Dissipative coupling of scalar conservation laws across an interface: theory and applications

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Abstract

We give a brief account on the theory of L^1-contractive solvers of the model conservation law with discontinuous flux:

$$(MP)\ \ u_t + (\mathfrak{f}(x,u))_x = 0, \quad \mathfrak{f}(x,\cdot) = f^l(\cdot)1\!\!1_{x<0} + f^r(\cdot)1\!\!1_{x>0},$$

constructed in the work [6] of K.H. Karlsen, N.H. Risebro and the author. We discuss the modifications that can be used for extending our approach to the multi-dimensional setting and curved flux discontinuity hypersurfaces; the vanishing viscosity case (see [7]) is presented as an illustration. Applications to a road traffic with point constraint and to a coupled particle-fluid interaction model, coming from the joint works [4] with P. Goatin, N. Seguin and [8, 5] with F. Lagoutière, N. Seguin, T. Takahashi, are presented.

1 Introduction

Well-posedness for conservation laws with discontinuous flux of the form $\mathfrak{f}(t,x;u)$ is an active area of research since the early 1990ies. Several notions of solution were proposed, specifically in the model case (MP). Extensive references on the subject are given in the paper [6] in which a general approach to the Cauchy problem (MP) was proposed; many of the ideas that we used appear in the original contributions to the subject (see in particular [1, 2, 9, 10, 11, 13, 14, 15, 16, 21] and references in [6]).

We describe the notion(s) of entropy solutions to (MP) that lead to an L^1 theory analogous to the classical Kruzhkov theory (namely, with the L^1 contraction principle in the domain of dependence). Each of these notions is fully determined by a subset $\mathcal{G}$ of $\mathbb{R}^2$, called *admissibility germ*; the role of $\mathcal{G}$ is to describe the coupling, across the interface $\{x=0\}$, of two conservation laws $u_t+(f^{l,r}(u))_x=0$ set up in the domains $\{\mp x>0\}$, respectively. The principal tools we use are the Kato inequalities away from the interface and near the interface, strong traces theory ([19]), adapted entropies ([10, 9, 11]), adapted vanishing viscosities, monotone finite volume schemes with the Godunov solver at the interface, variation bounds away from the interface ([11]), measure-valued solutions and compactification arguments. Section 2 briefly presents the theory of [6].

Establishing general properties of admissibility germs, we were able to classify many of the known uniqueness criteria for particular cases of the problem (MP); in particular, we gave a description of the *vanishing viscosity germ* $\mathcal{G}_{VV}$ which corresponds to solutions constructed by adding the standard viscosity εu_{xx} into the right-hand side of (MP) (see [6, 7]; cf. [13] for an equivalent description of $\mathcal{G}_{VV}$). Further, the theory of [6] (or rather, its straightforward generalizations, cf. Section 3) found new applications, which we briefly describe below (see Section 4 for details).

Namely, in [4] the road traffic model

$$u_t + f(u)_x = 0 \text{ with a (formal) constraint } f(u)(t,0) \leq F(t) \qquad (1.1)$$

is considered, following [12]. Although the flux here is continuous, the point constraint in (1.1) acts as an interface coupling condition and it may lead to introduction of a non-Kruzhkov shock at the interface. In this setting, we proved well-posedness in L^∞ and constructed a simple but efficient finite volume scheme for the problem (1.1).
Next, following [18], we looked at the Burgers equation

$$u_t + (u^2/2)_x = -u\delta_0(x) \qquad (1.2)$$

(here δ_0 is the Dirac delta function). Formally, the singular source $-u\delta_0(x)$ acts as an absorption term, thus it does not destruct the L^1-dissipativity; but it induces a non-conservative coupling. The coupling is interpreted in [18] using a regularization of δ_0. Dropping the Rankine-Hugoniot coupling restriction (cf. (2.1) below) , in [8, 5] we apply the previous theory to this case and establish well-posedness for solutions defined in [18]; once more, a simple numerical scheme for problem (1.2) is validated. Further, in the work [5] the properties of solutions to (1.2) are exploited in the fixed-point argument that allows to solve the coupled "particle-in-Burgers" problem of [18] (the Burgers equation with singular source $-(u-h'(t))\delta_0(x-h(t))$ for the fluid velocity u is coupled with an ODE for the evolution of the particle position h). In an on-going work, we eventually achieve a BV well-posedness theory for this free-interface problem with non-conservative interface coupling.

The model case (MP) made apparent a number of ideas that can be now applied to the general setting of multi-dimensional conservation laws with piecewise smooth flux. Actually, references [4, 5, 7, 8] (cf. Section 4) already contain a few extensions. This work is on-going; some generalizations to our definitions and tools are presented in Section 3.

