Renormalized solutions adapted to the fractal Laplace operator

B. Andreianov[1]

based on joint work with
N. Alibaud[1] & M. Bendahmane[2]

[1]Laboratoire de Mathématiques CNRS UMR 6623
Université de Franche-Comté
Besançon, France
[2]Université Victor Ségalen, Bordeaux, France.

Recent trends in differential equations:
analysis and approximation methods
Bielefeld, Oct. 31 – Nov. 2, 2011

Plan of the talk

WHY “INFINITE ENERGY SOLUTIONS” ?

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.

But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$ for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.

Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").

NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.

NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.
But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.
But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.
But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.
But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.
But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
NB: "Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

In some cases it is easy to give a partial answer: the "closure solutions" may happen to be solutions in the sense of distributions. But this answer is not satisfactory, because one fails to prove well-posedness in the distributional setting (uniqueness may fail: J. Serrin 's counterexample, etc.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$, and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Renormalization of PDEs.

Assume u is a variational solution of $u - \Delta u = f$,and let $S : \mathbb{R} \mapsto \mathbb{R}$ be a sufficiently regular function with bounded derivative $S'(\cdot) \geq 0$. The multiplication of the equation by $S'(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (this operation makes sense within the setting of variational solutions) leads to a "new equation":

$$S'(u)u - \Delta S(u) + S''(u)|\nabla u|^2 = S'(u)f.$$

This can be rewritten as $P(w) - \Delta w + Q(w)|\nabla w|^2 = R(w)f$ with $w := S(u)$ and nonlinearities $P(\cdot), Q(\cdot), R(\cdot)$ that only depend on $S(\cdot)$. Then:

- a "usual solution" verifies a family of "complementary PDEs" on $w = S(u)$ for different nonlinearities $S(\cdot)$
- some of these complementary PDEs also yield "estimates for large u"
- moreover, assuming that u is such that $w = S(u)$ verify these "complementary PDEs" *and* that u satisfies the "estimates for large u" *and* that $u \in H_0^1$, one deduces that u is a variational solution
- but: in general, saying that u satisfies "complementary PDEs"+"large u estimates" yields a weaker notion of solution (without asking $u \in H_0^1$)

This leads to two intrinsic interpretations of 'closure solutions":

- the renormalized solutions (P.L. Lions and F. Murat)
- the "BBGGPV entropy solutions" of elliptic problems (Ph. Bénilan et al.)

Truncations. Integrability constraints. Entropy (BBGGPV) and renormalized (Lions-Murat) solutions.

Consider the functions $T_k(\cdot)$ and $\varphi_k(\cdot)$ defined, for $k > 0$, by

$$T_k : r \mapsto \text{sign}\, r \, \min\{k, |r|\} \quad \text{and} \quad \varphi_k : r \mapsto T_k(r+1) - T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k > 0$.

If we formally take $T_k(u)$ and $\varphi_k(u)$ for the test functions in the equation $u - \Delta u = f$, we find the "estimates for large u"

$$\int_{[|u|<k]} |\nabla u|^2 \le k \int |f|, \quad \int_{[k<|u|<k+1]} |\nabla u|^2 \le \int_{[|u|>k]} |f| \to 0 \text{ as } k \to \infty.$$

- firstly, one gets the "renormalized formulation"

$$\int (u-f)S'(u)\phi + \nabla S(u) \cdot \nabla \phi + S''(u)|\nabla u|^2 \phi = 0 \text{ for } \phi \in H_0^1 \cap L^\infty.$$

- secondly, inspired by the idea of variational inequalities, one gets

$$\int (u-f)\, T_k(u-\phi) + \nabla u \cdot \nabla T_k(u-\phi) \le 0, \text{ for } \phi \in H_0^1 \cap L^\infty;$$

this leads to the notion of "BBGGPV entropy solutions".

Truncations. Integrability constraints. Entropy (BBGGPV) and renormalized (Lions-Murat) solutions.

Consider the functions $T_k(\cdot)$ and $\varphi_k(\cdot)$ defined, for $k > 0$, by

$$T_k : r \mapsto \operatorname{sign} r \, \min\{k, |r|\} \quad \text{and} \quad \varphi_k : r \mapsto T_k(r+1) - T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k > 0$.

If we formally take $T_k(u)$ and $\varphi_k(u)$ for the test functions in the equation $u - \Delta u = f$, we find the "estimates for large u"

$$\int_{[|u|<k]} |\nabla u|^2 \le k \int |f|, \quad \int_{[k<|u|<k+1]} |\nabla u|^2 \le \int_{[|u|>k]} |f| \to 0 \text{ as } k \to \infty.$$

• firstly, one gets the "renormalized formulation"

$$\int (u-f)S'(u)\phi + \nabla S(u) \cdot \nabla \phi + S''(u)|\nabla u|^2 \phi = 0 \ \text{ for } \phi \in H_0^1 \cap L^\infty.$$

• secondly, inspired by the idea of variational inequalities, one gets

$$\int (u-f)\, T_k(u-\phi) + \nabla u \cdot \nabla T_k(u-\phi) \le 0, \ \text{ for } \phi \in H_0^1 \cap L^\infty;$$

this leads to the notion of "BBGGPV entropy solutions".

Truncations. Integrability constraints. Entropy (BBGGPV) and renormalized (Lions-Murat) solutions.

Consider the functions $T_k(\cdot)$ and $\varphi_k(\cdot)$ defined, for $k > 0$, by

$$T_k : r \mapsto \mathrm{sign}\, r\ \min\{k, |r|\} \quad \text{and} \quad \varphi_k : r \mapsto T_k(r+1) - T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k > 0$.

If we formally take $T_k(u)$ and $\varphi_k(u)$ for the test functions in the equation $u - \Delta u = f$, we find the "estimates for large u"

$$\int_{[|u|<k]} |\nabla u|^2 \leq k \int |f|, \quad \int_{[k<|u|<k+1]} |\nabla u|^2 \leq \int_{[|u|>k]} |f| \to 0 \text{ as } k \to \infty.$$

• firstly, one gets the "renormalized formulation"

$$\int (u-f)S'(u)\phi + \nabla S(u) \cdot \nabla \phi + S''(u)|\nabla u|^2 \phi = 0 \ \text{ for } \phi \in H_0^1 \cap L^\infty.$$

• secondly, inspired by the idea of variational inequalities, one gets

$$\int (u-f)\, T_k(u-\phi) + \nabla u \cdot \nabla T_k(u-\phi) \leq 0, \ \text{ for } \phi \in H_0^1 \cap L^\infty;$$

this leads to the notion of "BBGGPV entropy solutions".

