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Uniqueness for an elliptic-parabolic problem with Neumann boundary condition

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Dedicated to the memory of Philippe Bénilan

Abstract. We consider the problem $b(u) - \Delta u + \operatorname{div} F(u) = f$ in a smooth boundary domain $\Omega \subset \mathbb{R}^N$, as well as the corresponding evolution equation $b(u)_t - \Delta u + \operatorname{div} F(u) = f$, $b(u(0,.)) = b^0$. For the stationary equation we show existence results, then we adapt the techniques of doubling of variables to the case of the homogeneous Neumann boundary conditions and obtain the appropriate L^1-contraction principle and uniqueness. Subsequently, we are able to apply the nonlinear semigroup theory and prove the L^1-contraction principle for the associated evolution equation.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. We consider the Neumann problems

$$\begin{cases} b(u) - \Delta u + \operatorname{div} F(u) = f & \text{in } \Omega \\ (\nabla u - F(u)) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \qquad P(b, F)(f)$$

and

$$\begin{cases} b(u)_t - \Delta u + \operatorname{div} F(u) = f & \text{in } Q = (0, T) \times \Omega \\ (\nabla u - F(u)) \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega \\ b(u)(t = 0) = b^o & \text{in } \Omega, \end{cases} \qquad E(b, F)(f, b^o)$$

where ν is the exterior unit normal vector to $\partial\Omega$.

Here,

$$b : \mathbb{R} \longrightarrow \mathbb{R} \text{ is increasing, normalised by } b(0) = 0,$$
$$F : \mathbb{R} \longrightarrow \mathbb{R}^N \text{ is continuous, normalised by } F(0) = 0.$$

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For the Dirichlet or mixed Dirichlet-Neumann boundary conditions, different uniqueness results for elliptic-parabolic equations of the form $b(u)_t + \operatorname{div} a(u, \nabla u) = f$ have been obtained by Alt-Luckhaus [2], Diaz-de Thélin [13], Otto [18], Bénilan-Wittbold [7], assuming in particular that $a(\cdot, \cdot)$ is Hölder continuous in the first argument of order $\alpha \geq \text{const} > 0$. These results cover the case of the equations we consider if $\alpha \geq 1/2$.

Results for the stationary problem and the corresponding abstract evolution problem are given by Simondon [19], Bénilan-Touré [5], [6] and by Bénilan-Wittbold [7].

Developping the Kruzhkov's ideas of doubling of variables ([15]), Carrillo proves in [10] uniqueness results for hyperbolic-elliptic-parabolic equations without any Hölder continuity assumption, in the case of homogeneous Dirichlet boundary conditions; further results on renormalized solution are given in [11]. In this paper, we show how one can use the doubling of variables in case of Neumann boundary conditions. We use the techniques of [10] and [7], the existence of solutions to the stationary problem which are regular up to the boundary, and the general nonlinear semigroup theory. To have regular solutions, we assume the local Hölder continuity of F of unrestricted order $\alpha > 0$ (here we use a result of Lieberman [16]).

Let us give the main definitions and results of this paper.

First we consider the stationary problem $P(b, F)(f)$.

DEFINITION 1. Let $f \in L^1(\Omega)$. A function $u \in H^1(\Omega)$ such that $b(u) \in L^1(\Omega)$ and $F(u) \in L^2(\Omega)$ is a weak solution of $P(b, F)(f)$ if for all $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$, one has

$$\int_\Omega b(u)\,\xi + \int_\Omega (\nabla u - F(u)) \cdot \nabla \xi = \int_\Omega f\,\xi. \tag{1}$$

An existence result for weak solutions of $P(b, F)(f)$ with $f \in L^2(\Omega)$ is shown in Section 2, under the assumptions

$$\begin{cases} \text{there exist } c > 0, \delta > 0 \text{ such that} \\ |F(z)|^2 \leq c(1 + |z|^2 + (zb(z))^{1-\delta}) \text{ for all } z \in \mathbb{R}, \end{cases} \tag{$H1$}$$

and

$$b(+\infty) = +\infty,\ b(-\infty) = -\infty.$$

We rewrite this last hypothesis under the equivalent form:

$$\begin{cases} \text{there exists a function } \beta\ :\ \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that} \\ \beta(r) \to +\infty \text{ as } r \to +\infty \text{ and } |b(z)| \geq \beta(|z|) \text{ for all } z \in \mathbb{R}. \end{cases} \tag{$H2$}$$

In Section 3, we prove the uniqueness of $b(u)$ such that u is a weak solution of $P(b, F)(f)$ and the L^1contraction principle (2) below. In case F is Hölder continuous with exponent $\alpha \geq 1/2$, (2) is easy to prove. In this paper, we assume

$$\begin{cases} \text{for all compact set } K \subset \mathbb{R} \text{ there exists } \alpha > 0 \text{ such that} \\ F \text{ is Hölder continuous of order } \alpha \text{ on } K. \end{cases} \tag{$H3$}$$

We show that a weak solution satisfies entropy inequalities in the spirit of Kruzhkov [15] and Carrillo [10], and adapt their techniques to establish:

THEOREM 1. *Suppose* $(H1), (H2), (H3)$ *hold, and* $\partial\Omega \in \mathcal{C}^2$. *Let* $f, g \in L^1(\Omega)$ *and* u, v *be weak solutions of* $P(b, F)(f)$ *and* $P(b, F)(g)$*, respectively. Then*

$$\|b(u) - b(v)\|_{L^1(\Omega)} \leq \|f - g\|_{L^1(\Omega)}. \tag{2}$$

In particular, there exists at most one function $b(u)$ *such that* u *is a weak solution of* $P(b, F)(f)$.

In the proof, we first show (2) in the case one of the two solutions is regular up to the boundary. The fact that there exists a weak solution v of $P(b, F)(g)$ which belongs to $\mathcal{C}^1(\overline{\Omega})$ is ensured by the hypotheses $(H1)$, $(H2)$, $(H3)$ whenever $g \in L^\infty(\Omega)$ (cf. Proposition 2). The result then follows by a density argument.

Next we consider the elliptic-parabolic problem $E(b, F)(f, b^o)$.

DEFINITION 2. Let $f \in L^1(Q)$, $b^o \in L^1(\Omega)$. A function $u \in L^2(0, T; H^1(\Omega))$ such that $b(u) \in L^1(Q)$ and $F(u) \in L^2(Q)$ is a weak solution of $E(b, F)(f, b^o)$ if for all $\xi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$ such that $\xi_t \in L^\infty(Q)$ and $\xi(T) = 0$, one has

$$\int_0^T \int_\Omega (b^o - b(u))\, \xi_t + \int_0^T \int_\Omega (\nabla u - F(u)) \cdot \nabla\xi = \int_0^T \int_\Omega f\, \xi\ . \tag{3}$$

Different existence results for weak solutions of $E(b, F)(f, b^o)$ under additional assumptions on f and b^o were obtained, using in particular the techniques of [2] (eg., cf. [14]).

For the uniqueness, we need one more hypothesis

$$\begin{cases} \text{there exist } c > 0, \delta > 0 \text{ such that} \\ |F(z)|^2 \leq c(1 + |z|^{2(1-\delta)} + zb(z)) \text{ for all } z \in \mathbb{R}. \end{cases} \tag{$H4$}$$

In this case, the results obtained for the stationary problem permit to apply the non-linear semigroup theory and obtain the existence and uniqueness for integral solutions of the associated abstract evolution problem (the problem $S(b, F)(f, b^o)$ in Section 4). This problem is closely related to $E(b, F)(f, b^o)$. More exactly, using again the techniques of doubling of variables in space, we show that if u is a weak solution of $E(b, F)(f, b^o)$, then $w = b(u)$ is an integral solution of $S(b, F)(f, b^o)$. These results, gathered in Section 4, yield

THEOREM 2. *Suppose* $(H1), (H2), (H3), (H4)$ *hold and* $\partial\Omega \in \mathcal{C}^2$. *Let* $f, \hat{f} \in L^1(Q), b^o, \hat{b}^o \in L^1(\Omega)$ *and* $u, \hat{u}$ *be weak solutions of* $E(b, F)(f, b^o)$ *and* $E(b, F)(\hat{f}, \hat{b}^o)$ *respectively. Then for a.a* $t \in (0, T)$,

$$\|b(u(t)) - b(\hat{u}(t))\|_{L^1(\Omega)} \leq \|b^o - \hat{b}^o\|_{L^1(\Omega)} + \int_0^t \|f(\tau) - \hat{f}(\tau)\|_{L^1(\Omega)} d\tau.$$

In particular, there exists at most one function $b(u)$ *such that* u *is a weak solution of* $E(b, F)(f, b^o)$.

At the present stage, it is not clear to the authors whether the uniqueness of $b(u)$ for $P(b, F)(f)$ and $E(b, F)(f, b^o)$ holds without the hypothesis $(H3)$, that is, for only continuous F. Note that this is true for $N = 1$, since all weak solution of $P(b, F)(f)$ is in $\mathcal{C}^1(\overline{\Omega})$ in this case.

2. Existence of weak and regular solutions to the stationary problem

First we establish some a priori estimates.

