Habilitation à diriger les recherches

ANALYSIS OF NONLINEAR PDES OF HYPERBOLIC AND DEGENERATE PARABOLIC TYPE, NUMERICAL APPROXIMATION BY FINITE VOLUME METHODS, AND APPLICATIONS

Boris P. Andreianov[1]

based on joint works with N. Alibaud, M. Bendahmane, Ph. Bénilan, F. Bouhsiss, F. Boyer, C. Cancès, R. Eymard, M. Gazibo Karimou, M. Ghilani, P. Goatin, M. Gutnic, F. Hubert, N. Igbida, K.H. Karlsen, S. Krell, S.N. Kruzhkov, H. Labani, F. Lagoutière, M. Maliki, N. Marharoui, S. Ouaro, Ch. Pierre, N.H. Risebro, R. Ruiz Baier, M. Saad, K. Sbihi, N. Seguin, T. Takahashi, G. Vallet, P. Wittbold

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Faculté des Sciences et Techniques, Besançon — 24 Novembre 2011

Plan of the talk

1. **What am I doing ? A sample problem**
2. **Highlights: kind of questions considered, techniques in use**
3. **A panorama of questions, works, and results**
   - Uniqueness for degenerate parabolic problems
   - Leray-Lions operators and their numerical approximation
   - Doubly and triply degenerate parabolic equations
   - Absorption in parabolic equations
   - Variable exponent problems, renormalized solutions
   - Renormalization for nonlocal diffusion operators
   - Entropy solutions of nonlocal conservation laws
   - General boundary conditions for conservation laws
   - Conservation laws with discontinuous flux and applications
   - Some applied problems (porous media, cross-diffusion, electrocardiology) and their finite volume approximation
   - Discrete functional analysis tools

APPROXIMATION OF AN ELLIPTIC-PARABOLIC PROBLEM "WITHOUT THE STRUCTURE CONDITION"

Well-posedness and structural stability for elliptic-parabolic problems...

The problem

$$\begin{cases} \partial_t b(u) + \operatorname{div} F(u) - \Delta u = f & \text{in the domain} \\ u = 0 & \text{on the boundary} \\ b(u) = b_0 & \text{at the initial moment,} \end{cases}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous non-strictly increasing function .

One application:
porous media (u is some pressure-like quantity, $b(u)$ is the saturation).

Questions:

- notion of solution to be used
- well-posedness: existence, uniqueness, continuous dependence on data
- approximation of solutions: structural stability (continuity in coeficients), convergent numerical schemes

Classical (non-degenerate) case: $b = Id$ (heat equation + convection term).
The bi-Lipschitz case ($\frac{1}{C} \leq b'(\cdot) \leq C$) is analogous (but: nonlinearities !).
Degenerate case: $b'(\cdot)$ may be zero at points (weak degeneracy)
or on intervals (strong degeneracy).

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Convergence of approximations. The structure condition.

Our problem is:
in the degenerate case, prove convergence of approximations.

A rough scheme of the proof:

- some approximation procedure (regularization or numerical scheme): problem (P) is approached by problems (P^h)
- construction of solutions u^h to problems (P^h) (simpler than (P)!)
- uniform in h estimates on u^h (e.g., estimates in some functional spaces)
- compactness of $(u^h)_h$ (extracting convergent subsequences: $u^h \to u$ as $h \to 0$)
- passage to the limit: "u^h solves (P^h)" + "$u = \lim u^h$" $\Longrightarrow$ "u solves (P) "..?

Often, the difficulty lies in "passage to the limit in nonlinear terms". For our problem, this is the term $\operatorname{div} F(u)$:
we need strong compactness of $(u^h)_h$, we only have the weak one.

Many methods are known for "upgrading weak to strong compactness" in different contexts (compensated compactness, Young measures,...); such methods use more of the PDE structure (not only functional analysis). But here, nothing of this works...

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For the problem $\partial_t b(u) + \operatorname{div} F(u) - \Delta u = f$,
the bottleneck of the method is the "time compactness" for $(u^h)_h$.

Indeed, natural (uniform) estimates for u^h are:

- an estimate of ∇u^h, which gives "space compactness" of $(u^h)_h$
  moreover, it gives also their space-time weak compactness
- an "estimate" of $\partial_t b(u^h)$, which gives "time compactness" of $(b(u^h))_h$.

The outcome is: assuming that , in a sense, (P^h) converges to (P),

- we can pass to the limit in the linear terms (e.g., Δu^h)
- we can pass to the limit in expressions of the form $\widetilde{F} \circ b(u^h)$.

