Strong boundary traces and well-posedness for scalar conservation laws with dissipative boundary conditions

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1 Introduction

The aim of this paper is to give sense to the following formal problem for a scalar conservation law with boundary condition (BC, in the sequel) :

$$(H)\begin{cases} u_t + \operatorname{div}\, \varphi(u) = f & \text{in } \; Q := (0,T)\times\Omega \\ u(0,\cdot) = u_0 & \text{on } \; \Omega \\ \varphi_\nu(u) := \varphi(u)\cdot\nu \in \beta(u) & \text{on } \; \Sigma := (0,T)\times\partial\Omega. \end{cases}$$

Here $\Omega = \mathbb{R}^+ \times \mathbb{R}^{N-1}$ $(N \geq 1)$, $T > 0$, ν is the unit outward normal vector on $\partial\Omega$, $\varphi : \mathbb{R} \to \mathbb{R}^N$ is continuous, and β is a maximal monotone graph on $\mathbb{R}$. Assume $\varphi_\nu(0) - \beta(0) \ni 0$ and normalize φ, β by $\varphi(0) = 0$, $0 \in \beta(0)$. The classical Neumann (zero-flux) and Dirichlet homogeneous BC correspond to the graphs $\beta = \mathbb{R}\times\{0\}$ and $\beta = \{0\}\times\mathbb{R}$, respectively.

To show existence, we restrict our attention to the case of Ω with flat boundary, to $L^1 \cap L^\infty$ data u_0, f (see (12) for the precise assumption on the data), and make the following simplifying assumptions:

$$\text{there exists a constant } C \text{ such that } |\beta(z)| \geq \operatorname{sign}(z)\,\varphi_\nu(z) \quad \forall |z| > C; \tag{1}$$

$$\varphi \text{ is Lipshitz continuous, and } \varphi_\nu = \varphi\cdot\nu \text{ is piecewise monotone;} \tag{2}$$

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \text{ the function } z \mapsto \xi\cdot\varphi(z) \text{ is non-constant on any interval.} \tag{3}$$

In the Conclusion, possible generalizations are indicated.

It is well known that existence for (H) generally fails if one interprets the Dirichlet BC literally (cf. [BLN]). This is also the case for general β, except for some particular situations (cf. e.g. [BFK]). If one approximates (H) by a sequence of problems (H^ϵ) with the BC understood literally (e.g. parabolic "viscous" approximations, or numerical schemes), a boundary layer can form in the corresponding solutions u^ϵ. The convergence of u^ϵ then takes place only locally inside the domain, and the limiting function u satisfies to the scalar conservation law with a different BC, which we call "effective BC".

It is the goal of this paper to investigate the form of the effective BC for a general BC given by a maximal monotone graph β. Note that the monotonicity is necessary, if we hope the boundary condition to be dissipative in L^1 (in particular, if we hope for an L^1 contraction principle for solutions of (H), as in the classical situations of $\Omega = \mathbb{R}^N$, or of bounded Ω with Dirichlet BC). We suggest that the effective BC is given by a monotone graph $\tilde{\beta}$ defined by

$$\tilde{\beta} := \left\{ (z, \varphi_\nu(z)); \begin{array}{l} \exists b \in \overline{\mathrm{Range}(\beta)} \text{ such that } \varphi_\nu(z) = b \text{ and} \\ \text{if } z < m := \inf \beta^{-1}(b), \text{ then } \varphi_\nu(k) \geq b \ \forall k \in [z, m[\\ \text{if } z > M := \sup \beta^{-1}(b), \text{ then } \varphi_\nu(k) \leq b \ \forall k \in]M, z] \end{array} \right\}, \quad (4)$$

which can be visualized as the horizontal projection of β on the graph of φ_ν. The graph $\tilde{\beta}$ is monotone, and $\tilde{\tilde{\beta}} = \tilde{\beta}$ (thus operation $\tilde{\cdot}$ is indeed a projection).