2 Admissibility germs and entropy solutions

A starting assumption is that, away from the flux discontinuities, one uses the Kruzhkov notion of entropy solutions ([17]). Thus, understanding the model case (MP) should reduce to understanding the coupling of two scalar conservation laws across the interface $\Sigma := \{x = 0\}$. We justify this reduction; moreover, thanks to the theory of strong boundary traces for conservation laws (see Panov [19]), the coupling across Σ can be encoded by the set $\mathcal{G}^*$ of the admissible couples $\big(\gamma^l u\,,\,\gamma^r u\big)$ of left- and right-sided traces of u at a.e. point of the interface Σ. To be precise, these strong traces do exist if $f^{l,r}(\cdot)$ are non-degenerately nonlinear (see [19]); more generally, introducing the appropriate singular mappings (see [6]), we can always rely on strong traces of $f^{l,r}(u)$ and of the associated Kruzhkov entropy fluxes $q^{l,r}(u,k) := \mathrm{sign}(u-k)(f^{l,r}(u)-f^{l,r}(k))$, $k \in \mathbb{R}$. To simplify the presentation, we assume in the sequel that $\gamma^{l,r}u$ exist.

We make apparent the properties of the set $\mathcal{G}^*$; namely, the coupling is conservative and L^1-dissipative (L^1D, for short) in the sense that

$$\forall (u^l,u^r),(v^l,v^r) \in \mathcal{G}^* \quad \begin{cases} f^l(u^l) = f^r(u^r), \\ q^l(u^l,v^l) \geq q^r(u^r,v^r), \end{cases} \tag{2.1}$$

moreover, the set $\mathcal{G}^* \subset \mathbb{R}^2$ is *maximal* L^1D (i.e. it has no extension that still satisfies the properties (2.1)). Further, in practice it is often enough to describe only a subset $\mathcal{G}$ of $\mathcal{G}^*$ that satisfies properties (2.1) and admits a unique maximal extension. In this situation, we say that $\mathcal{G}$ is a *definite L^1-dissipative (or L^1D) admissibility germ* and $\mathcal{G}^*$ is its maximal L^1D extension. For a definite L^1D germ, we define a $\mathcal{G}$-entropy solution as a juxtaposition of entropy solutions $u|_{x<0}$ (for the conservation law $u_t + f^l(u)_x = 0$) and $u|_{x>0}$ (for $u_t + f^r(u)_x = 0$) coupled via

$$\text{for a.e. } t \in \mathbb{R}^+ \quad \big((\gamma^l u)(t)\,,\,\gamma^r(u)(t)\big) \in \mathcal{G}^*. \tag{2.2}$$

In this way, definite germs $\mathcal{G}, \tilde{\mathcal{G}}$ with different maximal L^1D extensions $\mathcal{G}^*, \tilde{\mathcal{G}}^*$ lead to different notions of entropy solution for (MP) (cf. [1, 11]).

We introduce some other notions related to germs; in particular, $\mathcal{G}^*$ is a complete germ if every Riemann problem for (MP) admits a $\mathcal{G}$-entropy solution. The key statement of the theory is the following:

for every definite L^1D germ $\mathcal{G}$ such that $\mathcal{G}^*$ is complete,
problem (MP) is well-posed
in the framework of $\mathcal{G}$-entropy solutions with L^∞ initial data.

More precisely, uniqueness holds for general continuous fluxes $f^{l,r}$; existence is shown under some structure assumptions on $f^{l,r}$ (including e.g. the trivial requirement that the ranges of $f^l(\cdot)$ and $f^r(\cdot)$ intersect); the continuous dependence on the initial datum u_0 can be stated as the L^1 contraction in the domain of dependence (for $f^{l,r}(\cdot)$ locally Lipschitz).

The proof of uniqueness (continuous dependence is similar) for the Cauchy problem for (MP) is very straightforward from this definition. From the Kruzhkov theory away from the interface, the Kato inequality with test functions $\xi\xi_h$, $\xi \geq 0$, $\xi_h := \min\{|x|/h, 1\}$ is derived, namely,

$$\int_0^T \int_{\mathbb{R}} \big\{ -|u-\hat{u}|(\xi\xi_h)_t - \operatorname{sign}(u-\hat{u})(\mathfrak{f}(x;u)-\mathfrak{f}(x;\hat{u}))\,(\xi\xi_h)_x \big\} \leq 0 \quad (2.3)$$

for $u, \hat{u}$ two $\mathcal{G}$-entropy solutions associated with the same initial datum. Letting h desrease to zero and using the definition of strong traces, we generate the interface term that takes precisely the form

$$\int_0^T \big\{ q^l(\gamma^l u, \gamma^l \hat{u}) - q^r(\gamma^r u, \gamma^r \hat{u}) \big\}(t)\, \xi(t,0)\, dt. \quad (2.4)$$

Now the trace constraint (2.2) and the dissipativity property (2.1) of $\mathcal{G}^*$ make this term non-negative; it can therefore be dropped. Choosing ξ in the classical way of Kruzhkov, we find that $u = \hat{u}$.

Existence arguments require a definition of $\mathcal{G}$-entropy solution that is clearly stable by passage to the limit. We provide the following:

Definition 2.1. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is a $\mathcal{G}$-entropy solution of (MP) if it is a Kruzhkov entropy solution in the domains $\{\pm x > 0\}$, it is a weak solution in the whole domain (i.e. the Rankine-Hugoniot condition holds), and moreover, the adapted entropy inequalities

$$|u - c(x)|_t + (\mathfrak{q}(x; u, c(x)))_x \leq 0 \ \text{ in } \mathcal{D}'((0,+\infty)\times\mathbb{R}), \quad (2.5)$$

$(\mathfrak{q}(x;\cdot, c(x)) = q^l(\cdot, c^l)1\!\!1_{x<0} + q^r(\cdot, c^r)1\!\!1_{x>0}$ being the adapted entropy flux) hold for every function $c(x) = c^l 1\!\!1_{x<0} + c^r 1\!\!1_{x>0}$ with $(c^l, c^r) \in \mathcal{G}$.

