$ is the maximum diameter for $\mathfrak{D}$.
- $\sin\alpha(\mathcal{T}) = \inf_{\mathcal{D}\in\mathfrak{D}} |\sin\alpha_{\mathcal{D}}|$, $\alpha(\mathcal{T}) \in]0, \frac{\pi}{2}]$.

and

$$\mathrm{reg}(\mathcal{T}) = \max\left(\frac{1}{\alpha(\mathcal{T})}, \mathcal{N}_{\mathcal{T}}, \sup\frac{\mathrm{d}_{\mathcal{D}}^2}{m_{\mathcal{D}}}, \sup\frac{\mathrm{d}_{\mathcal{K}}}{\mathrm{d}_{\mathcal{D}}}, \sup\frac{\mathrm{d}_{\mathcal{K}^*}}{\mathrm{d}_{\mathcal{D}}}, \cdots\right).$$

$\mathrm{reg}(\mathcal{T})$ measures :

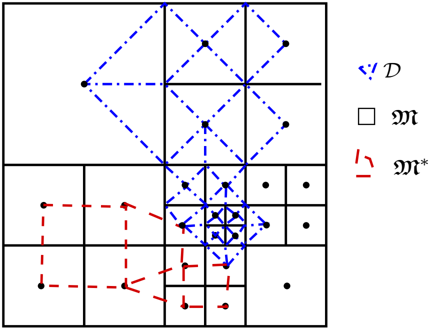
- how flat are the diamond cells.
- how large is the difference between the control volumes and the diamond cells.
- $\cdots$

Remark on the meshes

⇒ Non-conformal edges.

⇒ Locally refined meshes ($\mathrm{reg}(\mathcal{T}_n)$ bounded).

Example :



Discrete unknowns and boundary conditions

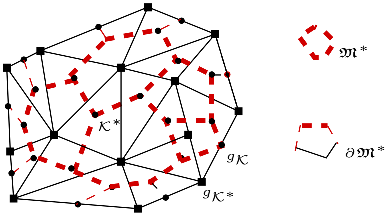
Discrete unknowns

$$u^{\mathcal{T}} = (u^{\mathfrak{M}}, u^{\mathfrak{M}^*}) \text{ where } u^{\mathfrak{M}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, u^{\mathfrak{M}^*} = (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$$

Boundary conditions

$$g_{\mathcal{K}} = \frac{1}{m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} g(s)\, ds, \ \forall \mathcal{K} \in \partial\mathfrak{M} \quad g_{\mathcal{K}^*} = \frac{1}{m_{B_{\mathcal{K}^*}}} \int_{B_{\mathcal{K}^*}} g(s^*)\, ds^*, \ \forall \mathcal{K}^* \in \partial\mathfrak{M}$$

with $B_{\mathcal{K}} = B(x_{\mathcal{K}}, \rho_{\mathcal{K}}) \cap \partial\Omega$ et $B_{\mathcal{K}^*} = B(x_{\mathcal{K}^*}, \rho_{\mathcal{K}^*}) \cap \partial\Omega$.



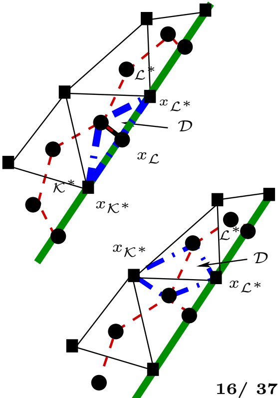
The discrete gradient

$$\nabla_{g^{\mathcal{T}}}^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \boldsymbol{\nu}^* \right)$$

Boundary diamonds :

$$\nabla_{g^{\mathcal{T}}}^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{g_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \boldsymbol{\nu} + \frac{g_{\mathcal{L}^*} - g_{\mathcal{K}^*}}{m_{\sigma}} \boldsymbol{\nu}^* \right)$$

$$\nabla_{g^{\mathcal{T}}}^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \boldsymbol{\nu} + \frac{g_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \boldsymbol{\nu}^* \right)$$



The scheme

$$-\sum_{\mathcal{D}\cap\mathcal{K}\neq\emptyset}\left(\int_{\sigma}\varphi(s,\nabla u_e(s))ds,\boldsymbol{\nu}\right) \;=\; \int_{\mathcal{K}} f(z)\,dz$$

$$-\sum_{\mathcal{D}\cap\mathcal{K}^*\neq\emptyset}\left(\int_{\sigma^*}\varphi(s^*,\nabla u_e(s^*))ds^*,\boldsymbol{\nu}^*\right) \;=\; \int_{\mathcal{K}^*} f(z)\,dz$$