Fortunately, in most applications, the structure condition holds:

$$\forall z, \widehat{z} \quad F(z) = F(\widehat{z}) \text{ whenever } b(z) = b(\widehat{z}).$$

If so, we have $F(u^h) = \widetilde{F} \circ b(u^h)$, and the whole procedure works.
These things are known since Alt, Luckhaus 83 ...

But what happens if the structure condition fails ??
Bénilan, Wittbold '96 :
existence is shown, but with a very different (not constructive) argument.
And if we want to "see" the solutions, in any reasonable sense ???

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Problem without the structure condition: solution, uniqueness, comparison.

Key idea on convergence of approximations: use monotone approximation ! Ammar, Wittbold '03 .
If the data b_0, f are approximated by b_0^h, f^h in a monotone way,
then $(u^h)_h$ is monotone and the strong compactness of $(u^h)_h$ is automatic .
Application: stability wrt perturbation of f follows , Andr., Wittbold '11, preprint.

Idea: get reduced to monotone $(f^h)_h$ using $\liminf - \limsup$ arguments;
further exploit the order-preservation feature of the PDE.

The notion of solution is very standard: weak formulation of the PDE.
We need in addition to define a kind of potential: $B(z) = \int z\, db(z)$.

Definition

Let $B_0 \in L^1(\Omega)$, $f \in L^2(Q)$; u is a weak (energy) solution of (P) if

(i) $u \in L^2(0,T;H_0^1(\Omega))$, $B(u) \in L^\infty(0,T;L^1(\Omega))$ ($\Rightarrow b(u) \in L^1(Q)$);

(ii) for all $\zeta \in L^2(0,T;H_0^1(\Omega))$ with $\zeta_t \in L^\infty(Q)$ and $\zeta(T) = 0$,

$$\iint_Q b(u)\zeta_t + \int_\Omega b(u_0)(\cdot)\zeta(0,\cdot) = \iint_Q (\nabla u + F(u)) \cdot \nabla\zeta - \iint_Q f\zeta.$$

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Problem (P) without the structure condition: comparison, stability.

Lemma (comparison result, F general)

Assume F is a merely continuous function. Assume that u_i, $i = 1, 2$, are weak solutions of (P) associated with data such that $b_0^1 \leq b_0^2$ and $f_1 < f_2$ (strictly!) a.e.. Then $u_1 \leq u_2$ a.e. on Q.

Entropy solutions + doubling of variables (Kruzhkov +Carrillo) are needed.
Idea of the proof: obtain (entropy sol.), then exploit the Kato inequality

$$-\iint_Q \Big(b(u_1) - b(u_2)\Big)^+ \zeta_t - \mathrm{sign}^+(u_1 - u_2)(\mathcal{F}_1 - \mathcal{F}_2) \cdot \nabla\zeta \leq \iint_Q \kappa\,(f_1 - f_2)\,\zeta,$$

where $\kappa(t, x) \in \mathrm{sign}^+(u_1 - u_2)$ and $\mathcal{F}_i = -F(u_i) + \nabla u_i$ are the fluxes.
New remark: if $(f_1 - f_2) < 0$ then the set $\big[u_1 > u_2\big]$ is of measure zero.

Proposition

If f^h converges to f in L^2 and if (P) with right-hand side f enjoys uniqueness , then u^h converge to u .

Arguments: set $\underline{f^n} := \inf_{h \leq 1/n} f^h - \frac{1}{n}$, use existence for $\underline{f^n}$, use the lemma.
We get $\underline{u} := \lim \underline{u^n} \leq \liminf u^h$, and $\underline{u}$ solves (P). In the same way,
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Problem (P): uniqueness, conclusion. Numerical approximation ?

Finally, to apply the Proposition, we just need uniqueness (it's quite natural!)

Lemma

Assume that F is Lipschitz continuous. Then uniqueness for (P) holds.

Arguments: we already know, from the Kato inequality, that $b(u_1) = b(u_2)$. The Kato inequality then reads

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To sum up: existence + uniqueness + monotonicity $\Rightarrow$ continuity in data.

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We have the following comparison arguments:

$$\begin{cases} \psi_+ > 0 \text{ on } \mathbb{R}, & \text{therefore } (u_{+,m})_m \text{ is non-decreasing;} \\ \psi_- < 0 \text{ on } \mathbb{R}, & \text{therefore } (u_{-,m})_m \text{ is non-increasing;} \\ \text{if } \psi_- < 0 < \psi_+ \text{ on } \mathbb{R}, & \text{therefore } u_{-,m} \geq u_{+,m} \text{ a.e. on } Q. \end{cases}$$

As a consequence, there exist $\underline{u}, \overline{u}$ solutions such that

$$\underline{u} = \lim_{m\to\infty} u_{+,m} \leq \lim_{m\to\infty} u_{-,m} = \overline{u}.$$

Assume we have a monotone discretization procedure for $(P_{\pm,m})$.
Translation estimates due to penalization yield $u_{\pm,m} = \lim_{h\to 0} u^h_{\pm,m}$, hence

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Finite volume schemes, comparison of discrete solutions and convergence.