Example 1. (i) If $\beta = \{0\} \times \mathbb{R}$, then $\tilde{\beta}$ is the Bardos-LeRoux-Nédélec graph:

$$\tilde{\beta} = \left\{ (z, \varphi_\nu(z)) \mid \mathrm{sign}(z)(\varphi_\nu(z) - \varphi_\nu(k)) \geq 0 \quad \forall k \in [0 \wedge z, 0 \vee z] \right\}. \qquad (5)$$

(ii) If $\beta = \mathbb{R} \times \{0\}$, then $\tilde{\beta} = \{(z, \varphi_\nu(z)) \mid \varphi_\nu(z) = 0\}$. Assumption (1) is restrictive in this case; a similar assumption is made in [BFK]. For the general case, one has to complete β to a maximal monotone graph on $[-\infty, +\infty]$ before defining $\tilde{\beta}$ by (4); see Conclusion and the forthcoming paper [AS].

2 Strong boundary traces for entropy solutions

The Bardos-LeRoux-Nédélec pointwise formulation of the Dirichlet BC, which can be expressed by means of the graph (5) was initially given for BV solutions (compare to the work of Otto in [MNRR], where a weak formulation of the Dirichlet boundary condition is given, valid for any L^∞ solution). It has recently been realized that any L^∞ entropy solution actually has strong L^1_{loc} initial and boundary traces in a fairly general situation (see Panov [P05, P06] and the previous works of Chen-Rascle and Vasseur). In particular, the non-degeneracy condition (3) is not needed. The concept of strong trace (in L^1_{loc}) is stated in the following definition. Set $x = (x_1, x'), x_1 \in \mathbb{R}^+, x' \in \mathbb{R}^{N-1}$.

Definition 1. *A function $\tilde{v} \in L^1_{loc}(\mathbb{R}^{N-1})$ is a strong trace of function $v \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^{N-1})$ on $\{x_1 = 0\}$ if for all $\xi \in C_c(\mathbb{R}^{N-1})$, $\xi \geq 0$*

$$ess\text{-}\lim_{x_1 \to 0} \int_{\mathbb{R}^{N-1}} \xi(x') |v(x_1, x') - \tilde{v}(x')| dx' = 0.$$

In the sequel $\gamma : L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^{N-1}) \to L^1_{loc}(\mathbb{R}^{N-1})$ is the strong trace operator in the sense of Definition 1. Traces of $L^1_{loc}(Q)$ functions on $\Sigma = (0,T) \times \mathbb{R}^{N-1}$ and on $\{t=0\}$ are defined similarly. Clearly, the strong trace operators are unbounded. The following proposition, inferred by the result of Panov [P05] on traces of entropy (quasi-)solutions at $\{t=0\}$, is therefore remarkable.

Proposition 1. *Assume (2). Let u be a quasi-solution for $u_t + \operatorname{div} \varphi(u)$[3]. Then there exists a strong trace $\gamma V_{\varphi_\nu}(u)$ of the function $V_{\varphi_\nu}(u)$ on Σ in the sense of Definition 1, where $V_{\varphi_\nu}(\cdot)$ is the variation function of φ_ν :*

$$V_{\varphi_\nu}(z) = \operatorname{sign}(z) \operatorname{Var}_{[0\wedge z, 0\vee z]} \varphi_\nu(\cdot).$$

In particular, $\operatorname{sign}^\pm(u-k)(\varphi(u)-\varphi(k)) \cdot \nu$, which can be rewritten as

$$Q^\pm(V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)) := \operatorname{sign}^\pm(V_{\varphi_\nu}(u) - V_{\varphi_\nu}(k))(\Psi_\nu(V_{\varphi_\nu}(u)) - \Psi_\nu(V_{\varphi_\nu}(k)))$$

with $\Psi_\nu := \varphi_\nu \circ V^{-1}_{\varphi_\nu}$, has the strong trace $Q^\pm(\gamma V_{\varphi_\nu}(u), V_{\varphi_\nu}(k))$ on Σ.

Notice that the above result does not depend on (1),(3). It still holds if Ω is a domain with piecewise C^1-smooth boundary and $\varphi \in BV$ (see Panov [P06]).

