Theory and numerics for locally constrained conservation laws.

B. Andreianov[1] P. Goatin [2] N. Seguin[3]

[1]Université de Franche-Comté
[2]ISITV, Université du Sud Toulon - Var
[3]Laboratoire J.-L. Lions, Université Paris 6

Université Paris-Sud, Décembre 3, 2009

Plan of the talk

MODEL AND MAIN RESULTS

Model and motivations

Think first of a standard road traffic model (a road modelled by $\mathbb{R}$, conservation law in the concentration u of cars on the road):

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

Model and motivations

Think first of a standard road traffic model (a road modelled by $\mathbb{R}$, conservation law in the concentration u of cars on the road):

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

In this context, one takes the hypotheses

- $u_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$ (measurable, with values in $[0,1]$)

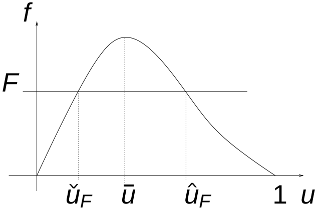
Model and motivations

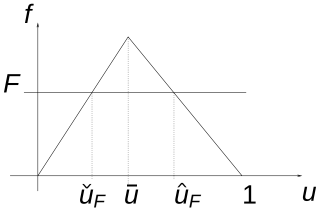
Think first of a standard road traffic model (a road modelled by $\mathbb{R}$, conservation law in the concentration u of cars on the road):

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

In this context, one takes the hypotheses

- $u_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ (measurable, with values in $[0,1]$)
- $f : [0,1] \to \mathbb{R}^+$ Lipschitz continuous, $f(0) = f(1) = 0$, $f \geqslant 0$
- (additional simplifying hypothesis) there exists $\bar{u}$ s.t. $f'(u)(\bar{u} - u) > 0$ for $u \neq \bar{u}$





Model and motivations (cont^{d})

Now think of an obstacle on the road: toll gate, construction site, exit...
One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \text{"}f(u(t,0))\text{"} \leqslant F(t) & t > 0, \end{cases}$$

Model and motivations (cont^{d})

Now think of an obstacle on the road: toll gate, construction site, exit...
One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \text{"} f(u(t,0)) \text{"} \leqslant F(t) & t > 0, \end{cases}$$

The hypothesis on F is minimal:

- $F \in \mathbf{L}^\infty(\mathbb{R}^+; [0, f(\bar{u})])$ (NB: $f(\bar{u}) \equiv \max_{[0,1]} f$).

Model and motivations (cont^{d})

Now think of an obstacle on the road: toll gate, construction site, exit...
One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \text{"}f(u(t,0))\text{"} \leqslant F(t) & t > 0, \end{cases}$$

The hypothesis on F is minimal:

- $F \in \mathbf{L}^\infty(\mathbb{R}^+; [0, f(\bar{u})])$ (NB: $f(\bar{u}) \equiv \max_{[0,1]} f$).

Rq: The meaning of "$f(u(t,0))$" has to be precised .

Model and motivations (cont[d])

Now think of an obstacle on the road: toll gate, construction site, exit...
One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \text{"} f(u(t,0)) \text{"} \leqslant F(t) & t > 0, \end{cases}$$

The hypothesis on F is minimal:

- $F \in \mathbf{L}^\infty(\mathbb{R}^+; [0, f(\bar{u})])$ (NB: $f(\bar{u}) \equiv \max_{[0,1]} f$).

Rq: The meaning of "$f(u(t,0))$" has to be precised .

– Indeed, a generic solution is L^∞ and not pointwise defined.

– If we think of a piecewise continuous solution with possible discontinuity across $\{x = 0\}$, the Rankine-Hugoniot condition ensures that

$$f(u(t,0^-)) = f(u(t,0^+)) \quad \text{for a.e. } t > 0.$$

Model and motivations (cont^{d})

Now think of an obstacle on the road: toll gate, construction site, exit...
One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \text{"}f(u(t,0))\text{"} \leqslant F(t) & t > 0, \end{cases}$$

The hypothesis on F is minimal:

- $F \in \mathbf{L}^\infty(\mathbb{R}^+; [0, f(\bar{u})])$ (NB: $f(\bar{u}) \equiv \max_{[0,1]} f$).

Rq: The meaning of "$f(u(t,0))$" has to be precised .

– Indeed, a generic solution is L^∞ and not pointwise defined.

– If we think of a piecewise continuous solution with possible discontinuity across $\{x = 0\}$, the Rankine-Hugoniot condition ensures that

$$f(u(t,0^-)) = f(u(t,0^+)) \quad \text{for a.e. } t > 0.$$

– In general, we have here $(u, f(u))$ which is a divergence-measure field in the domains $\{x > 0\}$ and $\{x < 0\}$ (cf. weak traces of Chen and Frid); therefore $x \mapsto [t \mapsto f(u(t,x))]$ is continuous from $\mathbb{R}$ to $L^1_{loc} - weak$.

– A stronger sense can be assigned to $f(u(t,0^\pm))$ thanks to the strong trace theory of Vasseur and Panov .

Main results

- adequate entropy formulation for BV data and solutions, well-posedness in BV (existence, uniqueness, continuous dependence) were developed by R. Colombo and P. Goatin

Main results

- adequate entropy formulation for *BV* data and solutions, well-posedness in *BV* (existence, uniqueness, continuous dependence) were developed by R. Colombo and P. Goatin

We continued this work and obtained

- a series of equivalent entropy formulations for $\mathbf{L}^\infty$ data and solutions (each formulation having advantages and drawbacks...)
- a well-posedness theory in the $\mathbf{L}^\infty$ framework (combining the different definitions)
- a definition of "entropy-process" (measure-valued) solutions
- formulation and a proof of convergence of a very simple adaptation of standard monotone finite volume schemes
- numerically, the convergence order observed is the same as for the unconstrained conservation law

NUMERICAL RESULTS

Numerical results: a Riemann problem

We pick $f(u) = u(1-u)$ and use the Rusanov numerical flux (known to be consistent and monotone). We first simulate the solution to a Riemann problem .

Numerical results: a Riemann problem

We pick $f(u) = u(1-u)$ and use the Rusanov numerical flux (known to be consistent and monotone). We first simulate the solution to a Riemann problem . The domain of computation is $[-1/2, 1/2]$ and the data are

$$u_0(x) = \begin{cases} 0.4 & \text{if } x < 0, \\ 0.5 & \text{if } x > 0, \end{cases} \quad \text{and} \quad F(t) \equiv const = 0.2 \,.$$

The exact solution is composed of a classical shock wave with a negative speed, of a non-classical stationary shock wave at $x = 0$ satisfying the constraint , and of another classical shock wave with a positive speed.

Numerical results: a Riemann problem

We pick $f(u) = u(1-u)$ and use the Rusanov numerical flux (known to be consistent and monotone). We first simulate the solution to a Riemann problem. The domain of computation is $[-1/2, 1/2]$ and the data are

$$u_0(x) = \begin{cases} 0.4 & \text{if } x < 0, \\ 0.5 & \text{if } x > 0, \end{cases} \quad \text{and} \quad F(t) \equiv const = 0.2 \, .$$

The exact solution is composed of a classical shock wave with a negative speed, of a non-classical stationary shock wave at $x = 0$ satisfying the constraint, and of another classical shock wave with a positive speed. The non-classical shock wave seems to be perfectly solved.

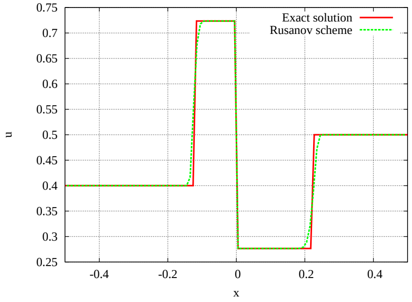


Figure: Comparison between the Rusanov scheme (100 cells, CFL=0.4) and the exact solution at time $t = 1$.

Numerical results: a Riemann problem (cont^d)

The next figure depicts the error with respect to the space step. We can easily see that the rate of convergence is 1 ; this means that the constraint does not affect the accuracy of the numerical scheme .

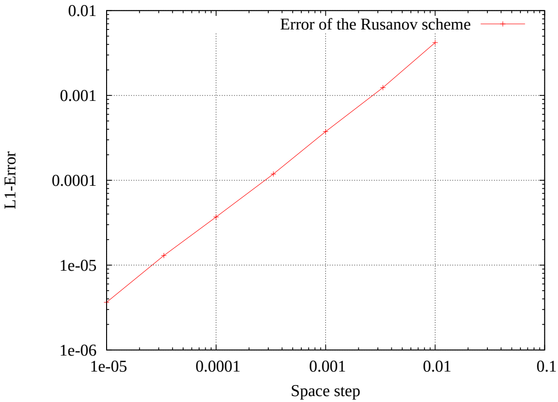


Figure: Convergence of the Rusanov scheme in the $\mathbf{L}^1$ norm.

