On conservation laws with discontinuous flux in the porous medium context.

B. Andreianov[1]

based on joint works with
K.H. Karlsen, N.N. Risebro
C. Cancès, P. Goatin, N. Seguin

[1] Laboratoire de Mathématiques CNRS UMR 6623
Université de Franche-Comté
Besançon, France

MAMERN 2011

Saidia, Morocco, May 2011

Plan of the talk

BUCKLEY-LEVERETT EQUATION IN FRACTURED POROUS MEDIUM

Two-phase porous medium model in one space dimension...

Consider a two-phase flow of phases ("oil", "water") of saturation u and $1-u$, respectively, with neglected (???) capillary pressure effects , in a vertical homogeneous porous medium with gravity. This is the Buckley-Leverett model, which is a hyperbolic conservation law

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0 \text{ with initial datum } u_0.$$

The Buckley-Leverett flux $f : [0,1] \mapsto \mathbb{R}$ is computed as follows:

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)}$$

and

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

Physical meaning:

K : permeability of the medium
k_o, k_w relative permeabilities per phase
μ_a, μ_b phase viscosities
ρ_a, ρ_b phase densities.

Two-phase porous medium model in one space dimension...

Consider a two-phase flow of phases ("oil", "water") of saturation u and $1-u$, respectively, with neglected (???) capillary pressure effects , in a vertical homogeneous porous medium with gravity. This is the Buckley-Leverett model, which is a hyperbolic conservation law

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0 \text{ with initial datum } u_0.$$

The Buckley-Leverett flux $f : [0,1] \mapsto \mathbb{R}$ is computed as follows:

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)}$$

and

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

Physical meaning:

K : permeability of the medium
k_o, k_w relative permeabilities per phase
μ_a, μ_b phase viscosities
ρ_a, ρ_b phase densities.

Two-phase porous medium model in one space dimension...

Consider a two-phase flow of phases ("oil", "water") of saturation u and $1-u$, respectively, with neglected (???) capillary pressure effects , in a vertical homogeneous porous medium with gravity. This is the Buckley-Leverett model, which is a hyperbolic conservation law

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0 \text{ with initial datum } u_0.$$

The Buckley-Leverett flux $f : [0,1] \mapsto \mathbb{R}$ is computed as follows:

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)}$$

and

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

Physical meaning:

K : permeability of the medium
k_o, k_w relative permeabilities per phase
μ_a, μ_b phase viscosities
ρ_a, ρ_b phase densities.

Buckley-Leverett equation as limit of degenerate parabolic pb...

A more precise model contains an additional second-order term, with a small effect due to capillary pressure :

$$(BL_\varepsilon) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\lambda_o(u)\, \partial_x \pi(u) \right)$$

with

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)},$$

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

and π an increasing function modelling the capillary pressure.

Typical form of f : $f(0) = 0$, $f(1) = q_{tot}$, and f is bell-shaped (i.e., f has one local maximum in $(0, 1)$).

Typical form of π: a strictly increasing on $(0, 1)$ function with any kind of behaviour at $u = 0$ and $u = 1$ (finite values or vertical asymptotes).

NB: (Cancès) in the case of finite values at 0 or at 1, we will consider π extended to a maximal monotone graph on $[0, 1]$ (i.e., a monotone function, multivalued at $u = 0$ at $u = 1$).

Buckley-Leverett equation as limit of degenerate parabolic pb...

A more precise model contains an additional second-order term, with a small effect due to capillary pressure :

$$(BL_\varepsilon) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\lambda_o(u)\, \partial_x \pi(u)\right)$$

with

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)},$$

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

and π an increasing function modelling the capillary pressure.

Typical form of f : $f(0) = 0$, $f(1) = q_{tot}$, and f is bell-shaped (i.e., f has one local maximum in $(0, 1)$).

Typical form of π: a strictly increasing on $(0, 1)$ function with any kind of behaviour at $u = 0$ and $u = 1$ (finite values or vertical asymptotes).

NB: (Cancès) in the case of finite values at 0 or at 1, we will consider π extended to a maximal monotone graph on $[0, 1]$ (i.e., a monotone function, multivalued at $u = 0$ at $u = 1$).

Buckley-Leverett equation as limit of degenerate parabolic pb...

A more precise model contains an additional second-order term, with a small effect due to capillary pressure :

$$(BL_\varepsilon) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\lambda_o(u) \, \partial_x \pi(u) \right)$$

with

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)},$$

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

and π an increasing function modelling the capillary pressure.

Typical form of f : $f(0) = 0$, $f(1) = q_{tot}$, and f is bell-shaped (i.e., f has one local maximum in $(0, 1)$).

Typical form of π: a strictly increasing on $(0, 1)$ function with any kind of behaviour at $u = 0$ and $u = 1$ (finite values or vertical asymptotes).

NB: (Cancès) in the case of finite values at 0 or at 1, we will consider π extended to a maximal monotone graph on $[0, 1]$ (i.e., a monotone function, multivalued at $u = 0$ at $u = 1$).

Buckley-Leverett equation as limit of degenerate parabolic pb...

A more precise model contains an additional second-order term, with a small effect due to capillary pressure :

$$(BL_\varepsilon) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\lambda_o(u) \, \partial_x \pi(u) \right)$$

with

$$f(u) = (\rho_o - \rho_w) g \lambda_o(u) + q_{tot} \frac{k_o(u)}{k_o(u) + \frac{\mu_o}{\mu_w} k_w(u)},$$

$$\lambda_o(u) = K \frac{k_o(u) k_w(u)}{\mu_w k_o(u) + \mu_o k_w(u)}.$$

and π an increasing function modelling the capillary pressure.