3 Entropy solutions and well-posedness

Definition 2. *A function $u \in L^\infty(Q)$ is said an entropy solution for Problem (H) if $\forall k \in \mathbb{R}$, $\forall \xi \in C^\infty_c(Q)$, $\xi \geq 0$ the local entropy inequalities hold :*

$$\int_Q (u-k)^\pm \xi_t + \int_Q \operatorname{sign}^\pm(u-k)(\varphi(u)-\varphi(k)) \cdot D\xi + \int_Q f \operatorname{sign}^\pm(u-k)\xi \geq 0, \quad (6)$$

u has the strong trace u_0 on $\{t=0\}$, and for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$ the strong traces $\tilde{w} = \gamma\varphi_\nu(u), \tilde{v} = \gamma V_{\varphi_\nu}(u)$ on Σ of the functions $\varphi_\nu(u), V_{\varphi_\nu}(u)$ verify

$$(\tilde{v}(t,x), \tilde{w}(t,x)) \in \tilde{\beta} \circ V^{-1}_{\varphi_\nu}. \quad (7)$$

Notice that if V_{φ_ν} is invertible (which is the case under assumption (3)), then requirement (7) is equivalent to the requirement that $(\tilde{u}(t,x), \tilde{w}(t,x)) \in \tilde{\beta}$, where $\tilde{u}$ is the strong trace of u on Σ, $\tilde{u} = V^{-1}_{\varphi_\nu}(\tilde{v})$.

Definition 2 can be reformulated so that to extend the entropy inequalities up to the boundaries Σ and $\{t=0\}$. Indeed, we have

Proposition 2. *A function $u \in L^\infty(Q)$ such that strong traces $\tilde{v} := \gamma V_{\varphi_\nu}(u)$, $\tilde{w} := \gamma\varphi_\nu(u)$ on Σ exist and satisfy (7) is an entropy solution for (H) if and only if it satisfies $\forall k \in \mathbb{R}$, $\forall \xi \in C^\infty_c([0,T) \times \mathbb{R}^N)$, $\xi \geq 0$:*

[3] A function $u \in L^\infty(Q)$ is called a quasi-solution if $\forall k \in \mathbb{R}$ $\eta^\pm_k(u)_t + \operatorname{div} q^\pm_k(u) = -\mu^\pm_k$ in $\mathcal{D}'(Q)$, where $(\eta^\pm_k, q^\pm_k)$ are the Kruzhkov entropy-flux pairs and $\mu^\pm_k$ are Borel measures on Q, locally finite up to the boundary; see [P05, P06].

$$0 \le \int_Q (u-k)^{\pm}\xi_t + \int_\Omega (u_0-k)^{\pm}\xi(0) + \int_Q \mathrm{sign}^{\pm}(u-k) f\xi$$
$$+ \int_Q \mathrm{sign}^{\pm}(u-k)(\varphi(u)-\varphi(k))\cdot D\xi - \int_\Sigma Q^{\pm}(\tilde{v}, V_{\varphi_\nu}(k))\xi. \qquad (8)$$

For the proof, one truncates $\xi \in C_c^\infty([0,T)\times\mathbb{R}^N)$ in a neighborhood of the boundaries, and passes to the limit using Definition 1.

Notice that both formulations (6),(7) and (8),(7) make sense for L^∞ (and even more general) data u_0, f, for general domain Ω with Lipschitz deformable boundary, and without assumptions (1)-(3). Uniqueness and comparison results of the next section remain valid in this general framework. For the existence part, we use a chain of approximations of (H) enjoying convenient compactness properties (this is the aim of our simplifying assumptions). We show that they converge to an entropy solution inside Q, then deduce the existence of strong traces $\tilde{v}$, $\tilde{w}$ by Proposition 1, and finally, we identify the couple $(\tilde{v},\tilde{w})$ as belonging to $\tilde{\beta}\circ V_{\varphi_\nu}^{-1}$.