Numerical results: the green wave

This test case is much more complicated. The space domain is $[0,100]$ and it involves five lights . They are modelled by the use of a constraint $F_i(t)$ for each light i. They are located at $x_i = (i+2)12.5$, $i = 1, ..., 5$. The constraint of the first light is defined by alternance of red and green light:

$$F_1(t) = \begin{cases} 0 & \text{if } t \in [0,50) \quad \text{(meaning red light)} \\ \max_u f(u) = 1/4 & \text{if } t \in [50,100) \quad \text{(meaning green light)} \end{cases}$$

it is periodic (with period equal to 100).

Numerical results: the green wave

This test case is much more complicated. The space domain is $[0, 100]$ and it involves five lights . They are modelled by the use of a constraint $F_i(t)$ for each light i. They are located at $x_i = (i+2)12.5$, $i = 1, ..., 5$. The constraint of the first light is defined by alternance of red and green light:

$$F_1(t) = \begin{cases} 0 & \text{if } t \in [0, 50) \quad \text{(meaning red light)} \\ \max_u f(u) = 1/4 & \text{if } t \in [50, 100) \quad \text{(meaning green light)} \end{cases}$$

it is periodic (with period equal to 100).

The other constraints are given by $F_i(t) = F_1(t - i\varphi)$, $i = 2, ..., 5$.
The real constant φ then corresponds to the time lag between the lights .

Numerical results: the green wave

This test case is much more complicated. The space domain is $[0,100]$ and it involves five lights . They are modelled by the use of a constraint $F_i(t)$ for each light i. They are located at $x_i = (i+2)12.5$, $i = 1, ..., 5$. The constraint of the first light is defined by alternance of red and green light:

$$F_1(t) = \begin{cases} 0 & \text{if } t \in [0,50) \quad \text{(meaning red light)} \\ \max_u f(u) = 1/4 & \text{if } t \in [50,100) \quad \text{(meaning green light)} \end{cases}$$

it is periodic (with period equal to 100).

The other constraints are given by $F_i(t) = F_1(t - i\varphi)$, $i = 2, ..., 5$.
The real constant φ then corresponds to the time lag between the lights .

Our aim is to minimize the influence of the lights i with $i \geqslant 2$ on the traffic .

Numerical results: the green wave

This test case is much more complicated. The space domain is $[0, 100]$ and it involves five lights . They are modelled by the use of a constraint $F_i(t)$ for each light i. They are located at $x_i = (i+2)12.5$, $i = 1, ..., 5$. The constraint of the first light is defined by alternance of red and green light:

$$F_1(t) = \begin{cases} 0 & \text{if } t \in [0, 50) \quad \text{(meaning red light)} \\ \max_u f(u) = 1/4 & \text{if } t \in [50, 100) \quad \text{(meaning green light)} \end{cases}$$

it is periodic (with period equal to 100).

The other constraints are given by $F_i(t) = F_1(t - i\varphi)$, $i = 2, ..., 5$.
The real constant φ then corresponds to the time lag between the lights .

Our aim is to minimize the influence of the lights i with $i \geqslant 2$ on the traffic .

The conditions of simulation are the following: the initial density is set to 0 on the whole domain, the left boundary condition is a Dirichlet condition, with $u(0, t) = 0.1$, and at $x = 100$, we impose an open boundary condition. The domain contains 1000 cells and the CFL number is set to 0.4.

Numerical results: the green wave (cont^{d})

Whatever the value of φ is, the results become periodic in time, at least for $t \geqslant 500$. As an example, the next figure represents the evolution of the total mass in the domain for several values of φ. The case of "One light" (in red) corresponds to $F_i \equiv 1/4$ (eternal green light) for $i \geqslant 2$; this is the ideal case .

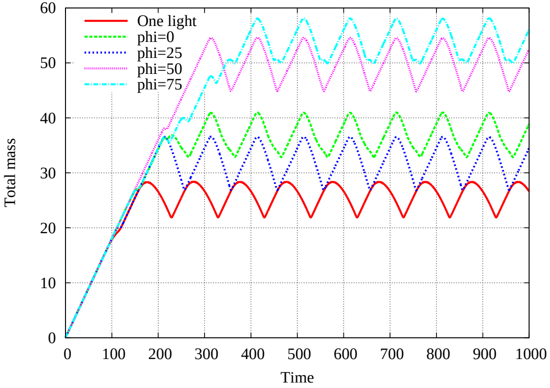


Figure: Time variation of the total mass, with different values of the time lag φ, compared to the ideal case (one light).

Numerical results: the green wave (cont^{d})

In Figure 4, we can see the average over the time interval $[500, 1000]$ of the total mass of cars and of the total variation (in space) of the velocity.
There seems to be an optimal value near $\varphi = 16.1$. Moreover, for this value, the average total mass and the average total variation of the velocity are very close to the values obtained in the ideal case of one light (horizontal lines).

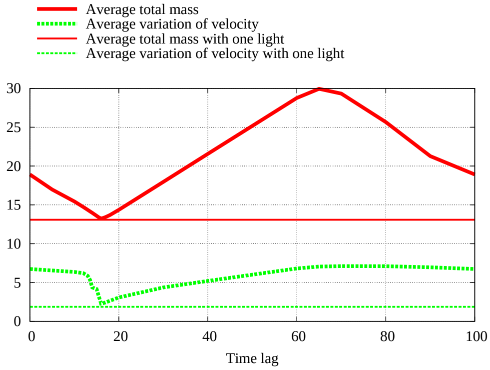


Figure: Variation of average quantities with respect to the time lag φ, compared to the ideal case (one light).

KRUZHKOV'S THEORY

A crash course through the Kruzhkov theory

Let us describe very briefly (a part of) the classical theory of SCL.

- for regular data, classical solutions can be constructed locally in time, by the method of characteristics; but the characteristics may cross in finite time, which results in a blow-up of the derivative of the solution or in shock creation. The classical solution ceases to exist...
- one can look for weak solutions (in the sense $\mathcal{D}'$) and recover existence, globally in time; but then the uniqueness is lost
- introducing the notion of Kruzhkov entropy solution , one recovers existence and uniqueness (and even L^1 contraction).

A crash course through the Kruzhkov theory

Let us describe very briefly (a part of) the classical theory of SCL.

- for regular data, classical solutions can be constructed locally in time, by the method of characteristics; but the characteristics may cross in finite time, which results in a blow-up of the derivative of the solution or in shock creation. The classical solution ceases to exist...
- one can look for weak solutions (in the sense $\mathcal{D}'$) and recover existence, globally in time; but then the uniqueness is lost
- introducing the notion of Kruzhkov entropy solution , one recovers existence and uniqueness (and even L^1 contraction).
- two entropy solutions verify the Kato inequality

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big(|u-v|\,\partial_t\varphi + \Phi(u,v)\,\partial_x\varphi \Big) + \int_{\mathbb{R}} |u_0-v_0|\,\varphi(0,x) \geqslant 0$$

  Here $\Phi(u,v) = \mathrm{sign}(u-v)(f(u)-f(v))$ and φ is a test function.
- the notion of an entropy solution itself is based upon the Kato inequality postulated with respect to a selected family of "elementary solutions" v; namely, one takes all the constant solutions $v \equiv \kappa$, $\kappa \in \mathbb{R}$.

A crash course through the Kruzhkov theory (cont[d])

- letting φ go to $1\!\!1_{[0,T)\times\mathbb{R}}$, from the Kato inequality

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big(|u-v|\,\partial_t\varphi+\Phi(u,v)\,\partial_x\varphi\Big)+\int_{\mathbb{R}}|u_0-v_0|\,\varphi(0,x)\geqslant 0$$

one recovers $\displaystyle\int_{\mathbb{R}}|u-v|(T)\leqslant\int_{\mathbb{R}}|u_0-v_0|+\mathit{sthg},\quad \mathit{sthg}\leqslant 0.$

This yields uniqueness and continuous dependence (L^1 contraction).

A crash course through the Kruzhkov theory (cont[d])

- letting φ go to $1\!\!1_{[0,T)\times\mathbb{R}}$, from the Kato inequality

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big(|u-v|\,\partial_t\varphi+\Phi(u,v)\,\partial_x\varphi\Big)+\int_{\mathbb{R}}|u_0-v_0|\,\varphi(0,x)\geqslant 0$$

one recovers $\displaystyle\int_{\mathbb{R}}|u-v|(T)\leqslant\int_{\mathbb{R}}|u_0-v_0|+stHg, \quad stHg\leqslant 0.$

This yields uniqueness and continuous dependence (L^1 contraction).