Typical form of f : $f(0) = 0$, $f(1) = q_{tot}$, and f is bell-shaped (i.e., f has one local maximum in $(0,1)$).

Typical form of π: a strictly increasing on $(0,1)$ function with any kind of behaviour at $u = 0$ and $u = 1$ (finite values or vertical asymptotes).

NB: (Cancès) in the case of finite values at 0 or at 1,
we will consider π extended to a maximal monotone graph on $[0,1]$
(i.e., a monotone function, multivalued at $u = 0$ at $u = 1$).

Buckley-Leverett equation: entropy solutions...

We forget about the motivations and look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

being understood that solutions are expected to be limits of the capillarity regularized parabolic problem (BL_ε) .

The solutions are "just L^∞" and can be characterized *intrinsically* by Kruzhkov entropy inequalities

$$\forall k \in \mathbb{R} \quad \partial_t |u-k| + \partial_x \operatorname{sign}(u-k)(f(u)-f(k)) \leqslant 0,$$

in the sense of distributions.
Notation : $q(u,k) := \operatorname{sign}(u-k)(f(u)-f(k))$ is the Kruzhkov entropy flux.

"Typical solutions" are piecewise smooth, with smoothness regions separated by jumps. In the regions of smoothness, entropy inequalities hold with "$=$" sign, but on the jumps, we truly have "$<$": this corresponds to dissipation processes taking place on the jumps.

NB: a capillary pressure π is "tacitly present" in the formulation,
its presence is manifested by the sign "<" of entropy inequalities;
but the notion of solution does not depend on the form of π !

Buckley-Leverett equation: entropy solutions...

We forget about the motivations and look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

being understood that solutions are expected to be limits of the capillarity regularized parabolic problem (BL_ε) .

The solutions are "just L^∞" and can be characterized *intrinsically* by Kruzhkov entropy inequalities

$$\forall k \in \mathbb{R} \quad \partial_t |u-k| + \partial_x \operatorname{sign}(u-k)(f(u)-f(k)) \leqslant 0,$$

in the sense of distributions.

Notation : $q(u,k) := \operatorname{sign}(u-k)(f(u)-f(k))$ is the Kruzhkov entropy flux.

"Typical solutions" are piecewise smooth, with smoothness regions separated by jumps. In the regions of smoothness, entropy inequalities hold with "$=$" sign, but on the jumps, we truly have "$<$": this corresponds to dissipation processes taking place on the jumps.

NB: a capillary pressure π is "tacitly present" in the formulation, its presence is manifested by the sign "$<$" of entropy inequalities; but the notion of solution does not depend on the form of π !

Buckley-Leverett equation: entropy solutions...

We forget about the motivations and look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

being understood that solutions are expected to be limits of the capillarity regularized parabolic problem (BL_ε) .

The solutions are “just L^∞” and can be characterized *intrinsically* by Kruzhkov entropy inequalities

$$\forall k \in \mathbb{R} \quad \partial_t |u-k| + \partial_x \operatorname{sign}(u-k)(f(u)-f(k)) \leqslant 0,$$

in the sense of distributions.

Notation : $q(u,k) := \operatorname{sign}(u-k)(f(u)-f(k))$ is the Kruzhkov entropy flux.

“Typical solutions” are piecewise smooth, with smoothness regions separated by jumps. In the regions of smoothness, entropy inequalities hold with "$=$" sign, but on the jumps, we truly have "$<$": this corresponds to dissipation processes taking place on the jumps.

NB: a capillary pressure π is “tacitly present” in the formulation,

its presence is manifested by the sign “$<$” of entropy inequalities;

but the notion of solution does not depend on the form of π !

Buckley-Leverett equation: entropy solutions...

We forget about the motivations and look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

being understood that solutions are expected to be limits of the capillarity regularized parabolic problem (BL_ε) .

The solutions are "just L^∞" and can be characterized *intrinsically* by Kruzhkov entropy inequalities

$$\forall k \in \mathbb{R} \quad \partial_t |u-k| + \partial_x \operatorname{sign}(u-k)(f(u)-f(k)) \leqslant 0,$$

in the sense of distributions.

Notation : $q(u,k) := \operatorname{sign}(u-k)(f(u)-f(k))$ is the Kruzhkov entropy flux.

"Typical solutions" are piecewise smooth, with smoothness regions separated by jumps. In the regions of smoothness, entropy inequalities hold with "$=$" sign, but on the jumps, we truly have "$<$": this corresponds to dissipation processes taking place on the jumps.

NB: a capillary pressure π is "tacitly present" in the formulation,
its presence is manifested by the sign "$<$" of entropy inequalities;
but the notion of solution does not depend on the form of π !

Buckley-Leverett equation: entropy solutions...

We forget about the motivations and look at the resulting hyperbolic equation:

$$(BL) \qquad \partial_t u + \partial_x f(u) = 0,$$

being understood that solutions are expected to be limits of the capillarity regularized parabolic problem (BL_ε) .

The solutions are “just L^∞” and can be characterized *intrinsically* by Kruzhkov entropy inequalities

$$\forall k \in \mathbb{R} \quad \partial_t |u-k| + \partial_x \operatorname{sign}(u-k)(f(u)-f(k)) \leqslant 0,$$

in the sense of distributions.

