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Partial Differential Equations

Scalar conservation laws with nonlinear boundary conditions

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Abstract

This Note deals with uniqueness and continuous dependence of solutions to the problem $u_t + \operatorname{div}\varphi(u) = f$ on $(0,T)\times\Omega$ with initial condition $u(0,\cdot) = u_0$ on Ω and with (formal) nonlinear boundary conditions $\varphi(u)\cdot\nu \in \beta(t,x,u)$ on $(0,T)\times\partial\Omega$, where $\beta(t,x,\cdot)$ stands for a maximal monotone graph on $\mathbb{R}$. We suggest an interpretation of the formal boundary condition which generalizes the Bardos–LeRoux–Nédélec condition, and introduce the corresponding notions of entropy and entropy process solutions using the strong trace framework of E.Yu. Panov. We prove uniqueness and provide some support for our interpretation of the boundary condition. ***To cite this article: B. Andreianov, K. Sbihi, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Lois de conservation scalaires avec des conditions non linéaires au bord. Cette Note est dédiée aux résultats d'unicité des solutions du problème $u_t+\operatorname{div}\varphi(u) = f$ sur $(0,T)\times\Omega$ avec la condition initiale $u(0,\cdot) = u_0$ sur Ω et les conditions non linéaires $\varphi(u)\cdot\nu \in \beta(t,x,u)$ sur $(0,T)\times\partial\Omega$; ici $\beta(t,x,\cdot)$ désigne un graphe maximal monotone sur $\mathbb{R}$. Nous proposons une interprétation de la condition formelle « $\varphi(u)\cdot\nu \in \beta(t,x,u)$ » qui généralise celle de Bardos–LeRoux–Nédélec ; nous introduisons les notions de solutions entropiques et solutions processus entropiques. Nous montrons l'unicité et argumentons en faveur de notre interprétation de la condition au bord. ***Pour citer cet article : B. Andreianov, K. Sbihi, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Consider the (formal) problem

$$(H_\beta)(u_0,f)\quad \begin{cases} u_t + \operatorname{div}\varphi(u) = f & \text{in } Q := (0,T)\times\Omega,\\ u(0,\cdot) = u_0 & \text{on } \Omega,\\ \beta(t,x,u) - \varphi_\nu(u) \ni 0 & \text{on } \Sigma := (0,T)\times\partial\Omega \quad \text{(condition abbreviated to 'BC')}.\end{cases}$$

Here $T>0$, $\nu=\nu(x)$ is the unit outward normal vector at the point $x\in\partial\Omega$ of a C^1 domain $\Omega\subset\mathbb{R}^N$ (the results remain valid for more general case of a locally Lipschitz deformable boundary in the sense of [5]). The flux $\varphi:\mathbb{R}\to\mathbb{R}^N$ is assumed Lipschitz continuous for the sake of simplicity, $\varphi_\nu(\cdot)$ denotes $\varphi(\cdot)\cdot\nu(x)$. We assume $u_0\in L^\infty(\Omega)$, $f\in L^1(Q)$ with $f(t,\cdot)\in L^\infty(\Omega)$ for a.e. $t\in(0,T)$, and $\int_0^T \|f(t,\cdot)\|_\infty\,\mathrm{d}t<\infty$. Finally, we assume that $\beta(\cdot,\cdot,r)$ is

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measurable for all $r \in \mathbb{R}$, and $\beta(t,x,\cdot)$ is a maximal monotone graph on $\mathbb{R}$ for a.e. $(t,x) \in \Sigma$. The classical Neumann (zero-flux) and Dirichlet BC correspond to the graphs $\beta = \mathbb{R} \times \{0\}$ and $\beta(t,x,\cdot) = \{u^D(t,x)\} \times \mathbb{R}$, respectively (see [3,2]). General nonlinear boundary conditions include in particular mixed Dirichlet–Neumann conditions, as well as conditions of the obstacle type.