Truncations. Integrability constraints. Entropy (BBGGPV) and renormalized (Lions-Murat) solutions.

Consider the functions $T_k(\cdot)$ and $\varphi_k(\cdot)$ defined, for $k > 0$, by

$$T_k : r \mapsto \text{sign}\, r \, \min\{k, |r|\} \quad \text{and} \quad \varphi_k : r \mapsto T_k(r+1) - T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k > 0$.

If we formally take $T_k(u)$ and $\varphi_k(u)$ for the test functions in the equation $u - \Delta u = f$, we find the "estimates for large u"

$$\int_{[|u|<k]} |\nabla u|^2 \leq k \int |f|, \quad \int_{[k<|u|<k+1]} |\nabla u|^2 \leq \int_{[|u|>k]} |f| \to 0 \text{ as } k \to \infty.$$

• firstly, one gets the "renormalized formulation"

$$\int (u-f)S'(u)\phi + \nabla S(u) \cdot \nabla \phi + S''(u)|\nabla u|^2 \phi = 0 \ \text{ for } \phi \in H_0^1 \cap L^\infty.$$

• secondly, inspired by the idea of variational inequalities, one gets

$$\int (u-f)\, T_k(u-\phi) + \nabla u \cdot \nabla T_k(u-\phi) \leq 0, \ \text{ for } \phi \in H_0^1 \cap L^\infty;$$

this leads to the notion of "BBGGPV entropy solutions".

FRACTAL LAPLACE OPERATOR AND RELATED PDES

Fractal Laplace operator...

Our prototype nonlocal operator is the fractal (fractional) laplacian on $\mathbb{R}^n$, denoted $\mathcal{L}$ or $(-\Delta)^{s/2}$, for $s \in (0,2)$. It can be defined via the Fourier transform:

$$\mathcal{L}u = \mathcal{F}^{-1}\Big[|\cdot|^s\,\mathcal{F}[u](\cdot)\Big].$$

Another definition is by the Lévy-Khintchin formula :

$$(\mathcal{L}u)(x) = -p.v.\int_{\mathbb{R}^n}\Big[u(x+z)-u(x)\Big]\,d\mu(z) \qquad \text{for a.e. } x \in \mathbb{R}^n,$$

where $d\mu(z)$ is the measure with the density $G_s|z|^{-(n+s)}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, G_s being a normalization constant. Notice the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^n)}$:

Proposition

Set $d\pi(x,y) := \frac{1}{2}g(x-y)\,dxdy$. For all $u, v \in \mathcal{D}(\mathbb{R}^n)$ (and more, by density) $\displaystyle\int_{\mathbb{R}^n}(\mathcal{L}u)\,v = \iint_{\mathbb{R}^{2n}}\Big(u(x)-u(y)\Big)\,\Big(v(x)-v(y)\Big)\,d\pi(x,y).$

Notation: $\delta_{x,y}v := v(x)-v(y), \qquad \theta_{x,y}v := \frac{v(x)+v(y)}{2}.$

Fractal Laplace operator...

Our prototype nonlocal operator is the fractal (fractional) laplacian on $\mathbb{R}^n$, denoted $\mathcal{L}$ or $(-\Delta)^{s/2}$, for $s \in (0,2)$. It can be defined via the Fourier transform:

$$\mathcal{L}u = \mathcal{F}^{-1}\Big[|\cdot|^s \, \mathcal{F}[u](\cdot)\Big].$$

Another definition is by the Lévy-Khintchin formula :

$$(\mathcal{L}u)(x) = -p.v. \int_{\mathbb{R}^n} \Big[u(x+z) - u(x)\Big] \, d\mu(z) \qquad \text{for a.e. } \; x \in \mathbb{R}^n,$$

where $d\mu(z)$ is the measure with the density $G_s|z|^{-(n+s)}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, G_s being a normalization constant.

Notice the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^n)}$:

Proposition

Set $d\pi(x,y) := \frac{1}{2} g(x-y)\, dxdy$. For all $u, v \in \mathcal{D}(\mathbb{R}^n)$ (and more, by density) $\displaystyle\int_{\mathbb{R}^n} (\mathcal{L}u)\, v = \iint_{\mathbb{R}^{2n}} \Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\, d\pi(x,y).$

Notation: $\delta_{x,y} v := v(x) - v(y), \qquad \theta_{x,y} v := \frac{v(x)+v(y)}{2}.$

Fractal Laplace operator...

Our prototype nonlocal operator is the fractal (fractional) laplacian on $\mathbb{R}^n$, denoted $\mathcal{L}$ or $(-\Delta)^{s/2}$, for $s \in (0,2)$. It can be defined via the Fourier transform:

$$\mathcal{L}u = \mathcal{F}^{-1}\Big[|\cdot|^s\,\mathcal{F}[u](\cdot)\Big].$$

Another definition is by the Lévy-Khintchin formula :

$$(\mathcal{L}u)(x) = -p.v.\int_{\mathbb{R}^n}\Big[u(x+z)-u(x)\Big]\,d\mu(z) \qquad \text{for a.e. } x \in \mathbb{R}^n,$$

where $d\mu(z)$ is the measure with the density $G_s|z|^{-(n+s)}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, G_s being a normalization constant.

Notice the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^n)}$:

Proposition

Set $d\pi(x,y) := \frac{1}{2}g(x-y)\,dxdy$. For all $u, v \in \mathcal{D}(\mathbb{R}^n)$ (and more, by density) $\displaystyle\int_{\mathbb{R}^n}(\mathcal{L}u)\,v = \iint_{\mathbb{R}^{2n}}\Big(u(x)-u(y)\Big)\,\Big(v(x)-v(y)\Big)\,d\pi(x,y).$

Notation: $\delta_{x,y}v := v(x)-v(y)$, $\theta_{x,y}v := \frac{v(x)+v(y)}{2}$.

Fractal Laplace operator...