- Rq: If u,v are piecewise continuous with a jump at $x=0$, then the jumps "contribute to sthg " with the term

$$\Phi(\gamma^l u,\gamma^l v)-\Phi(\gamma^r u,\gamma^r v),\quad \text{which is non-negative}.$$

Here and in the sequel, γ^l and γ^r denote one-sided traces on $\{x=0\}$.

A crash course through the Kruzhkov theory (cont^{d})

- letting φ go to $1\!\!1_{[0,T)\times\mathbb{R}}$, from the Kato inequality

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big(|u-v|\,\partial_t\varphi+\Phi(u,v)\,\partial_x\varphi\Big)+\int_{\mathbb{R}}|u_0-v_0|\,\varphi(0,x)\geqslant 0$$

  one recovers $\displaystyle\int_{\mathbb{R}}|u-v|(T)\leqslant\int_{\mathbb{R}}|u_0-v_0|+sthg,\qquad sthg\leqslant 0.$

  This yields uniqueness and continuous dependence (L^1 contraction).

- Rq: If u,v are piecewise continuous with a jump at $x=0$, then the jumps "contribute to sthg " with the term

$$\Phi(\gamma^l u,\gamma^l v)-\Phi(\gamma^r u,\gamma^r v),\qquad \text{which is non-negative}.$$

  Here and in the sequel, γ^l and γ^r denote one-sided traces on $\{x=0\}$.

- Rq: Assume u is has a jump at $x=0$, and is an entropy solution "away from $\{x=0\}$". Then (cf. Vol'pert) u is an entropy solution if and only if

$$\Phi(\gamma^l u,\kappa)-\Phi(\gamma^r u,\kappa)\quad \text{is non-negative, for all } \kappa.$$

A crash course through the Kruzhkov theory (cont^d)

- existence of an entropy solution can be shown in several ways.
  One of the most convenient ways to construct solutions is

  – to learn solving "Riemann problems" (that is, the Cauchy problems with simplest discontinuous initial data, kind of Heavyside functions)
  – to use the Riemann solvers as "building blocks" to construct solutions (wave-front tracking algorithms, or numerical schemes: Godunov,Glimm)

A crash course through the Kruzhkov theory (cont^d)

- existence of an entropy solution can be shown in several ways. One of the most convenient ways to construct solutions is

  – to learn solving "Riemann problems" (that is, the Cauchy problems with simplest discontinuous initial data, kind of Heavyside functions)
  – to use the Riemann solvers as "building blocks" to construct solutions (wave-front tracking algorithms, or numerical schemes: Godunov,Glimm)
- Depending on the procedure used to construct approximate solutions, we may have more or less strong compactness properties. Often, only a uniform L^∞ bound is available . It only gives weak compactness.

  This does not allow to pass to the limit in the equation (the equation is nonlinear !), unless one weakens the notion of solution (Young measures, entropy-process solutions) .

  But the careful use of the Kato inequality allows to prove that such measure-valued solution is the unique entropy solution .

A crash course through the Kruzhkov theory (cont^{d})

- existence of an entropy solution can be shown in several ways. One of the most convenient ways to construct solutions is

  – to learn solving "Riemann problems" (that is, the Cauchy problems with simplest discontinuous initial data, kind of Heavyside functions)
  – to use the Riemann solvers as "building blocks" to construct solutions (wave-front tracking algorithms, or numerical schemes: Godunov,Glimm)
- Depending on the procedure used to construct approximate solutions, we may have more or less strong compactness properties. Often, only a uniform L^∞ bound is available . It only gives weak compactness.
  This does not allow to pass to the limit in the equation (the equation is nonlinear !), unless one weakens the notion of solution (Young measures, entropy-process solutions) .
  But the careful use of the Kato inequality allows to prove that such measure-valued solution is the unique entropy solution .
- Entropy solutions "have strong traces" (Vasseur, Panov).
  E.g. if u is an entropy solution in $\{x > 0\}$, then

  – $f(u)$ and $\Phi(u,k)$ have strong traces on $\{x = 0\}$
  – if, in addition, f' does not vanish on intervals,
  then u has a strong trace $\gamma^r u$ on $\{x = 0\}$.

Key Ideas

Key ideas : the Riemann solver (fixing $F(t) \equiv F = const$)

• The first idea is: solving SCL in one space dimension is, roughly speaking, equivalent to solving the Riemann problem . Let us denote by $\mathcal{R}(u^l, u^r)$ the Riemann solver of the unconstrained pb.

$$\text{(RP)} \quad \partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x) := \begin{cases} u^l \text{ if } x < 0 \\ u^r \text{ if } x > 0 \end{cases}$$

Key ideas : the Riemann solver (fixing $F(t) \equiv F = const$)

• The first idea is: solving SCL in one space dimension is, roughly speaking, equivalent to solving the Riemann problem . Let us denote by $\mathcal{R}(u^l, u^r)$ the Riemann solver of the unconstrained pb.

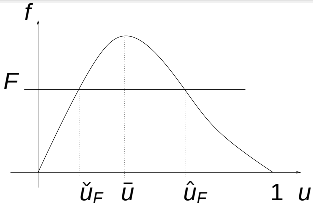
$$\text{(RP)} \quad \partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x) := \begin{cases} u^l & \text{if } x < 0 \\ u^r & \text{if } x > 0 \end{cases}$$

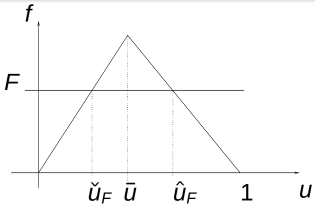
Then the Riemann solver for $(C-RP) := (RP) + "f(u(t,0)) \leqslant F"$ is given by

Definition (Colombo-Goatin '07)

If $f\left(\mathcal{R}(u^l, u^r))(0)\right) \leqslant F$, then $\mathcal{R}^F(u^l, u^r) = \mathcal{R}(u^l, u^r)$.

Otherwise, $\mathcal{R}^F(u^l, u^r)(x) = \begin{cases} \mathcal{R}(u^l, \hat{u}_F)(x) \text{ if } x < 0\,, \\ \mathcal{R}(\check{u}_F, u^r)(x) \text{ if } x > 0\,. \end{cases}$





Key ideas : the Riemann solver (fixing $F(t) \equiv F = const$)

• The first idea is: solving SCL in one space dimension is, roughly speaking, equivalent to solving the Riemann problem . Let us denote by $\mathcal{R}(u^l, u^r)$ the Riemann solver of the unconstrained pb.

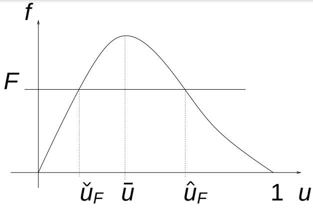
$$\text{(RP)} \quad \partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x) := \begin{cases} u^l \text{ if } x < 0 \\ u^r \text{ if } x > 0 \end{cases}$$

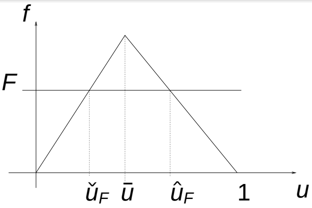
Then the Riemann solver for $(C-RP) := (RP) + \text{"} f(u(t,0)) \leqslant F \text{"}$ is given by

Definition (Colombo-Goatin '07)

If $f\left(\mathcal{R}(u^l, u^r))(0)\right) \leqslant F$, then $\mathcal{R}^F(u^l, u^r) = \mathcal{R}(u^l, u^r)$.

Otherwise, $\mathcal{R}^F(u^l, u^r)(x) = \begin{cases} \mathcal{R}(u^l, \hat{u}_F)(x) \text{ if } x < 0\,, \\ \mathcal{R}(\check{u}_F, u^r)(x) \text{ if } x > 0\,. \end{cases}$





$\Longrightarrow$ possible presence of a non-classical shock at $x = 0$

Elementary solutions and admissible traces.

Looking at the Riemann solver of the previous page, we see that it contains a family of stationary solutions $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$.

Elementary solutions and admissible traces.

Looking at the Riemann solver of the previous page, we see that it contains a family of stationary solutions $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$.

Firstly, this is the non-classical shock joining $\hat{u}_F$ on the left to $\check{u}_F$ on the right.
Secondly, these are the constant solutions $v \equiv \kappa$ with $f(\kappa) \leqslant F$.
Thirdly, these are the Kruzhkov stationary shocks satisfying the constraint.

Elementary solutions and admissible traces.

Looking at the Riemann solver of the previous page, we see that it contains a family of stationary solutions $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$.

Firstly, this is the non-classical shock joining $\hat{u}_F$ on the left to $\check{u}_F$ on the right.
Secondly, these are the constant solutions $v \equiv \kappa$ with $f(\kappa) \leqslant F$.
Thirdly, these are the Kruzhkov stationary shocks satisfying the constraint.