The scheme

$$-\sum_{\mathcal{D}\cap\mathcal{K}\neq\emptyset}\left(\frac{m_\sigma}{m_\mathcal{D}}\int_\mathcal{D}\varphi(z,\nabla^\mathcal{D}_{g^\mathcal{T}}u^\mathcal{T})dz,\boldsymbol{\nu}\right) = m(\mathcal{K})f_\mathcal{K}$$

$$-\sum_{\mathcal{D}\cap\mathcal{K}^*\neq\emptyset}\left(\frac{m_{\sigma^*}}{m_\mathcal{D}}\int_\mathcal{D}\varphi(z,\nabla^\mathcal{D}_{g^\mathcal{T}}u^\mathcal{T})dz,\boldsymbol{\nu}^*\right) = m(\mathcal{K}^*)f_{\mathcal{K}^*}$$

The scheme

$$-\sum_{\mathcal{D}\cap\kappa\neq\emptyset} m_\sigma\left(\varphi_{\mathcal{D}},\boldsymbol{\nu}\right) \;=\; m(\kappa)f_\kappa$$

$$-\sum_{\mathcal{D}\cap\kappa^*\neq\emptyset} m_{\sigma^*}\left(\varphi_{\mathcal{D}},\boldsymbol{\nu}^*\right) \;=\; m(\kappa^*)f_{\kappa^*}$$

$\Rightarrow \mathfrak{a}_g(u^{\mathcal{T}}) = \mathbb{P}_m^{\mathcal{T}} f$, where $\mathbb{P}_m^{\mathcal{T}} f$ is the mean projection of f

$$\mathbb{P}_m^{\mathcal{T}} f \;=\; \left((f_\kappa),(f_{\kappa^*})\right).$$

$\Rightarrow$ Existence and uniqueness of a solution

Convergence

Theorem

Let $\mathcal{T}_n$ be a family of meshes such that $\text{size}(\mathcal{T}_n) \to 0$ *and* $\text{reg}(\mathcal{T}_n)$ *is bounded. Under assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$,$(\mathcal{H}_3)$, *we have*

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} ue \quad \text{strongly in } L^p(\Omega)$$

$$\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} \nabla ue \quad \text{weakly in } (L^p(\Omega))^2$$

$$\text{strongly in } (L^q(\Omega))^2, \forall q < p$$

$$\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) \xrightarrow[n\to\infty]{} \varphi(\cdot, \nabla ue) \quad \text{weakly in } \left(L^{p'}(\Omega)\right)^2$$

$$\text{strongly in } (L^r(\Omega))^2, \forall r < p'.$$

Remark : Strong convergence of $\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}$ (resp. $\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n})$) in L^p (resp. in $L^{p'}$) under stronger monotonicity assumption :

- For $p > 2$,

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq C_3|\xi - \eta|^p. \qquad (\mathcal{H}_{1'b})$$

Error estimates

Further assumptions on φ : $\Rightarrow$ Regularity with respect to z

- For $1 < p \leq 2$,

$$\left|\varphi(z,\xi) - \varphi(z',\xi)\right| \leq C(1+|\xi|^{p-1})|z-z'|^{p-1} + \left|b_5(z) - b_5(z')\right|^{p-1} \quad (\mathcal{H}_{5a})$$

with $b_5 \in \left(W^{1,p}(\Omega)\right)^2$.

- For $p > 2$,

$$\left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C\left(b_6(z) + C|\xi|^{p-1}\right). \quad (\mathcal{H}_{5b})$$

with $b_6 \in L^{p'}(\Omega)$.

Remark : The previous assumptions are satisfied by

- The anisotropic laplacian for $K \in W^{1,\infty}(\Omega)$.
- $\varphi(z,\xi) = k(z)|F(z)+\xi|^{p-2}(F(z)+\xi)$ for $k \in W^{1,\infty}(\Omega)$, $F \in W^{1,p}(\Omega)$.

Error estimates

Theorem

Under assumptions $(\mathcal{H}_{1'})$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$, $(\mathcal{H}_5)$. *If* $ue \in W^{2,p}(\Omega)$, *for* $1 < p \leq 2$

$$\|ue - u^{\mathcal{T}}\|_{L^p} + \|\nabla ue - \nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}\|_{L^p} \leq C\,\text{size}(\mathcal{T})^{p-1}$$

for $p > 2$

$$\|ue - u^{\mathcal{T}}\|_{L^p} + \|\nabla ue - \nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}\|_{L^p} \leq C\,\text{size}(\mathcal{T})^{\frac{1}{p-1}}$$

where $C = C(f, g, b_i, \|ue\|_{W^{2,p}}, \text{reg}(\mathcal{T}))$.