LEMMA 1. *Suppose* $(H1)$, $(H2)$ *hold. Let* $f \in L^2(\Omega)$ *and* u *be a weak solution of* $P(b, F)(f)$. *Then* $u\, b(u) \in L^1(\Omega)$ *and there exists a constant* C *which depends only on* Ω, β, $\|f\|_{L^2(\Omega)}$, c *and* δ *such that* $\|u\|_{H^1(\Omega)} \leq C$ *and* $\|u\, b(u)\|_{L^1(\Omega)} \leq C$.

Throughout the paper, we abbreviate the notations for sets; for instance, the set $\{x \in \Omega,\ |u(x)| > k\}$ is denoted by $\{|u| > k\}$ and its characteristic function is denoted by $\chi_{\{|u|>k\}}$. For a set $\mathbb{E} \subset \mathbb{R}^N$, we denote by $|\mathbb{E}|$ its Lebesgue measure. The notation $H_\varepsilon(\cdot)$ is used for the approximation of $\operatorname{sign}(\cdot)$ given by

$$H_\varepsilon(r) = \begin{cases} 1 & \text{if} \quad r \geq \varepsilon \\ r/\varepsilon & \text{if} \quad -\varepsilon \leq r \leq \varepsilon \\ -1 & \text{if} \quad -\varepsilon \leq r. \end{cases}$$

We denote $H_\varepsilon{}'(r) = \frac{1}{\varepsilon}\chi_{\{|r|<\varepsilon\}}$; for all $w \in H^1(\Omega)$, one has $\nabla H_\varepsilon(w) = H_\varepsilon{}'(w)\nabla w$ in $L^2(\Omega)$.

Finally, we denote by 2^* the number $\frac{2N}{N-2}$ if $N > 2$ and an arbitrary number $2^* > 2$ if $N = 1$ or $N = 2$.

REMARK 1. Let $u,\ \xi \in H^1(\Omega)$ and $k \in \mathbb{R}$. Then

$$\int_\Omega (F(u) - F(k)) \cdot \nabla u\ H_\varepsilon{}'(u-k)\, \xi \ \to 0 \ \text{ as } \varepsilon \to 0.$$

Proof. Set $\psi_\varepsilon(r) = \int_{k-\varepsilon}^{r} (F(s) - F(k))\, H_\varepsilon{}'(s-k)\, ds$. Note that $|\psi_\varepsilon(r)| \leq 2\,\omega(\varepsilon)$, where $\omega : \varepsilon \in \mathbb{R} \mapsto \sup_{|s-k|<\varepsilon} |F(s) - F(k)|$ is the modulus of continuity of F at the point k.

It follows by the Green-Gauss Formula that

$$\left| \int_\Omega H_\varepsilon{}'(u-k)\, \nabla u \cdot (F(u) - F(k))\, \xi \right| = \left| \int_\Omega \operatorname{div}(\psi_\varepsilon(u))\, \xi \right|$$
$$\leq 2\,\omega(\varepsilon) \left(\int_\Omega |\nabla \xi| + \int_{\partial\Omega} |\xi| \right) \to 0 \ \text{ as } \varepsilon \to 0. \qquad \square$$

Proof of Lemma 1. We denote by C all constant depending only on Ω, $\|f\|_{L^2(\Omega)}$, β in $(H2)$ and c, δ in $(H1)$.

CLAIM 1. $|\{|u| > \kappa\}| \leq \frac{C}{\beta(\kappa)}$ for all $\kappa > 0$.

Take $H_\varepsilon(u) \in H^1(\Omega) \cap L^\infty(\Omega)$ as a test function in (1). We have

$$\int_\Omega b(u) H_\varepsilon(u) + \int_\Omega |\nabla u|^2 H_\varepsilon{}'(u) - \int_\Omega F(u) \cdot \nabla u \, H_\varepsilon{}'(u) = \int_\Omega f \, H_\varepsilon(u).$$

The second term is non-negative; in the other terms we use Remark 1, the Lebesgue dominated convergence theorem, and pass to the limit as $\varepsilon \to 0$ to get

$$\int_\Omega |b(u)| \leq \int_\Omega |f| \leq C. \tag{4}$$

Therefore by $(H2)$, for all $\kappa \geq 0$ we have $\beta(\kappa)|\{|u| > \kappa\}| \leq \int_\Omega |f| \leq C$. Hence Claim 1 follows.

CLAIM 2. for all $\kappa \geq 0$, one has

$$\delta \int_{\{|u|>\kappa\}} u \, b(u) + \int_{\{|u|>\kappa\}} |\nabla u|^2 \leq C \left(\kappa + 1 + \int_{\{|u|>\kappa\}} |u|^2 \right).$$

First note that for all increasing globally Lipschitz continuous function ϕ, we have $b(u)\phi(u) \in L^1(\Omega)$ and

$$\int_\Omega b(u) \, \phi(u) + \int_\Omega (\nabla u - F(u)) \cdot \nabla \phi(u) = \int_\Omega f \, \phi(u). \tag{5}$$

Indeed, one can take $T_k(\phi(u))$ as a test function in (1), where

$$T_k(r) = \begin{cases} r & \text{if } \; |r| < k \\ k \,\text{sign}(r) & \text{if } \; |r| \geq k, \end{cases} \tag{6}$$

and pass to the limit as $k \to \infty$ using Levy's and Lebesgue's theorems to obtain (5).

In particular, with $\phi(u) = (|u| - \kappa)^+ \text{sign}(u)$, we get

$$\int_{\{|u|>\kappa\}} |b(u)|(|u| - \kappa)^+ + \int_{\{|u|>\kappa\}} |\nabla u|^2 \leq \int_{\{|u|>\kappa\}} F(u) \cdot \nabla u + \int_{\{|u|>\kappa\}} |f| \, |u|.$$

Using the Young inequality, $(H1)$ and the Hölder inequality, we obtain

$$\int_{\{|u|>\kappa\}} |b(u)|(|u| - \kappa)^+ + \frac{1}{2} \int_{\{|u|>\kappa\}} |\nabla u|^2 \leq C \left(1 + \int_{\{|u|>\kappa\}} |u|^2 \right) + \int_{\{|u|>\kappa\}} \frac{c}{2} \, (u b(u))^{1-\delta}.$$

Applying the Young inequality in the last term and using (4), we finally get Claim 2.

CLAIM 3. $\int_\Omega |u|^{2^*} \leq C$.

For all $v \in H^1(\Omega)$, the Sobolev inequality (e.g., cf. [1]) can write

$$\int_\Omega |v|^{2^*} \leq C \left(\left(\int_\Omega |\nabla v|^2 \right)^{\frac{2^*}{2}} + \left(\int_\Omega |v|^2 \right)^{\frac{2^*}{2}} \right).$$

Take $v = \phi(u)$. By Claim 2 and the Hölder inequality, we have

$$I_\kappa = \int_\Omega ((|u| - \kappa)^+)^{2^*} \leq C \left(\kappa^{2^*} + 1 + \left(\int_{\{|u|>\kappa\}} |u|^{2^*} \right) |\{|u| > \kappa\}|^{\frac{2^*}{2}-1} \right).$$

Since $|u| \leq (|u| - \kappa)^+ + \kappa$, we get

$$I_\kappa \leq C(\kappa^{2^*} + 1 + (I_\kappa + \kappa^{2^*})|\{|u| > \kappa\}|^{\frac{2^*}{2}-1}).$$

By Claim 1, $|\{|u| > \kappa\}| \to 0$ as $\kappa \to +\infty$ and we can choose κ_o (depending on β) such that

$$\frac{1}{2} I_{\kappa_o} \leq C \left(1 + \frac{3}{2} (\kappa_o)^{2^*} \right) \leq C.$$

Since $|u| \leq (|u| - \kappa_o)^+ + \kappa_o$, Claim 3 follows.

Finally, we deduce from Claim 3 that $\int_\Omega |u|^2 \leq C$, and take $\kappa = 0$ in Claim 2 to obtain the desired estimates.

PROPOSITION 1. *Suppose* $(H1)$ *and* $(H2)$ *hold. Let* $f \in L^2(\Omega)$. *Then there exists a weak solution to* $P(b, F)(f)$.

In the proof, we use the following

LEMMA 2. *Let V be a reflexive separable Banach space, and A be an operator from V to its dual V'. Suppose that A is coercive, i.e,* $\frac{\langle Av, v\rangle_{V',V}}{\|v\|_V}$ *tends to* $+\infty$ *as* $\|v\|_V$ *tends to* $+\infty$*, and that A is continuous for the weak topologies of V and V'. Then A is surjective.*

This lemma can be proved as in [17] Chap. 2, Theorem 2.1, using Galerkin approximations. Also recall the De La Vallée Poussin Lemma:

LEMMA 3. *Let* $\Omega \subset \mathbb{R}^N$ *be of finite measure,* $f_n \to f$ *a.e. on* Ω *and* $|f_n| \leq \mathcal{L}(|g_n|)$ *for some sequence* $(g_n)_n$ *bounded in* $L^q(\Omega)$, $1 \leq q < \infty$ *and some function* $\mathcal{L} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *such that* $\mathcal{L}(r)/r \to 0$ *as* $r \to +\infty$. *Then* $f_n \to f$ *in* $L^q(\Omega)$.

The proof of this lemma consists in showing that the sequence $(|f_n|^q)_n$ is equi-integrable on Ω, and then applying the Egorov theorem.