$\Longrightarrow$"Admissibility germ" $\mathscr{G}(F) = \mathscr{G}_1(F) \cup \mathscr{G}_2(F) \cup \mathscr{G}_3(F)$, where

- $\mathscr{G}_1(F) = \{(c_l, c_r) \in [0,1]^2; c_l > c_r, f(c_l) = f(c_r) = F\} \equiv \{(\hat{u}_F, \check{u}_F)\}$,
- $\mathscr{G}_2(F) = \{(c,c) \in [0,1]^2; f(c) \leqslant F\}$,
- $\mathscr{G}_3(F) = \{(c_l, c_r) \in [0,1]^2; c_l < c_r, f(c_l) = f(c_r) \leqslant F\}$.

Elementary solutions and admissible traces.

Looking at the Riemann solver of the previous page, we see that it contains a family of stationary solutions $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$.

Firstly, this is the non-classical shock joining $\hat{u}_F$ on the left to $\check{u}_F$ on the right.
Secondly, these are the constant solutions $v \equiv \kappa$ with $f(\kappa) \leqslant F$.
Thirdly, these are the Kruzhkov stationary shocks satisfying the constraint.

$\Longrightarrow$"Admissibility germ" $\mathcal{G}(F) = \mathcal{G}_1(F) \cup \mathcal{G}_2(F) \cup \mathcal{G}_3(F)$, where

- $\mathcal{G}_1(F) = \{(c_l, c_r) \in [0,1]^2; c_l > c_r, f(c_l) = f(c_r) = F\} \equiv \{(\hat{u}_F, \check{u}_F)\}$,
- $\mathcal{G}_2(F) = \{(c, c) \in [0,1]^2; f(c) \leqslant F\}$,
- $\mathcal{G}_3(F) = \{(c_l, c_r) \in [0,1]^2; c_l < c_r, f(c_l) = f(c_r) \leqslant F\}$.

• The second idea is (mimicking the Kruzhkov's definition !):

– accept these "elementary solutions" as admissible;
– define entropy solutions as the functions satisfying the Kato inequality with respect to these elementary solutions.

Elementary solutions and admissible traces.

Looking at the Riemann solver of the previous page, we see that it contains a family of stationary solutions $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$.

Firstly, this is the non-classical shock joining $\hat{u}_F$ on the left to $\check{u}_F$ on the right.
Secondly, these are the constant solutions $v \equiv \kappa$ with $f(\kappa) \leqslant F$.
Thirdly, these are the Kruzhkov stationary shocks satisfying the constraint.

$\Longrightarrow$"Admissibility germ" $\mathscr{G}(F) = \mathscr{G}_1(F) \cup \mathscr{G}_2(F) \cup \mathscr{G}_3(F)$, where

- $\mathscr{G}_1(F) = \{(c_l, c_r) \in [0,1]^2; c_l > c_r, f(c_l) = f(c_r) = F\} \equiv \{(\hat{u}_F, \check{u}_F)\}$,
- $\mathscr{G}_2(F) = \{(c,c) \in [0,1]^2; f(c) \leqslant F\}$,
- $\mathscr{G}_3(F) = \{(c_l, c_r) \in [0,1]^2; c_l < c_r, f(c_l) = f(c_r) \leqslant F\}$.

• The second idea is (mimicking the Kruzhkov's definition !):

– accept these "elementary solutions" as admissible;
– define entropy solutions as the functions satisfying the Kato inequality with respect to these elementary solutions.

• The third idea is (think of the scaling argument !):

– $\mathscr{G}(F)$ encodes the possible traces on $\{x = 0\}$ of elementary solutions
– see an admissible solution for the constrained CL as a Kruzhkov solution of the CL in domains $\{x > 0\}$,$\{x < 0\}$ with traces $\gamma^{l,r} u$ on $\{x = 0\}$ satisfying $(\gamma^l u, \gamma^r u) \in \mathscr{G}(F)$.

Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have “the interface dissipation” in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathscr{G}(F)$,
the interface dissipation is a property of $\mathscr{G}(F)$.

Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have "the interface dissipation" in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathscr{G}(F)$, the interface dissipation is a property of $\mathscr{G}(F)$.

Rq.: If F varies with t, the set of elementary solutions and the set $\mathscr{G}(F)$ of their traces vary. We cope with this problem using

Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have "the interface dissipation" in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathscr{G}(F)$, the interface dissipation is a property of $\mathscr{G}(F)$.

Rq.: If F varies with t, the set of elementary solutions and the set $\mathscr{G}(F)$ of their traces vary. We cope with this problem using

- The fifth idea (heading to a global entropy formulation):

– define an admissible solution u as satisfying the Kato inequality with respect to any function of the form $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$

– we pay with a remainder term that vanishes when $(c_l, c_r) \in \mathscr{G}(F(t))$

Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have "the interface dissipation" in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathcal{G}(F)$, the interface dissipation is a property of $\mathcal{G}(F)$.

Rq.: If F varies with t, the set of elementary solutions and the set $\mathcal{G}(F)$ of their traces vary. We cope with this problem using

- The fifth idea (heading to a global entropy formulation):

– define an admissible solution u as satisfying the Kato inequality with respect to <u>any</u> function of the form $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$

– we pay with a remainder term that vanishes when $(c_l, c_r) \in \mathcal{G}(F(t))$

Finally, what about numerics ?? We have a very "cheap" solution !

Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have "the interface dissipation" in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathscr{G}(F)$, the interface dissipation is a property of $\mathscr{G}(F)$.

Rq.: If F varies with t, the set of elementary solutions and the set $\mathscr{G}(F)$ of their traces vary. We cope with this problem using

- The fifth idea (heading to a global entropy formulation):

– define an admissible solution u as satisfying the Kato inequality with respect to any function of the form $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$

– we pay with a remainder term that vanishes when $(c_l, c_r) \in \mathscr{G}(F(t))$

Finally, what about numerics ?? We have a very "cheap" solution !

- The last (but not least !!) idea is :

take any FV numerical scheme that works well on the unconstrained CL, and simply truncate the numerical flux at $x = 0$ (by replacing the given numerical flux $g(u_K, u_L)$ with $\min\{g(u_K, u_L), F(t)\}$).

DEFINITIONS

The Colombo-Goatin definition

Definition (R. Colombo and P. Goatin)

A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of the constrained SCL if

(i) for all nonnegative test function $\varphi \in \mathcal{C}_c^\infty(\Pi)$ and all $\kappa \in [0,1]$,

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(|u(t,x) - \kappa| \partial_t + \Phi(u(t,x),\kappa) \partial_x \right) \varphi(t,x)\, dx\, dt \\ + \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx + 2 \int_0^{+\infty} \left(1 - F(t)/f(\bar{u}) \right) f(\kappa)\, \varphi(t,0)\, dt \geqslant 0;$$

(ii) the constraint "$f(u)|_{x=0} \leqslant F$" is satisfied pointwise :

$$f((\gamma^l u)(t)) = f((\gamma^r u)(t)) \leqslant F(t) \quad \text{for a.e. } t > 0,$$

where $\gamma^{l,r}$ are the operators of left- and right-side strong traces on $\{x = 0\}$.

The Colombo-Goatin definition

Definition (R. Colombo and P. Goatin)

A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of the constrained SCL if

(i) for all nonnegative test function $\varphi \in \mathcal{C}_c^\infty(\Pi)$ and all $\kappa \in [0,1]$,

$$\int_0^{+\infty} \int_{\mathbb{R}} \big(|u(t,x) - \kappa| \partial_t + \Phi(u(t,x),\kappa)\partial_x \big) \varphi(t,x)\, dx\, dt$$
$$+ \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx + 2 \int_0^{+\infty} \big(1 - F(t)/f(\bar{u}) \big) f(\kappa)\, \varphi(t,0)\, dt \geqslant 0;$$

(ii) the constraint "$f(u)|_{x=0} \leqslant F$" is satisfied pointwise :

$$f((\gamma^l u)(t)) = f((\gamma^r u)(t)) \leqslant F(t) \quad \text{for a.e. } t > 0,$$

where $\gamma^{l,r}$ are the operators of left- and right-side strong traces on $\{x = 0\}$.