Notice that Definition 2.1 admits a straightforward extension to the case of measure-valued ($\mathcal{G}$-entropy-process) solutions. Unfortunately, in our theory we require existence of a $\mathcal{G}$-entropy solution $u(\cdot,\cdot)$ to prove that every $\mathcal{G}$-entropy-process solution $(\hat{u}(\cdot,\cdot,\alpha))_{\alpha\in[0,1]}$ is independent of α and coincides with u. Yet if existence of u can be obtained via some particular strongly convergent approximation procedure, this extension allows to prove convergence of the approximations possessing only a uniform L^∞ bound and compatible with (adapted) entropy inequalities.

Notice that smaller is $\mathcal{G}$, easier it is to check the constraints (2.5); e.g. in the cases considered in [1, 3, 4, 9, 11], one can pick for $\mathcal{G}$ a singleton (A, B), called "(A, B)-connection"; in this way, only one adapted entropy inequality has to be checked. Another advantage of restraining $\mathcal{G}^*$ as much as possible (which leads to the use of definite L^1D germs) is the following. Assume that the approximation procedure in use allows explicitly for some particular stationary solutions of (MP) of the form $u(x) = c^l_{expl}\mathbb{1}_{x<0} + c^r_{expl}\mathbb{1}_{x>0}$; assume that we know that the procedure converges, for each fixed initial datum, and that the resulting solver for (MP) is L^1 contractive (this is the case when we start with L^1 contractive approximate solvers; the vanishing viscosity method is one typical example, monotone finite volume methods provide another example). Then we know that the resulting notion of solution is a $\mathcal{G}^*$-entropy solution for some maximal L^1D germ $\mathcal{G}^*$, and $\mathcal{G}^*$ contains the couples (c^l_{expl}, c^r_{expl}) corresponding to the explicit stationary limits of the approximation scheme in use. Eventually, the set of all such couples (c^l_{expl}, c^r_{expl}) may form a definite L^1D germ $\mathcal{G}$ of which $\mathcal{G}^*$ is the unique maximal L^1D extension; $\mathcal{G}^*$ can be calculated starting from the set of all (c^l_{expl}, c^r_{expl}). The proofs of [3, 4, 7, 8] are based on this approach.

Let us explain why, *for a definite L^1D germ $\mathcal{G}$*, Definition 2.1 is equivalent to the previously given definition "with traces" of $\mathcal{G}$-entropy solutions. Firstly, the entropy inequalities (2.5) derive from splitting a test function ξ into $\xi\xi_h + \xi(1-\xi_h)$ with $\xi_h = \min\{|x|/h, 1\}$. Indeed, for the test function $\xi\xi_h$, the Kruzhkov entropy inequalities away from $\{x = 0\}$ can be used, with $k = c^{l,r}$; for the family $(\xi(1-\xi_h))_{h>0}$, we pass to the limit as $h \to 0$, use the definition of strong traces and generate the term (2.4) with $\gamma^{l,r}\hat{u} := c^{l,r}$. From the inclusion $\mathcal{G} \subset \mathcal{G}^*$ and the dissipativity property of the germ $\mathcal{G}^*$ (see (2.1)), this term has the good sign and we get (2.5) as $h \to 0$. Secondly, the same calculation made with the test function $\xi(1-\xi_h)$ ($\xi \geq 0$ being arbitrary) allows to go from the adapted entropy inequality (2.5) to the t-a.e. inequalities

$$\forall (c^l, c^r) \in \mathcal{G} \quad q^l((\gamma^l u)(t), c^l) - q^r((\gamma^r u)(t), c^r) \geq 0.$$

By assumption, the unique maximal L^1D extension of $\mathcal{G}$ is $\mathcal{G}^*$; it follows that the couple $\big((\gamma^l u)(t)\,, (\gamma^r u)(t)\big)$ belongs to $\mathcal{G}^*$. Thus (2.2) holds.

Well-chosen numerical scheme yields a rather general existence result:

Theorem 2.2 (see Andreianov, Karlsen and Risebro [6]).
Let $\mathcal{G}$ be definite L^1D germ with the maximal extension $\mathcal{G}^*$ that is a complete germ. Assume that the functions $f^{l,r}$ are locally Lipschitz continuous on $\mathbb{R}$. Then for any initial function $u_0 \in L^\infty(\mathbb{R})$ there exists a unique $\mathcal{G}$-entropy solution of the Cauchy problem (MP), $u|_{t=0} = u_0$.