Notation : $q(u,k) := \operatorname{sign}(u-k)(f(u)-f(k))$ is the Kruzhkov entropy flux.

“Typical solutions” are piecewise smooth, with smoothness regions separated by jumps. In the regions of smoothness, entropy inequalities hold with "$=$" sign, but on the jumps, we truly have "$<$": this corresponds to dissipation processes taking place on the jumps.

NB: a capillary pressure π is “tacitly present” in the formulation,
its presence is manifested by the sign “<” of entropy inequalities;
but the notion of solution does not depend on the form of π !

...Buckley-Leverett equation: entropy solutions.

MAIN FACT:
The (BL) equation is well posed in the framework of entropy solutions.

More precisely :

- For all L^∞ initial datum, there exists a unique entropy solution
- The solver $\mathcal{S}$ (i.e., the map $\mathcal{S} : u_0(x) \mapsto u(t,x)$ that associates the entropy solution to an initial datum) is L^1-contractive , in the sense that the Kato inequality holds:

$$\partial_t |u - \hat{u}| + \partial_x \operatorname{sign}(u - \hat{u})(f(u) - f(\hat{u})) \leqslant 0$$

  for $u, \hat{u}$ two entropy solutions.

  NB: if $\hat{u} \equiv k$, it's just the definition !
  (the Kato inequality is then one of the entropy inequalities)
- Every entropy solution of (BL) is the limit, as $\varepsilon \to 0$, of the solutions u_ε of the capillarity regularized parabolic problem (BL_ε).

...Buckley-Leverett equation: entropy solutions.

MAIN FACT:
The (BL) equation is well posed in the framework of entropy solutions.

More precisely :

- For all L^∞ initial datum, there exists a unique entropy solution
- The solver $\mathcal{S}$ (i.e., the map $\mathcal{S} : u_0(x) \mapsto u(t,x)$ that associates the entropy solution to an initial datum) is L^1-contractive , in the sense that the Kato inequality holds:

$$\partial_t |u - \hat{u}| + \partial_x \operatorname{sign}(u - \hat{u})(f(u) - f(\hat{u})) \leqslant 0$$

  for $u, \hat{u}$ two entropy solutions.

  NB: if $\hat{u} \equiv k$, it's just the definition !
  (the Kato inequality is then one of the entropy inequalities)
- Every entropy solution of (BL) is the limit, as $\varepsilon \to 0$, of the solutions u_ε of the capillarity regularized parabolic problem (BL_ε).

...Buckley-Leverett equation: entropy solutions.

MAIN FACT:
The (BL) equation is well posed in the framework of entropy solutions.

More precisely :

- For all L^∞ initial datum, there exists a unique entropy solution
- The solver $\mathcal{S}$ (i.e., the map $\mathcal{S} : u_0(x) \mapsto u(t,x)$ that associates the entropy solution to an initial datum) is L^1-contractive , in the sense that the Kato inequality holds:

$$\partial_t |u - \hat{u}| + \partial_x \operatorname{sign}(u - \hat{u})(f(u) - f(\hat{u})) \leqslant 0$$

  for $u, \hat{u}$ two entropy solutions.

  NB: if $\hat{u} \equiv k$, it's just the definition !
  (the Kato inequality is then one of the entropy inequalities)
- Every entropy solution of (BL) is the limit, as $\varepsilon \to 0$, of the solutions u_ε of the capillarity regularized parabolic problem (BL_ε).

...Buckley-Leverett equation: entropy solutions.

MAIN FACT:
The (BL) equation is well posed in the framework of entropy solutions.

More precisely :

- For all L^∞ initial datum, there exists a unique entropy solution
- The solver $\mathcal{S}$ (i.e., the map $\mathcal{S} : u_0(x) \mapsto u(t,x)$ that associates the entropy solution to an initial datum) is L^1-contractive , in the sense that the Kato inequality holds:

$$\partial_t |u - \hat{u}| + \partial_x \operatorname{sign}(u - \hat{u})(f(u) - f(\hat{u})) \leqslant 0$$

  for $u, \hat{u}$ two entropy solutions.

  NB: if $\hat{u} \equiv k$, it's just the definition !
  (the Kato inequality is then one of the entropy inequalities)
- Every entropy solution of (BL) is the limit, as $\varepsilon \to 0$, of the solutions u_ε of the capillarity regularized parabolic problem (BL_ε).

The case of a heterogeneous medium : model equation.

In order to study the Buckley-Leverett equation in a porous medium that consists of geologically different “blocks” separated by sharp fractures, we look at the following model problem :

$$\partial_t u + \partial_x \mathfrak{f}(x,u)_x = 0, \quad \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x<0, \\ f^r(u), & x>0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x=0\}$.

It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution can be used separately in each region $\{x<0\}$ and $\{x>0\}$;

– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(u^l) = f^r(u^r) \ \text{ where } u^{l,r} = (\gamma^{l,r} u)(t) \text{ are “traces”}$$

– But what dissipative processes take place on the interface ???

We answer the question and give a physically motivated notion of solution of the “discontinuous flux Buckley-Leverett” equation.

The case of a heterogeneous medium : model equation.

In order to study the Buckley-Leverett equation in a porous medium that consists of geologically different “blocks” separated by sharp fractures, we look at the following model problem :

$$\partial_t u + \partial_x \mathfrak{f}(x,u)_x = 0, \quad \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x<0, \\ f^r(u), & x>0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x=0\}$.