Look at one popular discretization strategy: Finite Volumes.

Space is partitioned into disjoint polygons (say, triangles) κ with centers x_κ.
Stationary pb.: discrete solution $u^h(x) = \sum_\kappa u_\kappa 1\!\!1_\kappa(x)$ is piecewise const.

Approximation of convection-diffusion operator: in each volume κ, write

$$\int_\kappa \Big(\mathrm{div}\,(F(u) - \nabla u\Big) = \int_{\partial\kappa} (F(u)\cdot n - \partial_n u) = \sum_{\kappa|L} |\kappa|L| \fint_{\kappa|L} (F(u) - \partial_n u);$$

now the right-hand side is approximated numerically in terms of u^h,
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Questions considered and techniques in use

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  - entropy solutions and the associated techniques for hyperbolic and degenerate parabolic problems
  - “new” notions of solution for “old” convection and diffusion operators: mild/integral, renormalized, entropy, kinetic
  - adaptation of known notions to “new” operators (non local, etc.)
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  - search for generality or for optimality
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A PANORAMA OF QUESTIONS, WORKS AND RESULTS

Degenerate quasilinear parabolic pb. in the whole space

Equation:

$$\partial_t v + \operatorname{div}\Big(F(v) - \nabla\varphi(v)\Big) = s$$

with $\varphi(\cdot)$ non-decreasing. If possible, F, φ merely continuous.

NB: this is the degenerate parabolic-hyperbolic case.

Key concepts: Kato inequality, moduli of continuity.

Goal: uniqueness proof. The Kato inequality is well known; exploit it.

$$(KI) \quad \int_0^T\int_\Omega \Big(-|v-\hat v|\partial_t\xi \;-\; \operatorname{sign}(w-\hat w)(F(w)-F(\hat w))\cdot\nabla\xi \\ +\operatorname{sign}(w-\hat w)(\nabla w - \nabla\hat w)\cdot\nabla\xi\Big) \le 0.$$

Difficulty: treating the infinity (let $\xi \to 1$).

Techniques that allow for new results:

– use of a new test function

– use of fine properties of moduli of continuity

Byproduct: New results for the very classical problem $u - \Delta\varphi(u) = f$: well-posedness in L^∞, in some weighted L^1 spaces, in L^1_{loc}.

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NB: this is the Stefan-like problem.
But, the final goal is the parabolic-hyperbolic case : $F = F(v)$.

Key concept: entropy solutions of Kruzhkov and Carrillo .

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Leray-Lions operators and their numerical approximation

Nonlinear Leray-Lions problems:

$$u_t - \mathrm{div}\, \mathfrak{a}(t,x,u;\nabla u) = s,$$

with $\mathfrak{a}$ satisfying "pseudomonotonicity", "coercivity" and "growth" conditions that permit to set up the problem in the duality framework of $W^{1,p} - W^{-1,p'}$ Sobolev spaces.

The prototype example is the p-laplacian: $-\Delta_p u = \mathrm{div}\, |\nabla u|^{p-1} \nabla u$

Use: these operators model (some) nonlinear effects in diffusion.

Mathematically: compatible with methods of entropy solutions, etc. $\Rightarrow$ the previous results can be extended to Leray-Lions diffusions

Difficulty: numerical approximation by finite volumes.
Goal: "structure-preserving" finite volume schemes for Leray-Lions

Necessary equipment: discrete functional analysis tools

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Co-volume schemes, schemes on cartesian meshes

First attempt: “complementary volumes” (now developed by Mikula).
Goal: convergence proof
Difficulty: go from the continuous to the discrete framework
Key technical idea: substitute discrete solutions by regularized ones

Second attempt: schemes on cartesian meshes.
Goal: (optimal) error estimates
Difficulties: recognize good schemes; go through heavy technics :-)

We produced a whole series of works, with the following contents:

– consistency lemmas in Sobolev setting (heavy...)
– standard error estimates (like FE) with “$W^{1+1,p}$” solutions
– even more regularity ⇒ superconvergence on uniform meshes
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2D and 3D DDFV (Discrete Duality) schemes