Proof of Proposition 1. For $n \in \mathbb{N}$, set $F_k(z) = F(T_k(z))$, where $T_k(\cdot)$ is the cut-off function in (3), and set

$$b_k(z) = \begin{cases} z + k + b(-k), & z \le -k, \\ b(z), & |z| < k, \\ z - k + b(k), & z \ge k. \end{cases}$$

Note that $F_k \to F$ and $b_k \to b$ uniformly on all compact subset of $\mathbb{R}$. In addition,

$$\min_{k \in \mathbb{N}} |b_k(z)| = \min_{|\zeta| \le |z|} (|z| - |\zeta| + |b(\zeta)|) \ge \min\{|z|/2, \min_{|z|/2 \le |\zeta| \le |z|} |b(\zeta)|\},$$

which tends to infinity as $z \to \infty$, because b satisfies $(H2)$. Therefore $(H2)$ is satisfied by $b_k(\cdot)$ with a function $\beta(\cdot)$ independent of k; also $(H1)$ is satisfied by $b_k(\cdot)$, $F_k(\cdot)$ with c, δ independent of k. Furthermore, there exist $M_k > 0$, $R_k > 0$ such that $|F_k(z)| \le M_k$ and $z b_k(z) \ge |z|^2 - R_k$, for all $z \in \mathbb{R}$.

Define $A_k : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ by

$$\langle A_k(u), \xi \rangle_{(H^1(\Omega))', H^1(\Omega)} = \int_\Omega b_k(u)\, \xi + (\nabla u - F_k(u)) \cdot \nabla \xi$$

for $u,\ \xi \in H^1(\Omega)$; note that the integral in the right-hand side always makes sense.

It can be easily checked that A_k satisfy the hypotheses of Lemma 2. Hence there exists $u_k \in H^1(\Omega)$ such that

$$\int_\Omega b_k(u_k)\xi + (\nabla u_k - F_k(u_k))\, \nabla \xi = \int_\Omega f\, \xi \quad \text{for all } \xi \in H^1(\Omega). \tag{7}$$

Since $u_k b_k(u_k) \in L^1(\Omega)$ and $F_k(u_k) \in L^2(\Omega)$, u_k is a weak solution to $P(b_k, F_k)(f)$. By Lemma 1, $\|u_k\|_{H^1(\Omega)}$ and $\|u_k b_k(u_k)\|_{L^1(\Omega)}$ are bounded by a constant independent of k. There exists $u \in H^1(\Omega)$ and a subsequence, which we still denote $(u_k)_k$, such that $\nabla u_k \to \nabla u$ weakly in $L^2(\Omega)$ and $u_k \to u$ a.e on Ω; moreover, by the Sobolev injection theorem, $(u_k)_k$ is bounded in $L^{2^*}(\Omega)$. Due to the hypothesis $(H1)$ and the uniform on all compact set convergence of F_k to F, we can apply Lemma 3 and obtain that $F_k(u_k) \to F(u)$ in $L^2(\Omega)$. Finally, note that $|b_k(z)| \le |z| + |b(z)|$ for all z, k; therefore there exists a function $\mathcal{L}(\cdot)$ such that $\mathcal{L}(r)/r \to 0$ as $r \to +\infty$ and $|b_k(z)| \le \mathcal{L}(z b_k(z))$ for all k, z. Hence by Lemma 3 we also have $b_k(u_k) \to b(u)$ in $L^1(\Omega)$.

Now for all $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$, we can pass to the limit as $k \to \infty$ in (7) to obtain that u is a weak solution of $P(b, F)(f)$.

Now let us consider the case of bounded function f.

LEMMA 4. *Suppose* $(H1)$ *and* $(H2)$ *hold. Let* $f \in L^{\infty}(\Omega)$ *and* u *be a weak solution of* $P(b, F)(f)$. *Then* $u \in L^{\infty}(\Omega)$*and there exists a constant* C *which depends only on* $\Omega, \beta, \|f\|_{L^{\infty}(\Omega)}, c$ *and* δ *such that* $\|u\|_{L^{\infty}(\Omega)} \leq C$.

Proof. We use the Moser iteration techniques.

CLAIM 1. Let $p \geq 2$. If $u \in L^p(\Omega)$, then $u \in L^{\gamma p}(\Omega)$, where $\gamma = 2^*/2 > 1$, and

$\|u\|_{L^{\gamma p}(\Omega)} \leq (C_o p^{1+1/\delta})^{1/p} \|u\|_{L^p(\Omega)}$ in case $\|u\|_{L^p(\Omega)} \geq 1$.

Take $T_k(|u|^{p-2}u)$ as a test function in (1), where T_k is the cut-off function defined in (6). Since $b(u)T_k(|u|^{p-2}u)$is non-negative, we get

$$\int_{\{|u|^{p-1}<k\}} |u|^{p-2}ub(u) + (p-1)\int_{\{|u|^{p-1}<k\}} |\nabla u|^2|u|^{p-2}$$
$$\leq (p-1)\int_{\{|u|^{p-1}<k\}} F(u)\cdot\nabla u|u|^{p-2} + \|f\|_{L^{\infty}(\Omega)}\int_{\Omega}|u|^{p-1}.$$

Since $F(u)\cdot\nabla u|u|^{p-2} \leq \frac{1}{2}|F(u)|^2|u|^{p-2} + \frac{1}{2}|\nabla u|^2|u|^{p-2}$, by $(H1)$ we have

$$\int_{\{|u|^{p-1}<k\}} |u|^{p-2}ub(u) + \frac{(p-1)}{2}\int_{\{|u|^{p-1}<k\}} |\nabla u|^2|u|^{p-2}$$
$$\leq C(p-1)\int_{\Omega}(|u|^{p-2} + |u|^{p-1} + |u|^p)$$
$$+ \frac{c}{2}(p-1)\int_{\{|u|^{p-1}<k\}} |u|^{p-2}(ub(u))^{1-\delta}.$$

Using the Young inequality for the last term, by the Fatou Lemma we deduce that $|\nabla u|^2|u|^{p-2} \in L^1(\Omega)$; since $p \geq 2$ and δ in $(H1)$ can be taken less than or equal to 1, we get

$$\frac{(p-1)}{2}\int_{\Omega}|\nabla u|^2|u|^{p-2} \leq C(p-1)^{1/\delta}\int_{\Omega}|u|^{p-2} + |u|^{p-1} + |u|^p, \tag{8}$$

where C is independent of p. Moreover, if $\|u\|_{L^p(\Omega)} \geq 1$, (8) implies that

$$\int_{\Omega}|\nabla u|^2|u|^{p-2} \leq Cp^{1/\delta}\int_{\Omega}|u|^p, \tag{9}$$

where C is still independent of p.

Note that $|\nabla u||u|^{\frac{p}{2}-1} = \frac{2}{p}|\nabla v|$, where $v = |u|^{\frac{p}{2}-1}u$. From (9) and the Sobolev injection $\|v\|^2_{L^{2^*}(\Omega)} \leq C(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})$, it follows that $u \in L^{\gamma p}(\Omega)$ and $\|u\|^p_{L^{\gamma p}(\Omega)} = \|v\|^2_{L^{2^*}(\Omega)} \leq C_o p^{2+1/\delta}\|u\|^p_{L^p(\Omega)}$ in case $\|u\|_{L^p(\Omega)} \geq 1$. This proves Claim 1.

Now suppose that $\operatorname{ess\,sup}|u| > 1$ (otherwise, there is nothing to prove). Then there exists $p_o \geq 2$ such that $\|u\|_{L^p(\Omega)} \geq 1$ for all $p \geq p_o$. Iterating the estimate in Claim 1, we get

$$\|u\|_{L^{\gamma^k p_o}(\Omega)} \leq \prod_{n=0}^{\infty} (C_o p_o^{2+1/\delta} \gamma^{(2+1/\delta)n})^{1/(p_o\gamma^n)} \|u\|_{L^{p_o}(\Omega)}$$

for all $k \in \mathbb{N}$ large enough. Since in the right-hand side the infinite product converges, we have

$$\operatorname{ess\,sup}|u| = \liminf_{p\to\infty} \|u\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \|u\|_{L^{\gamma^k p_o}(\Omega)} < \infty.$$

Thus $u \in L^\infty(\Omega)$.

Finally, iterating more carefully the estimate (8), one can obtain that $\|u\|_{L^\infty(\Omega)}$ is bounded by a constant depending only on Ω, $\|f\|_{L^\infty(\Omega)}$, c, δ and $\|u\|_{L^2(\Omega)}$. Therefore Lemma 4 follows from Lemma 1.

It follows from a result of Lieberman [16] that any bounded weak solution of $P(b, F)(f)$ is regular up to the boundary provided $(H3)$ holds. Combining Proposition 1 with Lemma 4, we have

PROPOSITION 2. *Suppose* $(H1)$, $(H2)$, $(H3)$ *hold, and* $\partial\Omega \in \mathcal{C}^{1,\gamma}$, $\gamma > 0$. *Let* $f \in L^\infty(\Omega)$. *Then there exists a weak solution to* $P(b, F)(f)$ *which belongs to* $\mathcal{C}^1(\overline{\Omega})$.

3. The contraction principle for the stationary problem

In this section we prove Theorem 1. Since we use the Kruzhkov's idea of doubling of variables ([15]), let us remark that weak solution of $P(b, F)(f)$ satisfies the appropriate entropy formulation.

