Conservation laws with discontinuous flux

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based on joint works with

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Plan of the talk

General framework and Model equation

General framework and the model equation...

Consider the Cauchy problem for a scalar conservation law

$$\begin{cases} u_t + \operatorname{div} \mathfrak{f}(t,x,u) = 0, & \text{on } [0,+\infty) \times \mathbb{R}^N \\ u|_{t=0} = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

with a Carathéodory flux $\mathfrak{f}$ $\big[$ measurable in (t,x), continuous in u $\big]$.

Main Question:

Well-posedness, for an appropriate generalization of the S.N. Kruzhkov notion of entropy solutions (known for regular in (t,x,u) flux).

Related:

Convergence of approximation methods :
vanishing viscosity, numerical schemes,...

Most of techniques are restricted to the case of piecewise regular dependency on (t,x) (yet, see the talk of Buliček); $\Rightarrow$ think of piecewise constant in (t,x) flux. Then, the main issue is to understand coupling of conservation laws across an interface .

General framework and the model equation...

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Playground: the 1D case with jump discontinuity at $\{x=0\}$:

$$u_t+\mathfrak{f}(x,u)_x=0,\ \mathfrak{f}(x,u)=\begin{cases} f^l(u), & x<0,\\ f^r(u), & x>0,\end{cases} = f^l(u)1\!\!1_{\{x<0\}}+f^r(u)1\!\!1_{\{x>0\}}$$

Many contributions since 1990:

T. Gimse and N.H. Risebro; S. Diehl; C. Klingenberg and Risebro;
F. Kaasschietter; P. Baiti and H.K. Jenssen; J. Towers,
then K.H. Karlsen, Towers and Karlsen, Risebro, Towers;
R. Bürger and al.; Adimurthi and G.D. Veerappa Gowda,
then Adimurthi, J. Jaffré, Gowda, and Adimurthi, S. Mishra, Gowda;
D. Ostrov; N. Seguin and J. Vovelle, then F. Bachmann and Vovelle;
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B. Piccoli and A. Terracina; G.Q. Chen, N. Even and Klingenberg;
J. Jimenez, then Jimenez and L.Lévi; D. Mitrovič; C. Cancès;...
We re-use and combine several key ideas that were introduced, and construct a "general theory" for the model equation.

Key fact: different solution semigroups for the same formal equation.
$\Rightarrow$ Which is(are) appropriate? How to approximate "the good one"?

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A fundamental example: Buckley-Leverett equation in two-rocks' porous medium

Two-phase porous medium model in one space dimension...

Consider a two-phase flow of phases ("oil", "water") of saturation u and $1-u$, respectively, with neglected (???) capillary pressure effects , in a vertical homogeneous porous medium with gravity.
This is the Buckley-Leverett model,
which is a hyperbolic conservation law

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0 \text{ with initial datum } u_0.$$

The Buckley-Leverett flux $f : [0,1] \mapsto \mathbb{R}$ is computed as follows:

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)}$$

and

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

Physical meaning:

K : permeability of the medium
k_o, k_w relative permeabilities per phase
μ_a, μ_b phase viscosities
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Buckley-Leverett equation as limit of degenerate parabolic pb...

A more precise model contains an additional second-order term, with a small effect due to capillary pressure :

$$(BL_\varepsilon) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\lambda_o(u)\, \partial_x \pi(u)\right)$$

where π an increasing function modelling the capillarity effects.

Now, when we look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

we expect that solutions are limits of solutions u^ε of the capillarity-regularized problem (BL_ε) .
But how to check that a given function is solution in this sense?

Intrinsic charcterization: vanishing capillarity limits coincide with Kruzhkov entropy solutions . The resulting notion of solution is independent of the choice of $\pi(\cdot)$!

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The case of a two-rocks' medium : model equation.

Study the Buckley-Leverett equation in a porous medium that consists of geologically different "rocks" separated by sharp fractures, $\Rightarrow$ our model problem:

$$\partial_t u + \partial_x \mathfrak{f}(x,u) = 0, \quad \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x < 0, \\ f^r(u), & x > 0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x = 0\}$.

It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution, stated separately in each region $\{x < 0\}$ and $\{x > 0\}$;

– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(\gamma^l u) = f^r(\gamma^r u) \ \text{ for a.e. } t, \text{ where } (\gamma^{l,r} u)(t) \text{ are "traces"}$$

– And now, what admissibility condition on the interface?