It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution can be used separately in each region $\{x<0\}$ and $\{x>0\}$;

– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(u^l) = f^r(u^r) \;\text{ where } u^{l,r} = (\gamma^{l,r} u)(t) \text{ are “traces”}$$

– But what dissipative processes take place on the interface ???

We answer the question and give a physically motivated notion of solution of the “discontinuous flux Buckley-Leverett” equation.

The case of a heterogeneous medium : model equation.

In order to study the Buckley-Leverett equation in a porous medium that consists of geologically different "blocks" separated by sharp fractures, we look at the following model problem :

$$\partial_t u + \partial_x \mathfrak{f}(x,u)_x = 0, \ \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x<0, \\ f^r(u), & x>0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x=0\}$.
It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution can be used separately in each region $\{x<0\}$ and $\{x>0\}$;
– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(u^l) = f^r(u^r) \ \text{ where } u^{l,r} = (\gamma^{l,r} u)(t) \text{ are "traces"}$$

– But what dissipative processes take place on the interface ???

We answer the question and give a physically motivated notion of solution of the "discontinuous flux Buckley-Leverett" equation.

The case of a heterogeneous medium : model equation.

In order to study the Buckley-Leverett equation in a porous medium that consists of geologically different "blocks" separated by sharp fractures, we look at the following model problem :

$$\partial_t u + \partial_x \mathfrak{f}(x,u)_x = 0, \ \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x < 0, \\ f^r(u), & x > 0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x = 0\}$.
It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution can be used separately in each region $\{x < 0\}$ and $\{x > 0\}$;
– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(u^l) = f^r(u^r) \text{ where } u^{l,r} = (\gamma^{l,r}u)(t) \text{ are "traces"}$$

– But what dissipative processes take place on the interface ???

We answer the question and give a physically motivated notion of solution of the "discontinuous flux Buckley-Leverett" equation.

The case of a heterogeneous medium : model equation.

In order to study the Buckley-Leverett equation in a porous medium that consists of geologically different “blocks” separated by sharp fractures, we look at the following model problem :

$$\partial_t u + \partial_x \mathfrak{f}(x,u)_x = 0, \ \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x<0, \\ f^r(u), & x>0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

where $f^{l,r}$ are two different nonlinearities . We can see it as two conservation laws coupled across the interface at $\{x=0\}$.
It is clear that understanding the problem is equivalent to understanding what coupling is allowed at the interface. One uses,

– the standard (Kruzhkov) notion of entropy solution can be used separately in each region $\{x<0\}$ and $\{x>0\}$;
– the Rankine-Hugoniot condition (flux conservation) on the interface:

$$f^l(u^l) = f^r(u^r) \ \text{ where } u^{l,r} = (\gamma^{l,r} u)(t) \text{ are “traces”}$$

– But what dissipative processes take place on the interface ???

We answer the question and give a physically motivated notion of solution of the “discontinuous flux Buckley-Leverett” equation.

AN ADMISSIBLE STATIONARY SOLUTION

Physical conditions on the interface: Rankine-Hugoniot condition.

First, look at the flux conservation at the interface :

$$f^l(u^l) = f^r(u^r) \text{ where } u^{l,r} = (\gamma^{l,r} u)(t) \text{ are traces .}$$

Indeed, it was shown recently (Vasseur, Panov) that solutions "in L^∞" are actually regular enough to have strong traces on the boundary , at least for "genuinely nonlinear" $f^{l,r}$.

In the configuration of fluxes $f^{l,r}$ we may have for the Buckley-Leverett equation , there holds

$$f^{l,r}(0) = 0, \quad f^{l,r}(1) = q_{tot}, \quad \text{and both functions } f^{l,r} \text{ are "bell-shaped" .}$$

which means that, roughly speaking, at every "flux level" F in $Range(f^l) \cap Range(f^r)$ we have one or two states u^l (with $f(u^l) = F$) and one or two states u^r (with f(u^r) $= F$) that we can "connect" across the interface. Let's call "connection" any such couple (u^l, u^r).

Some of such connections will not be "admissible": need to select the "good ones"! To select, we will "connect" capillary pressures.

Physical conditions on the interface: Rankine-Hugoniot condition.

First, look at the flux conservation at the interface :

$$f^l(u^l) = f^r(u^r) \text{ where } u^{l,r} = (\gamma^{l,r}u)(t) \text{ are traces .}$$

Indeed, it was shown recently (Vasseur, Panov) that solutions "in L^∞" are actually regular enough to have strong traces on the boundary , at least for "genuinely nonlinear" $f^{l,r}$.
In the configuration of fluxes $f^{l,r}$ we may have for the Buckley-Leverett equation , there holds

$$f^{l,r}(0) = 0, \quad f^{l,r}(1) = q_{tot}, \quad \text{and both functions } f^{l,r} \text{ are "bell-shaped" .}$$

which means that, roughly speaking, at every "flux level" F in $Range(f^l) \cap Range(f^r)$ we have one or two states u^l (with $f(u^l) = F$) and one or two states u^r (with f(u^r) $= F$) that we can "connect" across the interface. Let's call "connection" any such couple (u^l, u^r).

Some of such connections will not be "admissible": need to select the "good ones"! To select, we will "connect" capillary pressures.

Physical conditions on the interface: Rankine-Hugoniot condition.