REMARK 2. Let u be a weak solution of $P(b, F)(f)$. Then for all $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\xi \geq 0$, for all $k \in \mathbb{R}$ one has

$$\begin{aligned}&\int_\Omega \operatorname{sign}(u-k)\, b(u)\,\xi + \int_\Omega \operatorname{sign}(u-k)\{\nabla u - F(u) + F(k)\}\cdot\nabla\xi \\ &\quad\leq \int_\Omega \operatorname{sign}(u-k)\, f\xi + \int_{\partial\Omega} \operatorname{sign}(u-k)\, F(k)\cdot\nu\,\xi \qquad (10)\end{aligned}$$

Proof. Let $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\xi \geq 0$. Take $H_\varepsilon(u-k)\,\xi \in H^1(\Omega) \cap L^\infty(\Omega)$ for the test function in (1). Adding the term

$$\begin{aligned}0 = &\int_\Omega H_\varepsilon(u-k)\, F(k)\cdot\nabla\xi - \int_{\partial\Omega} H_\varepsilon(u-k)\, F(k)\cdot\nu\,\xi \\ &+ \int_\Omega H_\varepsilon{}'(u-k)\,\nabla u\cdot F(k)\,\xi,\end{aligned}$$

we obtain

$$\int_\Omega H_\varepsilon(u-k)\{b(u)-f\}\,\xi + \int_\Omega H_\varepsilon(u-k)\{\nabla u - F(u) + F(k)\}\cdot\nabla\xi$$
$$- \int_{\partial\Omega} H_\varepsilon(u-k)\,F(k)\cdot\nu\,\xi + \int_\Omega H_\varepsilon{}'(u-k)\,|\nabla u|^2\,\xi$$
$$- \int_\Omega H_\varepsilon{}'(u-k)(F(u)-F(k))\cdot\nabla u\,\xi = 0. \tag{11}$$

By Remark 1, the last term in (11) tends to zero as $\varepsilon \to 0$. Further, the fourth term in (11) is nonnegative. Finally, we pass to the limit in the first three terms using the Lebesgue dominated convergence theorem and obtain (10). □

In the sequel, let $f, g \in L^1(\Omega)$ and u, v be weak solutions of $P(b,F)(f)$ and $P(b,F)(g)$, respectively. For a function ξ of two variables $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ we denote by $\nabla_x\xi$, $\nabla_y\xi$ its gradients with respect to x and y, respectively. We have the following basic lemma.

LEMMA 5. *Suppose $v \in L^\infty(\Omega)$. Then for all $\xi \in \mathcal{D}(\overline{\Omega}\times\overline{\Omega})$, $\xi|_{\overline{\Omega}\times\partial\Omega} = 0$, $\xi \geq 0$, one has*

$$\iint_{\Omega\times\Omega} |b(u(x)) - b(v(y))|\,\xi + \iint_{\Omega\times\Omega} \mathrm{sign}\,(u(x)-v(y))$$
$$\times \{\nabla u(x) - \nabla v(y) - F(u(x)) + F(v(y))\}\cdot(\nabla_x\xi + \nabla_y\xi)$$
$$\leq \iint_{\Omega\times\Omega} \mathrm{sign}\,(u(x)-v(y))(f(x)-g(y))\,\xi$$
$$- \iint_{\partial\Omega\times\Omega} \mathrm{sign}\,(u(x)-v(y))\{\nabla v(y) - F(v(y))\}\cdot\nu_x\,\xi, \tag{12}$$

where for $(x,y) \in \partial\Omega\times\Omega$, ν_x is the exterior unit normal vector to $\partial\Omega$ at the point x.

Proof. Take $\xi \in \mathcal{D}(\overline{\Omega}\times\overline{\Omega})$, $\xi \geq 0$ such that $\xi|_{\overline{\Omega}\times\partial\Omega} = 0$. The function u is a weak solution of $P(b,F)(f)$; in particular, for a.e. $y \in \Omega$, u satisfies (1) with the test function $H_\varepsilon(u(\cdot) - v(y))\,\xi(\cdot, y) \in H^1(\Omega)\cap L^\infty(\Omega)$. Integrating in $y \in \Omega$, we obtain

$$\iint_{\Omega\times\Omega} H_\varepsilon(u(x)-v(y))\{b(u(x)) - f(x)\}\,\xi$$
$$+ \iint_{\Omega\times\Omega} H_\varepsilon(u(x)-v(y))\{\nabla u(x) - F(u(x))\}\cdot\nabla_x\xi$$
$$+ \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))|\nabla u(x)|^2\,\xi$$
$$- \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))F(u(x))\cdot\nabla u(x)\,\xi = 0. \tag{13}$$

In addition, we have by the Green-Gauss formula

$$\begin{aligned}
&\iint_{\Omega\times\Omega} H_\varepsilon(u(x)-v(y))(\nabla u(x) - F(u(x)))\cdot\nabla_y\xi \\
&\quad - \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)) - v(y))\nabla u(x)\cdot\nabla v(y)\ \xi \\
&\quad + \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))\, F(u(x))\cdot\nabla v(y)\ \xi \\
&= \iint_{\Omega\times\partial\Omega} H_\varepsilon(u(x)-v(y))(\nabla u(x) - F(u(x)))\cdot\nu_y\ \xi. \qquad (14)
\end{aligned}$$

Due to the special choice of the test function ξ, the right-hand side of (14) is zero. In (13) and (14), we can substitute f by g, exchange u with v and x with y. We obtain two equalities, which we add to (13) and (14) and rearrange the terms.

This yields

$$\begin{aligned}
&\iint_{\Omega\times\Omega} H_\varepsilon(u(x)-v(y))\{b(u(x))-b(v(y)) - f(x) + g(y)\}\,\xi \\
&\quad + \iint_{\Omega\times\Omega} H_\varepsilon(u(x)-v(y))\{\nabla u(x) - \nabla v(y) - F(u(x)) + F(v(y))\}\cdot(\nabla_x\xi+\nabla_y\xi) \\
&\quad + \iint_{\partial\Omega\times\Omega} H_\varepsilon(u(x)-v(y))(\nabla v(y) - F(v(y)))\cdot\nu_x\ \xi \\
&= \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))\{-|\nabla u(x)|^2 - |\nabla v(y)|^2 + 2\nabla u(x)\cdot\nabla v(y)\}\ \xi \\
&\quad + \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))(F(u(x))-F(v(y)))\cdot\nabla u(x)\ \xi \\
&\quad - \iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))(F(u(x))-F(v(y)))\cdot\nabla v(y)\ \xi. \qquad (15)
\end{aligned}$$

The first term in the right-hand side of (15) is negative or zero. Further, we have required that $\|v\|_{L^\infty(\Omega)} \le +\infty$. Proceeding for a.e. $y\in\Omega$ as in the proof of Remark 1, we get

$$\begin{aligned}
&\left|\iint_{\Omega\times\Omega} H_\varepsilon{}'(u(x)-v(y))(F(u(x))-F(v(y)))\cdot\nabla u(x)\,\xi\right| \\
&\quad \le \int_{y\in\Omega}\left(\int_{x\in\Omega}|\nabla_x\xi\,| + \int_{x\in\partial\Omega}|\,\xi\,|\right) 2\,\omega(\varepsilon)\to 0
\end{aligned}$$

as $\varepsilon\to 0$, where $\omega(\varepsilon) = \sup_{r,s\in[-M,M],|r-s|\le\varepsilon}|F(r) - F(s)|$ and $M = \|v\|_{L^\infty(\Omega)} + \varepsilon$. Similarly, the last term in (15) vanishes as $\varepsilon\to 0$. Now we pass to the limit as $\varepsilon\to 0$ in the left-hand side of (15), using the Lebesgue dominated convergence theorem. Hence (12) follows. □

Now we will make y tend to x. Note that heuristically, the last term in (12) tends to zero, due to the Neumann boundary condition on the solution v. We can justify it requiring additional regularity of v.

PROPOSITION 3. *Let* $\partial\Omega \in \mathcal{C}^2$ *and suppose* $v \in \mathcal{C}^1(\overline{\Omega})$. *Then for all* $\eta \in \mathcal{D}(\overline{\Omega})$, $\eta \geq 0$,

$$\int_\Omega |b(u) - b(v)|\eta + \int_\Omega \operatorname{sign}(u - v)\{\nabla u - \nabla v - F(u) + F(v)\} \cdot \nabla\eta \\ \leq \int_\Omega \operatorname{sign}(u - v)(f - g)\,\eta + \int_{\{u=v\}} |f - g|\,\eta. \tag{16}$$

Note that the right-hand side of (16) is exactly the bracket in $L^1(\Omega)$ between $(u - v)$ and $(f - g)\eta$. For $w, h \in L^1(\Omega)$, the bracket is defined by

$$[w, h]_{L^1(\Omega)} = \int_\Omega \operatorname{sign}(w)\,h + \int_{\{w=0\}} |h| \tag{17}$$

(cf. e.g. [4]). For the bracket in $\mathbb{R}^N$, we just use the notation $[\cdot, \cdot]$.

For $n \in \mathbb{N}$ and any $m \in \mathbb{N}$, we will abusively use the notation $\delta_n(s)$ for $n^m \prod_{i=1}^m \delta_1(ns_i)$ with $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$, where $\delta_1 \in C_0^\infty([-1, 1], \mathbb{R})$ is a nonnegative even function such that $\int_{\mathbb{R}} \delta_1(r)\,dr = 1$.