Could the notion of admissible solution be different for different capillarity profiles $\pi^{l,r}$ that we "neglect"? The answer is, yes !

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KRUZHKOV'S THEORY

A crash course through the Kruzhkov theory

Let us describe very briefly (a part of) the classical theory of SCL.

- for regular data, classical solutions can be constructed locally in time, by the method of characteristics. But the characteristics may cross in finite time $\Rightarrow$ blow-up of the derivative of the solution or shock creation. The classical solution ceases to exist...
- one can look for weak solutions (in the sense $\mathcal{D}'$) and recover existence, globally in time; but then the uniqueness is lost
- introducing the notion of Kruzhkov entropy solution, one recovers existence and uniqueness and even L^1 contraction.
- two entropy solutions verify the Kato inequality

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big(|u-v|\,\partial_t\varphi + \Phi(u,v)\,\partial_x\varphi \Big) + \int_{\mathbb{R}} |u_0-v_0|\,\varphi(0,x) \geqslant 0$$

Here $\Phi(u,v)=\mathrm{sign}(u-v)(f(u)-f(v))$ and $\varphi \geqslant 0$ is a test function.

- the notion of an entropy solution itself is based upon the Kato inequality postulated with respect to a selected family of "elementary solutions" v; namely, one takes all the constant solutions $v \equiv \kappa$, $\kappa \in \mathbb{R}$.

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A crash course through the Kruzhkov theory (cont^d)

- letting φ go to $1\!\!1_{[0,T)\times\mathbb{R}}$, from the Kato inequality

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one recovers $\displaystyle\int_{\mathbb{R}}|u-v|(T)\leqslant\int_{\mathbb{R}}|u_0-v_0|+sthg,\quad sthg\leqslant 0.$

This yields uniqueness and continuous dependence (L^1 contraction).

- Rq: If u,v are piecewise continuous with a jump at $x=0$, then the jumps "contribute to sthg " with the term

$$\Phi(\gamma^l u,\gamma^l v)-\Phi(\gamma^r u,\gamma^r v),\quad \text{which is non-negative .}$$

Here and in the sequel, γ^l and γ^r denote one-sided traces on $\{x=0\}$.

- Rq: Assume u is has a jump at $x=0$, and is an entropy solution "away from $\{x=0\}$". Then (cf. Vol'pert) u is an entropy solution if and only if

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A crash course through the Kruzhkov theory (cont[d])

- existence of an entropy solution can be shown in several ways.
  One of the most convenient ways to construct solutions is

  – to learn solving "Riemann problems" (that is, the Cauchy problems with simplest discontinuous initial data, kind of Heavyside functions)
  – to use the Riemann solvers as "building blocks" to construct solutions (wave-front tracking algorithms, or numerical schemes: Godunov,Glimm)
- The other standard way to construct solutions is vanishing viscosity: the solution of $u_t + f(u)_x = 0$ is approximated by those of $u_t + f(u)_x = \varepsilon u_{xx}$.
- Entropy solutions "have strong traces" (Vasseur, Panov).
  E.g. if u is an entropy solution in $\{x > 0\}$, then

  – $f(u)$ and $\Phi(u, k)$ have strong traces on $\{x = 0\}$
  – if, in addition, f' does not vanish on intervals,
  then u has a strong trace $\gamma^r u$ on $\{x = 0\}$.

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Another example:
Road traffic
with point constraint

Problem and the associated Riemann solver

Think of an obstacle on the road: road light, toll gate, construction site, exit... One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_0(x) \end{cases} \quad \text{and } "f(u(t,0))" \leqslant F(t).$$

Heuristics ⇒ solution of the Riemann problem . To get it, Colombo, Goatin modified the Riemann solver $\mathcal{R}(u^l, u^r)$ of the unconstrained pb.

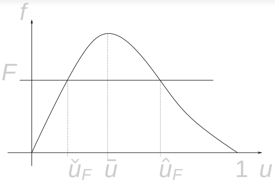
$$\text{(RP)} \quad \partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x) := \begin{cases} u^l \text{ if } x < 0 \\ u^r \text{ if } x > 0 \end{cases}$$

Definition (Colombo-Goatin '07)

For the constrained pb., if $f\left(\mathcal{R}(u^l, u^r))(0)\right) \leqslant F$, then $\mathcal{R}^F(u^l, u^r) = \mathcal{R}(u^l, u^r)$.