Anisotropic laplacian with a variable viscosity

$$-\mathrm{div}(K(x,y)\nabla u_e) = f,\ K(x,y) = \frac{1}{x^2+y^2}\begin{bmatrix} x^2+2y^2 & -xy \\ -xy & 2x^2+y^2 \end{bmatrix}$$

- For H^2 solutions :

  Order 1 in H^1 norm.

  Order 2 in L^2 norm.

- For a radial solution $u_e(x,y) = (x^2+y^2)^{0.25}, u_e \in H^{\frac{3}{2}+\varepsilon}$

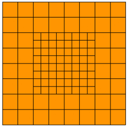
  Order 0.4 in H^1 norm.

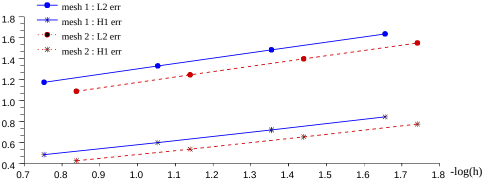
  Order 0.5 in L^2 norm.

Anisotropic laplacian with a variable viscosity

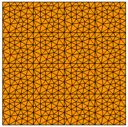
$$-\mathrm{div}(K\nabla ue) = f,\ K = \frac{1}{x^2+y^2}\begin{bmatrix} x^2+2y^2 & -xy \\ -xy & 2x^2+y^2 \end{bmatrix},\ ue = (x^2+y^2)^{0.25}$$

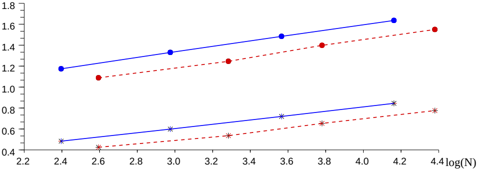
Mesh 1





Mesh 2



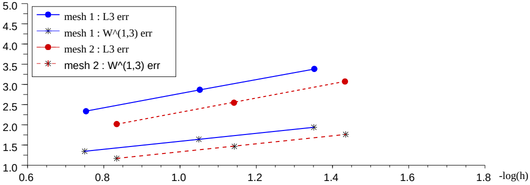


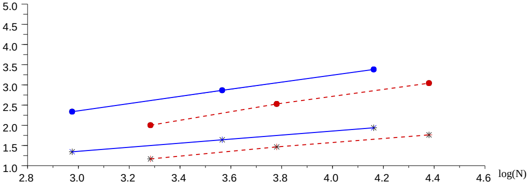
The fully nonlinear operator

$$-\mathrm{div}\left(|F+\nabla ue|^{p-2}(F+\nabla ue)\right)=f,\ F(x,y)=\begin{bmatrix} y \\ -x \end{bmatrix},\ ue=(x^2+y^2)^{0.675}$$

Here $p=3$, $ue \in W^{2,3}$

- Order 0.98 in $W^{1,3}$ norm.
- Order 1.73 in L^3 norm.



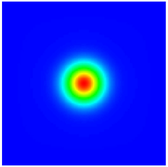


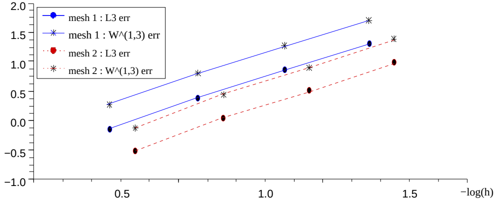
Mesh refinement

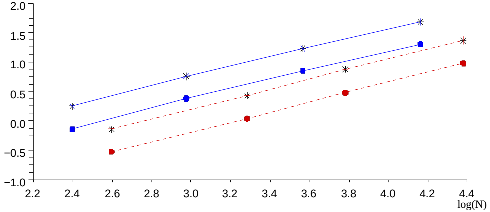
$$-\mathrm{div}\left(|\nabla ue|^{p-2}\nabla ue\right) = f,\; ue = \exp\left(-\frac{x^2+y^2}{0.05}\right)$$

Here $p = 3$, ue is regular

- Order 1.66 in $W^{1,3}$ norm.
- Order 1.67 in L^3 norm.