First, look at the flux conservation at the interface :

$$f^l(u^l) = f^r(u^r) \text{ where } u^{l,r} = (\gamma^{l,r}u)(t) \text{ are traces .}$$

Indeed, it was shown recently (Vasseur, Panov) that solutions “in L^∞” are actually regular enough to have strong traces on the boundary , at least for “genuinely nonlinear” $f^{l,r}$.

In the configuration of fluxes $f^{l,r}$ we may have for the Buckley-Leverett equation , there holds

$$f^{l,r}(0) = 0, \quad f^{l,r}(1) = q_{tot}, \quad \text{and both functions } f^{l,r} \text{ are “bell-shaped” .}$$

which means that, roughly speaking, at every “flux level” F in $Range(f^l) \cap Range(f^r)$ we have one or two states u^l (with $f(u^l) = F$) and one or two states u^r (with f(u^r) $= F$) that we can “connect” across the interface. Let's call “connection” any such couple (u^l, u^r).

Some of such connections will not be “admissible”: need to select the “good ones”! To select, we will “connect” capillary pressures.

“Non-Lax connections”.

For a short moment, look at the case $f^l = f^r$: this is the Kruzhkov case !!!
But, then it is well known that the “admissible connections” (u^l, u^r) must satisfy the so-called Lax condition :

$$(*) \qquad (f^l)'(u^l) \geqslant 0 \text{ and } (f^r)'(u^r) \leqslant 0.$$

(here f^l and f^r are one and the same function).

It was understood in the 1990th that in the discontinuous flux case,

– some non-Lax connections should appear (heuristically, “the interface creates information !”).
– and some Lax connections will become non admissible.

Those non-Lax connections for which *both* signs in $(*)$ are inverted will be called totally non-Lax connections .

Let us concentrate on finding admissible totally non-Lax connections:
we will see later that there is one and only one such connection (u^l_π, u^r_π) , and it determines completely the interface coupling.

For bell-shaped fluxes, the set $\mathcal{L}_f$ defined by

$$\{(u^l, u^r) \mid f^l(u^l) \text{ and } f^r(u^r) \text{ are connected by a totally non-Lax shock}\}$$

is a portion of decreasing curve in $[0, 1] \times [0, 1]$.

"Non-Lax connections".

For a short moment, look at the case $f^l = f^r$: this is the Kruzhkov case !!!
But, then it is well known that the "admissible connections" (u^l, u^r) must satisfy the so-called Lax condition :

$$(*) \qquad (f^l)'(u^l) \geqslant 0 \;\text{ and }\; (f^r)'(u^r) \leqslant 0.$$

(here f^l and f^r are one and the same function).

It was understood in the 1990th that in the discontinuous flux case,

– some non-Lax connections should appear (heuristically, "the interface creates information !").
– and some Lax connections will become non admissible.

Those non-Lax connections for which *both* signs in $(*)$ are inverted will be called totally non-Lax connections .

Let us concentrate on finding admissible totally non-Lax connections:
we will see later that there is one and only one such connection (u^l_π, u^r_π) , and it determines completely the interface coupling.

For bell-shaped fluxes, the set $\mathcal{L}_f$ defined by

$$\{(u^l, u^r) \,|\, f^l(u^l) \text{ and } f^r(u^r) \text{ are connected by a totally non-Lax shock}\}$$

is a portion of decreasing curve in $[0,1] \times [0,1]$.

“Non-Lax connections”.

For a short moment, look at the case $f^l = f^r$: this is the Kruzhkov case !!!
But, then it is well known that the “admissible connections” (u^l, u^r) must satisfy the so-called Lax condition :

$$(*) \qquad (f^l)'(u^l) \geqslant 0 \;\text{ and }\; (f^r)'(u^r) \leqslant 0.$$

(here f^l and f^r are one and the same function).

It was understood in the 1990th that in the discontinuous flux case,

– some non-Lax connections should appear (heuristically, “the interface creates information !”).
– and some Lax connections will become non admissible.

Those non-Lax connections for which *both* signs in $(*)$ are inverted will be called totally non-Lax connections .

Let us concentrate on finding admissible totally non-Lax connections:
we will see later that there is one and only one such connection (u^l_π, u^r_π) , and it determines completely the interface coupling.

For bell-shaped fluxes, the set $\mathcal{L}_f$ defined by

$$\{(u^l, u^r) \,|\, f^l(u^l) \text{ and } f^r(u^r) \text{ are connected by a totally non-Lax shock}\}$$

is a portion of decreasing curve in $[0, 1] \times [0, 1]$.

“Non-Lax connections”.

For a short moment, look at the case $f^l = f^r$: this is the Kruzhkov case !!!
But, then it is well known that the “admissible connections” (u^l, u^r) must satisfy the so-called Lax condition :

$$(*) \qquad (f^l)'(u^l) \geqslant 0 \;\text{ and }\; (f^r)'(u^r) \leqslant 0.$$

(here f^l and f^r are one and the same function).

It was understood in the 1990th that in the discontinuous flux case,

– some non-Lax connections should appear (heuristically, “the interface creates information !”).
– and some Lax connections will become non admissible.

Those non-Lax connections for which *both* signs in $(*)$ are inverted will be called totally non-Lax connections .

Let us concentrate on finding admissible totally non-Lax connections:
we will see later that there is one and only one such connection (u^l_π, u^r_π) , and it determines completely the interface coupling.