3.1 Comparison and uniqueness of entropy solutions

Theorem 1 (The Kato inequality). *For $i=1,2$, let u_i be an entropy solution for Problem (H) with data $(u_0^i, f_i)\in L^1(\Omega)\times L^1(Q)$. Then for all $\xi\in C_c^\infty([0,T)\times\mathbb{R}^N), \xi \ge 0$*

$$\int_Q (u_1-u_2)^+\xi_t + \int_Q \mathrm{sign}^+(u_1-u_2)(\varphi(u_1)-\varphi(u_2))\cdot D\xi$$
$$+ \int_Q \mathrm{sign}^+(u_1-u_2)(f_1-f_2)\xi \ge \int_\Sigma (\tilde{w}_1-\tilde{w}_2)^+\xi. \qquad (9)$$

Proof. We use the Kruzhkov method of doubling of variables. As $u_1(t,x)$, resp., $u_2(s,y)$ is an entropy solution with data $u_0^1(x)$ and $f_1(t,x)$, resp., $u_0^2(y)$ and $f_2(s,y)$, then for all $\phi=\phi(t,x,s,y)\in C_c^\infty(Q\times Q)$ one has

$$\int_{Q\times Q} (u_1-u_2)^+(\phi_t+\phi_s) + \mathrm{sign}^+(u_1-u_2)(\varphi(u_1)-\varphi(u_2))\cdot (D_x\phi + D_y\phi)$$
$$+ \int_{Q\times Q} \mathrm{sign}^+(u_1-u_2)(f_1-f_2) \ge 0. \qquad (10)$$

Let $\xi\in C_c^\infty([0,T)\times\mathbb{R}^N), \xi\ge 0$ and ρ_n, resp., ρ_m, be a classical sequence of mollifiers in $\mathbb{R}^N$, resp., in $\mathbb{R}$. Define $\phi(t,x,s,y)=\mu_\delta(x)\mu_\eta(y)\rho_n(x-y)\rho_m(t-s)$. Using ϕ as a test function in (10) and passing to the limit with $\delta,\eta\to 0$ yields

$$0 \le \int_{Q\times Q} \rho_m\rho_n(u_1-u_2)^+\xi_t + \int_{Q\times Q} \rho_m\rho_n \mathrm{sign}^+(u_1-u_2)(\varphi(u_1)-\varphi(u_2))\cdot D_x\xi$$
$$+ \int_{Q\times Q} \mathrm{sign}^+(u_1-u_2)(f_1-f_2)\rho_m\rho_n\xi$$
$$- \int_{\Sigma\times Q} Q^+(\tilde{v}_1, V_{\varphi_\nu}(u_2))\rho_m\rho_n\xi - \int_{Q\times\Sigma} Q^+(V_{\varphi_\nu}(u_1),\tilde{v}_2)\rho_m\rho_n\xi. \qquad (11)$$

By Proposition 1, each of the last two terms converges to $\frac{1}{2}\int_\Sigma Q^+(\tilde v_1,\tilde v_2)$ as $n,m\to\infty$. Relations (7) for u_1,u_2 and the definitions of Q^+ and $\tilde\beta$ yield (10).

Corollary 1. *Assume (2). For data (u_0^i,f_i), $i=1,2$, satisfying*

$$\begin{cases} u_0\in(L^1\cap L^\infty)(\Omega),\ \ f\in L^1(Q)\ \ \textit{with}\\ f(t,\cdot)\in L^\infty(\Omega)\ \ \textit{a.e.}\ t\in(0,T),\ \ \int_0^T\|f(t,\cdot)\|_\infty dt<\infty, \end{cases} \tag{12}$$

let u_i an entropy solution for Problem (H). Then for all $t\in(0,T)$,

$$\int_0^t\int_{\partial\Omega}(\tilde w_1-\tilde w_2)^+ + \int_\Omega (u_1-u_2)^+(t)\le\int_\Omega(u_0^1-u_0^2)^+ + \int_0^T\int_\Omega(f_1-f_2)^+.$$

Thus if $u_0^1\le u_0^2$ a.e. on Ω and if $f_1\le f_2$ a.e. on Q, then $u_1\le u_2$ a.e. on Q. In particular, an entropy solution of (H) with data (12) is unique.

Proof. Take in (9), $\xi(t,x)=\xi_\alpha(x)\kappa(t)$, where $\kappa\in C_c^\infty([0,T)),\kappa\ge0$, $\xi_\alpha\to1$ in Ω, $|D\xi_\alpha|\le C$ and $\mathrm{Supp}(D\xi_\alpha)\subset\{x\,|\,\alpha<|x|<\alpha+1\}$. As $\alpha\to0$ the claim follows, because $u_i\in L^1(Q)$ under assumption (12), and φ is Lipschitz.