Otherwise, $\mathcal{R}^F(u^l, u^r)(x) = \begin{cases} \mathcal{R}(u^l, \hat{u}_F)(x) \text{ if } x < 0\,, \\ \mathcal{R}(\check{u}_F, u^r)(x) \text{ if } x > 0\,. \end{cases}$ (see picture)

"(A,B)-connection" ! (Adimurthi et al.)



Problem and the associated Riemann solver

Think of an obstacle on the road: road light, toll gate, construction site, exit... One can model it by a point constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_0(x) \end{cases} \quad \text{and } "f(u(t,0))" \leqslant F(t).$$

Heuristics $\Rightarrow$ solution of the Riemann problem . To get it, Colombo, Goatin modified the Riemann solver $\mathcal{R}(u^l, u^r)$ of the unconstrained pb.

$$\text{(RP)} \quad \partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x) := \begin{cases} u^l \text{ if } x < 0 \\ u^r \text{ if } x > 0 \end{cases}$$

Definition (Colombo-Goatin '07)

For the constrained pb., if $f\,(\mathcal{R}(u^l, u^r))(0)) \leqslant F$, then $\mathcal{R}^F(u^l, u^r) = \mathcal{R}(u^l, u^r)$.

Otherwise, $\mathcal{R}^F(u^l, u^r)(x) = \begin{cases} \mathcal{R}(u^l, \hat{u}_F)(x) \text{ if } x < 0\,, \\ \mathcal{R}(\check{u}_F, u^r)(x) \text{ if } x > 0\,. \end{cases}$ (see picture)

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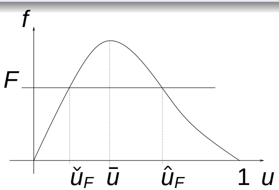

Elementary solutions and admissible traces.

• The main object we introduce is the following.
The Riemann solver of the previous page contains
a family of stationary solutions $c(x) = c^l \mathbb{1}_{\{x<0\}} + c^r \mathbb{1}_{\{x>0\}}$
$\Longrightarrow$"Admissibility germ" $\mathcal{G}(F)$ = the set of such couples (c^l, c^r) .

• The second idea is (mimicking the Kruzhkov's definition !):

– accept these "elementary solutions" as admissible;
– define entropy solutions as the functions satisfying
the Kato inequality with respect to these elementary solutions.

• The third idea is (think of the scaling argument !):

– $\mathcal{G}(F)$ encodes possible traces on $\{x = 0\}$ of elementary solutions
– see an admissible solution for the constrained CL
as a Kruzhkov solution of the CL in domains $\{x > 0\}$,$\{x < 0\}$
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Interface dissipation. Global entropy formulation. Flux constraint.

- The fourth idea is (mimicking the Kruzhkov case):

– if for two solutions u, v we have "the interface dissipation" in the sense

$$\Phi(\gamma^l u, \gamma^l v) - \Phi(\gamma^r u, \gamma^r v) \geqslant 0,$$

then from the Kato inequality in domains $\{\pm x > 0\}$ we get uniqueness.

– since the traces $(\gamma^l u, \gamma^r u)$ and $(\gamma^l v, \gamma^r v)$ both belong to $\mathscr{G}(F)$, the interface dissipation is a property of $\mathscr{G}(F)$.

Finally, what about numerics ?? We have a "cheap" solution !

- The idea is :take any FV numerical scheme that works well on the unconstrained CL, and truncate the numerical flux at $x = 0$ (replacing given numerical flux $g(u_K, u_L)$ with $\min\{g(u_K, u_L), F(t)\}$).

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GERMS AND $\mathcal{G}$-ENTROPY SOLUTIONS. UNIQUENESS.

L^1-dissipative germs and $\mathcal{G}$-entropy solutions; uniqueness...

Set $\Phi^{l,r}(.,k) := sign(.-k)\big(f^{l,r}(.) - f^{l,r}(k)\big)$ (Kruzhkov entropy-fluxes for $f^{l,r}$).