Theorem (Adapted from Panov; cf. Vasseur)

Let f be a continuous, non constant on any nontrivial interval of $[0,1]$.
Let u be an entropy solution of $\partial_t u + \partial_x f(u) = 0$ in $(0,+\infty) \times (0,+\infty)$.
Then there exists a strong trace $\gamma^r u$ on the boundary $\{x = 0\}$, in the sense

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^{+\infty} \int_0^h |u(t,x) - (\gamma^r u)(t)|\, \xi(t)\, dxdt = 0 \quad \forall \xi \in \mathcal{C}_c^\infty([0,+\infty)).$$

The Colombo-Goatin definition (cont$^{\mathrm{d}}$)

Rq. The definition is based upon the approximation of our pb. by

$$\partial_t u^\varepsilon(t,x) + \partial_x(k^\varepsilon(t,x) f(u^\varepsilon)) = 0, \quad k^\varepsilon(t,x) := \begin{cases} 1 & \text{if } |x| \geqslant \varepsilon, \\ F(t)/f(\bar{u}) & \text{if } |x| < \varepsilon. \end{cases}$$

This kind of problems (SCL with discontinuous flux) is well studied by now, and the adequate definition of entropy solution passes to the limit $\varepsilon \to 0$.

The Colombo-Goatin definition (cont^{d})

Rq. The definition is based upon the approximation of our pb. by

$$\partial_t u^\varepsilon(t,x) + \partial_x(k^\varepsilon(t,x) f(u^\varepsilon)) = 0, \quad k^\varepsilon(t,x) := \begin{cases} 1 & \text{if } |x| \geqslant \varepsilon, \\ F(t)/f(\bar{u}) & \text{if } |x| < \varepsilon. \end{cases}$$

This kind of problems (SCL with discontinuous flux) is well studied by now, and the adequate definition of entropy solution passes to the limit $\varepsilon \to 0$.

This notion leads to well-posedness in BV:

Theorem (R. Colombo and P. Goatin)

Assume that $u_0 \in BV(\mathbb{R}, [0,1])$ and $F \in BV(\mathbb{R}^+, [0, f(\bar{u})])$. Then there exists one and only one entropy solution $u \in BV(\mathbb{R}^+ \times \mathbb{R})$ to Constrained SCL . Moreover, given two initial data $u_0, v_0 \in BV(\mathbb{R}, [0,1])$ such that $(u_0 - v_0) \in \mathbf{L}^1(\mathbb{R})$, the corresponding entropy solutions u, v satisfy the following $\mathbf{L}^1$-contraction property : $\displaystyle\int_{\mathbb{R}} |u - v|(t) \leqslant \int_{\mathbb{R}} |u_0 - v_0|$.

The existence proof is by convergence of the wave-front tracking algorithm; the algorithm is based on the special Riemann solver at $\{x = 0\}$, with the Kruzhkov Riemann solver used elsewhere.

The Colombo-Goatin definition (cont^{d})

It is not easy (and, in a more general context, not possible) to adapt the BV techniques to numerical schemes. Quite often, one needs the L^{∞} framework and, furthermore, the measure-valued (entropy process) solutions. Straightforward attempts to generate such a notion from the Colombo-Goatin definition failed.

The Colombo-Goatin definition (cont^{d})

It is not easy (and, in a more general context, not possible) to adapt the BV techniques to numerical schemes. Quite often, one needs the L^∞ framework and, furthermore, the measure-valued (entropy process) solutions. Straightforward attempts to generate such a notion from the Colombo-Goatin definition failed.

But a series of re-formulations leads to a notion of entropy-process solution.

The Colombo-Goatin definition (cont^{d})

It is not easy (and, in a more general context, not possible) to adapt the *BV* techniques to numerical schemes. Quite often, one needs the L^∞ framework and, furthermore, the measure-valued (entropy process) solutions. Straightforward attempts to generate such a notion from the Colombo-Goatin definition failed.

But a series of re-formulations leads to a notion of entropy-process solution.

The first step is to characterize the possible traces of entropy solutions:

Lemma

If u is an entropy solution or Constrained SCL, then

$$\text{for a.e. } t>0, \qquad \big(\,(\gamma^l u)(t)\,,\,(\gamma^r u)(t)\,\big) \in \mathscr{G}(F(t)).$$

Here $\mathscr{G}(F(t))$ is the “admissibility germ” defined previously.

Rq: the Lemma is just a rigorous statement of the fact that possible traces of all admissible solutions correspond to traces of all admissible elementary solutions.

$\mathscr{G}$-entropy solutions

Theorem/Definition (Three equivalent definitions)

The assertions **(A)**, **(B)** *and* **(C)** *below are equivalent:*

$\mathscr{G}$-entropy solutions

Theorem/Definition (Three equivalent definitions)

The assertions **(A)**, **(B)** *and* **(C)** *below are equivalent:*

(A) (A_1) *u is a Kruzhkov entropy solution for* $x < 0$ *and* $x > 0$ *, i.e., for all nonnegative test functions* $\varphi \in \mathcal{C}_c^\infty(\Pi \backslash \{x=0\})$ *and all* $\kappa \in [0,1]$,

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}} \big(\, |u(t,x)-\kappa|\, \partial_t + \Phi(u(t,x),\kappa)\, \partial_x \,\big)\, \varphi(t,x)\, dx\, dt + \int_{\mathbb{R}} |u_0(x)-\kappa|\, \varphi(0,x)\, dx \geqslant 0;$$

(A_2) *in addition, for a.e.* $t>0$, $\big(\, (\gamma^l u)(t)\,,\, (\gamma^r u)(t)\,\big) \in \mathscr{G}(F(t))$.

$\mathcal{G}$-entropy solutions

Theorem/Definition (Three equivalent definitions)

The assertions **(A)**, **(B)** *and* **(C)** *below are equivalent:*

(A) (A_1) *u is a Kruzhkov entropy solution for* $x < 0$ *and* $x > 0$, *i.e., for all nonnegative test functions* $\varphi \in \mathcal{C}_c^\infty(\Pi \setminus \{x = 0\})$ *and all* $\kappa \in [0, 1]$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(|u(t,x) - \kappa|\, \partial_t + \Phi(u(t,x),\kappa)\, \partial_x \right) \varphi(t,x)\, dx\, dt + \int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx \geqslant 0;$$

(A_2) *in addition, for a.e.* $t > 0$, $\left((\gamma^l u)(t)\, ,\, (\gamma^r u)(t) \right) \in \mathcal{G}(F(t))$.

(B) (B_1) *u is a Kruzhkov entropy solution for* $x < 0$ *and* $x > 0$, *as above;*

(B_2) *u is a weak solution of the SCL*, *i.e., for all* $\varphi \in \mathcal{C}_c^\infty(\Pi)$, $\varphi(0, x) = 0$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(u(t,x)\, \partial_t + f(u(t,x))\, \partial_x \right) \varphi(t,x)\, dx\, dt = 0;$$

($\Longrightarrow$ *the Rankine-Hugoniot condition* $f(\gamma^l u) = f(\gamma^r u)$)

(B_3) *in addition, for a.e.* $t > 0$, *the interface* $\{x = 0\}$ *dissipates, i.e.*

$$\forall (c_l, c_r) \in \mathcal{G}(F(t)) \quad \Phi((\gamma^l u)(t), c_l) \geqslant \Phi((\gamma^r u)(t), c_r).$$

$\mathscr{G}$-entropy solutions (cont^{d})

Theorem/Definition (continued) (Three equivalent definitions)

(C) *u satisfies the following "global" entropy inequalities:*

(C_1) *there exists* $M > 0$ *such that for all* $(c_l, c_r) \in [0,1]^2$ *and all nonnegative test functions* $\varphi \in \mathcal{C}_c^\infty(\Pi)$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \big(|u(t,x) - c(x)|\, \partial_t + \Phi(u(t,x), c(x))\, \partial_x \big)\, \varphi(t,x)\, dx\, dt + \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx \geqslant -M \int_0^{+\infty} \mathrm{dist}\,\big((c_l, c_r)\,,\, \mathcal{G}(F(t)) \big)\, \varphi(t,0)\, dt,$$

where $c(x)$ *is the piecewise constant function given by*

$$c(x) := c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}} \equiv \begin{cases} c_l & \text{if } x < 0, \\ c_r & \text{if } x > 0, \end{cases}$$

and dist *refers to a distance function on* $\mathbb{R}^2$.

Rq: formulation **(C)** generalizes to measure-valued solutions.

$\mathcal{G}$-entropy solutions (cont^{d})

Rq: In the case $F(t) \equiv F = const$, requiring the global entropy inequalities

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}} \big(|u(t,x)-c(x)|\,\partial_t + \Phi(u(t,x),c(x))\,\partial_x \big)\,\varphi(t,x)\,dx\,dt + \int_{\mathbb{R}} |u_0(x)-c(x)|\,\varphi(0,x)\,dx \geqslant -M \int_0^{+\infty} \mathrm{dist}\,\big((c_l,c_r)\,,\,\mathcal{G}(F(t))\,\big)\,\varphi(t,0)\,dt,$$

of **(C)** for all φ and all (c_l,c_r) is equivalent to require them (cf. Carrillo)

$$\begin{cases} \text{only for } \varphi\in\mathcal{C}_c^\infty(\mathbb{R}^+\times\{x\neq 0\}) \text{ with any } (c_l,c_r)\in[0,1]^2, \\ \text{for all } \varphi\in\mathcal{C}_c^\infty(\mathbb{R}^+\times\mathbb{R}) \text{ with only } (c_l,c_r)\in\mathcal{G}(F). \end{cases}$$

Notice that in both cases, the remainder term vanishes; we get

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}} \big(|u(t,x)-c(x)|\,\partial_t + \Phi(u(t,x),c(x))\,\partial_x \big)\,\varphi(t,x)\,dx\,dt + \int_{\mathbb{R}} |u_0(x)-c(x)|\,\varphi(0,x)\,dx \geqslant 0,$$

which is merely a set of Kato inequalities !