For bell-shaped fluxes, the set $\mathcal{L}_f$ defined by

$$\{(u^l, u^r) \,|\, f^l(u^l) \text{ and } f^r(u^r) \text{ are connected by a totally non-Lax shock}\}$$

is a portion of decreasing curve in $[0,1] \times [0,1]$.

Two particular (extremal) totally non-Lax connections.

In the previous works (Kaasschieter; Adimurthi et al.; Cancès), two connections were used in the context of porous media: the optimal and the barrier connections.

- The optimal connection maximizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the largest possible. We call this connection (u^l_{opt}, u^r_{opt}).
- The barrier connection minimizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the smallest possible among all totally non-Lax connections. We call this connection (u^l_{bar}, u^r_{bar}). In practice, we have $u^l_{bar} = 1$ if $q_{tot} \geqslant 0$ and $u^r_{bar} = 0$ if $q_{tot} \leqslant 0$.

The optimal connection “facilitates” the transport of the “oil” phase at the interface. The barrier connection corresponds to “trapping” of the ‘oil” phase at the interface.

Both phenomena appear in practice !! But we will show that all intermediate cases can also appear.

The choice depends on the FORM of capillary pressure graphs $\pi^{l,r}$.

The totally non-Lax connection we'll select will be called (u^l_π, u^r_π).

Two particular (extremal) totally non-Lax connections.

In the previous works (Kaasschieter; Adimurthi et al.; Cancès), two connections were used in the context of porous media: the optimal and the barrier connections.

- The optimal connection maximizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the largest possible. We call this connection (u^l_{opt}, u^r_{opt}).
- The barrier connection minimizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the smallest possible among all totally non-Lax connections. We call this connection (u^l_{bar}, u^r_{bar}). In practice, we have $u^l_{bar} = 1$ if $q_{tot} \geqslant 0$ and $u^r_{bar} = 0$ if $q_{tot} \leqslant 0$.

The optimal connection "facilitates" the transport of the "oil" phase at the interface. The barrier connection corresponds to "trapping" of the 'oil" phase at the interface.

Both phenomena appear in practice !! But we will show that all intermediate cases can also appear.

The choice depends on the FORM of capillary pressure graphs $\pi^{l,r}$.

The totally non-Lax connection we'll select will be called (u^l_{π}, u^r_{π}).

Two particular (extremal) totally non-Lax connections.

In the previous works (Kaasschieter; Adimurthi et al.; Cancès), two connections were used in the context of porous media: the optimal and the barrier connections.

- The optimal connection maximizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the largest possible. We call this connection (u^l_{opt}, u^r_{opt}).
- The barrier connection minimizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the smallest possible among all totally non-Lax connections. We call this connection (u^l_{bar}, u^r_{bar}). In practice, we have $u^l_{bar} = 1$ if $q_{tot} \geqslant 0$ and $u^r_{bar} = 0$ if $q_{tot} \leqslant 0$.

The optimal connection "facilitates" the transport of the "oil" phase at the interface. The barrier connection corresponds to "trapping" of the 'oil" phase at the interface.

Both phenomena appear in practice !! But we will show that all intermediate cases can also appear.

The choice depends on the FORM of capillary pressure graphs $\pi^{l,r}$.

The totally non-Lax connection we'll select will be called (u^l_π, u^r_π).

Two particular (extremal) totally non-Lax connections.

In the previous works (Kaasschieter; Adimurthi et al.; Cancès), two connections were used in the context of porous media: the optimal and the barrier connections.

- The optimal connection maximizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the largest possible. We call this connection (u^l_{opt}, u^r_{opt}).
- The barrier connection minimizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the smallest possible among all totally non-Lax connections. We call this connection (u^l_{bar}, u^r_{bar}). In practice, we have $u^l_{bar} = 1$ if $q_{tot} \geqslant 0$ and $u^r_{bar} = 0$ if $q_{tot} \leqslant 0$.

The optimal connection "facilitates" the transport of the "oil" phase at the interface. The barrier connection corresponds to "trapping" of the 'oil" phase at the interface.

Both phenomena appear in practice !! But we will show that all intermediate cases can also appear.

The choice depends on the FORM of capillary pressure graphs $\pi^{l,r}$.

The totally non-Lax connection we'll select will be called (u^l_π, u^r_π).

Two particular (extremal) totally non-Lax connections.

In the previous works (Kaasschieter; Adimurthi et al.; Cancès), two connections were used in the context of porous media: the optimal and the barrier connections.

- The optimal connection maximizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the largest possible. We call this connection (u^l_{opt}, u^r_{opt}).
- The barrier connection minimizes the flux at the interface : $F = f^l(u^l) = f^r(u^r)$ in the smallest possible among all totally non-Lax connections. We call this connection (u^l_{bar}, u^r_{bar}). In practice, we have $u^l_{bar} = 1$ if $q_{tot} \geqslant 0$ and $u^r_{bar} = 0$ if $q_{tot} \leqslant 0$.

The optimal connection “facilitates” the transport of the “oil” phase at the interface. The barrier connection corresponds to “trapping” of the ‘oil” phase at the interface.

Both phenomena appear in practice !! But we will show that all intermediate cases can also appear.
The choice depends on the FORM of capillary pressure graphs $\pi^{l,r}$.

The totally non-Lax connection we'll select will be called (u^l_π, u^r_π).

Physical conditions on the interface: capillary pressures.

To select connections , we need to know what information on the interface can be "inherited" from the idea that a solution u is a limit of solutions u_ε of the parabolic, capillarity regularized problem (here $\pi^{l,r}$, not present in (BL), enter the stage!).