It is well known that existence of entropy solutions for (H) generally fails if one interprets the Dirichlet BC literally [2]. This is also the case for general β. Following [1,8], we consider that the 'formal' BC given by the graph β gives rise to the 'effective' BC given in terms of the monotone graph $\tilde\beta$ defined by

$$\tilde\beta := \left\{ \big(z, \varphi_\nu(z)\big) \;\middle|\; \begin{array}{l} \text{if } z < m := \inf \beta^{-1}(\varphi_\nu(z)), \text{ then } \varphi_\nu(k) \geqslant \varphi_\nu(z)\ \forall k \in [z, m[\\ \text{if } z > M := \sup \beta^{-1}(\varphi_\nu(z)), \text{ then } \varphi_\nu(k) \leqslant \varphi_\nu(z)\ \forall k \in]M, z] \end{array} \right\}. \tag{1}$$

Here, as usual, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. If $\bar\beta$ denotes the maximal monotone extension of β on $[-\infty, +\infty]$, then $\tilde\beta$ can be visualized as the horizontal projection of $\bar\beta$ on the graph of φ_ν. In the Dirichlet case $\beta = \{u^D(t,x)\} \times \mathbb{R}$, the condition '$(u, \varphi_\nu(u)) \in \tilde\beta$ on Σ' is an equivalent way to state the celebrated Bardos–LeRoux–Nédélec condition [2]. Generally, the graph $\tilde\beta$ is a monotone subgraph of φ_ν which is maximal, i.e., it does not possess a nontrivial monotone extension which is still a subgraph of φ_ν. Notice that (1) defines the mapping $\beta \mapsto \tilde\beta$ which is a projection: indeed, one checks that $\tilde{\tilde\beta} = \tilde\beta$.

In Section 3 we provide some support for our interpretation of the 'effective BC'.

2. Definitions and main results

In the following, γ (resp. γ_w) denotes the strong trace operator in the sense of [7] and [1, Definition 1] (resp. the weak trace operator in the sense of [5]). The graph $\tilde\beta \circ V_{\varphi_\nu}^{-1}$ is denoted by Γ; notice that Γ depends in a measurable way on (t,x), through $\varphi_{\nu(x)}$ and $\beta(t,x,\cdot)$. Further, $V_{\varphi_\nu}(\cdot)$ denotes the singular mapping defined by $V_{\varphi_\nu}(0) = 0$, $V'_{\varphi_\nu}(z) = \mathbb{1}_{\{\varphi'(z) \neq 0\}}$. It follows from the result of Panov in [7] that if u is an entropy solution of $u_t + \operatorname{div} \varphi(u) = f$ inside Q, then the strong trace of $V_{\varphi_\nu}(u)$ exists; thus the normal component of the entropy flux

$$\operatorname{sign}^\pm(u-k)\big(\varphi_\nu(u) - \varphi_\nu(k)\big) \equiv \operatorname{sign}^\pm\big(V_{\varphi_\nu}(u) - V_{\varphi_\nu}(k)\big)\big(\Psi_\nu\big(V_{\varphi_\nu}(u)\big) - \Psi_\nu\big(V_{\varphi_\nu}(k)\big)\big) =: Q^\pm\big(V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)\big)$$

(here $\Psi_\nu := \varphi_\nu \circ V_{\varphi_\nu}^{-1}$ and $Q^\pm$ are continuous functions) has the strong trace $Q^\pm(\gamma V_{\varphi_\nu}(u), V_{\varphi_\nu}(k))$ on Σ.

Definition 2.1. Denote $q^\pm(z,k) = \operatorname{sign}^\pm(z-k)(\varphi(z) - \varphi(k))$. A function $u \in L^\infty(Q)$ is called an entropy solution for Problem $(H_\beta)(u_0, f)$ if for all $(k,\xi) \in \mathbb{R} \times C_c^\infty(Q)$, $\xi \geqslant 0$, the local entropy inequalities hold :

$$\int_Q \big((u-k)^\pm \xi_t + q^\pm(u,k) \cdot \nabla\xi + \operatorname{sign}^\pm(u-k) f \xi\big) \geqslant 0, \tag{2}$$

if u has the strong trace u_0 on $\{t=0\}$, and if the strong traces $\tilde w = \gamma\varphi_\nu(u)$, $\tilde v = \gamma V_{\varphi_\nu}(u)$ on Σ of the functions $\varphi_\nu(u)$, $V_{\varphi_\nu}(u)$ verify

$$\big(\tilde v(t,x), \tilde w(t,x)\big) \in \Gamma(t,x) \quad \mathcal{H}^N\text{-a.e. on } \Sigma. \tag{3}$$

This definition can be reformulated so that to extend the test functions up to the boundaries:

Definition 2.2. A function $u \in L^\infty(Q)$ such that strong traces $\tilde v := \gamma V_{\varphi_\nu}(u)$, $\tilde w := \gamma\varphi_\nu(u)$ on Σ exist and satisfy (3) is called an entropy solution for $(H_\beta)(u_0, f)$ if for all $(k,\xi) \in \mathbb{R} \times C_c^\infty([0,T) \times \mathbb{R}^N)$, $\xi \geqslant 0$,

$$\int_Q \big((u-k)^\pm \xi_t + q^\pm(u,k) \cdot \nabla\xi + \operatorname{sign}^\pm(u-k) f\xi\big) + \int_\Omega (u_0 - k)^\pm \xi(0) - \int_\Sigma Q^\pm\big(\tilde v, V_{\varphi_\nu}(k)\big)\xi \geqslant 0.$$

The next definition presents an equivalent solution concept in the spirit of Carrillo [4]. If the dependence of Γ on (t,x) is not too irregular, it suffices to require (4) below for all $(k,\xi) \in \mathbb{R} \times C_c^\infty([0,T) \times \mathbb{R}^N)$, $\xi \geqslant 0$ such that $\xi = 0$ $\mathcal{H}^N$-a.e. on the set $\{(t,x) \in \Sigma \,|\, k \notin \operatorname{Dom}(\tilde\beta(t,x,\cdot))\}$; then the last term in (4) vanishes.

Definition 2.3. Let $u \in L^\infty(Q)$; for $k \in \mathbb{R}$, set $M_k = \max\{|k|, \|u\|_\infty\}$. Let $\omega_{M_k}(\cdot)$ be a modulus of continuity $Q^\pm(\cdot\,,\cdot)$ on $[V_{\varphi_\nu}(-M_k), V_{\varphi_\nu}(M_k)]^2$. Then u is called an entropy solution for Problem $(H_\beta)(u_0, f)$ if

$$\int_Q \big((u-k)^\pm \xi_t + q^\pm(u,k)\cdot\nabla\xi + \operatorname{sign}^\pm(u-k) f\xi\big) + \int_\Omega (u_0-k)^\pm \xi(0,\cdot) \geqslant -\int_\Sigma \xi R_k \tag{4}$$

for all $k \in \mathbb{R}$, for all $\xi \in C_c^\infty([0,T)\times\mathbb{R}^N)$, $\xi \geqslant 0$, where $R_k(t,x) = \omega_{M_k}(\operatorname{dist}(V_{\varphi_\nu}(k), \operatorname{Dom}(\Gamma(t,x,\cdot))))$.

It is proved in [1] that Def. 2.1 and Def. 2.2 are equivalent. To prove that Def. 2.2 implies Def. 2.3, one uses the monotonicity of Γ and the definition of R_k. To show that Def. 2.3 implies Def. 2.1, first note that (4) implies (2) and the initial trace condition. Since u is an entropy solution inside Q, then by [7], the strong traces $\tilde v, \tilde w$ of the functions $V_{\varphi_\nu}(u)$, $\varphi_\nu(u)$ on Σ exist, and $(\tilde v, \tilde w) \in \varphi_\nu \circ V_{\varphi_\nu}^{-1}$. To prove that $\tilde v \in \operatorname{Dom}(\Gamma)$, one takes ξ_h supported in the h-neighbourhood of $\partial\Omega$ such that $(1-\xi_h) \in C_c^\infty(\Omega)$ and $\nabla\xi_h$ converges as $h \to 0$ to the Hausdorff measure $\mathcal{H}^{N-1}$ on $\partial\Omega$ multiplied by the exterior unit normal vector ν to $\partial\Omega$ (such sequence $(\xi_h)_h$ is constructed using a partition of unity on $\bar\Omega$ and local coordinates near the boundary). As $h \to 0$, (4) with test function $\xi\xi_h$ yields $\gamma_w Q^\pm(V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)) \geqslant -R_k$ for all $k \in \mathbb{Q}$, for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$. The map $k \mapsto R_k$ being continuous, $\gamma_w Q^\pm(V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)) \geqslant 0$ for all $k \in \operatorname{Dom}(\tilde\beta(t,x))$. Now,

$$\gamma_w Q^\pm\big(V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)\big) = Q^\pm\big(\gamma V_{\varphi_\nu}(u), V_{\varphi_\nu}(k)\big) = \operatorname{sign}^\pm\big(\tilde v - V_{\varphi_\nu}(k)\big)\big(\Psi_\nu(\tilde v) - \Psi_\nu\big(V_{\varphi_\nu}(k)\big)\big);$$

since Γ is a maximal monotone subgraph of $\Psi_\nu = \varphi_\nu \circ V_{\varphi_\nu}^{-1}$, it follows that $\tilde v(t,x) \in \operatorname{Dom}(\Gamma(t,x))$ $\mathcal{H}^N$-a.e. on Σ.