$\mathscr{G}$-entropy solutions (cont^{d})

Definition ($\mathscr{G}$-entropy solution)

If any of the properties **(A)**, **(B)** or **(C)** holds, u is called a $\mathcal{G}$-entropy solution.

As expected from the construction, we have

Theorem

A function u is a $\mathscr{G}$-entropy solution of the Constrained SCL if and only if it is an entropy solution in the sense of Colombo and Goatin.

$\mathcal{G}$-entropy solutions (cont[d])

Definition ($\mathcal{G}$-entropy solution)

If any of the properties **(A)**, **(B)** or **(C)** holds, u is called a $\mathcal{G}$-entropy solution.

As expected from the construction, we have

Theorem

A function u is a $\mathcal{G}$-entropy solution of the Constrained SCL if and only if it is an entropy solution in the sense of Colombo and Goatin.

All the proofs are based:
– on localization/splitting techniques (zoom on $\{x = 0\}$, use of traces);
– on the two following properties of the "admissibility germ" $\mathcal{G}(F)$:

Lemma

(i) ("dissipativity" of $\mathcal{G}(F)$) If $(b_l, b_r) \in \mathcal{G}(F)$, then

$$\forall (c_l, c_r) \in \mathcal{G}(F), \quad \Phi(b_l, c_l) \geqslant \Phi(b_r, c_r). \qquad (*)$$

(ii) ("maximality" of $\mathcal{G}(F)$) The converse is true, in the sense that

if $(*)$ holds and the Rankine-Hugoniot condition $f(b_l) = f(b_r)$ is satisfied, then $(b_l, b_r) \in \mathcal{G}(F)$.

SCL WITH DISCONTINUOUS FLUX

A general framework

The above formulations are inspired by a number of previous works on the model "discontinuous flux" SCL :

$$u_t + (\mathfrak{f}(x,u))_x = 0, \qquad \mathfrak{f}(x,u) = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}.$$

First formulations were given by Gimse and Risebro and Diehl ;
the most successful formulation is due to Towers, Karlsen and Risebro .
Other formulations were given: Seguin-Vovelle, Bachmann, Jimenez... .

All these authors were thinking of solutions attainable by vanishing viscosity approximations.

A general framework

The above formulations are inspired by a number of previous works on the model "discontinuous flux" SCL :

$$u_t + (\mathfrak{f}(x,u))_x = 0, \qquad \mathfrak{f}(x,u) = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}.$$

First formulations were given by Gimse and Risebro and Diehl ;
the most successful formulation is due to Towers, Karlsen and Risebro .
Other formulations were given: Seguin-Vovelle, Bachmann, Jimenez... .

All these authors were thinking of solutions attainable by vanishing viscosity approximations.

Then Baiti and Jenssen and Audusse and Perthame suggested to use the "adapted entropies" (e.g., $\eta(u,x) := |u - c(x)|$, $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$).
Their formulation excludes some constant weak solutions !

A general framework

The above formulations are inspired by a number of previous works on the model "discontinuous flux" SCL :

$$u_t + (\mathfrak{f}(x,u))_x = 0, \qquad \mathfrak{f}(x,u) = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}.$$

First formulations were given by Gimse and Risebro and Diehl ;
the most successful formulation is due to Towers, Karlsen and Risebro .
Other formulations were given: Seguin-Vovelle, Bachmann, Jimenez... .

All these authors were thinking of solutions attainable by vanishing viscosity approximations.

Then Baiti and Jenssen and Audusse and Perthame suggested to use the "adapted entropies" (e.g., $\eta(u,x) := |u - c(x)|$, $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$).
Their formulation excludes some constant weak solutions !

Adimurty, Mishra and Gowda discovered that there can be infinitely many L^1-contractive semigroups of solutions for SCL with discontinuous bell-shaped flux .

A general framework

The above formulations are inspired by a number of previous works on the model "discontinuous flux" SCL :

$$u_t + (\mathfrak{f}(x,u))_x = 0, \qquad \mathfrak{f}(x,u) = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}.$$

First formulations were given by Gimse and Risebro and Diehl ;
the most successful formulation is due to Towers, Karlsen and Risebro .
Other formulations were given: Seguin-Vovelle, Bachmann, Jimenez... .

All these authors were thinking of solutions attainable by vanishing viscosity approximations.

Then Baiti and Jenssen and Audusse and Perthame suggested to use the "adapted entropies" (e.g., $\eta(u,x) := |u - c(x)|$, $c(x) = c_l 1\!\!1_{\{x<0\}} + c_r 1\!\!1_{\{x>0\}}$).
Their formulation excludes some constant weak solutions !

Adimurty, Mishra and Gowda discovered that there can be infinitely many L^1-contractive semigroups of solutions for SCL with discontinuous bell-shaped flux .
They introduced the notion of (A,B)-connection (cf. Bürger,Karlsen,Towers).

(A,B)-connection is a selected couple of traces ; its peculiarity is that it pre-determines the whole set $\mathscr{G}$ of admissible traces .
For the bell-shaped flux f, each connection gives one notion of solution.

A general framework (cont^{d})

For us, $f^l = f^r = f$; and the interesting connection is $(A, B) = (\hat{u}_F, \check{u}_F)$ (the non-Kruzhkov shock !).

The whole theory presented above is an adaptation of the general theory of L^1-contractive semigroups of solutions for the discont. flux SCL (A., Karlsen and Risebro).

The idea is,

admissibility is fully defined by a "dissipative" and "maximal" germ $\mathscr{G}$.

A general framework (cont[d])

For us, $f^l = f^r = f$; and the interesting connection is $(A, B) = (\hat{u}_F, \check{u}_F)$ (the non-Kruzhkov shock !).

The whole theory presented above is an adaptation of the general theory of L^1-contractive semigroups of solutions for the discont. flux SCL (A., Karlsen and Risebro).

The idea is,

admissibility is fully defined by a "dissipative" and "maximal" germ $\mathscr{G}$.

In fact, admissibility is fully defined by a germ $\mathscr{G}$ which has one and only one "dissipative" and "maximal" extension .

For the Constrained flux problem, the connection $(\hat{u}_F, \check{u}_F)$ corresponding to the constraint level F fully defines the notion of $\mathscr{G}(F)$-entropy solution.

A general framework (cont^d)

For us, $f^l = f^r = f$; and the interesting connection is $(A, B) = (\hat{u}_F, \check{u}_F)$ (the non-Kruzhkov shock !).

The whole theory presented above is an adaptation of the general theory of L^1-contractive semigroups of solutions for the discont. flux SCL (A., Karlsen and Risebro).

The idea is,

admissibility is fully defined by a "dissipative" and "maximal" germ $\mathscr{G}$.

In fact, admissibility is fully defined by a germ $\mathscr{G}$ which has one and only one "dissipative" and "maximal" extension .

For the Constrained flux problem, the connection $(\hat{u}_F, \check{u}_F)$ corresponding to the constraint level F fully defines the notion of $\mathscr{G}(F)$-entropy solution.

Rq. We see that for all bell-shaped flux f, there exist infinitely many non-Kruzhkov L^1-contractive semigroups of solutions to the standard scalar conservation law (one per level F).
(cf. the "kinetic relations" of LeFloch and al. , for monotone fluxes with one inflexion point)

WELL-POSEDNESS

Existence, uniqueness, L^1 contraction for $\mathscr{G}$-entropy solutions

Lemma

Assume that $F^1, F^2 \in \mathbf{L}^\infty$, and $u_0, v_0 \in \mathbf{L}^\infty$ such that $(u_0 - v_0) \in L^1(\mathbb{R})$. Assume that u, v are entropy solutions of Constrained SCL corresponding to the initial data u_0, v_0 and to the constraints F^1, F^2, respectively. Then, for a.e. $T > 0$,

$$\int_{\mathbb{R}} |u - v|(T, x)\, dx \leqslant 2 \int_0^T |F^1 - F^2|(t)\, dt + \int_{\mathbb{R}} |u_0 - v_0|(x)\, dx.$$

Proof: by combination of the Kato inequality and the comparison of $\mathscr{G}(F_1)$,$\mathscr{G}(F_2)$.