Our prototype nonlocal operator is the fractal (fractional) laplacian on $\mathbb{R}^n$, denoted $\mathcal{L}$ or $(-\Delta)^{s/2}$, for $s \in (0,2)$. It can be defined via the Fourier transform:

$$\mathcal{L}u = \mathcal{F}^{-1}\Big[|\cdot|^s \mathcal{F}[u](\cdot)\Big].$$

Another definition is by the Lévy-Khintchin formula :

$$(\mathcal{L}u)(x) = -p.v. \int_{\mathbb{R}^n} \Big[u(x+z) - u(x)\Big]\, d\mu(z) \qquad \text{for a.e. } \; x \in \mathbb{R}^n,$$

where $d\mu(z)$ is the measure with the density $G_s|z|^{-(n+s)}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, G_s being a normalization constant.

Notice the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^n)}$:

Proposition

Set $d\pi(x,y) := \frac{1}{2} g(x-y)\, dxdy$. For all $u, v \in \mathcal{D}(\mathbb{R}^n)$ (and more, by density) $\displaystyle\int_{\mathbb{R}^n} (\mathcal{L}u)\, v = \iint_{\mathbb{R}^{2n}} \Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\, d\pi(x,y).$

Notation: $\delta_{x,y} v := v(x) - v(y), \qquad \theta_{x,y} v := \frac{v(x)+v(y)}{2}.$

SAMPLE NONLOCAL PROBLEM

A sample nonlocal problem...

In view of the Lévy-Khinthine formula, the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions:

$$d\mu(z) = g(z)dz \quad \text{with } g \geq 0, \;\; g(z) = g(-z), \;\; \int_{\mathbb{R}^n} \min\{1, |z|^2\}\, g(z)\, dz < +\infty.$$

Fix such a Lévy operator. Fix a continuous function β on $\mathbb{R}$ with $\beta(0) = 0$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$\beta(u) + \mathcal{L}u = f \quad \text{with source } f \in L^1(\mathbb{R}^n).$$

Our results (Alibaud, Andr., Bendahmane, C.R. Acad. Sci. Paris'10) are:

- a definition of renormalized solutions
- the well-posedness: existence, uniqueness, L^1 contraction $\|\beta(u) - \beta(\widehat{u})\|_{L^1} \leq \|f - \widehat{f}\|_{L^1}$
- consequence: existence of a unique mild solution e.g. for the fractal heat equation $u_t = (-\Delta)^{s/2}u$.

A sample nonlocal problem...

In view of the Lévy-Khinthine formula, the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions:

$$d\mu(z) = g(z)dz \quad \text{with } g \geq 0, \quad g(z) = g(-z), \quad \int_{\mathbb{R}^n} \min\{1, |z|^2\}\, g(z)\, dz < +\infty.$$

Fix such a Lévy operator. Fix a continuous function β on $\mathbb{R}$ with $\beta(0) = 0$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$\beta(u) + \mathcal{L}u = f \quad \text{with source } f \in L^1(\mathbb{R}^n).$$

Our results (Alibaud, Andr., Bendahmane, C.R. Acad. Sci. Paris'10) are:

- a definition of renormalized solutions
- the well-posedness: existence, uniqueness, L^1 contraction $\|\beta(u) - \beta(\widehat{u})\|_{L^1} \leq \|f - \widehat{f}\|_{L^1}$
- consequence: existence of a unique mild solution e.g. for the fractal heat equation $u_t = (-\Delta)^{s/2}u$.

A sample nonlocal problem...

In view of the Lévy-Khinthine formula, the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions:

$$d\mu(z) = g(z)dz \quad \text{with } g \geq 0, \;\; g(z) = g(-z), \;\; \int_{\mathbb{R}^n} \min\{1, |z|^2\}\, g(z)\, dz < +\infty.$$

Fix such a Lévy operator. Fix a continuous function β on $\mathbb{R}$ with $\beta(0) = 0$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$\beta(u) + \mathcal{L}u = f \quad \text{with source } f \in L^1(\mathbb{R}^n).$$

Our results (Alibaud, Andr., Bendahmane, C.R. Acad. Sci. Paris'10) are:

- a definition of renormalized solutions
- the well-posedness: existence, uniqueness, L^1 contraction $\|\beta(u) - \beta(\widehat{u})\|_{L^1} \leq \|f - \widehat{f}\|_{L^1}$
- consequence: existence of a unique mild solution e.g. for the fractal heat equation $u_t = (-\Delta)^{s/2}u$.

A sample nonlocal problem...

In view of the Lévy-Khinthine formula, the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions:

$$d\mu(z) = g(z)dz \quad \text{with } g \geq 0, \;\; g(z) = g(-z), \;\; \int_{\mathbb{R}^n} \min\{1, |z|^2\}\, g(z)\, dz < +\infty.$$

Fix such a Lévy operator. Fix a continuous function β on $\mathbb{R}$ with $\beta(0) = 0$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$\beta(u) + \mathcal{L}u = f \quad \text{with source } f \in L^1(\mathbb{R}^n).$$

Our results (Alibaud, Andr., Bendahmane, C.R. Acad. Sci. Paris'10) are:

- a definition of renormalized solutions
- the well-posedness: existence, uniqueness, L^1 contraction $\|\beta(u) - \beta(\widehat{u})\|_{L^1} \leq \|f - \widehat{f}\|_{L^1}$
- consequence: existence of a unique mild solution e.g. for the fractal heat equation $u_t = (-\Delta)^{s/2} u$.

A sample nonlocal problem...

In view of the Lévy-Khinthine formula, the fractional Laplacian falls within the wide class of Lévy integral diffusion operators. Our framework is the one of Lévy operators with even density functions:

$$d\mu(z) = g(z)dz \quad \text{with } g \geq 0, \;\; g(z) = g(-z), \;\; \int_{\mathbb{R}^n} \min\{1, |z|^2\}\, g(z)\, dz < +\infty.$$

Fix such a Lévy operator. Fix a continuous function β on $\mathbb{R}$ with $\beta(0) = 0$ and $\beta(\mathbb{R}) = \mathbb{R}$. Consider the problem

$$\beta(u) + \mathcal{L}u = f \quad \text{with source } f \in L^1(\mathbb{R}^n).$$

Our results (Alibaud, Andr., Bendahmane, C.R. Acad. Sci. Paris'10) are:

- a definition of renormalized solutions
- the well-posedness: existence, uniqueness, L^1 contraction $\|\beta(u) - \beta(\widehat{u})\|_{L^1} \leq \|f - \widehat{f}\|_{L^1}$
- consequence: existence of a unique mild solution e.g. for the fractal heat equation $u_t = (-\Delta)^{s/2} u$.