The main subject of this Note is the uniqueness and comparison result for entropy solutions of $(H_\beta)(u_0, f)$ given below. In order to allow for a perturbation of the graph β, we introduce the following order relation:

$$\beta_1 \succeq \beta_2 \quad \text{if } d^-(\tilde\beta_1, \tilde\beta_2) := \sup\big\{\big(\varphi_\nu(a) - \varphi_\nu(b)\big)^- \big| a \in \operatorname{Dom}(\tilde\beta_1), b \in \operatorname{Dom}(\tilde\beta_2), a > b\big\} \text{ equals zero.}$$

Theorem 2.4. *Let u_i be an entropy solution for Problem $(H_{\beta_i})(u_0^i, f_i)$, $i = 1, 2$. Then for a.e. $t \in (0,T)$*

$$\int_\Omega (u_1-u_2)^+(t) \leqslant \int_\Omega (u_0^1 - u_0^2)^+ + \int_0^t\!\!\int_\Omega (f_1-f_2)^+ + \int_0^t\!\!\int_{\partial\Omega} d^-(\tilde\beta_1, \tilde\beta_2). \tag{5}$$

In particular, if $u_0^1 \leqslant u_0^2$ a.e. on Ω, $f_1 \leqslant f_2$ a.e. on Q and if $\beta_1(t,x,\cdot) \succeq \beta_2(t,x,\cdot)$ $\mathcal{H}^N$-a.e. on Σ, then one has $u_1 \leqslant u_2$ a.e. on Q. In particular, there exists at most one entropy solution to $(H_\beta)(u_0, f)$.

For the proof, by the Kruzhkov's doubling of variables argument one deduces from (2) the inequality

$$\int_0^t\!\!\int_\Omega \big(q^+(u_1,u_2)\cdot\nabla\xi + \operatorname{sign}^+(u_1-u_2)(f_1-f_2)\xi\big) \geqslant \int_\Omega (u_1-u_2)^+(t)\xi - \int_\Omega (u_0^1-u_0^2)^+\xi(0,\cdot)$$

for all $\xi \in C_c^\infty([0,t]\times\Omega)$. With $\xi(t,x) = 1 - \xi_h(x)$, where the sequence $(\xi_h)_h$ is described hereabove,

$$\int_0^t\!\!\int_\Omega q^+(u_1,u_2)\cdot\nabla(1-\xi_h) \underset{h\to 0}{\longrightarrow} \int_0^t\!\!\int_{\partial\Omega} -\gamma_w Q^+\big(V_{\varphi_\nu}(u_1), V_{\varphi_\nu}(u_2)\big) = \int_0^t\!\!\int_{\partial\Omega} -Q^+(\tilde v_1, \tilde v_2),$$

because $Q^+(\tilde v_1, \tilde v_2)$ is the strong trace of the function $q^+(u_1,u_2)\cdot\nu \equiv Q^+(V_{\varphi_\nu}(u_1), V_{\varphi_\nu}(u_2))$ on Σ. Since by (3), $(\tilde v_i, \tilde w_i) \in \tilde\beta_i \circ V_{\varphi_\nu}^{-1}$, which are subgraphs of $\varphi_\nu \circ V_{\varphi_\nu}^{-1}$, using the definition of $d^-(\cdot\,,\cdot)$ one gets $Q^+(\tilde v_1, \tilde v_2) = \operatorname{sign}^+(\tilde v_1 - \tilde v_2)(\tilde w_1 - \tilde w_2) \geqslant -d^-(\tilde\beta_1, \tilde\beta_2)$. Hence the claims of the theorem follow.

Notice that Definition 2.3 permits to define a notion of entropy process solutions, following [6]:

Definition 2.5. Let $\mu \in L^\infty((0,1)\times Q)$; take R_k of Def. 2.3 with $M_k = \max\{|k|, \|\mu\|_\infty\}$. Then μ is called an entropy process solution to Problem $(H_\beta)(u_0, f)$ if for all $k \in \mathbb{R}$, for all $\xi \in C_c^\infty([0,T)\times\mathbb{R}^N)$, $\xi \geqslant 0$,

$$\int_0^1\!\!\int_Q \big((\mu-k)^\pm\xi_t + q^\pm(\mu,k)\cdot\nabla\xi + f\operatorname{sign}^\pm(\mu-k)\xi\big) + \int_\Omega (u_0-k)^\pm\xi(0,\cdot) \geqslant -\int_\Sigma \xi R_k. \tag{6}$$

It turns out that if $\mu(\alpha,t,x)$ is an entropy process solution for Problem $(H_\beta)(u_0,f)$, and $u(t,x)$ is an entropy solution of $(H_\beta)(u_0,f)$, then $\mu(\alpha,t,x)=u(t,x)$ for almost every $(\alpha,t,x)\in(0,1)\times Q$.