Definition (Germs; maximal, definite, and complete germs)

- Any set $\mathcal{G}$ of couples $(c^l, c^r) \in \mathbb{R}\times\mathbb{R}$ satisfying the Rankine-Hugoniot relation $f^l(c^l) = f^r(c^r)$ and the L^1-dissipativity relation

$$(L^1D) \qquad \forall (c^l,c^r),(b^l,b^r)\in\mathcal{G}, \quad \Phi^l(c^l,b^l) \geqslant \Phi^r(c^r,b^r).$$

  is called L^1D admissibility germ (an L^1D-germ, for short) associated with the couple of fluxes (f^l, f^r).
- We say that $\mathcal{G}'$ is an extension of an L^1D-germ $\mathcal{G}$ if $\mathcal{G}\subset\mathcal{G}'$ and $\mathcal{G}'$ still satisfies the L^1-dissipation property (L^1D) and the Rankine-Hugoniot condition.
- L^1D-germ $\mathcal{G}$ is called maximal, if it does not admit a nontrivial extension.
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- L^1D-germ $\mathcal{G}$ is called complete if all Riemann problem admits a self-similar solution u such that $(\gamma^l u, \gamma^r u)\in\mathcal{G}$, where $\gamma^l u, \gamma^r u$ are the one-sided traces of u on the interface.

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Definition (dual of a germ)

Let $\mathcal{G}$ be a subset of $\mathbb{R}\times\mathbb{R}$. The dual of $\mathcal{G}$ is the set

$$\mathcal{G}^* := \left\{ (b^l, b^r) \in \mathbb{R}\times\mathbb{R} \;\middle|\; \begin{array}{l} f^l(b^l) = f^r(b^r) \quad \text{and} \\ \forall\,(c^l, c^r) \in \mathcal{G} \quad \Phi^l(c^l, b^l) \geq \Phi^r(c^r, b^r) \end{array} \right\}.$$

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If $\mathcal{G}$ is a definite germ, then $\mathcal{G}^*$ is the unique maximal extension of $\mathcal{G}$.

Heuristically:

– Complete germs are good for existence

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– Yet a definite germ $\mathcal{G}$ determines completely its unique maximal extension $\mathcal{G}^*$; therefore our main object is a definite germ $\mathcal{G}$ of which the dual $\mathcal{G}^*$ is complete

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Given $f^{l,r} \in C(\mathbb{R},\mathbb{R})$, let $\mathcal{G}$ be a definite germ.
A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is called a $\mathcal{G}$-entropy solution of

$$u_t + \big(\mathfrak{f}(x;u)\big)_x = 0, \quad u|_{t=0} = u_0$$

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(i) the restriction of u on $\{x>0\}$ (resp., on $\{x<0\}$) is a Kruzhkov entropy solution of the pb. with flux f^r (resp., with flux f^l);

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Theorem (uniqueness, comparison, L^1 contraction)

Assume that $\mathcal{G}^*$ is an L^1D germ.
If u and $\hat{u}$ are two $\mathcal{G}$-entropy solutions of the model problem, then the comparison principle and the L^1-contractivity property hold .
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Equivalent definitions,

Existence,

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Reformulations, Existence&Convergence of approximations...

Let us give another formulation, which does not involve boundary traces of u.
For all $(c^l, c^r) \in \mathbb{R}^2$, consider $c(x) = c^l 1\!\!1_{\{x<0\}} + c^r 1\!\!1_{\{x<0\}}$.

Equivalent Definition (global adapted entropy inequalities)

(Still $\mathfrak{f}(x,.) = f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$ *with* $\mathcal{G}$ *a definite germ associated to* $f^{l,r}$ *)*

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· *firstly, for all* $k \in \mathbb{R}$*, the Kruzhkov entropy inequalities hold on each side from the interface (with test functions zero on the interface);*

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· *and thirdly,* $\forall\, (c^l, c^r) \in \mathcal{G}$ *and* $c(x)$ *defined above, the adapted entropy inequalities hold globally (with general test functions* $\xi \in \mathcal{D}(\mathbb{R})$*):*

$$\int_{\mathbb{R}} |u(t,x) - c(x)|\xi(x) - \int_0^t \int_{\mathbb{R}} \mathfrak{q}(x; u, c(x))\, \xi_x \leqslant \int_{\mathbb{R}} |u_0(x) - c(x)|\xi(x)$$