We have seen that in the parabolic (regularized) problem, the quantity connected through the interface is the capillary pressure. Moreover, it was shown by Cancès that the right way to connect capillary pressures π^l and π^r is the following :

$$\pi^l(\gamma^l u_\varepsilon) \text{ "} = \text{" } \pi^r(\gamma^r u_\varepsilon) \quad \text{in the generalized sense:}$$

namely, at $u = 0$ and at $u = 1$, $\pi^{l,r}$ are extended to multi-valued monotone functions and "equality" actually means

$$\pi^l(\gamma^l u_\varepsilon) \cap \pi^r(\gamma^r u_\varepsilon) \text{ is non-empty .}$$

Because $\pi^{l,r}$ are increasing, the set $\mathcal{L}_\pi$ defined as

$$\{(u^l_\varepsilon, u^r_\varepsilon) \,|\, \pi^l(u^l_\varepsilon) \text{ and } \pi^r(u^r_\varepsilon) \text{ are connected}\}$$

is a monotone graph in $[0,1] \times [0,1]$.

Physical conditions on the interface: capillary pressures.

To select connections , we need to know what information on the interface can be "inherited" from the idea that a solution u is a limit of solutions u_ε of the parabolic, capillarity regularized problem (here $\pi^{l,r}$, not present in (BL), enter the stage!).

We have seen that in the parabolic (regularized) problem, the quantity connected through the interface is the capillary pressure. Moreover, it was shown by Cancès that the right way to connect capillary pressures π^l and π^r is the following :

$$\pi^l(\gamma^l u_\varepsilon) \text{ "} = \text{" } \pi^r(\gamma^r u_\varepsilon) \quad \text{in the generalized sense:}$$

namely, at $u = 0$ and at $u = 1$, $\pi^{l,r}$ are extended to multi-valued monotone functions and "equality" actually means

$$\pi^l(\gamma^l u_\varepsilon) \cap \pi^r(\gamma^r u_\varepsilon) \text{ is non-empty .}$$

Because $\pi^{l,r}$ are increasing, the set $\mathcal{L}_\pi$ defined as

$$\{(u^l_\varepsilon, u^r_\varepsilon) \,|\, \pi^l(u^l_\varepsilon) \text{ and } \pi^r(u^r_\varepsilon) \text{ are connected}\}$$

is a monotone graph in $[0,1] \times [0,1]$.

Physical conditions on the interface: capillary pressures.

To select connections , we need to know what information on the interface can be "inherited" from the idea that a solution u is a limit of solutions u_ε of the parabolic, capillarity regularized problem (here $\pi^{l,r}$, not present in (BL), enter the stage!).

We have seen that in the parabolic (regularized) problem, the quantity connected through the interface is the capillary pressure. Moreover, it was shown by Cancès that the right way to connect capillary pressures π^l and π^r is the following :

$$\pi^l(\gamma^l u_\varepsilon) \text{ "} = \text{" } \pi^r(\gamma^r u_\varepsilon) \quad \text{in the generalized sense:}$$

namely, at $u = 0$ and at $u = 1$, $\pi^{l,r}$ are extended to multi-valued monotone functions and "equality" actually means

$$\pi^l(\gamma^l u_\varepsilon) \cap \pi^r(\gamma^r u_\varepsilon) \ \text{ is non-empty .}$$

Because $\pi^{l,r}$ are increasing, the set $\mathcal{L}_\pi$ defined as

$$\{(u^l_\varepsilon, u^r_\varepsilon) \,|\, \pi^l(u^l_\varepsilon) \text{ and } \pi^r(u^r_\varepsilon) \text{ are connected}\}$$

is a monotone graph in $[0,1] \times [0,1]$.

capillarity regularized problem, its well-posedness

Definition (Cancès, Gallouët, Porretta)

An L^∞ function u_ε on $(0,T)\times\mathbb{R} =: Q = Q^l \cup Q^r$ and such that "$\lambda^{l,r}(u^\varepsilon)\,\partial_x\,\pi^{l,r}(u^\varepsilon)$" $= \partial_x\,\varphi^{l.r}(u_\varepsilon) \in L^2(Q^{l,r})$ is a solution of (BL_ε) if

- for all $\xi \in L^2(0,T;H^1(\mathbb{R}))$ with $\partial_t\xi \in L^\infty(Q)$,

$$-\iint_Q u_\varepsilon\,\partial_t\,\xi - \int_{\mathbb{R}} u_0\xi(0,\cdot) - \iint_{Q^l} (f^l(u_\varepsilon) - \lambda^l(u_\varepsilon)\,\partial_x\,\pi^l(u_\varepsilon))\,\partial_x\,\xi$$
$$- \iint_{Q^r} (f^r(u_\varepsilon) - \lambda^r(u_\varepsilon)\,\partial_x\,\pi^r(u_\varepsilon))\,\partial_x\,\xi = 0;$$

- at the interface $\{x=0\}$, the capillary pressures are connected:

$$\pi^l(\gamma^l u_\varepsilon)\ \cap\ \pi^r(\gamma^r u_\varepsilon)\ \neq\ \emptyset.$$

capillarity regularized problem, its well-posedness

Definition (Cancès, Gallouët, Porretta)