Hermeline; Domelevo and Omnès :
a new strategy of FV discretization on “double” meshes
Goal: approximation of anisotropic linear diffusions
Difficulty: reconstruct all the directions of the gradient
Key technical idea: see the picture (2D) :-)

Remarkable feature: Discrete Duality

$$\left[\!\!\left[-\operatorname{div}^{\mathfrak{T}} \vec{\mathcal{F}},\, u^{\mathfrak{T}} \right]\!\!\right] = \left\{\!\!\left\{ \vec{\mathcal{F}},\, \nabla^{\mathfrak{T}} u^{\mathfrak{T}} \right\}\!\!\right\}.$$

Consequences: the scheme is structure-preserving for Leray-Lions

We produced a timely work:

– discrete duality and the associated “discrete calculus” formalism
– as a consequence, a “readable” convergence proof
– a scheme that found many applications
– reasonable convergence behaviour; good gradient approximation

Generalization to 3D ? It took time !
Several ideas were introduced ; among them, 3D CeVe-DDFV.
Applications: doubly nonlinear problems; electrocardiology

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Doubly and triply degenerate parabolic problems

By triply nonlinear, we mean a problem of the form

$$(TNL)\quad \partial_t b(v) - \operatorname{div} \mathfrak{a}(v, \nabla\varphi(v)) = s, \quad b(v)|_{t=0} = b^0$$

that we supplement by the homogeneous Dirichlet boundary condition, "for the sake of simplicity".

Key concepts: entropy solutions, monotonicity, chain rules.

Goals: existence, uniqueness, structural stability; FV schemes.
Difficulty: treating "everything" at the same time.

Techniques that allow for new results:

– a new lemma "cutting sets with small variation of $\varphi(v)$
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Parabolic equations with irregular absorption term

The background on this topic is provided by Wittbold (elliptic case), using convex analysis results of Bouchitté . The associated parabolic problem:

$$(AbsPb)\ \ \partial_t v - \operatorname{div} \mathfrak{a}(v, \nabla v) + \beta(x, v) \ni s, \quad v|_{t=0} = v_0$$

where $\beta(x, \cdot)$ is a maximal monotone graph, for all x.

Key concepts: diffuse measures, capacity, renormalized solutions.

Goals: notion of solution; well-posedness.
Difficulty: giving sense to the absorption term. Parabolic capacity (Pierre ; Droniou, Porretta, Prignet) ?

Techniques that allow for new results:

– a choice of functional setting :
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– a kind of maximal regularity result :
$\partial_t v + \beta(\cdot, v) \in L^1(0, T; \mathcal{M}_0) \Rightarrow$ each term is in the same space
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Variable exponent Leray-Lions problems, renormalized solutions

Generalization of Leray-Lions problems: variable exponent pbs

$$(PxPb)\;\; u - \mathrm{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right) = f.$$

Key concepts:
variable exponent Sobolev spaces, (vector) Young measures, renormalized solutions.

Goals: structural stability, numerical approx.; coupled problems.
Difficulty: e.g. in numerical approximation, treating $p_n(\cdot) \to p(\cdot)$

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– not being maniac about functional spaces :-)
– (vector) Young measures replace Minty-Browder argument (nice!)
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– $p(u)$ exponent !
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Renormalized solutions of nonlocal diffusion operators

Recent models in applied sciences, old subject in stochastic processes: “fractional diffusion” operators.
A prototype fractional (non-local) diffusion PDE:

$$(FDPb)\quad b(v)+(-\Delta)^{\lambda/2}v=f\quad \text{in } \mathbb{R}^n.$$

Key concept: notion of renormalized solution ?

Goal: extend the notion of renormalized sol. to fractional diffusions.
Difficulty: avoid (replace) chain rules

Techniques that allow for a new definition:

– observation of natural a priori estimates
– use of integration-by-parts in terms of bilinear forms :

$$\int(-\Delta)^{\frac{\lambda}{2}}u\,v=\int\!\!\int(u(x)-u(y))(v(x)-v(y))\,d\pi(x,y)$$

– a symmetrization of the nonlocal quantities:
$\delta_{x,y}u=u(x)-u(y),\ \theta_{x,y}u=\frac{u(x)+u(y)}{2}$
– simplification of proofs combining different hints on renormal. sol.

Renormalized solutions of nonlocal diffusion operators

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Entropy solution of nonlocal conservation laws

A theory due to Alibaud :

$$(FrCL) \quad u_t + \operatorname{div} f(u) + (-\Delta)^{\lambda/2} u = 0 \quad \text{in } \mathbb{R}^n.$$

The Kruzhkov theory of entropy solutions extended to encompass fractional diffusions.