Existence, uniqueness, L^1 contraction for $\mathscr{G}$-entropy solutions

Lemma

Assume that $F^1, F^2 \in \mathbf{L}^\infty$, and $u_0, v_0 \in \mathbf{L}^\infty$ such that $(u_0 - v_0) \in L^1(\mathbb{R})$. Assume that u, v are entropy solutions of Constrained SCL corresponding to the initial data u_0, v_0 and to the constraints F^1, F^2, respectively. Then, for a.e. $T > 0$,

$$\int_{\mathbb{R}} |u - v|(T, x)\, dx \leqslant 2 \int_0^T |F^1 - F^2|(t)\, dt \, + \int_{\mathbb{R}} |u_0 - v_0|(x)\, dx.$$

Proof: by combination of the Kato inequality and the comparison of $\mathscr{G}(F_1)$,$\mathscr{G}(F_2)$.

Theorem

For any $u_0 \in \mathbf{L}^\infty$ and $F \in \mathbf{L}^\infty$ there exists one and only one entropy solution.

Proof: The uniqueness claim is contained in the above Lemma.

To prove the existence, we truncate and regularize the data:

$$u_n^0 \in BV,\ F_n \in BV,\quad u_0^n \to u_0 \text{ in } L^1_{loc}(\mathbb{R}) \text{ and a.e.};\quad F^n \to F \text{ in } L^1_{loc}(\mathbb{R}^+) \text{ and a.e.}.$$

Solutions u_n exist, by the BV result. From the previous Lemma, we infer that the sequence $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1_{loc}(\Pi)$. Further, for all $(c_l, c_r) \in [0,1]^2$,

$$\text{for a.e. } t > 0,\ \operatorname{dist}\big((c_l, c_r), \mathcal{G}(F^n(t))\big) \longrightarrow \operatorname{dist}\big((c_l, c_r), \mathcal{G}(F(t))\big) \text{ as } n \to +\infty.$$

Thus the global entropy formulation passes to the limit.

Finite Volume Scheme

The Scheme

A FV scheme for SCL is determined by a mesh and a numerical flux.

The Scheme

A FV scheme for SCL is determined by a mesh and a numerical flux.

For the mesh, the things are standard:

Definition

A mesh $\mathscr{T}$ of $\mathbb{R}$ is given by an increasing sequence of real values $(x_{i+1/2})_{i\in\mathbb{Z}}$ (thus $\cup_{i\in\mathbb{Z}}[x_{i-1/2}, x_{i+1/2}]$ is a partition of $\mathbb{R}$). We fix $x_{1/2} = 0$.

The space step is $h_i = x_{i+1/2} - x_{i-1/2}$, and $h = \text{size}(\mathscr{T}) = \sup_{i\in\mathbb{Z}} h_i$.

The Scheme

A FV scheme for SCL is determined by a mesh and a numerical flux.

For the mesh, the things are standard:

Definition

A mesh $\mathscr{T}$ of $\mathbb{R}$ is given by an increasing sequence of real values $(x_{i+1/2})_{i\in\mathbb{Z}}$ (thus $\cup_{i\in\mathbb{Z}}[x_{i-1/2}, x_{i+1/2}]$ is a partition of $\mathbb{R}$). We fix $x_{1/2} = 0$.

The space step is $h_i = x_{i+1/2} - x_{i-1/2}$, and $h = \text{size}(\mathscr{T}) = \sup_{i\in\mathbb{Z}} h_i$.

The numerical flux g is defined by

$$g(u, v, F) = \min(h(u, v), F),$$

where h is a classical numerical flux (see Eymard, Gallouët and Herbin), i.e. it obeys the three following properties:

- Regularity: h is Lipschitz continuous, with L as Lipschitz constant.
- Consistency: $h(s, s) = f(s)$ for any $s \in [0, 1]$.
- Monotonicity: h is nondecreasing with respect to (w.r.t.) its first argument and nonincreasing w.r.t. its second argument.

The Scheme (cont$^{\mathrm{d}}$). Estimates.

The scheme is explicit in time, written under the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}\left(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\right)$$

with $F_{1/2}^n := \dfrac{1}{\Delta t}\displaystyle\int_{n\Delta t}^{(n+1)\Delta t} F(s)ds$ (taking into account the constraint at $\{x=0\}$) and $F_{1/2}^n = f(\bar{u})$ otherwise (no constraint elsewhere).

The Scheme (cont[d]). Estimates.

The scheme is explicit in time, written under the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}\left(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\right)$$

with $F_{1/2}^n := \dfrac{1}{\Delta t}\displaystyle\int_{n\Delta t}^{(n+1)\Delta t} F(s)ds$ (taking into account the constraint at $\{x=0\}$) and $F_{1/2}^n = f(\bar{u})$ otherwise (no constraint elsewhere).

Under the CFL condition, the scheme enjoys the standard FV estimates :

- the $\mathbf{L}^\infty$ estimate (more exactly, the confinement within $[0,1]$)
- the "weak BV" estimate (see Eymard, Gallouët, Herbin)
- "discrete contraction" between two discrete solutions.

The Scheme (cont^d). Estimates.

The scheme is explicit in time, written under the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}\big(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\big)$$

with $F_{1/2}^n := \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} F(s)ds$ (taking into account the constraint at $\{x=0\}$) and $F_{1/2}^n = f(\bar{u})$ otherwise (no constraint elsewhere).

Under the CFL condition, the scheme enjoys the standard FV estimates :

- the $\mathbf{L}^\infty$ estimate (more exactly, the confinement within $[0,1]$)
- the "weak BV" estimate (see Eymard, Gallouët, Herbin)
- "discrete contraction" between two discrete solutions.

The scheme preserves the two (out of three) kinds of elementary solutions :

– the non-Kruzhkov shock (i.e., the part $\mathscr{G}_1(F)$ of the germ $\mathscr{G}(F)$) is preserved, by construction

– the constant solutions (i.e., the part $\mathscr{G}_2(F)$ of the germ $\mathscr{G}(F)$) are preserved by any admissible scheme.

NB: The Godunov scheme (the one associated with the Colombo and Goatin Riemann solver) also preserves $\mathscr{G}_3(F)$, because it preserves all stationary solutions. Yet this property is restrictive, and we bypassed its use .

Towards entropy-process solution

$\Longrightarrow$ discrete entropy inequalities with $(c_l, c_r) \in \mathcal{G}_1(F) \cup \mathcal{G}_2(F)$!

Towards entropy-process solution

$\Longrightarrow$ discrete entropy inequalities with $(c_l, c_r) \in \mathcal{G}_1(F) \cup \mathcal{G}_2(F)$!

$\Longrightarrow$ continuous "global entropy inequalities" with $(c_l, c_r) \in \mathcal{G}_1(F) \cup \mathcal{G}_2(F)$ and with a remainder term controlled by the weak BV estimate :

$$\int_0^{+\infty} \int_{\mathbb{R}} \Big(|u_h(t,x) - c(x)|\, \partial_t + \Phi(u_h(t,x), c(x)) \partial_x \Big)\, \varphi(t,x)\, dx\, dt + \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx \geqslant -M \int_0^{+\infty} \mathrm{dist}\,\Big((c_l, c_r)\,,\, \mathcal{G}_1(F(t)) \cup \mathcal{G}_2(F(t)) \Big)\, \varphi(t,0)\, dt + \overline{\overline{o}}_h,$$

Towards entropy-process solution

$\Longrightarrow$ discrete entropy inequalities with $(c_l, c_r) \in \mathscr{G}_1(F) \cup \mathscr{G}_2(F)$!

$\Longrightarrow$ continuous "global entropy inequalities" with $(c_l, c_r) \in \mathscr{G}_1(F) \cup \mathscr{G}_2(F)$ and with a remainder term controlled by the weak BV estimate :

$$\int_0^{+\infty} \int_{\mathbb{R}} \big(|u_h(t,x) - c(x)|\, \partial_t + \Phi(u_h(t,x), c(x))\partial_x \big)\, \varphi(t,x)\, dx\, dt$$
$$+ \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx$$
$$\geqslant -M \int_0^{+\infty} \mathrm{dist}\,\big((c_l, c_r)\,,\, \mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t)) \big)\, \varphi(t,0)\, dt + \overline{\overline{o}}_h,$$

The $\mathbf{L}^\infty$ estimate is enough for "nonlinear weak-* compactness" of the approximations: if u_h is the discrete solution, then $u_h(t,x) \to \mu(t,x;\alpha)$, in the sense

$$\int_\Omega G(u_h(y))\varphi(y)\, dy \xrightarrow[h\to 0]{} \int_\Omega \int_0^1 G(\mu(y,\alpha))\varphi(y)\, dy\, d\alpha$$

(up to a subsequence) for all $\varphi \in L^1(\Omega)$ and for all $G \in \mathcal{C}(\mathbb{R})$.