Notice that the comparison and uniqueness result of Corollary 1 holds true also in the L^∞ framework (see the results of Kruzhkov-Panov, Bénilan-Kruzhkov).

3.2 Existence of entropy solutions

We infer the existence of an entropy solution to Problem (H) using the tools of the nonlinear semigroup theory (cf. e.g. [BCP]). We first study the boundary problem in Ω associated with (H) (known as the "stationary" problem)

$$(S)(f)\begin{cases} u+\mathrm{div}\,\varphi(u)=f & \text{on}\ \ \Omega\\ \varphi(u)\cdot\nu\in\beta(u) & \text{on}\ \ \partial\Omega.\end{cases}$$

Definition 3. *A function $u\in L^\infty(\Omega)$ is an entropy solution of Problem $(S)(f)$ if $\forall k\in\mathbb{R},\forall\phi\in C_c^\infty(\Omega),\phi\ge0$ the local entropy inequalities*

$$\int_\Omega \mathrm{sign}^\pm(u-k)(\varphi(u)-\varphi(k))\cdot D\phi+\int_\Omega\mathrm{sign}^\pm(u-k)(f-u)\phi\ge0$$

hold, and traces $\tilde v=\gamma V_{\varphi_\nu}(u)$, $\tilde w=\gamma\varphi_\nu(u)$ verify (7) for $\mathcal{H}^{N-1}-$a.e. $x\in\partial\Omega$.

Notice that Definition 3 can be reformulated in the way of Proposition 2. Define the operator $\mathcal{A}$ associated with the problem $(S)(f)$ by its graph :

$$(u,f)\in\mathcal{A}\Leftrightarrow u \text{ is an entropy solution of } (S)(f+u).$$

Theorem 2. *Let (1), (2) and (3) hold. Then the operator $\mathcal{A}$ is T-accretive with dense domain in $L^1(\Omega)$, and we have $(L^1\cap L^\infty)(\Omega)\subset Range(I+\mathcal{A})$. Moreover, for all $(u_i,f_i)\in\mathcal{A},\ i=1,2$ we have*

$$\int_{\partial\Omega}(\tilde{w}_1-\tilde{w}_2)^\pm+\int_\Omega(u_1-u_2)^\pm\leq\int_\Omega \mathrm{sign}^\pm(u_1-u_2)(f_1-f_2)+\int_{[u_1=u_2]}(f_1-f_2)^\pm. \quad (13)$$

A proof of this theorem is given in [S] (see also [AS]).

Proof: The T-accretivity and (13) are obtained in the same way as Corollary 1. Let us sketch the proof of existence with data $f\in(L^1\cap L^\infty)(\Omega)$. We use the standard vanishing viscosity approximation of the equation in $(S)(f)$ together with a Lipschitz regularization β^ε of the graph β. Let $(u^\varepsilon)_\varepsilon$ denote the corresponding sequence of approximate solutions; $\beta^\varepsilon(u^\varepsilon)$ are the corresponding normal fluxes on the boundary. Assumption (1) yields a uniform L^∞ bound on u^ε. Assumption (3) is sufficient for the strong precompactness of $(u^\varepsilon)_\varepsilon$ in $L^1_{loc}(\Omega)$, by the result of Panov [P94] (see also the well-known result of Lions-Perthame-Tadmor). This is sufficient to get (6) and deduce (by the "stationary" analogue of Proposition 1) the existence of boundary traces $\tilde{v},\tilde{w}$ of $V_{\varphi_\nu}(u),\varphi_\nu(u)$, respectively, where u is an accumulation point of $(u^\varepsilon)_\varepsilon$. It remains to show (7). Due to the flatness of $\partial\Omega$, problem $(S)(f)$ is invariant (taking into account translations of f) with respect to translations in directions $(0,h')$, $h'\in\mathbb{R}^{N-1}$. Using the analogue of (13) which holds true for $\varepsilon>0$, by the Fréchet-Kolmogorov theorem we deduce that the sequence $(\beta^\varepsilon(u^\varepsilon))_\varepsilon$ is strongly compact in $L^1(\partial\Omega)$. We then deduce (7) by using the argument sketched in the proof of Lemma 3 below (with $\tilde{\tilde{\beta}}$ replaced by $\tilde{\beta}$, and $\tilde{\beta}$ replaced by β). Finally, $\overline{D(\mathcal{A})}=L^1(\Omega)$, since we show that the solution u_α to $u+\alpha\,\mathcal{A}(u)=f\in C_c^\infty(\Omega)$ converges to f in $L^1(\Omega)$ as $\alpha\to0$.