DEFINITION OF RENORMALIZED SOLUTIONS

Definition of renormalized solutions...

Recall that $d\pi(x,y)$ denotes the measure $\frac{1}{2}g(x-y)\,dxdy$ on $\mathbb{R}^{2n}$.

Definition

Let $f \in L^1_{loc}(\mathbb{R}^n)$. A measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called renormalized solution of the problem $\beta(u) + \mathcal{L}u = f$ if $b = \beta(u) \in L^1_{loc}(\mathbb{R}^n)$ and

(i) for all $k > 0$, $\displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(T_k u(x) - T_k u(y)\right) d\pi(x,y) < +\infty$;

$$\lim_{k\to+\infty} \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\varphi_k u(x) - \varphi_k u(y)\right) d\pi(x,y) = 0;$$

(ii) for all compactly supported renormalization function $H \in W^{1,\infty}(\mathbb{R})$, for all test function $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} b\,Hu\,\phi + \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(Hu(x)-Hu(y)\right)\frac{\phi(x)+\phi(y)}{2}\,d\pi(x,y)$$
$$+\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\phi(x)-\phi(y)\right)\frac{Hu(x)+Hu(y)}{2}\,d\pi(x,y) = \int_{\mathbb{R}^n} f\,Hu\,\phi.$$

Definition of renormalized solutions...

Recall that $d\pi(x,y)$ denotes the measure $\frac{1}{2}g(x-y)\,dxdy$ on $\mathbb{R}^{2n}$.

Definition

Let $f \in L^1_{loc}(\mathbb{R}^n)$. A measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called renormalized solution of the problem $\beta(u) + \mathcal{L}u = f$ if $b = \beta(u) \in L^1_{loc}(\mathbb{R}^n)$ and

(i) for all $k > 0$, $\displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(T_k u(x) - T_k u(y)\right) d\pi(x,y) < +\infty;$

$$\lim_{k\to+\infty} \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\varphi_k u(x) - \varphi_k u(y)\right) d\pi(x,y) = 0;$$

(ii) for all compactly supported renormalization function $H \in W^{1,\infty}(\mathbb{R})$, for all test function $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} b\,Hu\,\phi + \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(Hu(x)-Hu(y)\right)\frac{\phi(x)+\phi(y)}{2}\,d\pi(x,y)$$
$$+\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\phi(x)-\phi(y)\right)\frac{Hu(x)+Hu(y)}{2}\,d\pi(x,y) = \int_{\mathbb{R}^n} f\,Hu\,\phi.$$

Definition of renormalized solutions...

Recall that $d\pi(x,y)$ denotes the measure $\frac{1}{2}g(x-y)\,dxdy$ on $\mathbb{R}^{2n}$.

Definition

Let $f \in L^1_{loc}(\mathbb{R}^n)$. A measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called renormalized solution of the problem $\beta(u) + \mathcal{L}u = f$ if $b = \beta(u) \in L^1_{loc}(\mathbb{R}^n)$ and

(i) for all $k > 0$, $\displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(T_k u(x) - T_k u(y)\right) d\pi(x,y) < +\infty;$

$$\lim_{k\to+\infty} \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\varphi_k u(x) - \varphi_k u(y)\right) d\pi(x,y) = 0;$$

(ii) for all compactly supported renormalization function $H \in W^{1,\infty}(\mathbb{R})$, for all test function $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} b\,Hu\,\phi + \iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(Hu(x)-Hu(y)\right)\frac{\phi(x)+\phi(y)}{2}\,d\pi(x,y)$$
$$+\iint_{\mathbb{R}^{2n}} (u(x)-u(y))\left(\phi(x)-\phi(y)\right)\frac{Hu(x)+Hu(y)}{2}\,d\pi(x,y) = \int_{\mathbb{R}^n} f\,Hu\,\phi.$$

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \left\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \right\}$; recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\, d\pi(x, y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \le k \|f\|_{L^1} \quad \text{and} \quad \lim_{k \to +\infty} \iint_{[(u(x),u(y)) \in A_k]} |u(x) - u(y)|\, d\pi(x, y) = 0,$$

$$\text{where } A_k := \left\{ (u, v) \in \mathbb{R}^2 \,\middle|\, k+1 \le \max\{|u|, |v|\} \text{ and } (\min\{|u|, |v|\} \le k \text{ or } uv < 0) \right\}.$$

Further, denote the region $\{ (u, v) \in \mathbb{R}^2 \mid |u| \le k,\ |v| \le k \}$ by B_k; finally, the "remainder region" C_k is: $\{ (u, v) \in \mathbb{R}^2 \mid |u| \ge k,\ |v| \ge k,\ uv > 0 \}$.

Heuristically: due to the "estimates for large u", at each truncation level k,

– we control $\delta_{x,y} u$ in L^1 in the region $\{(x, y) \mid (u(x), u(y)) \in A_k\}$;

– we control $\delta_{x,y} u$ in L^2 in the region $\{(x, y) \mid (u(x), u(y)) \in B_k\}$.

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \left\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \right\}$; recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\, d\pi(x,y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \le k \|f\|_{L^1} \quad \text{and} \quad \lim_{k \to +\infty} \iint_{[(u(x),u(y)) \in A_k]} |u(x) - u(y)|\, d\pi(x,y) = 0,$$

$$\text{where } A_k := \Big\{ (u,v) \in \mathbb{R}^2 \;\Big|\; k+1 \le \max\{|u|,|v|\} \text{ and } (\min\{|u|,|v|\} \le k \text{ or } uv < 0) \Big\}.$$

Further, denote the region $\{ (u,v) \in \mathbb{R}^2 \mid |u| \le k,\ |v| \le k \}$ by B_k; finally, the "remainder region" C_k is: $\{ (u,v) \in \mathbb{R}^2 \mid |u| \ge k,\ |v| \ge k,\ uv > 0 \}$.

Heuristically: due to the "estimates for large u", at each truncation level k,

– we control $\delta_{x,y} u$ in L^1 in the region $\{(x,y) \mid (u(x),u(y)) \in A_k\}$;

– we control $\delta_{x,y} u$ in L^2 in the region $\{(x,y) \mid (u(x),u(y)) \in B_k\}$.