Indeed, following the lines of the proof of Theorem 2.4, we need to show that

$$\gamma_w\int_0^1 Q^+\big(V_{\varphi_\nu}(\mu),V_{\varphi_\nu}(u)\big)\equiv\gamma_w\int_0^1 Q^+\big(V_{\varphi_\nu}(\mu),\gamma V_{\varphi_\nu}(u)\big)$$

is nonnegative. This is true since $\gamma V_{\varphi_\nu}(u)\in\mathrm{Dom}(\Gamma)$ and because (6) gives $\gamma_w\int_0^1 Q^+(V_{\varphi_\nu}(\mu),\ V_{\varphi_\nu}(k))\geqslant 0$ $\forall k\in\mathrm{Dom}(\tilde\beta(t,x))\}$, in the same way as (4) implies (3).

3. Justification of the 'effective' boundary condition

Now we argument in favor of our interpretation of the BC in $(H_\beta)(u_0,f)$. The idea is, as usual, to accept as solutions the limits of "natural" approximations of $(H_\beta)(u_0,f)$ (such as the vanishing viscosity limit, solutions of well-behaving numerical schemes, limits of various perturbed problems). In [8,1], the standard vanishing viscosity approximation combined with the nonlinear semigroup techniques was considered. Under many simplifying assumptions on Ω, φ and β, it was proved that the 'formal BC' graph β does transform into the 'effective BC' graph $\tilde\beta$ of (1) (this phenomenon, explained by the presence of a boundary layer, is well known for the vanishing viscosity approximation of $(H_\beta)(u_0,f)$ with Dirichlet BC). In order to get a uniform L^∞ bound on the sequence of approximate solutions, we assumed that

$$\text{there exists a constant } C \text{ such that } \big|\beta(z)\big|\geqslant \mathrm{sign}(z)\varphi_\nu(z)\ \forall|z|>C, \tag{7}$$

which excludes e.g. the zero-flux BC even for linear φ (similar restrictive assumptions are made in [3]).

Let us support definition (1) of $\tilde\beta$ in some cases where (7) fails. Assume that one can approximate the maximal monotone extension of β on $\bar{\mathbb{R}}$ by graphs $\beta_{m,n}$ such that: $\beta_{m+1,n}\succeq\beta_{m,n}\succeq\beta_{m,n+1}$; for all $k\in\mathrm{Dom}(\tilde\beta)$ and m,n large enough, $k\in\mathrm{Dom}(\tilde\beta_{m,n})$; and (7) holds with β replaced by $\tilde\beta_{m,n}$, uniformly in m,n. Such approximation is possible e.g. if φ_ν is monotone on $(-\infty,C]$ and on $[C,+\infty)$, since the choice of $\beta_{m,n}:=\beta+I_{[-m,n]}$ yields $\tilde\beta_{m,n}\equiv\tilde\beta$ as soon as $m,n>C$. With the arguments of [8,1] there exists a uniformly bounded on Q sequence $(u_{m,n})_{m,n}$ of solutions of $(H_{\beta_{m,n}})(u_0,f)$ in the sense of Definitions 2.1–2.3. By Theorem 2.4, one has $u_{m,n+1}\leqslant u_{m,n}\leqslant u_{m+1,n}$ a.e. on Q. One concludes that $u_{m,n}$ converges in $L^1(Q)$ to a function $u\in L^\infty(Q)$, as $m\to+\infty$ and then $n\to+\infty$; passing to the limit in inequalities (4) corresponding to $(H_{\beta_{m,n}})(u_0,f)$ one deduces that u is an entropy solution of Problem $(H_\beta)(u_0,f)$.

Example 1. The reader can check easily that the "effective BC" graphs corresponding to the simplest zero-flux problem $u_t+u_x=0$ in $(0,T)\times(0,1)$, $u(t,0)\nu(0)=0=u(t,1)\nu(1)$, are given by $\tilde\beta(t,1,\cdot)=\{(z,z)|z\in\mathbb{R}\}$, $\tilde\beta(t,0,\cdot)=\{0\}\times\mathbb{R}$. This interpretation of the BC is consistent with the classical approach by characteristics.

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