LEMMA 6. *Let for all* $s \in \mathbb{R}^N$, $w_n(\cdot, s) \to w(\cdot)$ *and* $h_n(\cdot, s) \to h(\cdot)$ *in* $L^1(\mathbb{R}^N)$ *as* $n \to \infty$. *If in addition* $\|h_n(\cdot, s)\|_{L^1(\mathbb{R}^N)}$ *is bounded uniformly in n and s, then*

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \operatorname{sign} w_n(x, s)\, h_n(x, s)\delta_1(s) \leq [w, h].$$

Moreover, if for all $n \in \mathbb{N}$ *and a.e.* $s \in \mathbb{R}^N$ $h_n(\cdot, s) = 0$ *a.e. on* $\{w_n(\cdot, s) = 0\}$ *and if* $h = 0$ *a.e. on* $\{w = 0\}$, *then there exists*

$$\lim_{n\to\infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \operatorname{sign} w_n(x, s)\, h_n(x, s)\delta_1(s) = \int_{\mathbb{R}^N} \operatorname{sign}(w)\,h.$$

Proof. The first claim follows from the definition and the upper semicontinuity of the bracket (cf. [4]), the definition of δ_1 and the Fatou lemma. The second claim follows by applying the first one to w_n, h_n and to $-w_n$, h_n. □

It is well known that for $g \in L^1(\mathbb{R}^N)$, $g(x + \sigma_n) \to g(x)$ in $L^1(\mathbb{R}^N)$, if $\sigma_n \to 0$ as $n \to +\infty$. We need the following generalisation of this property:

LEMMA 7. *Let* $g \in L^1(\mathbb{R}^N)$ *and* $g_n(x', x_N) = g(x' + \sigma_n, x_N + \phi_n(x'))$, *where* $\sigma_n \in \mathbb{R}^{N-1}$, $\phi_n : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ *are such that* $|\sigma_n| \to 0$ *and* $\phi_n(x' - \sigma_n) \to 0$ *as* $n \to \infty$, *for a.e.* $x' \in \mathbb{R}^{N-1}$. *Then* $\|g_n\|_{L^1(\mathbb{R}^N)} = \|g\|_{L^1(\mathbb{R}^N)}$ *for all n, and* $g_n \to g$ *in* $L^1(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. Setting $z = x' + \sigma_n$, we have

$$\iint_{\mathbb{R}^{N-1}\times\mathbb{R}} |g - g_n| \le \iint_{\mathbb{R}^{N-1}\times\mathbb{R}} |g(x', x_N) - g(x' + \sigma_n, x_N)|$$
$$+ \iint_{\mathbb{R}^{N-1}\times\mathbb{R}} |g(z, x_N) - g(z, x_N + \phi_n(z - \sigma_n))| = I_n^1 + I_n^2.$$

Clearly, $I_n^1 \to 0$ as $n \to \infty$. Also $I_n^2 \to 0$ as $n \to \infty$ by the Lebesgue dominated convergence theorem. Indeed, $I_n^2 = \int_{\mathbb{R}^{N-1}} \Delta_n(z)$, where

$$\Delta_n(z) = \int_{\mathbb{R}} |g(z, x_N) - g(z, x_N + \phi_n(z - \sigma_n))|.$$

One has $\Delta_n(z) \le 2\int_{\mathbb{R}} |g(z, x_N)| \in L^1(\mathbb{R}^{N-1})$ and $\Delta_n(z) \to 0$ as $n \to \infty$ for a.e. $z \in \mathbb{R}^{N-1}$, by the standard translation property. □

Proof of Proposition 3. Take $\eta \in \mathcal{D}(\overline{\Omega})$, $\eta \ge 0$. We construct a sequence $\xi_n \in \mathcal{D}(\overline{\Omega}\times\Omega)$ of test functions in (12) as follows.

Since $\partial\Omega$ is of class $\mathcal{C}^2$, we can cover $\partial\Omega$ by the union of open sets $\mathcal{O}_1, \dots, \mathcal{O}_l \subset \mathbb{R}^N$ such that, upon rotating and relabeling the coordinate axes for each $i = 1, \dots, l$, there exist $\Phi_i \in \mathcal{C}_0^2(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\partial\Omega \cap \mathcal{O}_i = \{x = (x', x_N) \in \mathcal{O}_i,\ x_N = \Phi_i(x')\}$ and $\Omega \cap \mathcal{O}_i = \{x = (x', x_N) \in \mathcal{O}_i,\ x_N > \Phi_i(x')\}$. In addition, there exists an open set $\mathcal{O}_0 \subset \Omega$ such that $\overline{\Omega} \subset \cup_{i=0}^l \mathcal{O}_i$. Take nonnegative functions $\alpha_0, \dots, \alpha_l \in \mathcal{D}(\mathbb{R}^N, \mathbb{R})$ such that $\operatorname{supp}\alpha_i \subset \mathcal{O}_i$ for $i = 0, \dots, l$ and $\sum_{i=0}^l \alpha_i(x) = 1$ for all $x \in \overline{\Omega}$.

Let $x, y \in \mathbb{R}^N$; set $\rho_n^0(x, y) = \delta_n(x - y)$, and for $i = 1, \dots, l$, set

$$\rho_n^i(x, y) = \delta_n(x' - y')\delta_n\left(x_N - \Phi_i(x') - y_N + \Phi_i(y') + \frac{2}{n}\right).$$

Here and below we abusively keep the same notation for different coordinate systems corresponding to $i = 0, \dots, l$. Take

$$\xi_n(x, y) = \sum_{i=0}^{l} \eta(x)\alpha_i(x)\rho_n^i(x, y). \tag{18}$$

CLAIM 1. $\xi_n \in \mathcal{D}(\overline{\Omega} \times \overline{\Omega})$, $\xi_n \ge 0$ and $\xi_n = 0$ on $\overline{\Omega} \times \partial\Omega$ for all n sufficiently large.

The first two properties are evident. Next, observe that there exists $d > 0$ such that for $i = 0, \dots, l$ one has $\rho_n^i(x, y) = 0$ for $|x - y| > d/n$. Let $n > d \times \min_{i=0,\dots,l} \operatorname{dist}(\operatorname{supp}\alpha_i, \mathbb{R}^N \backslash \mathcal{O}_i)$. We have $\rho_n^i(x, y) = 0$ whenever $x \in \operatorname{supp}\alpha_i$ and $y \notin \mathcal{O}_i$. In particular, for $(x, y) \in \overline{\Omega} \times \partial\Omega$ the sum in (18) reduces to the sum over $i \in \{1, \dots, l\}$ such that both x and y belong to $\mathcal{O}_i$. But in this case, $x_N - \Phi_i(x') \ge 0$ and $y_N - \Phi_i(y') = 0$. Thus $\rho_n^i(x, y) = 0$, which ends the proof of Claim 1.

Denote $\eta(x)\alpha_i(x)$ by $\eta_i(x)$ for $i = 0, \dots, l$. Taking $\xi = \xi_n$ in (12), we get

$$\sum_{i=0}^{l} I_n^i + \sum_{i=1}^{l} J_n^i \le \sum_{i=0}^{l} K_n^i + \sum_{i=0}^{l} L_n^i, \tag{19}$$

where

- for $i = 0, \dots, l$,
$$I_n^i = \iint_{\Omega\times\Omega} \operatorname{sign}(u(x)-v(y))\{(b(u(x))-b(v(y)))\eta_i(x) \\ + (\nabla u(x) - \nabla v(y) - F(u(x))+F(v(y))) \cdot \nabla\eta_i(x)\}\, \rho_n^i(x,y);$$
- for $i = 1, \dots, l$,
$$J_n^i = \iint_{\Omega\times\Omega} \operatorname{sign}(u(x)-v(y))\{(\nabla u(x) - \nabla v(y) - F(u(x)) + F(v(y)))\eta_i(x)\} \\ \cdot (\nabla_x + \nabla_y)\rho_n^i(x,y);$$
- for $i = 0, \dots, l$,
$$K_n^i = \iint_{\Omega\times\Omega} \operatorname{sign}(u(x)-v(y))\{(f(x)-g(y))\eta_i(x)\}\, \rho_n^i(x,y);$$
- for $i = 0, \dots, l$,
$$L_n^i = \iint_{\Omega\times\partial\Omega} \operatorname{sign}(u(x)-v(y))\{\eta_i(x)(\nabla v(y) - F(v(y))) \cdot \nu_x\}\, \rho_n^i(x,y).$$

In I_n^i, J_n^i, K_n^i we extend the domain of integration to $\mathbb{R}^N \times \mathbb{R}^N$ by formally setting u, v, ∇u, ∇v, f, g to be zero outside Ω.