Theorem 2 means that the closure of $\mathcal{A}$ is an m-T-accretive densely defined operator in $L^1(\Omega)$ (see e.g. [BCP]). By the Crandall-Liggett theorem, it generates a T-contractive semigroup on $L^1(\Omega)$; more exactly, we have

Theorem 3. *Let (1), (2) and (3) hold. Then for any $f\in L^1(Q)$, $u_0\in L^1(\Omega)$ there exists a unique mild solution of the abstract Cauchy problem*

$$u_t+\mathcal{A}u\ni f,\ u(0)=u_0. \quad (14)$$

We deduce existence for Problem (H) by showing that any mild solution of problem (14) is also a solution of (H) in the sense of Definition 2.

Theorem 4. *Assume (1), (2), (3). If data (u_0,f) satisfy (12), then the mild solution of the Cauchy problem (14) is an entropy solution of Problem (H).*

Proof. For $m\in\mathbb{N}^*$, set $\varepsilon=\frac{T}{m}$. For $i=0,1,...,m$, set $t_i=\varepsilon i$ and $f_i^\varepsilon(x)=\frac{1}{\varepsilon}\int_{t_{i-1}}^{t_i}f(t,x)dt$ a.e. $x\in\Omega$. We have $\varepsilon\sum_{i=1}^m\|f_i^\varepsilon\|_{L^\infty(\Omega)}\leq\int_0^T\|f(t)\|_{L^\infty(\Omega)}dt$ and $\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\|f(t)-f_i^\varepsilon\|_{L^1(\Omega)}\to0$ as $\varepsilon\to0$. Take $u_0^\varepsilon\in D(\mathcal{A})$ such that $\|u_0^\varepsilon-u_0\|_{L^1}\leq\varepsilon$. Let u_i^ε be the solution of $\varepsilon f_i^\varepsilon+u_{i-1}^\varepsilon\in(I+\varepsilon\mathcal{A})(u_i^\varepsilon)$,

$i = 1, ..., m$. Set $u^\varepsilon(t) = u_i^\varepsilon, f^\varepsilon(t) = f_i^\varepsilon, \tilde{v}^\varepsilon(t) = \tilde{v}_i^\varepsilon, \tilde{w}^\varepsilon(t) = \tilde{w}_i^\varepsilon$ if $t_{i-1} \le t \le t_i$ ($1 \le i \le m$). By the nonlinear semigroup theory (see e.g. [BCP]), the sequence $(u_\varepsilon)_\varepsilon$ is precompact in $L^\infty(0,T;L^1(\Omega))$. Hence there exist a subsequence, still denoted by u_ε, and a function $u \in C(0,T;L^1(\Omega))$ such that $\|u^\varepsilon - u\|_{L^\infty(0,T;L^1(\Omega))} \to 0$ as $\varepsilon \to 0$. By construction, u admits u_0 for the strong trace on $\{t = 0\}$. Thanks to (12) and Assumption (1), $(u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(Q)$. The continuation of the proof is divided into three lemmas.