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \left\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \right\};$ recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\, d\pi(x, y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \le k \|f\|_{L^1} \quad \text{and} \quad \lim_{k\to+\infty} \iint_{[(u(x),u(y))\in A_k]} |u(x) - u(y)|\, d\pi(x, y) = 0,$$

$$\text{where } A_k := \Big\{ (u, v) \in \mathbb{R}^2 \;\Big|\; k+1 \le \max\{|u|, |v|\} \text{ and } (\min\{|u|, |v|\} \le k \text{ or } uv < 0) \Big\}.$$

Further, denote the region $\{ (u, v) \in \mathbb{R}^2 \mid |u| \le k,\ |v| \le k \}$ by B_k; finally, the "remainder region" C_k is: $\{ (u, v) \in \mathbb{R}^2 \mid |u| \ge k,\ |v| \ge k,\ uv > 0 \}$.
Heuristically: due to the "estimates for large u", at each truncation level k,
– we control $\delta_{x,y} u$ in L^1 in the region $\{(x, y) \mid (u(x), u(y)) \in A_k\}$;
– we control $\delta_{x,y} u$ in L^2 in the region $\{(x, y) \mid (u(x), u(y)) \in B_k\}$.

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \Big\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \Big\}$; recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\; d\pi(x,y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \leq k \|f\|_{L^1} \quad \text{and} \quad \lim_{k\to+\infty} \iint_{[(u(x),u(y))\in A_k]} |u(x)-u(y)|\; d\pi(x,y) = 0,$$

$$\text{where } A_k := \Big\{ (u,v) \in \mathbb{R}^2 \;\Big|\; k+1 \leq \max\{|u|,|v|\} \text{ and } \big(\min\{|u|,|v|\} \leq k \text{ or } uv < 0\big) \Big\}.$$

Further, denote the region $\big\{ (u,v) \in \mathbb{R}^2 \mid |u| \leq k,\ |v| \leq k \big\}$ by B_k; finally, the "remainder region" C_k is: $\big\{ (u,v) \in \mathbb{R}^2 \mid |u| \geq k,\ |v| \geq k,\ uv > 0 \big\}$.

Heuristically: due to the "estimates for large u", at each truncation level k,

– we control $\delta_{x,y} u$ in L^1 in the region $\{(x,y) \mid (u(x),u(y)) \in A_k\}$;

– we control $\delta_{x,y} u$ in L^2 in the region $\{(x,y) \mid (u(x),u(y)) \in B_k\}$.

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \left\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \right\}$; recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\, d\pi(x, y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \leq k \|f\|_{L^1} \quad \text{and} \quad \lim_{k \to +\infty} \iint_{[(u(x),u(y)) \in A_k]} |u(x) - u(y)|\, d\pi(x, y) = 0,$$

$$\text{where } A_k := \Big\{ (u, v) \in \mathbb{R}^2 \,\Big|\, k+1 \leq \max\{|u|, |v|\} \text{ and } (\min\{|u|, |v|\} \leq k \text{ or } uv < 0) \Big\}.$$

Further, denote the region $\{ (u, v) \in \mathbb{R}^2 \mid |u| \leq k,\ |v| \leq k \}$ by B_k; finally, the "remainder region" C_k is: $\{ (u, v) \in \mathbb{R}^2 \mid |u| \geq k,\ |v| \geq k,\ uv > 0 \}$.
Heuristically: due to the "estimates for large u", at each truncation level k,

– we control $\delta_{x,y} u$ in L^1 in the region $\{(x, y) \mid (u(x), u(y)) \in A_k\}$;

– we control $\delta_{x,y} u$ in L^2 in the region $\{(x, y) \mid (u(x), u(y)) \in B_k\}$.

"Phase plane" splitting and equivalent form of the constraint...

Set $H_\mu := \left\{ v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2n}, d\pi) \right\}$; recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2n}} (\delta_{x,y} v)\,(\delta_{x,y} w)\, d\pi(x, y)$.

A close examination shows that the "estimates for large u" in the Definition are equivalent to the properties

$$\|T_k u\|_{H_\mu} \leq k \|f\|_{L^1} \quad \text{and} \quad \lim_{k\to+\infty} \iint_{[(u(x),u(y))\in A_k]} |u(x) - u(y)|\, d\pi(x, y) = 0,$$

$$\text{where } A_k := \left\{ (u, v) \in \mathbb{R}^2 \;\middle|\; k+1 \leq \max\{|u|, |v|\} \text{ and } (\min\{|u|, |v|\} \leq k \text{ or } uv < 0) \right\}.$$

Further, denote the region $\left\{ (u, v) \in \mathbb{R}^2 \mid |u| \leq k,\ |v| \leq k \right\}$ by B_k; finally, the "remainder region" C_k is: $\left\{ (u, v) \in \mathbb{R}^2 \mid |u| \geq k,\ |v| \geq k,\ uv > 0 \right\}$.

Heuristically: due to the "estimates for large u", at each truncation level k,

– we control $\delta_{x,y} u$ in L^1 in the region $\{(x, y) \mid (u(x), u(y)) \in A_k\}$;

– we control $\delta_{x,y} u$ in L^2 in the region $\{(x, y) \mid (u(x), u(y)) \in B_k\}$.

(HINTS AND) PROOF

A glimpse at the ideas and hints of the proof...

The proof is made in several steps; we combine uniqueness and existence arguments to make the proof technically simpler.

- Existence of variational solutions : e.g., for data in $L^\infty_c(\mathbb{R}^n)$ the solutions are obtained by the classical minimization technique for the energy functional

$$J[v] := \int_{\mathbb{R}^n} \Big[\int_0^v \beta(r)\,dr - vf \Big] + \frac{1}{2} \iint_{R^{2n}} \Big(\delta_{x,y} v \Big)^2 d\pi(x,y).$$

- Estimates on the solutions : taking $T_k(u)$ and $\varphi_k(u)$ for test functions, we find

$$\int_{\mathbb{R}^n} |b\,T_k u| + \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} T_k u \right) d\pi(x,y) \le k \int_{\mathbb{R}^n} |f|;$$

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} \varphi_k u \right) d\pi(x,y) \le \int_{[|u| \ge k]} |f|.$$

- A variational solution is also a renormalized one : it is enough to take $Hu\,\phi$ as a test function in the variational formulation to get (ii) of the Definition; and (i) comes from the above estimates.

A glimpse at the ideas and hints of the proof...