Nonetheless, necessity of entropy solutions' notion is unclear.
E.g., it is useless if $\lambda > 1$ (Droniou, Gallouet, Vovelle).
Shock creation is possible for $\lambda < 1$ (Alibaud, Droniou, Vovelle)...

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Goal: prove non-uniqueness of a weak solution.
Difficulty: no explicit calculation possible

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General boundary conditions for conservation laws

Back to scalar conservation laws... theory complete in 1970'th ???
Not really...

$$u_t + \operatorname{div} f(u) = 0 \quad \text{in } \Omega.$$

Questions remained/appeared; new techniques created.
E.g., for boundary-value problems, only few works are available:
Bardos, LeRoux, Nédélec , then Otto ; and Bürger. Frid, Karlsen .

Key concepts: boundary dissipation, singular limit, strong traces.

Goal: find a formulation suitable for "general" boundary conditions.
Difficulty: find... caracterize... prove well-posedness

Ideas that allow for a notion of solution:

– the boundary condition of the viscous case is projected
(typical singular limit situation)
– the "formal BC" should become some "efffective BC"
– every BC encoded by a maximal monotone subgraph of $f(\cdot)\cdot n$
– caracterize the projection procedure
– entropy inequalities with remainder terms
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Discontinuous flux

A problem in scalar conservation laws (origin: layered porous media?)

$$\partial_t u + \partial_x \mathfrak{f}(x;u) = 0 \ \text{ with } \ \mathfrak{f}(x;\cdot) = f^l(\cdot)1\!\!1_{[x<0]} + f^r(\cdot)1\!\!1_{[x>0]}.$$

Seems very similar to standard Kruzhkov, but strangely...

Adimurthi, Mishra, Veerappa Gowda : multiple notions of solution !

Towers; Karlsen, Risebro, Towers,. . . : successful attempts to apply Kruzhkov

Audusse, Perthame : notion of adapted entropies

Bachmann, Vovelle : combining kinetic and entropy-process solutions

Panov : strong traces are available; look at the interface, pointwise !

Key concepts: adapted entropies, admissibility germ, “remainder terms”

Goals: Initial: drop the “crossing condition”.
Final: “clean up” the area. Provide a unifying framework.

Difficulty: be at the good place at the appropriate time :-)

Ideas that allow to construct a kind of “theory”:

– look at the interface coupling: strong traces
– Observe that the set of couples $(\gamma_l u, \gamma_r u)$ has an “algebraic structure”
– explore this structure
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Applications of theory for discont. flux

- Road traffic model: cons. law with a point constraint

  Model: $\partial_t u + \partial_x f(u) = 0$ with "$f(u)|_{x=0}$"$\leq F(t)$ (Colombo, Goatin).

  Turns out to be a particular case of discontinuous-flux problem !
  $\Rightarrow$ Non-Kruzhkov solutions of the classical conservation law

  Goals: solution (general) ? well-posedness ? simple FV scheme

  Difficulties: find the good scheme (Seguin)

  Key technical point: the germ is completely determined by a singleton

  Byproduct: the idea of flux limitation useful for the porous medium pb.

- "particle-in-Burgers" : fluid-particle interact. via drag force

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = -\lambda(u - h'(t))\,\delta_0(x - h(t)) \\ h''(t) = \lambda(u(t, h(t) - h'(t)) \end{cases}$$

  Lagoutière, Seguin, Takahashi : sense of product $(u - h'(t))\,\delta_0(\ldots)$

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Some applied problems and their finite volume approximation

- A singular limit of two-phase flow equations and robust FV scheme
  Two-phase flow, air mobility going to ∞... is the limit Richards ?
  Key ideas of Eymard : ingenious flux discretization
  Difficulties: singular limit; robust FV scheme
- Some (weak) cross-diffusion systems. Example: the squirrel war .
  A more precise model with different "survival prediction"
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  Byproduct:
  discrete versions of Sobolev embeddings + compactness lemmas
- Electrocardiology model: the bidomain problem
  A system of two equations: one parabolic, one elliptic .
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Discrete functional analysis tools

While searching to prove “academic” convergence on the “academic” p-laplacian ... while working on the above applied problems ...

progressively, I took part in development of discrete versions of classical arguments of functional analysis (Gallouët school)

Important point: have a good formalism ! $\left\{\!\!\left\{ \, , \, \right\}\!\!\right\}$, $[\![\, , \,]\!]$, etc.

Results obtained and applied:

- a “magical” reconstruction formula (vertex-centered FV)
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