Lemma 1. *The function u verifies inequality (6).*

Proof. By Definition 3, for all $\phi \in C_c^\infty(Q),\ \phi \ge 0,\ k \in \mathbb{R}$, u_ε satisfies

$$0 \le \int_Q \Big(\frac{1}{\varepsilon}(u^\varepsilon(t-\varepsilon) - u^\varepsilon(t)) + f^\varepsilon(t)\Big)\mathrm{sign}^\pm(u^\varepsilon(t) - k)\phi(t) + \int_Q \mathrm{sign}^\pm(u^\varepsilon(t) - k)(\varphi(u^\varepsilon(t)) - \varphi(k)) \cdot D\phi(t). \tag{15}$$

As $\varepsilon \to 0$ the result follows, because $\mathrm{sign}^\pm(\cdot - k)$ belongs to $\partial(\cdot - k)^\pm$:

$$\mathrm{sign}^\pm(u^\varepsilon(t) - k)(u^\varepsilon(t-\varepsilon) - u^\varepsilon(t)) \le (u^\varepsilon(t-\varepsilon) - k)^\pm - (u^\varepsilon(t) - k)^\pm.$$

Lemma 2. *The sequence $(\tilde{w}^\varepsilon(t,x))_\varepsilon$ of strong traces of $\varphi_\nu(u^\varepsilon)$ on Σ converges (up to a subsequence) $\mathcal{H}^N$-a.e. on Σ and in $L^1_{loc}(\Sigma)$ to $b(t,x) \in \overline{Range(\tilde{\beta})}$.*

Proof. We prove that the sequence $(\tilde{w}^\varepsilon)_\varepsilon$ is bounded in $L^1(\Sigma)$, and

$$\int_0^{T-\Delta t} \|\tilde{w}^\varepsilon(t+\Delta t) - \tilde{w}^\varepsilon(t)\|_{L^1(\partial\Omega)} \le \tilde{\psi}(\Delta t), \quad \lim_{h\to 0} \tilde{\psi}(h) = 0.$$

A similar estimate of the space translates of $\tilde{w}^\varepsilon(t,x')$ follows from (13) and the translation invariance of $\partial\Omega$. We then apply the Fréchet-Kolmogorov theorem.

Lemma 3. *Strong traces $\tilde{v} = \gamma V_{\varphi_\nu}(u)$, $\tilde{w} = \gamma\varphi_\nu(u)$ on Σ of functions $V_{\varphi_\nu}(u)$ and $\varphi_\nu(u)$ exist. Moreover, $(\tilde{v}(t,x), \tilde{w}(t,x)) \in \tilde{\beta} \circ V_{\varphi_\nu}^{-1}$ $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$.*

Proof. By Proposition 1, strong traces of functions $V_{\varphi_\nu}(u)$ and $\varphi_\nu(u)$ exist. Now let us prove that $(\tilde{v}, \tilde{w}) \in \tilde{\beta} \circ V_{\varphi_\nu}^{-1}$ a.e. on Σ. By choosing in inequality (3) the test function $\phi(1 - \mu_\delta)$ with $\phi \in C_c^\infty([0,T) \times \mathbb{R}^N), \phi \ge 0, (\mu_\delta)_\delta \in C^2(\overline{\Omega}), \mu_\delta \to 1$ on Ω, $\mu_\delta = 0$ on $\partial\Omega$, and by letting $\varepsilon \to 0$ then $\delta \to 0$ we get

$$\int_\Sigma Q^\pm(\tilde{v}, V_{\varphi_\nu}(k))\, \phi \ge \int_\Sigma (b - \varphi \circ V_{\varphi_\nu}^{-1}(V_{\varphi_\nu}(k)))\, \theta_k^\pm\, \phi, \tag{16}$$

where $\theta_k^\pm, k \in \mathbb{Q}$ is the weak-star limit in $L^\infty(\Sigma)$ of $\mathrm{sign}^\pm(u_\varepsilon - k)$. But for a.e. $(t,x) \in \Sigma$, we have $b(t,x) = \lim_{\varepsilon\to 0} \tilde{w}^\varepsilon(t,x)$. By construction, $(\tilde{v}^\varepsilon, \tilde{w}^\varepsilon) \in \tilde{\beta} \circ V_{\varphi_\nu}^{-1}$; therefore for all $k \notin \tilde{\beta}^{-1}(b(t,x))$ we can identify $\theta_k^\pm(t,x)$ with $\mathrm{sign}^\pm(b(t,x) - \tilde{\beta}(k))$. For simplicity, let us use assumption (3) (notice that Lemma 3 holds without (3)). We can use $\tilde{u} = V_{\varphi_\nu}^{-1}(\tilde{v})$; then (16) implies

$$\mathrm{sign}^\pm(\tilde{u}-k)(\tilde{w}-\varphi_\nu(k)) \ge \mathrm{sign}^\pm(b-k)(b-\varphi_\nu(k))\ \forall k \notin \tilde{\beta}^{-1}(b),\ \mathcal{H}^N\text{-a.e. on } \Sigma.$$

Considering different values of k, we get $(\tilde{u}, \tilde{w}) \in \tilde{\tilde{\beta}}$. Since $\tilde{\tilde{\beta}} = \tilde{\beta}$, (7) holds.