(i.e. $|u - c(x)|_t + \mathfrak{q}(x; u, c(x))_x \leqslant 0$ *in the sense of distributions).*

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(Still $\mathfrak{f}(x,.) = f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$ with $\mathcal{G}$ a definite germ associated to $f^{l,r}$)
A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is called a $\mathcal{G}$-entropy solution of the problem if,

- *firstly, for all $k \in \mathbb{R}$, the Kruzhkov entropy inequalities hold on each side from the interface (with test functions zero on the interface);*
- *secondly, it is a solution in the sense of distributions (RH condition);*
- *and thirdly, $\forall\ (c^l, c^r) \in \mathcal{G}$ and $c(x)$ defined above, the adapted entropy inequalities hold globally (with general test functions $\xi \in \mathcal{D}(\mathbb{R})$):*

$$\int_{\mathbb{R}} |u(t,x) - c(x)|\xi(x) - \int_0^t \int_{\mathbb{R}} \mathfrak{q}(x; u, c(x))\, \xi_x \leqslant \int_{\mathbb{R}} |u_0(x) - c(x)|\xi(x)$$

(i.e. $|u - c(x)|_t + \mathfrak{q}(x; u, c(x))_x \leqslant 0$ in the sense of distributions).

This formulation is stable by passage-to-the limit $\Longrightarrow$ it is suited for existence!

NB. The definitions are equivalent because $\mathcal{G}^*$ is maximal .

Reformulations, Existence&Convergence of approximations...

Let us give another formulation, which does not involve boundary traces of u. For all $(c^l, c^r) \in \mathbb{R}^2$, consider $c(x) = c^l 1\!\!1_{\{x<0\}} + c^r 1\!\!1_{\{x<0\}}$.

Equivalent Definition (global adapted entropy inequalities)

(Still $\mathfrak{f}(x,.) = f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$ *with* $\mathcal{G}$ *a definite germ associated to* $f^{l,r}$ *)*
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Existence results follow; the most general one is

Theorem (well-posedness for complete maximal germs)

(Still $\mathfrak{f}(x,.)$ given by $f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$; let $f^{l,r}$ be Lipschitz continuous)
Assume $\mathcal{G}$ is a definite germ of which the dual $\mathcal{G}^*$ is complete .
Then for all L^∞ initial datum there exists a unique $\mathcal{G}$-entropy solution.
This solution can be obtained as the limit of a monotone FV scheme with the Godunov choice at the interface.

Well, using the Godunov scheme at the interface requires a deep knowledge of the Riemann solver at the interface, in particular, the germ $\mathcal{G}^*$ should be explicitly known. What about convergence of "more naive" schemes?

It turned out (A.&Goatin&Seguin , for the constrained traffic pb. above) that very "naive" schemes can converge, but the analysis uses non-generic arguments . Other "naive" schemes: Particle-in-Burgers pb (see below).

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Convergence: back to the Buckley-Leverett equation in two-rocks' medium

What's the use of all this theory ???

Let's go back to the two-rocks' Buckley-Leverett problem:
characterize intrinsically u the limit of u^ε solutions of

$$(BL_\varepsilon) \qquad \partial_t u^\varepsilon + \partial_x f^{l,r}(u^\varepsilon) = \varepsilon \partial_x \left(\lambda^{l,r}(u^\varepsilon)\, \partial_x \pi^{l,r}(u^\varepsilon)\right)$$

where the connection at the interface is:

– flux transmission (conservativity)
– capillarities connected: $\pi^l(\gamma^l u^\varepsilon) = \pi^r(\gamma^r u^\varepsilon)$
(this is physical: Cancès; Schweizer et al.).

The key information is as follows :

– for each $\varepsilon > 0$, L^1-contractivity for u^ε (Kato inequality)
– for given data, family $(u^\varepsilon)_\varepsilon$ has an accumulation point u as $\varepsilon \to 0$
– given $\pi^{l,r}$, some functions $c(x) = c^l_\pi(.)1\!\!1_{\{x<0\}} + c^r_\pi(.)1\!\!1_{\{x>0\}}$
are evident vanishing capillarity limits
– these couples (c^l_π, c^r_π) form a definite germ $\mathcal{G}_\pi$!