The proof is made in several steps; we combine uniqueness and existence arguments to make the proof technically simpler.

- Existence of variational solutions : e.g., for data in $L^\infty_c(\mathbb{R}^n)$ the solutions are obtained by the classical minimization technique for the energy functional

$$J[v] := \int_{\mathbb{R}^n} \Big[\int_0^v \beta(r)\,dr \;-\; vf \Big] + \frac{1}{2} \iint_{R^{2n}} \Big(\delta_{x,y} v \Big)^2 \, d\pi(x,y).$$

- Estimates on the solutions : taking $T_k(u)$ and $\varphi_k(u)$ for test functions, we find

$$\int_{\mathbb{R}^n} |b\,T_k u| + \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \Big(\delta_{x,y} T_k u \Big) \, d\pi(x,y) \le k \int_{\mathbb{R}^n} |f|;$$

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \Big(\delta_{x,y} \varphi_k u \Big) \, d\pi(x,y) \le \int_{[|u| \ge k]} |f|.$$

- A variational solution is also a renormalized one : it is enough to take $Hu\,\phi$ as a test function in the variational formulation to get (ii) of the Definition; and (i) comes from the above estimates.

A glimpse at the ideas and hints of the proof...

The proof is made in several steps; we combine uniqueness and existence arguments to make the proof technically simpler.

- Existence of variational solutions : e.g., for data in $L^\infty_c(\mathbb{R}^n)$ the solutions are obtained by the classical minimization technique for the energy functional

$$J[v] := \int_{\mathbb{R}^n} \Big[\int_0^v \beta(r)\, dr \; - \; vf \Big] + \frac{1}{2} \iint_{R^{2n}} \Big(\delta_{x,y} v \Big)^2 d\pi(x,y).$$

- Estimates on the solutions : taking $T_k(u)$ and $\varphi_k(u)$ for test functions, we find

$$\int_{\mathbb{R}^n} |b\, T_k u| + \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \Big(\delta_{x,y} T_k u \Big)\, d\pi(x,y) \le k \int_{\mathbb{R}^n} |f|;$$

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \Big(\delta_{x,y} \varphi_k u \Big)\, d\pi(x,y) \le \int_{[|u| \ge k]} |f|.$$

- A variational solution is also a renormalized one : it is enough to take $Hu\,\phi$ as a test function in the variational formulation to get (ii) of the Definition; and (i) comes from the above estimates.

A glimpse at the ideas and hints of the proof...

The proof is made in several steps; we combine uniqueness and existence arguments to make the proof technically simpler.

- Existence of variational solutions : e.g., for data in $L^\infty_c(\mathbb{R}^n)$ the solutions are obtained by the classical minimization technique for the energy functional

$$J[v] := \int_{\mathbb{R}^n} \Big[\int_0^v \beta(r)\,dr \; - \; vf \Big] + \frac{1}{2} \iint_{R^{2n}} \Big(\delta_{x,y} v \Big)^2 \, d\pi(x,y).$$

- Estimates on the solutions : taking $T_k(u)$ and $\varphi_k(u)$ for test functions, we find

$$\int_{\mathbb{R}^n} |b\,T_k u| + \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \, \Big(\delta_{x,y} T_k u \Big) \, d\pi(x,y) \leq k \int_{\mathbb{R}^n} |f|;$$

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \, \Big(\delta_{x,y} \varphi_k u \Big) \, d\pi(x,y) \leq \int_{[|u|\geq k]} |f|.$$

- A variational solution is also a renormalized one : it is enough to take $Hu\,\phi$ as a test function in the variational formulation to get (ii) of the Definition; and (i) comes from the above estimates.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \xi \leq 1$. Then $\xi \in H_\mu$ and for all $k > 0$,
  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$
  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \leq \xi \leq 1$
  $$\int_{\mathbb{R}^n} (\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(b - \widehat{b}) \leq \int_{\mathbb{R}^n} (\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(f - \widehat{f})$$
  Letting ξ go to $\text{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \leq \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \le \xi \le 1$. Then $\xi \in H_\mu$ and for all $k > 0$,
  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$
  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \le \xi \le 1$
  $$\int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(b - \widehat{b}) \le \int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(f - \widehat{f})$$
  Letting ξ go to $\mathrm{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \le \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \le \xi \le 1$. Then $\xi \in H_\mu$ and for all $k > 0$,
  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$
  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \le \xi \le 1$
  $$\int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(b - \widehat{b}) \le \int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(f - \widehat{f})$$
  Letting ξ go to $\mathrm{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \le \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \xi \leq 1$. Then $\xi \in H_\mu$ and for all $k > 0$,

  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$

  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$.
  Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \leq \xi \leq 1$

  $$\int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(b - \widehat{b}) \leq \int_{\mathbb{R}^n} (\mathrm{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(f - \widehat{f})$$

  Letting ξ go to $\mathrm{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \leq \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \le \xi \le 1$. Then $\xi \in H_\mu$ and for all $k > 0$,
  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$
  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \le \xi \le 1$
  $$\int_{\mathbb{R}^n} \left(\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]}\right)(b - \widehat{b}) \le \int_{\mathbb{R}^n} \left(\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]}\right)(f - \widehat{f})$$
  Letting ξ go to $\text{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \le \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- A partial comparison inequality :
  Let $u, \widehat{u}$ be two renormalized solutions. Assume that one of them is in $L^\infty(\mathbb{R}^n)$; then $T_k u \equiv u$ for large k, thus this solution also belongs to H_μ.
  Let $\xi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \xi \leq 1$. Then $\xi \in H_\mu$ and for all $k > 0$,
  $$\phi := \frac{1}{k} T_k(u - \widehat{u} + k\xi) \in L^\infty(\mathbb{R}^n) \cap H_\mu.$$
  By approximation, take ϕ for the test function in the renormalized formulations for both u and $\widehat{u}$. Using the fact that one of the solutions is bounded, exploiting the constraint for we can let $H(\cdot)$ go to 1 on $\mathbb{R}$. Then we subtract the so obtained equalities, drop the nonnegative terms, let k go to zero, and derive $\forall \xi \in \mathcal{D}(\mathbb{R}^n), 0 \leq \xi \leq 1$
  $$\int_{\mathbb{R}^n} (\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(b - \widehat{b}) \leq \int_{\mathbb{R}^n} (\text{sign}^+(u - \widehat{u}) + \xi 1\!\!1_{[u=\widehat{u}]})\,(f - \widehat{f})$$
  Letting ξ go to $\text{sign}^+(b - \widehat{b})$, we get $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$. As a byproduct, we get $\int_{\mathbb{R}^n} |b| \leq \int_{\mathbb{R}^n} |f|$.
- The maximum principle : A constant $c \in \mathbb{R}$ can be seen as a bounded renormalized solution of (10) with datum $f = \beta(c)$.
  Due to the above comparison principle, the previously constructed variational (and thus, renormalized) solutions lie in L^∞ if data are L^∞.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L^∞_c data

$$f^{l,m} = \min\{f^+, l\} 1\!\!1_{|x|<l} - \min\{f^-, m\} 1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leq |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".