Existence results can also derive from different vanishing viscosity approaches: this permits to bypass the tedious check of completeness of germs. We refer to [7, 6] for the standard viscosity case. If $\mathcal{G}$ is a singleton $\{(A, B)\}$, adapted "artificial" viscosities, explicitly allowing for the stationary solution $A\mathbb{1}_{x<0} + B\mathbb{1}_{x>0}$, can be used (see [6] and Section 4). Also the physically relevant "vanishing capillarity" regularization for the case of Buckley-Leverett equation with discontinuous flux becomes easy to analyze: we refer to the work [3] of the author with C. Cancès.

Let us give an insight into the proof of Theorem 2.2. Uniqueness was already justified, using the definition "with traces"; to prove existence, we construct a family of approximate solutions using a finite volume scheme with any monotone consistent numerical flux away from the interface, and with the Godunov numerical flux at $\{x=0\}$ (this Godunov flux is constructed from the Riemann solver at the interface $\{x=0\}$; the Riemann solver is well defined, due to the completeness assumption on $\mathcal{G}^*$). Then we prove that, firstly, the approximate solver is L^1-contractive (more generally, it satisfies approximate Kato inequalities); and secondly, that the solutions $c^l 1\!\!1_{x<0} + c^r 1\!\!1_{x>0}$ with $(c^l, c^r) \in \mathcal{G}^*$ are explicit limits of the scheme (this is evident because the Godunov scheme is well-balanced). If an L^1_{loc} limit u of the approximations $(u_n)_n$ exists, we "inherit" the adapted entropy inequalities (2.5) for u; the Kruzhkov entropy inequalities away from $\{x=0\}$ and the Rankine-Hugoniot condition are justified in the analogous (a more standard) way. Eventually, we have to justify the L^1_{loc} compactness of $(u_n)_n$. To this end, we use the BV_{loc} estimate device in the way of [11], and a uniform L^∞ estimate on u_n coming from comparison principle with well-chosen stationary solutions (at this point, the completeness of $\mathcal{G}^*$ is used in an indirect way). In this way, solutions with BV initial data are constructed; using localized L^1 contraction for these solutions, we deduce existence for L^∞ data.

It should be noticed that the different properties of L^1D germs are not independent; some relations are listed in [6] and in the Appendix of [7]. To give an example, complete L^1D germs are automatically maximal ones. The slight difference between the definition of the entropy solution in [15] and the Γ-condition of [13] is easily removed making the closure operation on germs (see [6]). The notion of closure of a germ turns out to be useful in the analysis of the vanishing viscosity approximation of (MP), starting from the viscous profiles for (MP) (see [7, 6]). Structural properties of germs may allow to establish that the set of stationary solutions admissible according to some admissibility criteria in not L^1D, which means that the criterion fails. For instance, we show in [6] that the *crossing condition* imposed in the works of [16, 21] is necessary for the uniqueness of the entropy solutions in the sense of inequalities $(\forall k \in \mathbb{R})$

$$\int_0^T\!\!\int_{\mathbb{R}} \big\{ |u-k|\,\xi_t + \mathfrak{q}(x,u,k)\,\xi_x \big\} \geq -\int_0^T |f^r(k)-f^l(k)|\,\xi(\cdot,0). \qquad (2.6)$$

Indeed, (2.6) uses the Kruzhkov entropies ([17]), whereas constants are, in general, not solutions to (MP). Thus an error term should be incorporated into the Kato inequality (2.3) (now let us take $\xi_h \equiv 1$ in (2.3)) written for u and $\hat{u} \equiv k$; the right-hand side of (2.6) lower bounds this error term. In Section 3, we use the idea of [21, 16] with k replaced by $c(x) = c^l 1\!\!1_{x<0} + c^r 1\!\!1_{x>0}$, but now with couples (c^l, c^r) that are not necessarily in $\mathcal{G}$, and with an *ad hoc* error term in the entropy inequalities.

3 Generalization to variable germs

The theory of Section 2 assumes that the admissibility germ $\mathcal{G}$ governs the interface coupling at a.e. point $(t,0)$ of the interface $\{x=0\}$. One could naturally ask what happens if a family $(\mathcal{G}(t))_{t>0}$ (of definite L^1D germs) is given. Such "variable germs" occur for instance in the problem (1.1) considered in [4], and in the free-interface problem of [5] (see Section 4). Moreover, dependence of $\mathcal{G}(\cdot)$ on the point of the interface naturally occurs whenever the interface is curved (indeed, the normal direction to the interface enters the definition of the germ, see [7]).

In these cases, the definition "with traces" of a $\mathcal{G}(\cdot)$-entropy solution carries on without difficulty, provided the interface Σ is regular enough and a non-degeneracy assumption guarantees the existence of one-sided traces $(\gamma^{l,r}u)(\cdot)$ on Σ of an entropy solution u in $(\mathbb{R}^+\times\mathbb{R}^N)\setminus\Sigma$. We simply have to put $\mathcal{G}^*=\mathcal{G}(t)^*$ in (2.2) (more generally, t has to be replaced with a local coordinate σ on Σ). Clearly, existence of solutions satisfying the so modified interface coupling constraint (2.2) requires some kind of measurability of the family $\mathcal{G}(\cdot)$. The families that appear in practice, such as the vanishing viscosity family $(\mathcal{G}_{VV}(\sigma))_{\sigma\in\Sigma}$ associated with a curved interface Σ, do satisfy this constraint (this can be understood from the fact that the existence of a solution is guaranteed, as a consequence of convergence of the vanishing viscosity method; see [7]).