Towards entropy-process solution

$\Longrightarrow$ discrete entropy inequalities with $(c_l, c_r) \in \mathscr{G}_1(F) \cup \mathscr{G}_2(F)$!

$\Longrightarrow$ continuous "global entropy inequalities" with $(c_l, c_r) \in \mathscr{G}_1(F) \cup \mathscr{G}_2(F)$ and with a remainder term controlled by the weak BV estimate :

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(|u_h(t,x) - c(x)|\, \partial_t + \Phi(u_h(t,x), c(x)) \partial_x \right) \varphi(t,x)\, dx\, dt$$

$$+ \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx$$

$$\geqslant -M \int_0^{+\infty} \mathrm{dist}\left((c_l, c_r)\, ,\, \mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t)) \right) \varphi(t,0)\, dt + \overline{\overline{o}}_h,$$

The $\mathbf{L}^\infty$ estimate is enough for "nonlinear weak-* compactness" of the approximations: if u_h is the discrete solution, then $u_h(t,x) \to \mu(t,x;\alpha)$, in the sense

$$\int_\Omega G(u_h(y))\varphi(y)\, dy \underset{h\to 0}{\longrightarrow} \int_\Omega \int_0^1 G(\mu(y,\alpha))\varphi(y)\, dy\, d\alpha$$

(up to a subsequence) for all $\varphi \in L^1(\Omega)$ and for all $G \in \mathcal{C}(\mathbb{R})$.

In fact, we'll show that $\mu(t,x,\alpha) \equiv u(t,x)$, that the convergence of u_h to u is pointwise a.e., and that u is a $\mathscr{G}(F)$-entropy solution of our problem.

Entropy-process formulation

Thus passing to the limit, we get the following “entropy process formulation”:

$$\int_0^1 \int_0^{+\infty} \int_{\mathbb{R}} \left(|\mu(t,x,\alpha) - c(x)|\, \partial_t + \Phi(\mu(t,x,\alpha), c(x))\partial_x \right) \varphi(t,x)\, dx\, dt\, d\alpha$$
$$+ \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx$$
$$\geqslant -M \int_0^{+\infty} \mathrm{dist}\left((c_l, c_r)\,,\, \mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t)) \right) \varphi(t,0)\, dt,$$

Then by going into finer details, we deduce that $\mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t))$ can be replaced with $\mathscr{G}(F(t))$ in the above formulation.

Entropy-process formulation

Thus passing to the limit, we get the following “entropy process formulation”:

$$\int_0^1 \int_0^{+\infty} \int_{\mathbb{R}} \big(|\mu(t,x,\alpha) - c(x)|\, \partial_t + \Phi(\mu(t,x,\alpha), c(x))\partial_x \big)\, \varphi(t,x)\, dx\, dt\, d\alpha$$
$$+ \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx$$
$$\geqslant -M \int_0^{+\infty} \operatorname{dist}\big((c_l, c_r)\, ,\, \mathcal{G}_1(F(t)) \cup \mathcal{G}_2(F(t)) \big)\, \varphi(t,0)\, dt,$$

Then by going into finer details, we deduce that
$\mathcal{G}_1(F(t)) \cup \mathcal{G}_2(F(t))$ can be replaced with $\mathcal{G}(F(t))$ in the above formulation.

$\Longrightarrow$ the limit u satisfies the “entropy-process version”
of the global entropy inequality that defines $\mathcal{G}$-entropy solutions

Entropy-process formulation

Thus passing to the limit, we get the following "entropy process formulation":

$$\int_0^1 \int_0^{+\infty} \int_{\mathbb{R}} \big(|\mu(t,x,\alpha) - c(x)|\, \partial_t + \Phi(\mu(t,x,\alpha), c(x))\partial_x \big)\, \varphi(t,x)\, dx\, dt\, d\alpha$$
$$+ \int_{\mathbb{R}} |u_0(x) - c(x)|\, \varphi(0,x)\, dx$$
$$\geqslant -M \int_0^{+\infty} \mathrm{dist}\,\big((c_l, c_r)\,,\, \mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t)) \big)\, \varphi(t,0)\, dt,$$

Then by going into finer details, we deduce that
$\mathscr{G}_1(F(t)) \cup \mathscr{G}_2(F(t))$ can be replaced with $\mathscr{G}(F(t))$ in the above formulation.
$\Longrightarrow$ the limit u satisfies the "entropy-process version"
of the global entropy inequality that defines $\mathscr{G}$-entropy solutions

It remains to show, "as usual", that the entropy-process solution $\mu(t,x,\alpha)$ is unique and independent of α... and here we get a not-so-nice surprise.

Uniqueness of an entropy-process solution ?

Let us look at the interface dissipation property for entropy-process solutions.

For $\mathscr{G}$-entropy solutions, we had (see Def.(B)) the inequalities

$$\forall (c_l, c_r) \in \mathscr{G}(F) \quad \Phi(\gamma^l u, c_l) \geqslant \Phi(\gamma^r u, c_r).$$

Because for any entropy solution v, we have $(\gamma^l v, \gamma^r v) \in \mathscr{G}(F)$,
this gives a sign to the interface term $\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v)$.

Uniqueness of an entropy-process solution ?

Let us look at the interface dissipation property for entropy-process solutions.

For $\mathscr{G}$-entropy solutions, we had (see Def.(B)) the inequalities

$$\forall (c_l, c_r) \in \mathscr{G}(F) \quad \Phi(\gamma^l u, c_l) \geqslant \Phi(\gamma^r u, c_r).$$

Because for any entropy solution v, we have $(\gamma^l v, \gamma^r v) \in \mathscr{G}(F)$, this gives a sign to the interface term $\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v)$.

Now for $\mathscr{G}$-entropy process solutions, we have the inequalities

$$\forall (c_l, c_r) \in \mathcal{G}(F) \quad \gamma^l_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), c_l)\, d\alpha \Big] \geqslant \gamma^r_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), c_r)\, d\alpha \Big];$$

here $\gamma_w^{l,r}$ are the operators of <u>weak</u> trace (Chen and Frid, Otto...).

Indeed, only weak boundary traces of the integrated $\int_0^1 \dots d\alpha$ quantities are easily available for entropy-process solutions.

Difficult to compare two entropy-process solutions !

Uniqueness of an entropy-process solution ?

Let us look at the interface dissipation property for entropy-process solutions.

For $\mathscr{G}$-entropy solutions, we had (see Def.(B)) the inequalities

$$\forall (c_l, c_r) \in \mathscr{G}(F) \quad \Phi(\gamma^l u, c_l) \geqslant \Phi(\gamma^r u, c_r).$$

Because for any entropy solution v, we have $(\gamma^l v, \gamma^r v) \in \mathscr{G}(F)$, this gives a sign to the interface term $\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v)$.

Now for $\mathscr{G}$-entropy process solutions, we have the inequalities

$$\forall (c_l, c_r) \in \mathcal{G}(F) \quad \gamma^l_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), c_l)\, d\alpha \Big] \geqslant \gamma^r_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), c_r)\, d\alpha \Big];$$

here $\gamma^{l,r}_w$ are the operators of <u>weak</u> trace (Chen and Frid, Otto...).

Indeed, only weak boundary traces of the integrated $\int_0^1 \ldots d\alpha$ quantities are easily available for entropy-process solutions.

Difficult to compare two entropy-process solutions !

But there is a way out: we still can compare an entropy-process solution μ with an entropy (non process !) solution v. The hint is here:

$$\gamma^{l,r}_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), v(\cdot)) d\alpha \Big] = \gamma^{l,r}_w \Big[\int_0^1 \Phi(\mu(\cdot,\alpha), \gamma^{l,r} v) d\alpha \Big],$$

thanks to the fact that the trace of v is strong !

Conclusion.

Thus in order to conclude the proof, we need the existence of a $\mathscr{G}$-entropy solution. Fortunately, we already have this existence result (thanks to the wave-front tracking algorithm and the BV estimates for this algorithm).

Conclusion.

Thus in order to conclude the proof, we need the existence of a $\mathscr{G}$-entropy solution. Fortunately, we already have this existence result (thanks to the wave-front tracking algorithm and the BV estimates for this algorithm).

Conclusions:

- The method can be applied in the general setting of conservation laws with discontinuous flux (we are able to describe different notions of solution that enjoy the L^1-contractivity).
- Yet the finite volume scheme should be adapted to each case; our investigation of the Constrained SCL problem has shown that, except for the Godunov scheme, the convergence proof can be delicate !
- The existence of strong traces of entropy solutions is a difficult result. But it simplifies very much the formulation and study of boundary-value and interface problems.

Oufff !!!

MERCI !