On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.

By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.

Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..

We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L_c^∞ data

$$f^{l,m} = \min\{f^+, l\} 1\!\!1_{|x|<l} - \min\{f^-, m\} 1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leq |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".

On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.
By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.
Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..
We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L_c^∞ data

$$f^{l,m} = \min\{f^+, l\} 1\!\!1_{|x|<l} - \min\{f^-, m\} 1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leq |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".

On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.
By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.
Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..
We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L^∞_c data

$$f^{l,m} = \min\{f^+, l\} 1\!\!1_{|x|<l} - \min\{f^-, m\} 1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \le |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".
On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.
By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.
Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..
We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L^∞_c data

$$f^{l,m} = \min\{f^+, l\}1\!\!1_{|x|<l} - \min\{f^-, m\}1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leq |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".

On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.
By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.
Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..
We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Bi-monotone sequence of approximate solutions, compactness : (Ammar and Wittbold) Given $f \in L^1(\mathbb{R}^n)$, consider the L^∞_c data

$$f^{l,m} = \min\{f^+, l\}1\!\!1_{|x|<l} - \min\{f^-, m\}1\!\!1_{|x|<m};$$

the sequence $(f^{l,m})_{l,m}$ is monotone in l and in m, and $|f^{l,m}| \leq |f|$.

Associated variational solutions $u^{l,m}$ obey uniform estimates given above. By the above results, $u^{l,m} \in L^\infty(\mathbb{R}^n)$; thus we use the partial comparison principle and prove that $\left(b^{l,m}\right)_{l,m}$ is "bi-monotone".

On can show (by approximation of b) that $\left(u^{l,m}\right)_{l,m}$ is also bi-monotone.

We deduce the a.e. convergence of $b^{l,m}$ to an L^1 function b.
By the a priori estimates, there exists an $\bar{\mathbb{R}}$-valued u such that for all k, $T_k u^{l,m}$ converge to $T_k u$ weakly in H_μ and a.e..

By passage to the limit, it follows that $b = \beta(u)$.
Since $b(\mathbb{R}) = \mathbb{R}$, then u is finite a.e..
We deduce that $\int_{[|u|>k]} |f| \to 0$; with the Fatou lemma, we get the "constraint" of the renormalized formulation.

A glimpse at the ideas and hints of the proof...

- Passage to the limit :
  We already have a renormalized formulation for $u^{l,m}$. Since $|f^{l,m}| \leq |f|$ and $u^{l,m}$ converge pointwise to an a.e. finite limit u, one shows that the H_μ bounds for $u^{l,m}$ are uniform. By the Fatou lemma, the limit u fulfills (i). To find (ii), first we get a "rough" version of the renormalized equation: namely, the term

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} Hu\right) (\theta_{x,y}\phi) \, d\pi(x,y)$$

  is replaced by $\overline{\overline{o}}_H(1)$ as $H(\cdot)$ goes to 1.

  Within, we take $\phi = T_k u$ for the test function; and we take $\phi = T_k u^{l,m}$ for the test function in the renormalized formulation for l, m. Letting $H(\cdot)$ go to 1 on $\mathbb{R}$ (here the bounds and constraints are used), we find that

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} T_k u\right) d\pi(x,y) \geq \overline{\lim_{l\to\infty}} \; \overline{\lim_{m\to\infty}} \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u^{l,m}) \left(\delta_{x,y} T_k u^{l,m}\right) d\pi(x,y).$$

  Hence the sequence $\left((\delta_{x,y} u^{l,m}) \left(\delta_{x,y} T_k u^{l,m}\right)\right)_{l,m}$ converges in $L^1(\mathbb{R}^{2n}, d\pi)$; therefore, it is equi-integrable. Now we can to pass to the limit in all the terms of the renormalized equation for $u^{l,m}$, and get (ii).

A glimpse at the ideas and hints of the proof...

- Passage to the limit :
  We already have a renormalized formulation for $u^{l,m}$. Since $|f^{l,m}| \leq |f|$ and $u^{l,m}$ converge pointwise to an a.e. finite limit u, one shows that the H_μ bounds for $u^{l,m}$ are uniform. By the Fatou lemma, the limit u fulfills (i). To find (ii), first we get a "rough" version of the renormalized equation: namely, the term

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} Hu\right) (\theta_{x,y}\phi)\, d\pi(x,y)$$

  is replaced by $\overline{\overline{o}}_H(1)$ as $H(\cdot)$ goes to 1.

  Within, we take $\phi = T_k u$ for the test function; and we take $\phi = T_k u^{l,m}$ for the test function in the renormalized formulation for l, m. Letting $H(\cdot)$ go to 1 on $\mathbb{R}$ (here the bounds and constraints are used), we find that

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u)\, (\delta_{x,y} T_k u)\, d\pi(x,y) \geq \overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}} \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m})\, d\pi(x,y).$$

  Hence the sequence $\left((\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m})\right)_{l,m}$ converges in $L^1(\mathbb{R}^{2n}, d\pi)$; therefore, it is equi-integrable. Now we can to pass to the limit in all the terms of the renormalized equation for $u^{l,m}$, and get (ii).

A glimpse at the ideas and hints of the proof...

- Passage to the limit :
  We already have a renormalized formulation for $u^{l,m}$. Since $|f^{l,m}| \leq |f|$ and $u^{l,m}$ converge pointwise to an a.e. finite limit u, one shows that the H_μ bounds for $u^{l,m}$ are uniform. By the Fatou lemma, the limit u fulfills (i). To find (ii), first we get a "rough" version of the renormalized equation: namely, the term

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} Hu\right) (\theta_{x,y}\phi)\, d\pi(x,y)$$

  is replaced by $\overline{\overline{o}}_H(1)$ as $H(\cdot)$ goes to 1.