Let us give a generalization of Definition 2.1 to the case of a variable germ; the idea is to relax the constraint $(c^l,c^r)\in\mathcal{G}$, and to "pay" with a remainder term concentrated on Σ. Assume that Σ is a regular enough hypersurface of $\mathbb{R}^+\times\mathbb{R}^N$ separating it into two domains $\Omega^{l,r}$, and

$$\mathfrak{f}(t,x;\cdot)=f^l(\cdot)1\!\!1_{\Omega^l}(t,x)+f^r(\cdot)1\!\!1_{\Omega^r}(t,x). \tag{3.1}$$

In the definition of the remainder R, we tacitly assume that $(\mathcal{G}(\sigma))_{\sigma\in\Sigma}$ is a measurable family of germs on Σ, in the *ad hoc* sense.

Definition 3.1. A function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R}^N)$ is a $\mathcal{G}(\cdot)$-entropy solution of $u_t+\mathrm{div}(\mathfrak{f}(t,x;u))=0$ (with $\mathfrak{f}$ given by (3.1)) if it is a Kruzhkov entropy solution in the domains $\Omega^{l,r}$, it is a weak solution in the whole domain (i.e. the Rankine-Hugoniot condition holds), and moreover, the adapted entropy inequalities with remainder term $R\left(\cdot;\left(c^l,c^r\right)\right)$:

$$|u-c(x)|_t+\mathrm{div}\,\mathfrak{q}(x;u,c(x))\leq R\left(\sigma;\left(c^l,c^r\right)\right)d\mathcal{H}(\sigma)\ \text{ in } \mathcal{D}'((0,+\infty)\times\mathbb{R}^N)$$

hold for every function $c(x)=c^l1\!\!1_{x<0}+c^r1\!\!1_{x>0}$ with $(c^l,c^r)\in\mathbb{R}^2$.
Here $d\mathcal{H}(\sigma)$ is the N-dimensional Hausdorff measure on Σ, so that the remainder term R is supported by the interface Σ; moreover, it is assumed that $R:\Sigma\times\mathbb{R}^2\to\mathbb{R}^+$ is a Carathéodory function such that for a.e. $\sigma\in\Sigma$

$$\forall\ \left(c^l,c^r\right)\in\mathcal{G}(\sigma),\ \lim_{r\downarrow 0}\fint_{B_r(\sigma)\cap\Sigma}R(\sigma';(c^l,c^r))\,d\mathcal{H}(\sigma')=0 \tag{3.2}$$

and
$$\forall\,(c^l,c^r)\in\mathbb{R}^2 \text{ and } \forall\,\left(a^l,a^r\right)\in\mathcal{G}(\sigma)$$
$$q^r\left(\sigma;a^r,c^r\right)-q^l\left(\sigma;a^l,c^l\right)\le R\left(\sigma;\left(c^l,c^r\right)\right). \tag{3.3}$$

Roughly speaking, property (3.2) means that the remainder vanishes whenever $(c^l,c^r)\in\mathcal{G}(\sigma)$ (thus providing a link with Definition 2.1). Property (3.3) leads to the term R in the entropy inequalities, since R dominates the quantity (2.4) for an entropy solution u and for $\hat{u}=c(x)$.

When $f^{l,r}$ are Lipschitz continuous, the simplest choice of R is

$$R\left(\sigma;\left(c^l,c^r\right)\right):=const\ \operatorname{dist}\big(\left(c^l,c^r\right)\,,\,\mathcal{G}(\sigma)\big). \tag{3.4}$$

In the cases considered in [4, 5], the measurability of $R(\cdot,(c^l,c^r))$ given by (3.4) is clear; in a more general setting, we prefer a subtler definition of R based upon the oscillation functions $Osc_{[a^{l,r},c^{l,r}]}f^{l,r}$ associated with $f^{l,r}|_{[a^{l,r},c^{l,r}]}$ (see [6]), where (a^l,a^r) are chosen in $\mathcal{G}(\cdot)$, as in (3.3).

Equivalence of the definition "with traces" and Definition 3.1 is established as in the model case (MP). Uniqueness is straightforward. For examples of existence proofs, we refer to [7] and to Section 4.

To end this section, assume that $u,\hat{u}$ are entropy solutions of (MP) corresponding to different families of germs $\mathcal{G}(\cdot)$ et $\hat{\mathcal{G}}(\cdot)$, respectively (this includes the case of two fixed but different germs). Then we can compare $u,\hat{u}$ in terms of an error term containing a "distance" between the two germs. Indeed, we define the "distance" between two elements in $\mathbb{R}^2$ as $\rho\big(\,(c^l,c^r)\,,\,(\hat{c}^l,\hat{c}^r)\,\big):=\big(q^r(c^r,\hat{c}^r)-q^l(c^l,\hat{c}^l)\big)^+$; then for $\mathcal{G},\hat{\mathcal{G}}\subset\mathbb{R}^2$,

$$\rho\big(\,\mathcal{G}\,,\,\hat{\mathcal{G}}\,\big):=\sup\big\{\,\rho\big(\,(c^l,c^r)\,,\,(\hat{c}^l,\hat{c}^r)\,\big)\mid(c^l,c^r)\in\mathcal{G},\,(\hat{c}^l,\hat{c}^r)\in\hat{\mathcal{G}}\,\big\}. \tag{3.5}$$