Possible extensions

- Possible changes in the definition of the scheme
  - If g is sufficiently smooth

$$\mathbb{P}_m^{\mathcal{T}} g \longrightarrow (g(x_{\mathcal{K}}), g(x_{\mathcal{K}^*}))$$

  - If f is sufficiently smooth

$$\mathbb{P}_m^{\mathcal{T}} f \longrightarrow (f(x_{\mathcal{K}}), f(x_{\mathcal{K}^*}))$$

  - If φ is sufficiently smooth

$$\varphi_{\mathcal{D}} \longrightarrow \varphi(z_0, \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} u^{\mathcal{T}})$$

- Neumann boundary conditions on a part of $\partial\Omega$.
- Extension to 3D.

Existence and uniqueness

$$\llbracket \mathfrak{a}_g(u^{\mathcal{T}}), v^{\mathcal{T}} \rrbracket = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}), \nabla^{\mathcal{D}}_0 v^{\mathcal{T}} \right)$$

where $\llbracket u^{\mathcal{T}}, v^{\mathcal{T}} \rrbracket = \frac{1}{2}\sum_{\mathcal{K}} m(\mathcal{K}) u_{\mathcal{K}} v_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*} m(\mathcal{K}^*) u_{\mathcal{K}^*} v_{\mathcal{K}^*}$.

Existence and uniqueness

$$[\![\mathfrak{a}_g(u^{\mathcal{T}}), v^{\mathcal{T}}]\!] = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}), \nabla^{\mathcal{D}}_{0} v^{\mathcal{T}}\right)$$

where $[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] = \frac{1}{2}\sum_{\mathcal{K}} m(\mathcal{K}) u_{\mathcal{K}} v_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*} m(\mathcal{K}^*) u_{\mathcal{K}^*} v_{\mathcal{K}^*}$.

$\Rightarrow$ Coercivity

$$[\![\mathfrak{a}_g(u^{\mathcal{T}}) - f^{\mathcal{T}}, u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}}_m \mathcal{R}(g)]\!] \geq C_1 \|\nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}\|^p - C\,(g, f, b_1, b_2)$$

where $\mathcal{R}(g)$ is the lift of g.

$\Rightarrow$ Monotonicity (Assumption $(\mathcal{H}_1)$)

$$[\![\mathfrak{a}_g(u^{\mathcal{T}}) - \mathfrak{a}_g(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}}]\!] > 0, \text{ si } u^{\mathcal{T}} \neq v^{\mathcal{T}}$$

Existence and uniqueness

$$\llbracket \mathfrak{a}_g(u^{\mathcal{T}}), v^{\mathcal{T}} \rrbracket = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}), \nabla^{\mathcal{D}}_0 v^{\mathcal{T}} \right)$$

where $\llbracket u^{\mathcal{T}}, v^{\mathcal{T}} \rrbracket = \frac{1}{2}\sum_{\mathcal{K}} m(\mathcal{K}) u_{\mathcal{K}} v_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*} m(\mathcal{K}^*) u_{\mathcal{K}^*} v_{\mathcal{K}^*}$.

$\Rightarrow$ Coercivity

$$\llbracket \mathfrak{a}_g(u^{\mathcal{T}}) - f^{\mathcal{T}}, u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}}_m \mathcal{R}(g) \rrbracket \geq C_1 \|\nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}\|^p - C\,(g, f, b_1, b_2)$$

where $\mathcal{R}(g)$ is the lift of g.

$\Rightarrow$ Monotonicity (Assumption $(\mathcal{H}_1)$)

$$\llbracket \mathfrak{a}_g(u^{\mathcal{T}}) - \mathfrak{a}_g(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}} \rrbracket > 0, \text{ si } u^{\mathcal{T}} \neq v^{\mathcal{T}}$$

$\Rightarrow$ Existence and uniqueness of $u^{\mathcal{T}}$

The fondamental lemma

Lemma

For $q \geq 1$, for any bounded domain $\mathcal{P}_\sigma \subset \mathbb{R}^2$, any segment $\sigma \subset \widehat{\mathcal{P}_\sigma}$, and all $v \in W^{1,q}(\mathbb{R}^2)$

$$|v_{\mathcal{P}_\sigma} - v_\sigma|^q \leq C \, \frac{\mathrm{d}_{\mathcal{P}_\sigma}^{q+1}}{m_\sigma m_{\mathcal{P}_\sigma}} \int_{\widehat{\mathcal{P}_\sigma}} |\nabla v\,(z)|^q \; dz$$

where

- $v_{\mathcal{P}_\sigma}$ *is the mean value of* v *on* $\mathcal{P}_\sigma$
- v_σ *is the mean value of* v *on* σ
- $\widehat{\mathcal{P}_\sigma}$ *is the convex hull of* $\mathcal{P}_\sigma \cup \sigma$

See : Eymard, Gallouët, Herbin, Droniou, ...