Conclusion and generalizations

We justify that the entropy formulation (6),(7) or, equivalently, (8),(7) is an adequate interpretation of the formal boundary value problem (H). Uniqueness, comparison and L^1 contraction properties hold for this formulation.

Furthermore, the proof of Lemma 3 actually shows how the "effective BC" graph $\tilde\beta$ appears from the "formal BC" graph β (or β^ε, if the graphs are perturbed). In our proof, this passage requires strong L^1 compactness of the sequence $(\beta^\varepsilon(u^\varepsilon))_\varepsilon$ of the associated boundary fluxes.

Using the same techniques and the fact that $\tilde{\tilde\beta}=\tilde\beta$, one deduces existence of a solution verifying (6),(7) under weaker assumptions. For instance, if we approximate $u_0\in L^\infty(\Omega)$ by $u_0^{m,n}=u_0^+1\!\!1_{\{\|x\|<n\}}+u_0^-1\!\!1_{\{\|x\|<m\}}\in(L^1\cap L^\infty)(\Omega)$, the sequence of the resulting solutions $u^{m,n}$ of (H) is monotone in each of the indices m,n, by (13). Thus we have convergence of (a subsequence of) $u^{m,n}$ in $L^1_{loc}(Q)$, of $\tilde\beta(u^{m,n})$ in $L^1_{loc}(\Sigma)$. Whence existence for (H) in the L^∞ framework follows (cf. [S]). Similarly, perturbation of φ by piecewise strictly monotone φ^ε permits to bypass (3), by using the techniques of L^1 continuous dependence of entropy solutions on the flux function (see e.g. the papers by Bouchut-Perthame and Chen-Karlsen). Further, approximation of β by quickly growing at infinity graphs β^ε such that $|\beta^\varepsilon|\downarrow|\beta|$ permits to get rid of hypothesis (1) and justify the remark on the extension of β in Example 1(ii).

Fine translation techniques permit to consider inhomogeneous and mixed BC.

Finally, in order to extend the well-posedness results to the L^1 data, one has to apply the same techniques to renormalized solutions of (H) (see Bénilan-Carrillo-Wittbold) instead of the entropy solutions.

References

[AS] Andreianov, B., Sbihi, K.: On boundary conditions for scalar conservation laws. In preparation

[BLN] Bardos, C., Le Roux, A.Y., Nédélec, J.C.: First order quasilinear equations with boundary conditions. Comm. Partial Diff. Equ., **4**, 1017–1034 (1979)

[BCP] Bénilan, Ph., Crandall, M.G., Pazy, A.: Evolution equations governed by accretive operators. Preprint book

[BFK] Bürger, R., Frid, H., Karlsen, K.H.: On the well-posedness of entropy solutions to conservation laws with a zero-flux boundary condition. To appear in J. Math. Anal. Appl.

[MNRR] Malek, J., Nečas, J., Rokyta, M., Ružička, M.: Weak and Measure-valued Solutions to Evolutionary PDEs., Chapman&Hall, 1996.

[P94] Panov, E. Yu.: On sequences of measure-valued solutions of a first-order quasilinear equation. Mat.Sb., **185**,no.2, 87–106(1994)

[P05] Panov, E. Yu.: Existence of strong traces for generalized solutions of multidimensional scalar conservation laws. J.Hyp.Diff.Eq., **4**, 885–908(2005)

[P06] Panov, E. Yu.: communication at HYP06 Lyon, paper in preparation

[S] Sbihi, K.: Etude de quelques E.D.P. non linéaires dans L^1 avec des conditions générales sur le bord. Thesis, Strasbourg University, France (2006)