Consequently,
– the limits u of $(u^\varepsilon)_\varepsilon$ form an L^1-consractive semigroup
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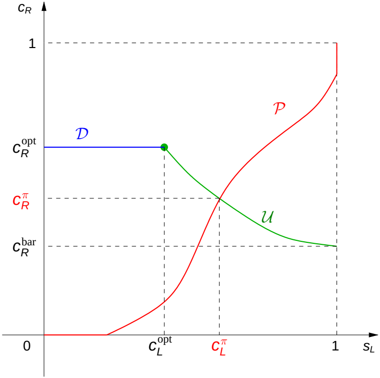
Capillarity chooses the connection...

Due to the work of Adimurthi, Mishra, Veerappa Gowda the germs are fully classified in the case of Buckley-Leverett kind fluxes $f^{l,r}$ ("bell-shaped" fluxes). Namely, it is enough to consider the definite germs $\mathcal{G}_{(A,B)} := \{(A,B)\}$ (singletons !) where (A,B) are "overcompressive shocks".



$\Rightarrow$ the germ $\mathcal{G}_{(c^l_\pi, c^r_\pi)}$ is selected by capillarity limit

Figure: The trivial vanishing capillarity limit $c(x) = c^l_\pi \mathbb{1}_{\{x<0\}} + c^r_\pi \mathbb{1}_{\{x>0\}}$ is found at the intersection of the two curves $\mathcal{P}$ (fluxes connected) and $\mathcal{U} \cup \mathcal{D}$ (capillarities connected).

Capillarity chooses the flux limitation level !

- For bell-shaped fluxes, the set $\mathcal{U}$ defined by

$$\{(u^l, u^r) \,|\, \text{the fluxes } f^l(u^l) \text{ and } f^r(u^r) \text{ connected by overcompr. shock}\}$$

is a portion of decreasing curve in $[0,1] \times [0,1]$.
The curve $\mathcal{U}$ has to be complemented by a segment $\mathcal{D}$
($\Rightarrow$ interface layer in u_ε !).

- Because $\pi^{l,r}$ are increasing, the set $\mathcal{P}$ defined as

$$\{(u^l_\varepsilon, u^r_\varepsilon) \,|\, \text{the capillarities } \pi^l(u^l_\varepsilon) \text{ and } \pi^r(u^r_\varepsilon) \text{ connected}\}$$

is a monotone graph in $[0,1] \times [0,1]$.

Moreover, one can interprete the coupling in terms of the flux limitation level $\bar{F}_\pi := f^{l,r}(c^{l,r}_\pi)$, as in the road traffic case !
Highest level $\Rightarrow$ "optimal" connection
Smallest level $\Rightarrow$ "barrier" connection (oil reservoirs created!)

Moreover, to get the Godunov flux for solutions corresponding to connection of level $\bar{F}_\pi$, it is enough to replace the Adimurthi, Jaffré, Veerappa Gowda flux $g(.,.)$ by the limited flux $\min\{\bar{F}_\pi, g(.,.)\}$.

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A new application: Particle-in-Burgers problem (non-conservative)

Coupling of a Burgers fluid with point particle

A model for the interaction, *via* a drag force, of a point particle with a Burgers fluid writes

$$u_t + (u^2/2)_t = (h'(t) - u)\,\delta_0(x - h(t)),$$
$$mh''(t) = u(t, h(t)) - h'(t)$$

- u, the velocity of the fluid
- h, the position of the solid particle (then h' and h'' respectively denote its velocity and acceleration).

Auxiliary problem:

$$u_t + (u^2/2)_x = -u(t,x)\,\delta_0(x)$$

i.e., the particle is decoupled from the fluid and fixed at zero.

Coupling non-conservative $\Rightarrow$ a study of nonconservative product (by approximation with regularized Dirac function) : Lagoutière, Seguin, Takahashi $\Rightarrow$ a "non-conservative" germ $\mathcal{G}$! $\Rightarrow$ a notion of entropy solution.

With much more work (interpretation of the ODE for the particle, finite volume approximation, splitting, fixed-point techniques, wave-front tracking) $\Rightarrow$ well-posedness and simple convergent finite volume scheme.

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Thank you for your attention !

Dziękuję – Thank you