Compactness theorem

Theorem

Let $u^{\mathcal{T}_n}$ *be defined on* $\mathcal{T}_n$ *with* $\mathrm{reg}(\mathcal{T}_n)$ *bounded and* $\mathrm{size}(\mathcal{T}_n) \to 0$. *Let* $g \in W^{1-\frac{1}{p}}(\partial\Omega)$. *If* $\|\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^p}$ *is bounded, then there exists* $u \in W^{1,p}(\Omega)$ *with* $\gamma(u) = g$ *and*

$$\begin{aligned} u^{\mathcal{T}_n} & \xrightarrow[n\to\infty]{} u \quad \text{in } L^p(\Omega) \\ \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} & \xrightharpoonup[n\to\infty]{} \nabla u \text{ weakly in } \left(L^p(\Omega)\right)^2 \end{aligned}$$

up to a subsequence.

Convergence

1. *a priori* estimates $\Rightarrow \|\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^p}$ is bounded

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1. *a priori* estimates $\Rightarrow$ $\|\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^p}$ is bounded
2. Compactness theorem $\Rightarrow$ up to a subsequence

$$\begin{aligned}
u^{\mathcal{T}_n} &\xrightarrow[n\to\infty]{} u \ \text{ in } L^p(\Omega) \\
\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} &\underset{n\to\infty}{\rightharpoonup} \nabla u \text{ weakly in } \left(L^p(\Omega)\right)^2 \\
\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) &\underset{n\to\infty}{\rightharpoonup} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2
\end{aligned}$$

Convergence

1. *a priori* estimates $\Rightarrow$ $\|\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^p}$ is bounded
2. Compactness theorem $\Rightarrow$ up to a subsequence

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} u \quad \text{in } L^p(\Omega)$$

$$\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \xrightharpoonup[n\to\infty]{} \nabla u \text{ weakly in } (L^p(\Omega))^2$$

$$\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) \xrightharpoonup[n\to\infty]{} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

3. The Minty-Browder trick
   - Pass to the limit in the scheme $\Rightarrow$ $\forall v \in W^{1,p}(\Omega)$, $\gamma(v) = g$

$$\int_\Omega f(z)(u(z) - v(z))\, dz = \int_\Omega (\zeta(z), \nabla u(z) - \nabla v(z))\, dz$$

   - Monotonicity $\Rightarrow$ $\forall v \in W^{1,p}(\Omega)$, $\gamma(v) = g$

$$\int_\Omega \Big(\zeta(z) - \varphi(z, \nabla v), \nabla u(z) - \nabla v(z)\Big)\, dz \geq 0.$$

Convergence

1. *a priori* estimates $\Rightarrow$ $\|\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^p}$ is bounded
2. Compactness theorem $\Rightarrow$ up to a subsequence

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} u \text{ in } L^p(\Omega)$$

$$\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \underset{n\to\infty}{\rightharpoonup} \nabla u \text{ weakly in } (L^p(\Omega))^2$$

$$\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) \underset{n\to\infty}{\rightharpoonup} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

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$\Rightarrow$ $\mathrm{div}\varphi(\cdot, \nabla u) = \mathrm{div}\zeta = f$ and $u = u_e$.

Center-value projection

Definition

For all $v \in \mathcal{C}(\bar{\Omega})$, $\mathbb{P}_c^{\mathcal{T}} v = \left((v(x_{\mathcal{K}}))_{\mathcal{K} \in \mathfrak{M}}, (v(x_{\mathcal{K}^*}))_{\mathcal{K}^* \in \mathfrak{M}^*} \right)$.