  Within, we take $\phi = T_k u$ for the test function; and we take $\phi = T_k u^{l,m}$ for the test function in the renormalized formulation for l, m. Letting $H(\cdot)$ go to 1 on $\mathbb{R}$ (here the bounds and constraints are used), we find that

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u)\, (\delta_{x,y} T_k u)\, d\pi(x,y) \geq \overline{\lim_{l\to\infty}}\; \overline{\lim_{m\to\infty}} \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m})\, d\pi(x,y).$$

  Hence the sequence $\left((\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m}) \right)_{l,m}$ converges in $L^1(\mathbb{R}^{2n}, d\pi)$; therefore, it is equi-integrable. Now we can to pass to the limit in all the terms of the renormalized equation for $u^{l,m}$, and get (ii).

A glimpse at the ideas and hints of the proof...

- Passage to the limit :
  We already have a renormalized formulation for $u^{l,m}$. Since $|f^{l,m}| \leq |f|$ and $u^{l,m}$ converge pointwise to an a.e. finite limit u, one shows that the H_μ bounds for $u^{l,m}$ are uniform. By the Fatou lemma, the limit u fulfills (i). To find (ii), first we get a "rough" version of the renormalized equation: namely, the term

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u) \left(\delta_{x,y} Hu\right) (\theta_{x,y}\phi)\, d\pi(x,y)$$

  is replaced by $\overline{\overline{o}}_H(1)$ as $H(\cdot)$ goes to 1.

  Within, we take $\phi = T_k u$ for the test function; and we take $\phi = T_k u^{l,m}$ for the test function in the renormalized formulation for l, m. Letting $H(\cdot)$ go to 1 on $\mathbb{R}$ (here the bounds and constraints are used), we find that

$$\iint_{\mathbb{R}^{2n}} (\delta_{x,y} u)\, (\delta_{x,y} T_k u)\, d\pi(x,y) \geq \overline{\lim_{l\to\infty}}\ \overline{\lim_{m\to\infty}} \iint_{\mathbb{R}^{2n}} (\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m})\, d\pi(x,y).$$

  Hence the sequence $\left((\delta_{x,y} u^{l,m})\, (\delta_{x,y} T_k u^{l,m}) \right)_{l,m}$ converges in $L^1(\mathbb{R}^{2n}, d\pi)$; therefore, it is equi-integrable. Now we can to pass to the limit in all the terms of the renormalized equation for $u^{l,m}$, and get (ii).

A glimpse at the ideas and hints of the proof...

- Extension of the comparison property :
  Let $f \in L^1(\mathbb{R}^n)$; let $\widehat{u}$ be a renormalized solution with datum f.

  A renormalized solution u was constructed as the limit of bounded solutions $u^{l,m}$.

  Passing to the limit in the previously obtained comparison inequality for $u^{l,m}$ and $\widehat{u}$, we find that $b = \widehat{b}$. Further analysis shows that $\iint_{\mathbb{R}^{2n}} \left|\delta_{x,y}(u - \widehat{u})\right|^2 = 0$, and we get $u = \hat{u}$.
  Hence by the density argument, one justifies the L^1 contraction and comparison property in full generality.

NB: the theory for "BBGGPV entropy solutions" is even simpler.
Moreover, both kinds of solutions can be seen as "closure solutions" with respect to the one-and-the-same set of variational solutions; therefore, renormalized and BBGGPV entropy solutions actually coincide.

A glimpse at the ideas and hints of the proof...

- Extension of the comparison property :
  Let $f \in L^1(\mathbb{R}^n)$; let $\widehat{u}$ be a renormalized solution with datum f.

  A renormalized solution u was constructed as the limit of bounded solutions $u^{l,m}$.

  Passing to the limit in the previously obtained comparison inequality for $u^{l,m}$ and $\widehat{u}$, we find that $b = \widehat{b}$. Further analysis shows that $\iint_{\mathbb{R}^{2n}} \left|\delta_{x,y}(u - \widehat{u})\right|^2 = 0$, and we get $u = \hat{u}$.
  Hence by the density argument, one justifies the L^1 contraction and comparison property in full generality.

NB: the theory for "BBGGPV entropy solutions" is even simpler.
Moreover, both kinds of solutions can be seen as "closure solutions" with respect to the one-and-the-same set of variational solutions; therefore, renormalized and BBGGPV entropy solutions actually coincide.

A glimpse at the ideas and hints of the proof...

- Extension of the comparison property :
  Let $f \in L^1(\mathbb{R}^n)$; let $\widehat{u}$ be a renormalized solution with datum f.

  A renormalized solution u was constructed as the limit of bounded solutions $u^{l,m}$.

  Passing to the limit in the previously obtained comparison inequality for $u^{l,m}$ and $\widehat{u}$, we find that $b = \widehat{b}$. Further analysis shows that $\iint_{\mathbb{R}^{2n}} \left|\delta_{x,y}(u-\widehat{u})\right|^2 = 0$, and we get $u = \hat{u}$.
  Hence by the density argument, one justifies the L^1 contraction and comparison property in full generality.

NB: the theory for "BBGGPV entropy solutions" is even simpler.
Moreover, both kinds of solutions can be seen as "closure solutions" with respect to the one-and-the-same set of variational solutions; therefore, renormalized and BBGGPV entropy solutions actually coincide.

A glimpse at the ideas and hints of the proof...

- Extension of the comparison property :
  Let $f \in L^1(\mathbb{R}^n)$; let $\widehat{u}$ be a renormalized solution with datum f.

  A renormalized solution u was constructed as the limit of bounded solutions $u^{l,m}$.

  Passing to the limit in the previously obtained comparison inequality for $u^{l,m}$ and $\widehat{u}$, we find that $b = \widehat{b}$. Further analysis shows that $\iint_{\mathbb{R}^{2n}} \left|\delta_{x,y}(u - \widehat{u})\right|^2 = 0$, and we get $u = \hat{u}$.
  Hence by the density argument, one justifies the L^1 contraction and comparison property in full generality.

NB: the theory for "BBGGPV entropy solutions" is even simpler.
Moreover, both kinds of solutions can be seen as "closure solutions" with respect to the one-and-the-same set of variational solutions; therefore, renormalized and BBGGPV entropy solutions actually coincide.

Thank you — Merci — Danke

DANKE SCHÖN !