In I_n^0, let us perform the change of variables $(x,y)\to(x,s)$, where $s = n(x-y)$. We obtain

$$I_n^0 = \iint_{\Omega\times\Omega} \operatorname{sign} w_n(x,s)\, h_n(x,s)\delta_1(s)$$

with $w_n(x,s) = u(x) - v(x-\frac{s}{n})$ and

$$h_n(x,s) = \left(b(u(x)) - b\left(v\left(x-\frac{s}{n}\right)\right)\right)\eta_0(x) \\ + \left\{\nabla u(x) - \nabla v\left(x-\frac{s}{n}\right) - F(u(x)) + F\left(v\left(x-\frac{s}{n}\right)\right)\right\} \cdot \nabla\eta_0(x).$$

Note that for all $s \in \mathbb{R}^N$, $h_n(\cdot,s) = 0$ a.e. on $\{w_n(\cdot,s) = 0\}$. Hence by Lemma 6, there exists

$$\lim_{n\to\infty} I_n^0 = \int_\Omega \operatorname{sign}(u-v)\{(b(u)-b(v))\eta_0 + (\nabla u - \nabla v - F(u) + F(v)) \cdot \nabla\eta_0\}. \tag{20}$$

Similarly, we get

$$\lim_{n\to\infty} K_n^0 \le [u-v,(f-g)\eta_0]. \tag{21}$$

For $i = 1, \dots, l$ we use the change of variables $(x, y) = (x', x_N, y', y_N) \to (x, s) = (x', x_N, s', s_N)$, where $s' = n(x'-y')$ and $s_N = n(x_N - \Phi_i(x') - y_N + \Phi_i(y') + \frac{2}{n})$. It yields

$$K_n^i = \iint_{\Omega\times\Omega} \operatorname{sign} w_n(x,s)\, h_n(x,s)\delta_1(s)$$

with

$$w_n(x,s) = u(x', x_N) - v\left(x' - \frac{s'}{n}, x_N - \frac{s_N}{n} - \Phi_i(x') + \Phi_i\left(x' - \frac{s'}{n}\right) + \frac{2}{n}\right)$$

and

$$h_n(x,s) = \left\{ f(x', x_N) - g\left(x' - \frac{s'}{n}, x_N - \frac{s_N}{n} - \Phi_i(x') + \Phi_i\left(x' - \frac{s'}{n}\right) + \frac{2}{n}\right)\right\} \eta_i(x', x_N).$$

Fix $s \in \mathbb{R}^N$; it follows by Lemma 7 that $w_n(\cdot, s) \to u - v$ and $h_n(\cdot, s) \to f - g$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$ and $\|h_n(\cdot,s)\|_{L^1(\mathbb{R}^N)} = \|f-g\|_{L^1(\mathbb{R}^N)}$. Hence we can apply Lemma 6 to obtain

$$\limsup_{n\to\infty} K_n^i \le [u-v,(f-g)\eta_i]. \tag{22}$$

Similarly, we show that there exists

$$\lim_{n\to\infty} I_n^i = \int_\Omega \operatorname{sign}(u-v)\{(b(u)-b(v))\eta_i + (\nabla u - \nabla v - F(u) + F(v))\cdot\nabla\eta_i\}. \tag{23}$$

CLAIM 2. There exists $\lim_{n\to\infty} J_n^i = 0$ for $i = 1, \dots, l$.

For simplicity, let us treat only the first of $(N-1)$ terms in the scalar product in J_n^i (remark that the Nth term is zero). This term equals

$$J_n^{i,1} = -\iint_{\Omega\times\Omega} \operatorname{sign}(u(x)-v(y))\{(\partial_1 u(x) - \partial_1 v(y) - F_1(u(x)) + F_1(v(y)))\eta_i(x)\}$$
$$\times\, \delta_n(x'-y'){\delta_n}'\left(x_N - \Phi_i(x') - y_N + \Phi_i(y') + \frac{2}{n}\right)(\partial_1\Phi_i(x) - \partial_1\Phi_i(y)),$$

where ∂_1 means the derivation with respect to the first variable, and F_1 means the first component of F. Note that for all x', y' there exists ξ' such that $\partial_1\Phi_i(x') - \partial_1\Phi_i(y') = \partial_1(\nabla\Phi)\cdot(x'-y')$ and $|\xi' - x'| \le |y'-x'|$. With the same change of variables as above, we calculate that

$$J_n^{i,1} = -\int {\delta_1}'(s_N)\delta_1(s')\, s'\cdot R_n(s', s_N),$$

where

$$R_n(s', s_N) = \int_{x\in\mathbb{R}^N}$$

$$\cdot \operatorname{sign}\left(u(x', x_N) - v\left(x' - \frac{s'}{n}, x_N - \frac{s_N}{n} - \Phi_i(x') + \Phi_i\left(x - \frac{s}{n}\right) + \frac{2}{n}\right)\right)$$

$$\times \left\{\left(\partial_1 u(x', x_N) - \partial_1 v\left(x' - \frac{s'}{n}, x_N - \frac{s_N}{n} - \Phi_i(x') + \Phi_i\left(x' - \frac{s'}{n}\right) + \frac{2}{n}\right)\right.\right.$$

$$-F_1(u(x', x_N)) + F_1\left(v\left(x' - \frac{s'}{n}, x_N - \frac{s_N}{n} - \Phi_i(x')\right.\right.$$

$$\left.\left.\left.\left. + \Phi_i\left(x' - \frac{s'}{n}\right) + \frac{2}{n}\right)\right)\right)\eta_i(x', x_N)\right\} \partial_1(\nabla\Phi_i)(\xi').$$

As in the proof of Lemma 6, it follows by Lemma 7 and the continuity of $\partial_1(\nabla\Phi_i)$ that $R_n(s', s_N)$ converges in $L^1(\mathbb{R}^N, \mathbb{R}^{N-1})$ to the bracket

$$[u - v, (\partial_1 u - \partial_1 v - F_1(u) + F_1(v))\eta_i \partial_1(\nabla\Phi_i)],$$

which does not depend on s. Hence $J_n^{i,1} \to 0$ because $\int \delta_1{}'(s_N)\delta_1(s')s' = 0$. This proves Claim 2.

Finally, due to the fact that $v \in \mathcal{C}^1(\overline{\Omega})$, we have

CLAIM 3. There exists $\lim_{n\to\infty} L_n^i = 0$ for $i = 0, \dots, l$.

Indeed, since the function $\nabla v - F(v)$ is continuous on $\overline{\Omega}$, it is easy to show that the Neumann condition $(\nabla v - F(v)) \cdot \nu = 0$ is satisfied pointwise on $\partial\Omega$. Set $\omega(\varepsilon) = \sup_{x,y\in\overline{\Omega}, |x-y|\le\varepsilon} |\nabla v(y) - F(v(y)) - \nabla v(x) + F(v(x))|$. We have

$$|L_n^i| = \left|\iint_{\partial\Omega\times\Omega} \{(\nabla v(y) - F(v(y))) \cdot \nu_x - (\nabla v(x) - F(v(x))) \cdot \nu_x\}\eta_i(x)\rho_n^i(x, y)\right|$$

$$\le \max_{x\in\partial\Omega} |\eta_i(x)| \int_{x\in\partial\Omega}\int_{y\in\Omega} \omega(|x-y|)\rho_n^i(x, y) \le \max_{x\in\partial\Omega} |\eta_i(x)|\, \omega\left(\frac{d}{n}\right) \operatorname{meas}(\partial\Omega),$$

where $\operatorname{meas}(\partial\Omega)$ is the $(N-1)$–dimensional Hausdorff measure of $\partial\Omega$, and the constant d is introduced in the proof of Claim 1. Thus Claim 3 follows.

Now since $\eta = \sum_{i=0}^{l} \eta_i$, (16) follows from Claim 2, Claim 3 and (20)–(23). □

Proof of Theorem 1. Let $f, g \in L^1(\Omega)$. There exists a sequence $(g_n)_n \subset L^\infty(\Omega)$ which converges to g in $L^1(\Omega)$. By virtue of Proposition 2, for all n there exists $v_n \in \mathcal{C}^1(\overline{\Omega})$ a weak solution of $P(b, F)(g_n)$. Applying Proposition 3 with the test function $\eta \equiv 1$ on Ω to u, v_n and to v, v_n, we obtain

$$\int_\Omega |b(u) - b(v)| \le \int_\Omega |b(u) - b(v_n)| + \int_\Omega |b(v) - b(v_n)| \le \int_\Omega |f - g_n| + \int_\Omega |g - g_n|.$$

At the limit as $n \to \infty$, we get (2).

Finally we can state the well-posedness result for $P(b, F)(f)$.

COROLLARY 1. *Assume* $\partial\Omega \in \mathcal{C}^2$, $(H1)$, $(H2)$ *and* $(H3)$ *hold. Then*

(i) *for all* $f \in L^2(\Omega)$, *there exists a unique* $b(u)$ *such that* u *is a weak (entropy) solution of* $P(b, F)(f)$*;*
(ii) *the corresponding mapping* $f \mapsto b(u)$ *is a contraction for the* L^1 *norm on* Ω.

4. Uniqueness for the evolution problem

In this section we prove Theorem 2, applying the nonlinear semigroup theory.

Let us define the (possibly multivalued) operator $A_{b,F} : L^1(\Omega) \longrightarrow L^1(\Omega)$ by its graph:

$$(\beta, g) \in A_{b,F} \iff \begin{cases} \text{there exists } v \in \mathcal{C}^1(\overline{\Omega}) \text{ such that } \beta = b(v) \text{ and} \\ v \text{ is a weak solution of } P(b, F)(g + \beta). \end{cases} \tag{24}$$

For an operator $A : L^1(\Omega) \longrightarrow L^1(\Omega)$, denote by $R(A)$ its range, by $D(A)$ its domain and by $\overline{R(A)}$, $\overline{D(A)}$ their closures in $L^1(\Omega)$ respectively. Recall (cf. [4]) that A is accretive if

$$[\beta - \hat{\beta}, g - \hat{g}]_{L^1(\Omega)} \ge 0, \quad \text{for all } (\beta, g), (\hat{\beta}, \hat{g}) \in A,$$

where the bracket $[\cdot, \cdot]_{L^1(\Omega)}$ is given by (17).