Then, essentially in the same way as the uniqueness of a $\mathcal{G}$-entropy solution is proved, in the case $u(0,\cdot)=\hat{u}(0,\cdot)$ we establish

$$\int_{\mathbb{R}}|u-\hat{u}|(t,\cdot)\le\int_0^t\rho\big(\,\mathcal{G}(t)\,,\,\hat{\mathcal{G}}(t)\,\big)\,dt. \tag{3.6}$$

Notice that definition (3.5) and the associated notion of a ρ-neighbourhood of a given germ $\mathcal{G}$ are also useful to understand the issue of measurability of families of germs $(\mathcal{G}(\sigma))_{\sigma\in\Sigma}$.

4 Applications

One important application was to understand the vanishing viscosity limit for (MP); this was done in [6] (cf. [13]), and extended in [7] to the multi-dimensional setting with a smooth curved interface. The Buckley-Leverett vanishing capillarity limit studied in [3] and the two original applications below allowed, in a sense, to validate the theory (notice that most of the examples given in [6] were already understood in the previous contributions to the subject). The two new applications came from the works [12] and [18], respectively, where the respective notions of admissible solution were already fixed; uncovering the germs underlying these notions, one gets well-posedness quite easily, with the tools of [6]. The second application shows that the theory extends without difficulty to the case of non-conservative (but L^1-dissipative) coupling.

For both applications, we construct convergent "naive" finite volume schemes, and it has to be stressed that the issue of convergence cannot be settled uniquely by the general arguments of Theorem 2.2; additional rather delicate observations and arguments are needed for finer schemes.

• **Road traffic with point constraint**

Consider (1.1) with a flux f with $f(0) = 0 = f(1)$, f nonnegative, Lipschitz, with $f'(\cdot)$ changing sign at $\bar{u} = \operatorname{argmax} f$; take initial data satisfying $0 \le u_0(\cdot) \le 1$. For $F \in [0, f(\bar{u})]$, the F-level set of $f(\cdot)$ consists of two points: $B_F \le \bar{u} \le A_F$. Then we define $\mathcal{G}(t) := \{(A_{F(t)}, B_{F(t)})\}$ and compute that the unique maximal L^1D extension of $\mathcal{G}(t)$ is

$$\mathcal{G}(t)^* := \mathcal{G}(t) \cup \{\text{couples } (c^l, c^r) \text{ such that } f(c^{l,r}) \le F \text{ and } c^l \le c^r\}$$

(in fact, all the couples in $\mathcal{G}(t)^* \setminus \mathcal{G}(t)$ correspond to Kruzhkov admissible stationary solutions of $u_t + f(u)_x = 0$ satisfying $f(u) \le F$, while the stationary solution $c(x) = A_F 1\!\!1_{x<0} + B_F 1\!\!1_{x<0}$ is a non-Kruzhkov shock). Thus for all $t > 0$, $\mathcal{G}(t)$ is a definite germ.

Then we define the associated $\mathcal{G}(\cdot)$-entropy solutions (here the family of germs is defined via an L^∞ function $F(\cdot)$, and the measurability of R in Definition 3.1 is straightforward). Uniqueness and L^1 contraction are immediate; for existence, one first works with piecewise constant $F(\cdot)$, then extends the result using (3.6). While Theorem 2.2 applies for $F(t) \equiv const$, let us sketch a proof by converging vanishing viscosity approximation. We study the approximation $u_t + f(u)_x = \varepsilon(k(x,u))_{xx}$ with $k(x, \cdot) := a(\cdot)1\!\!1_{x<0} + b(\cdot)1\!\!1_{x>0}$ for $a(\cdot)$ defined from A_F by

$$a : r \in [0,1] \mapsto \tfrac{1}{2A_F} \min\{r, A_F\} + \tfrac{1}{2(1-A_F)}(r - A_F)^+,$$

and $b(\cdot)$ defined accordingly from B_F. The point is that $a(\cdot), b(\cdot)$ are increasing, $a(0) = b(0)$, $a(A_F) = b(B_F)$, and $a(1) = b(1)$; therefore $u_-^\varepsilon := 0$, $u_+^\varepsilon := 1$ and $c^\varepsilon(x) := A_F 1\!\!1_{x<0} + B_F 1\!\!1_{x>0}$ are explicit solutions, independent of $\varepsilon > 0$. Moreover, introducing the new unknown $w(t,x) := k(x, u(t,x))$, it is easy to check that the *ad hoc* Kato inequality holds for solutions of the problem. This implies comparison principle that allows to bound the solutions u^ε of the viscous equation (with initial datum u_0) by $0 \equiv u_-^\varepsilon \le u^\varepsilon \le u_+^\varepsilon \equiv 1$. Assuming that the level sets of $f'(\cdot)$ do not contain intervals, we can use the compactification results (see e.g. [20]) and deduce that u^ε converge a.e. to a limit u which is a weak solution of $u_t + f(u) = 0$ and also an entropy solution away from the interface. Moreover, the passage to the limit in the Kato inequality written for u^ε and $\hat{u} := c^\varepsilon(x)$ yields the adapted entropy inequality (2.5). Thus u is indeed a $\mathcal{G}$-entropy solution of our problem, because $\{(A_F, B_F)\}$ is a definite germ with the same maximal extension as $\mathcal{G}$.