Lemma

For all $v \in W^{1,p}(\Omega)$, $\gamma(v) = g$

- *For* $p > 2$, $\|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p} \leq C \|\nabla v\|_{L^p}$
- *For* $1 < p \leq 2$, *if moreover* $v \in W^{2,p}(\Omega)$ *then*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p} \leq C \left(\|\nabla v\|_{L^p} + \mathrm{size}(\mathcal{T}) \|\nabla v\|_{W^{1,p}} \right)$$

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$$\Rightarrow \quad \|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v - \nabla v\|_{L^p} \xrightarrow[\mathrm{size}(\mathcal{T}) \to 0]{} 0$$

Boundary conditions

- Boundary conditions :

Definition (Mean value BC)

$$\mathbb{P}_m^{\mathcal{T}} g = \left(\left(\frac{1}{m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} g(s)\, ds \right)_{\mathcal{K} \in \partial\mathfrak{M}}, \left(\frac{1}{m_{B_{\mathcal{K}^*}}} \int_{B_{\mathcal{K}^*}} g(s^*)\, ds^* \right)_{\mathcal{K}^* \in \partial\mathfrak{M}^*} \right).$$

- If g is smooth :

Definition (Center value BC)

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$$\Rightarrow \quad \| \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g - \mathbb{P}_c^{\mathcal{T}} g} 0^{\mathcal{T}} \|_{L^p} \xrightarrow[\text{size}(\mathcal{T}) \to 0]{} 0$$

Error estimates

$$\|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq \|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue\|_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g - \mathbb{P}^{\mathcal{T}}_c g} 0^{\mathcal{T}}\|_{L^p}$$

$$+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p}$$

Error estimates

$$\|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq \|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue\|_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g - \mathbb{P}^{\mathcal{T}}_c g} 0^{\mathcal{T}}\|_{L^p}$$

$$+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p}$$

- $\|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue\|_{L^p} \leq C(p, \Omega, \mathrm{reg}(\mathcal{T}), \|ue\|_{W^{2,p}})\,\mathrm{size}(\mathcal{T})$

Error estimates

$$\|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq \|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue\|_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g - \mathbb{P}^{\mathcal{T}}_c g} 0^{\mathcal{T}}\|_{L^p}$$

$$+ \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p}$$

- $\|\nabla ue - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue\|_{L^p} \leq C(p, \Omega, \mathrm{reg}(\mathcal{T}), \|ue\|_{W^{2,p}})\, \mathrm{size}(\mathcal{T})$
- $\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g - \mathbb{P}^{\mathcal{T}}_c g} 0^{\mathcal{T}}\|_{L^p} \leq C(p, \Omega, \mathrm{reg}(\mathcal{T}), \|g\|_{\widetilde{W}^{2-\frac{1}{p},p}})\, \mathrm{size}(\mathcal{T})$

where $\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega) = \gamma(W^{2,p}(\Omega))$.

Error estimates

- For $p > 2$

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u e - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq C\left(\sum_{\mathcal{D}} m_{\mathcal{D}} |R_\sigma|^{\frac{p}{p-1}}\right)^{\frac{1}{p}} + C\left(\sum_{\mathcal{D}} m_{\mathcal{D}} |R_{\sigma^*}|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}$$

where

$$R_\sigma = \left|\frac{1}{m_\sigma}\int_\sigma (R_{\mathcal{D}}(s), \boldsymbol{\nu})\, ds\right|,\; R_{\sigma^*} = \left|\frac{1}{m_\sigma}\int_{\sigma^*} (R_{\mathcal{D}}(s), \boldsymbol{\nu}^*)\, ds^*\right|$$

with

$$R_{\mathcal{D}}(z) = \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} \varphi(z', \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u e)\, dz' - \varphi(z, \nabla u e(z))$$

Consistancy error

$$R_{\mathcal{D}}(z) = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \left(\varphi(z', \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} \mathbb{P}^{\mathcal{T}}_c ue) - \varphi(z, \nabla ue(z)) \right) \, dz'$$

can be splitted in

$$R_{\mathcal{D}}(z) \quad = \quad R^{bound}_{\mathcal{D}} + R^{grad}_{\mathcal{D}} + R^{\varphi}_{\mathcal{D}}(z),$$

with

$$\begin{aligned}
R^{bound}_{\mathcal{D}} &= \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \left(\varphi \left(z', \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} \mathbb{P}^{\mathcal{T}}_c ue \right) dz' - \varphi \left(z', \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue \right) \right) \, dz' \\
R^{grad}_{\mathcal{D}} &= \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \left(\varphi \left(z', \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_c g} \mathbb{P}^{\mathcal{T}}_c ue \right) - \varphi \left(z', \nabla ue(z') \right) \right) \, dz' \\
R^{\varphi}_{\mathcal{D}}(z) &= \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \left(\varphi \left(z', \nabla ue(z') \right) - \varphi \left(z, \nabla ue(z) \right) \right) \, dz'
\end{aligned}$$