If A is accretive and $R(I + \lambda A) = L^1(\Omega)$ for some $\lambda > 0$, then A is m-accretive.

PROPOSITION 4. *Suppose* $(H1)$, $(H2)$, $(H3)$, $(H4)$ *hold and* $\partial\Omega \in \mathcal{C}^2$. *Then*

(i) $A_{b,F}$ *is accretive in* $L^1(\Omega)$
(ii) $R(I + \lambda A_{b,F}) \supset L^\infty(\Omega)$ *for all* λ *sufficiently small;*
(iii) $\overline{D(A_{b,F})} = L^1(\Omega)$.

Proof. (i) Let (β, g), $(\hat{\beta}, \hat{g}) \in A_{b,F}$. There exist v, $\hat{v} \in \mathcal{C}^1(\overline{\Omega})$ such that $\beta = b(v)$, $\hat{\beta} = b(\hat{v})$, and v, $\hat{v}$ are weak solutions to $P(b, F)(g + \beta)$, $P(b, F)(\hat{g} + \hat{\beta})$ respectively. Applying Proposition 3 with $\eta \equiv 1$ on Ω, we have

$$\|b(v) - b(\hat{v})\|_{L^1(\Omega)} \le [v - \hat{v}, (g - \hat{g}) + b(v) - b(\hat{v})]_{L^1(\Omega)}.$$

Using (17), we deduce that $[b(v) - b(\hat{v}), g - \hat{g}]_{L^1(\Omega)} \ge [v - \hat{v}, g - \hat{g}]_{L^1(\Omega)} \ge 0$.

(ii) For $g \in L^\infty(\Omega)$ and $\lambda > 0$, consider the problem $P(\frac{b}{\lambda}, F)(\frac{g}{\lambda})$. By Proposition 2, there exists $u_\lambda \in \mathcal{C}^1(\overline{\Omega})$ a weak solution to $P(\frac{b}{\lambda}, F)(\frac{g}{\lambda})$. By (24), $(b(u_\lambda), \frac{g-b(u_\lambda)}{\lambda}) \in A_{b,F}$. Hence $g \in R(I + \lambda A_{b,F})$.

(iii) First take $g \in L^\infty(\Omega)$. By $(H2)$, there exists $u \in L^\infty(\Omega)$ such that $g = b(u)$. One can approach u a.e on Ω by a bounded in $L^\infty(\Omega)$ sequence of $H^1(\Omega)$ functions. Therefore the set $\widetilde{D} = \{g \in L^\infty(\Omega),\ g = b(u),\ u \in H^1(\Omega) \cap L^\infty(\Omega)\}$ is dense in $L^\infty(\Omega)$ for the L^1norm; hence it is dense in $L^1(\Omega)$.

It remains to show that $D(A_{b,F})$ is dense in $\widetilde{D}$. We have already shown that for all $g \in \widetilde{D}$ there exists a weak solution u_λ to $P(\frac{b}{\lambda}, F)(\frac{g}{\lambda})$, and $b(u_\lambda) \in D(A_{b,F})$.

In the sequel, C denotes a constant independent of λ.

CLAIM 1. for all λ sufficiently small, $\int_\Omega u_\lambda\, b(u_\lambda) \le C$.

Here we can relax $(H4)$ by taking $\delta = 0$. We argue as in the proof of Lemma 1, using in addition that $|g\, u_\lambda| \le C + \frac{1}{2} u_\lambda b(u_\lambda)$ since $g \in L^\infty(\Omega)$ and $(H2)$ holds. For all λ sufficiently small, we get $\|u_\lambda\|_{L^{2^*}(\Omega)} \le C/\sqrt{\lambda}$ and, finally, $\int_\Omega u_\lambda b(u_\lambda) \le C$.

CLAIM 2. forallλ sufficiently small, $\|u_\lambda\|_{H^1(\Omega)} \le C$.

Take $(u_\lambda - u) \in H^1(\Omega) \cap L^\infty(\Omega)$ as a test function for $P(\frac{b}{\lambda}, F)(\frac{g}{\lambda})$. Since $g = b(u)$, we get

$$\frac{1}{\lambda}\int_\Omega (b(u_\lambda) - b(u))(u_\lambda - u) + \int_\Omega (\nabla u_\lambda - F(u_\lambda)) \cdot \nabla(u_\lambda - u) = 0.$$

The first term is non-negative; by the Young inequality, we deduce that

$$\int_\Omega |\nabla(u_\lambda - u)|^2 \le \int_\Omega |F(u_\lambda) - \nabla u|^2.$$

Using $(H4)$ and Claim 1, we obtain that

$$\int_\Omega |\nabla u_\lambda|^2 \le C\left(1 + \int_\Omega |u_\lambda|^{2(1-\delta)}\right) \tag{25}$$

for all λ sufficiently small. By Claim 1 and $(H2)$, we also have $\int_\Omega |u_\lambda| \le C$. Substitute (25) in the Sobolev inequality $\int_\Omega |u_\lambda|^2 \le C(\int_\Omega |\nabla u_\lambda|^2 + (\int_\Omega |u_\lambda|)^2)$; by the Young inequality we deduce $\int_\Omega |u_\lambda|^2 \le C$. Now Claim 2 follows from (25).

By Claim 2, there exist $v \in L^2(\Omega)$ and a sequence $\lambda_m \to 0$ such that $u_{\lambda_m} \to v$ a.e. on Ω. Hence $b(u_{\lambda_m}) \to b(v)$ a.e. on Ω. By Claim 1 and Lemma 3 we get $b(u_{\lambda_m}) \to b(v)$ in $L^1(\Omega)$. Moreover, by Claim 1, Claim 2 and $(H4)$, $\lambda(\nabla u_\lambda - F(u_\lambda)) \to 0$ in $L^2(\Omega)$ as $\lambda \to 0$. Taking $\xi \in \mathcal{D}(\Omega)$ as a test function for $P(\frac{b}{\lambda}, F)(\frac{g}{\lambda})$, and passing to the limit as $\lambda \to 0$, we obtain that $b(u_\lambda) \to g$ in $\mathcal{D}'(\Omega)$. Hence $b(u_{\lambda_m}) \to g$ in $L^1(\Omega)$, which concludes the proof. □

DEFINITION 3. Let $f \in L^1(Q)$, $b^o \in L^1(\Omega)$. A function $w \in L^1(Q)$ is an integral solution of the problem

$$w_t + A_{b,F}(w) \ni f,\ w(t=0) = b^o, \quad S(b, F)(f, b^o)$$

if there exists $\operatorname{ess\,lim}_{t\to 0} w(t) = b^o$ in $L^1(\Omega)$ and for all $(\beta, g) \in A_{b,F}$

$$\frac{d}{dt}\|w(t) - \beta\|_{L^1(\Omega)} \le [w(t) - \beta, f(t) - g]_{L^1(\Omega)} \text{ in } \mathcal{D}'(0, T). \tag{26}$$

By Proposition 4, the closure of the operator $A_{b,F}$ is m-accretive densely defined in $L^1(\Omega)$. By the general theory of non-linear semigroups (cf. [12], [8], [4] and [3], Theorem 4), we have the following result:

COROLLARY 2. *Suppose* $(H1)$, $(H2)$, $(H3)$, $(H4)$ *hold and* $\partial\Omega \in \mathcal{C}^2$. *Then for all* $f \in L^1(Q)$, $b^o \in L^1(\Omega)$, *there exists a unique integral solution of* $S(b, F)(f, b^o)$. *Moreover, if* $\hat{f} \in L^1(Q)$, $b^o \in L^1(\Omega)$ *and* w, $\hat{w}$ *are integral solutions of* $S(b, F)(f, b^o)$, $S(b, F)(\hat{f}, \hat{b}^o)$, *respectively, then for a.e.* $t \in (0, T)$

$$\|w(t)-\hat{w}(t)\|_{L^1(\Omega)} \le \|b^o-\hat{b}^o\|_{L^1(\Omega)}+\int_0^t \|f(\tau)-\hat{f}(\tau)\|_{L^1(\Omega)}d\tau.$$

Now Theorem 2 follows immediately from (iii) of Proposition 4 and

PROPOSITION 5. *Suppose* $\partial\Omega \in \mathcal{C}^2$. *Let* $f \in L^1(Q)$, $b^o \in \overline{D(A_{b,F})}$. *If* u *is a weak solution of* $E(b, F)(f, b^o)$, *then* $w = b(u)$ *is an integral solution of* $S(b, F)(f, b^o)$.

Proof. We first argue as in [7], Theorem 4.4.

Let us define the functions

$$B_{\varepsilon,k} : r \in \mathbb{R} \mapsto \int_k^r H_\varepsilon(r-k)\, db(s)$$

for all $\varepsilon > 0$, $k \in \mathbb{R}$.