In [4], we have proved existence for (1.1) by making converge a finite volume scheme with a monotone consistent numerical flux $g(\cdot,\cdot)$: we simply re-define the flux at the interface $\{x = 0\}$ with $g_F := \min\{g, F\}$. Thus, the scheme avoids the intricate Godunov solver used in the proof of Theorem 2.2. Although seemingly naive, the scheme of [4] is partially well-balanced in the sense that it preserves a sufficiently large part of the stationary solutions described by $\mathcal{G}^*$ (namely, constant functions and the key function $c(x) = A_F 1\!\mathrm{l}_{x<0} + B_F 1\!\mathrm{l}_{x<0}$ are solutions of the scheme). Using the measure-valued (entropy-process) solution techniques (the above existence result is needed), we deduce convegence of the "naive" scheme.

• **Particle-in-Burgers model**

The interface coupling and the associated Riemann solver for the non-conservative problem (1.2) were fully described in [18]. In [8], we observed that the set of admissible one-sided traces fulfills the "L^1D" inequality of (2.1) (the Rankine-Hugoniot condition in (2.1) has to be dropped); moreover, it admits no L^1D extension, thus it is maximal. As previously, we denote it $\mathcal{G}$ (here $\mathcal{G}^* = \mathcal{G}$). Then $\mathcal{G}$-entropy solutions enjoy uniqueness, in the same way as in Section 2. Also existence can be shown with the Godunov scheme, as in Theorem 2.2, but once more we prefer a "naive" scheme that is by far simpler to implement (indeed, the Riemann solver of [18] involves a bunch of different cases). We have

$$\mathcal{G} = \big\{(c^l, c^r) \mid c^l - c^r = 1 \text{ or } (\, c^l \geq 0,\ c^r \leq 0,\ -1 \leq c^l + c^r \leq 1\,)\big\};$$

it should be stressed that, in a sense, the line $\mathcal{L}$: "$c^l - c^r = 1$" of $\mathbb{R}^2$ is the key part of $\mathcal{G}$. Unfortunately, this part itself is not a definite germ, but adjoining the square $\mathcal{Q} := [0,1] \times [-1,0]$ to the line $\mathcal{L}$, we get a definite part of the germ $\mathcal{G}$. The finite volume scheme with a consistent monotone numerical flux $g(\cdot,\cdot)$ satisfying some additional constraint (which is non-restrictive in practice) is constructed by modifying $g(\cdot,\cdot)$ at the interface. Recall that the conservation at the interface is lost; we define the left interface flux by $g_0^l(u_-, u_+) := g(u_-, u_+ + 1)$ (as if we wanted to connect the right state u_+ to the associated state on the line $\mathcal{L}$), and the right interface flux by $g_0^r(u_-, u_+) := g(u_- - 1, u_+)$ (as if we wanted to connect the left state u_- to the associated state on the line $\mathcal{L}$). By construction of $g_0^{l,r}$, the scheme is partially well-balanced: it preserves the stationary solutions $c(x) = c^l 1\!\mathrm{l}_{x<0} + c^r 1\!\mathrm{l}_{x>0}$ for $(c^l, c^r) \in \mathcal{L}$. The part $\mathcal{L}$ of $\mathcal{G}$ is not definite, that's why we prove in addition that the stationary solutions with $(c^l, c^r) \in \mathcal{Q}$ are preserved asymptotically (they are obtained at the limit of our "naive" scheme). Then we use the arguments analogous to those of Theorem 2.2 to pass to the limit in the scheme.

Problem (1.2) is a particular case of the Burgers equation with source term $-\lambda(u-h'(t))\,\delta_0(x-h(t))$, where $h \in W^{2,\infty}(0,T)$ defines an internal interface. We first consider the interface $\{x = h(t)\}$ as being fixed; later on, we couple the conservation law to the interface equation that formally reads as $mh''(t) = \lambda(u(t,h(t))-h'(t))$. The latter relation means that the particle of mass m is driven by the difference $u - h'(t)$ of fluid and particle velocities, with a viscosity coefficient λ (see [18, 5] for details); we fix $\lambda = 1$. The rigorous interpretation of the particle-driving equation relates $h''(t)$ to the jump of the normal traces on the interface $\{x = h(t)\}$ of the flux $(u, u^2/2)(t,x)$ of the Burgers equation (see [18] and [5]).