Note that for all r, $\hat{r} \in \mathbb{R}$,

$$(b(r)-b(\hat{r}))H_\varepsilon(r-k) \ge B_{\varepsilon,k}(r)-B_{\varepsilon,k}(\hat{r}), \tag{27}$$

and for all $r \in \mathbb{R}$,

$$\begin{cases} B_{\varepsilon,k}(r) \to |b(r)-b(k)| \text{ as } \varepsilon \to 0, \\ 0 \le B_{\varepsilon,k}(r) \le |b(r)-b(k)|. \end{cases} \tag{28}$$

Take u a weak solution of $E(b, F)(f, b^o)$; we extend u by u^o for $t \le 0$, where u^o is a measurable function such that $b^o = b(u^o)$.

Take positive functions $\eta \in \mathcal{D}(\Omega)$, $\mu \in \mathcal{D}(-\infty, T)$, and $h > 0$ such that $\operatorname{supp}\mu \subset (-\infty, T-h)$. Set $\phi = H_\varepsilon(u-k)\eta\mu$ and take $\phi^h = \frac{1}{h}\int_t^{t+h} \phi(s)\,ds$ as a test function in (3). For the first term, using the change of variable $t \to t+h$ and (27), we get

$$\int_0^T\!\!\int_\Omega (b^o-b(u))\phi_t^h = \int_0^T\!\!\int_\Omega (b^o-b(u))\frac{\phi(t+h)-\phi(t)}{h}$$
$$= \frac{1}{h}\int_0^T\!\!\int_\Omega (b(u(t-h)-b(u(t)) \times H_\varepsilon(u(t)-k)\mu(t)\eta$$
$$\geq \frac{1}{h}\int_0^T\!\!\int_\Omega (B_{\varepsilon,k}(u(t))-B_{\varepsilon,k}(u(t-h)))\mu(t)\eta.$$

Using the inverse change of variable, we obtain

$$\int_0^T\!\!\int_\Omega (b^o-b(u))\phi_t^h \geq \int_0^T\!\!\int_\Omega (B_{\varepsilon,k}(u^o)-B_{\varepsilon,k}(u))\frac{\mu(t+h)-\mu(t)}{h}\eta. \tag{29}$$

By (3) and (29), since $\phi^h \to \phi$ in $L^2(0,T;H^1(\Omega))$ and $\frac{1}{h}(\mu(\cdot+h)-\mu(\cdot)) \to \mu_t$ in $L^\infty(0,T)$ as $h \to 0$, we get

$$\int_0^T\int_\Omega (B_{\varepsilon,k}(u^o)-B_{\varepsilon,k}(u))\mu_t\eta \leq -\int_0^T\!\!\int_\Omega (\nabla u - F(u))\cdot\nabla(H_\varepsilon(u-k)\eta)\mu$$
$$+\int_0^T\int_\Omega f H_\varepsilon(u-k)\eta\mu. \tag{30}$$

We are now in a position to perform the doubling of variables. As in Lemma 5, take $\xi \in \mathcal{D}(\overline{\Omega}\times\overline{\Omega})$ such that $\xi|_{\overline{\Omega}\times\partial\Omega} = 0$, $\xi \geq 0$.

Consider $g \in L^1(\Omega)$ and $v \in \mathcal{C}^1(\overline{\Omega})$ such that v is a weak solution of $P(b,F)(g+b(v))$. For a.e. $(t,x) \in Q$, take $H_\varepsilon(v(\cdot)-u(t,x))\xi(x,\cdot)\mu(t)$ as a test function for $P(b,F)$ $(g+b(v))$. Integrating in $(t,x) \in Q$, we get

$$\int_0^T\!\!\iint_{\Omega\times\Omega} (\nabla v(y)-F(v(y)))\cdot\nabla_y(H_\varepsilon(v(y)-u(t,x))\xi(x,y))\mu(t)$$
$$= \int_0^T\!\!\iint_{\Omega\times\Omega} g(y)H_\varepsilon(v(y)-u(t,x))\xi(x,y)\mu(t). \tag{31}$$

Similarly, for a.e. $y \in \Omega$ let us take $k = v(y)$ and $\eta(\cdot) = \xi(\cdot,y)$ in (30) and integrate in $y \in \Omega$. Summing the obtained inequality with (31), arguing as in Lemma 5, we pass to the limit as $\varepsilon \to 0$. Note that by (28) and the Lebesque dominated convergence theorem we have

$$\lim_{\varepsilon\to 0}\int_0^T\!\!\iint_{\Omega\times\Omega} (B_{\varepsilon,v(y)}(u^o(x)) - B_{\varepsilon,v(y)}(u(t,x)))\mu_t\,\xi$$
$$= \int_0^T\!\!\iint_{\Omega\times\Omega} (|b^o(x)-b(v(y))|-|b(u(t,x))-b(v(y))|)\mu_t\,\xi.$$

We infer

$$\begin{aligned}
&\int_0^T\iint_{\Omega\times\Omega}(|b^o(x)-b(v(y))|-|b(u(t,x))-b(v(y))|)\mu_t\,\xi\\
&\quad\le-\int_0^T\iint_{\Omega\times\Omega}\operatorname{sign}(u(t,x)-v(y))\{\nabla u(t,x)-\nabla v(y)-F(u(t,x))+F(v(y))\}\cdot\\
&\qquad\cdot(\nabla_x\xi+\nabla_y\xi)\mu+\int_0^T\iint_{\Omega\times\Omega}\operatorname{sign}(u(t,x)-v(y))(f(t,x)-g(y))\xi\mu\\
&\qquad-\int_0^T\iint_{\partial\Omega\times\Omega}\operatorname{sign}(u(t,x)-v(y))\{\nabla v(y)-F(v(y))\}\cdot\nu_x\,\xi\mu. \qquad (32)
\end{aligned}$$

Now we make y tend to x.

CLAIM 1. For all weak solution of $E(b,F)(f,b^o)$, $g\in L^1(\Omega)$ and $v\in\mathcal{C}^1(\overline{\Omega})$ weak solution of $P(b,F)(g+b(v))$ we have

$$\begin{aligned}
&\int_0^T\int_\Omega(|b^o-b(v)|-|b(u)-b(v)|)\mu_t\eta\\
&\quad+\int_0^T\int_\Omega\operatorname{sign}(u-v)(\nabla u-\nabla v-F(u)+F(v))\cdot\nabla\eta\,\mu\\
&\quad\le\int_0^T\mu\,[u-v,(f-g)\eta]_{L^1(\Omega)}
\end{aligned}$$

for all positive $\eta\in\mathcal{D}(\overline{\Omega})$, $\mu\in\mathcal{D}(-\infty,T)$.

We should slightly modify the Proof of Proposition 3, because of the extra integral in t. Take $\xi=\xi_n$ in (32), where ξ_n is defined in (18). Consider for instance the left-hand side of (32). As in the Proof of Proposition 3, for a.e. $t\in(0,T)$ we get

$$\begin{aligned}
I_n(t)&=\iint_{\Omega\times\Omega}(|b^o(x)-b(v(y))|-|b(u(t,x))-b(v(y))|)\xi_n(x,y)\\
&\longrightarrow\int_\Omega(|b^o(x)-b(v(x))|-|b(u(t,x))-b(v(x))|)\eta(x)
\end{aligned}$$

as $n\to\infty$.

Moreover,

$$|I_n(t)|\le(\|b^o\|_{L^1(\Omega)}+2\|b(v)\|_{L^1(\Omega)}+\|b(u(t,\cdot)\|_{L^1(\Omega)})\|\eta\|_{L^\infty(\Omega)}\in L^1(0,T).$$

Multiplying by μ_t and applying the Lebesque dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_0^T I_n(t)\mu_t=\int_0^T(|b^o-b(v)|-|b(u)-b(v)|)\eta\mu_t.$$

In the same way, we pass to the limit as $n\to\infty$ in the right-hand side of (32) and obtain Claim 1.

Now for all $(\beta, g) \in A_{b,F}$ there exists $v \in \mathcal{C}^1(\overline{\Omega})$ such that $\beta = b(v)$ and v is a weak solution to $P(b, F)(g+b(v))$. Since $[u-v, f-g]_{L^1(\Omega)} \le [b(u)-b(v), f-g]_{L^1(\Omega)}$, applying Claim 1 with $\eta \equiv 1$ on Ω, we get (26).

Finally, consider $\hat{t} \to 0^+$. Take $h \in (0, T - \hat{t})$ and $\mu^h \in \mathcal{D}(-\infty, T)$ such that $\mu^h \equiv 1$ on $(-\infty, \hat{t})$, $\mu^h \equiv 0$ on $(\hat{t} + h, T)$ and $0 \le -\mu^h_t \le 2/h$. Applying Claim 1 with $\mu = \mu^h$ and $\eta \equiv 1$ and passing to the limit as $h \to 0$, for a.e. $\hat{t} \in (0, T)$ we get

$$\int_\Omega |b(u(\hat{t})) - \beta| - |b^o - \beta| \le \int_0^{\hat{t}} |f(\tau) - g|\, d\tau. \tag{33}$$

Since $b^o \in \overline{D(A_{b,F})}$, for all $\varepsilon > 0$ there exists $(\beta, g) \in A_{b,F}$ such that $\|b^o - \beta\|_{L^1(\Omega)} \le \varepsilon/3$. For $\hat{t}$ sufficiently small, the right-hand side of (33) is less than $\varepsilon/3$. Hence $\|b(u(\hat{t})) - b^o\|_{L^1(\Omega)} \le \varepsilon$ for a.a $\hat{t}$ sufficiently small, which concludes the proof. □

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