In [8, 5] we start with the case of a straight particle path $h(t)=Vt$; actually, a simple change of variables reduces the problem to the case $V = 0$, i.e. to problem (1.2), and the associated germ $\mathcal{G}_V$ is $\mathcal{G}+(V,V)$ ($\mathcal{G}$ being the germ for problem (1.2)). We get well-posedness for the singular conservation law driven by the source located at $\{x = Vt\}$. Next, we construct solutions to $u_t + (u^2/2)_x = -(u-h'(t))\,\delta_0(x-h(t))$ by approximating a general given path $h(\cdot)$ by piecewise affine continuous functions. These solutions have to be interpreted in the way of Section 3, as $\mathcal{G}(\cdot)$-entropy solutions with the family of germs $\mathcal{G}(t) := \mathcal{G} + (h'(t), h'(t))$. Uniqueness is ensured by the general theory (cf. Section 3). Moreover, in a work in progress of F. Lagoutière, N. Seguin and T. Takahashi and the author it is shown that, constructing solutions with the wave-front tracking algorithm, we can ensure a uniform BV bound on the solution in terms of the variation of the initial datum u_0 and the variation of $h'(\cdot)$ (as the Godunov scheme, the WFT algorithm is more difficult to use in practice because a perfect knowledge of the Riemann solver is required; yet in our case, it yields finer uniform estimates). At this point, fixed-point arguments can be used for the coupled problem of [18]. In [5], we give a sketch of existence proof for L^∞ data. Moreover, exploiting the techniques of continuous dependence of BV solutions of $u_t + f(u)_x = s$ on the flux f, thanks to the Gronwall inequality we deduce uniqueness of BV solutions of the free-interface coupled problem introduced in [18].

References

[1] Adimurthi, S. Mishra, and G. D. Veerappa Gowda. Optimal entropy solutions for conservation laws with discontinuous flux-functions. *J. Hyp. Diff. Equ.*, 2 (2005), no.4, pp.783–837.

[2] Adimurthi and G. D. Veerappa Gowda. Conservation laws with discontinuous flux. *J. Math. Kyoto University*, 43 (2003), no.1, pp.27–70.

[3] B. Andreianov and C. Cancès. Vanishing capillarity solutions of Buckley-Leverett equation with gravity in two-rocks' medium. *In preparation.*

[4] B. Andreianov, P. Goatin and N. Seguin. Finite volume schemes for locally constrained conservation laws. *Numer. Math.* 115 (2010), pp.609-645.

[5] B. Andreianov, F. Lagoutière, N. Seguin, and T. Takahashi. Small solids in an inviscid fluid. *Netw. Heter. Media*, 5 (2010), no.3, pp.385–404.

[6] B. Andreianov, K.H. Karlsen, and N.H. Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. *Arch. Rat. Mech. Anal.* 2011, DOI: 10.1007/s00205-010-0389-4.

[7] B. Andreianov, K.H. Karlsen, and N.H. Risebro. On vanishing viscosity approximation of conservation laws with discontinuous flux. *Netw. Heter. Media*, 5 (2010), no.3, pp.617-633.

[8] B. Andreianov and N. Seguin. Well-posedness of a singular balance law. *Preprint 2011, http://hal.archives-ouvertes.fr/hal-00576959*

[9] E. Audusse and B. Perthame. Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies. *Proc. Roy. Soc. Edinburgh A*, 135 (2005), no.2, pp.253–265.

[10] P. Baiti and H. K. Jenssen. Well-posedness for a class of 2×2 conservation laws with L^∞ data. *J. Differ. Equ.*, 140 (1997), no.1, pp.161–185.

[11] R. Bürger, K. H. Karlsen, and J. Towers. An Engquist-Osher type scheme for conservation laws with discontinuous flux adapted to flux connections. *SIAM J. Numer. Anal.*, 47 (2009), pp.1684-1712.

[12] R. Colombo and P. Goatin. A well posed conservation law with a variable unilateral constraint. *J. Differ. Equ.*, 234 (2007), no.2, pp.654–675.

[13] S. Diehl. A uniqueness condition for nonlinear convection-diffusion equations with discontinuous coefficients. *J. Hyp. Diff. Eq.* 6(2009), pp.127-159.

[14] M. Garavello, R. Natalini, B. Piccoli, and A. Terracina. Conservation laws with discontinuous flux. *Netw. Heter. Media*, 2 (2007), pp.159–179.

[15] T. Gimse and N. H. Risebro. Solution of the Cauchy problem for a conservation law with a discontinuous flux function. *SIAM J. Math. Anal.*, 23 (1992), no.3, pp.635–648.

[16] K. H. Karlsen, N. H. Risebro, and J. D. Towers. L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients. *Skr. K. Nor. Vidensk. Selsk.*, 3 (2003).

[17] S. N. Kruzhkov. First order quasilinear equations with several independent variables, *Mat. Sb.(N.S)* 81(123) (1970), pp.228-255.

[18] F. Lagoutière, N. Seguin, and T. Takahashi. A simple 1D model of inviscid fluid-solid interaction. *J. Differ. Equ.*, 245 (2008), no.11, pp.3503-3544.

[19] E. Yu. Panov. Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyp. Diff. Equ.*, 4 (2007), no.4, pp.729-770.

[20] E. Yu. Panov. Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux. *Arch. Rat. Mech. Anal.* 195 (2009), no.2, pp.643-673.

[21] J. D. Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. An.*, 38 (2000), pp.681-698.