An L^∞ function u_ε on $(0,T)\times\mathbb{R} =: Q = Q^l \cup Q^r$ and such that "$\lambda^{l,r}(u^\varepsilon)\,\partial_x\,\pi^{l,r}(u^\varepsilon)$" $= \partial_x\,\varphi^{l.r}(u_\varepsilon) \in L^2(Q^{l,r})$ is a solution of (BL_ε) if

- for all $\xi \in L^2(0,T;H^1(\mathbb{R}))$ with $\partial_t\xi \in L^\infty(Q)$,

$$-\iint_Q u_\varepsilon\,\partial_t\,\xi - \int_{\mathbb{R}} u_0\xi(0,\cdot) - \iint_{Q^l}(f^l(u_\varepsilon)-\lambda^l(u_\varepsilon)\,\partial_x\,\pi^l(u_\varepsilon))\,\partial_x\,\xi$$
$$-\iint_{Q^r}(f^r(u_\varepsilon)-\lambda^r(u_\varepsilon)\,\partial_x\,\pi^r(u_\varepsilon))\,\partial_x\,\xi = 0;$$

- at the interface $\{x=0\}$, the capillary pressures are connected:

$$\pi^l(\gamma^l u_\varepsilon)\ \cap\ \pi^r(\gamma^r u_\varepsilon)\ \neq\ \emptyset.$$

capillarity regularized problem, its well-posedness

Definition (Cancès, Gallouët, Porretta)

An L^∞ function u_ε on $(0,T)\times\mathbb{R} =: Q = Q^l \cup Q^r$ and such that "$\lambda^{l,r}(u^\varepsilon)\,\partial_x\,\pi^{l,r}(u^\varepsilon)$" $= \partial_x\,\varphi^{l.r}(u_\varepsilon) \in L^2(Q^{l,r})$ is a solution of (BL_ε) if

- for all $\xi \in L^2(0,T;H^1(\mathbb{R}))$ with $\partial_t\xi \in L^\infty(Q)$,

$$-\iint_Q u_\varepsilon\,\partial_t\,\xi - \int_{\mathbb{R}} u_0\xi(0,\cdot) - \iint_{Q^l}(f^l(u_\varepsilon) - \lambda^l(u_\varepsilon)\,\partial_x\,\pi^l(u_\varepsilon))\,\partial_x\,\xi$$
$$- \iint_{Q^r}(f^r(u_\varepsilon) - \lambda^r(u_\varepsilon)\,\partial_x\,\pi^r(u_\varepsilon))\,\partial_x\,\xi = 0;$$

- at the interface $\{x=0\}$, the capillary pressures are connected:

$$\pi^l(\gamma^l u_\varepsilon)\ \cap\ \pi^r(\gamma^r u_\varepsilon)\ \neq\ \emptyset.$$

capillarity regularized problem, its well-posedness

Theorem (Cancès, Gallouët, Porretta)

There exists a solver $\mathcal{S}_\varepsilon$, i.e., a map that associates to a $[0,1]$-valued function u_0 on $\mathbb{R}$ a solution of (BL_ε) such that the Kato inequality holds: for $\xi \geqslant 0$, ξ regular,

$$-\iint_Q |u_\varepsilon - \hat{u}_\varepsilon|\,\partial_t \xi - \int_{\mathbb{R}} |u_0 - \hat{u}_0|\,\xi(0,\cdot)$$
$$-\iint_{Q^l} \left(q^l(u_\varepsilon,\hat{u}_\varepsilon) - \varepsilon\partial_x\left|\varphi^l(u_\varepsilon) - \varphi^l(\hat{u}_\varepsilon)\right|\right)\partial_x \xi$$
$$-\iint_{Q^r} \left(q^r(u_\varepsilon,\hat{u}_\varepsilon) - \varepsilon\partial_x\left|\varphi^r(u_\varepsilon) - \varphi^r(\hat{u}_\varepsilon)\right|\right)\partial_x \xi \leqslant 0.$$

The Kato inequality means, in the sense of distributions:

$$\partial_t |u_\varepsilon - \hat{u}_\varepsilon| + \partial_x q(\cdot; u_\varepsilon, \hat{u}_\varepsilon) \leqslant \varepsilon\partial_x \left|\lambda(\cdot; u_\varepsilon)\,\partial_x \pi(\cdot; u_\varepsilon) - \lambda(\cdot; \hat{u}_\varepsilon)\,\partial_x \pi(\cdot; \hat{u}_\varepsilon)\right|.$$

In particular $\mathcal{S}_\varepsilon$ is L^1-contractive:

$$\|u_\varepsilon(t,\cdot) - \hat{u}_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leqslant \|u_0 - \hat{u}_0\|_{L^1(\mathbb{R})}.$$

capillarity regularized problem, its well-posedness

Theorem (Cancès, Gallouët, Porretta)

There exists a solver $\mathcal{S}_\varepsilon$, i.e., a map that associates to a $[0,1]$-valued function u_0 on $\mathbb{R}$ a solution of (BL_ε) such that the Kato inequality holds: for $\xi \geqslant 0$, ξ regular,

$$-\iint_Q |u_\varepsilon - \hat{u}_\varepsilon|\,\partial_t\xi - \int_{\mathbb{R}} |u_0 - \hat{u}_0|\,\xi(0,\cdot) - \iint_{Q^l} \left(q^l(u_\varepsilon,\hat{u}_\varepsilon) - \varepsilon\partial_x\left|\varphi^l(u_\varepsilon) - \varphi^l(\hat{u}_\varepsilon)\right|\right)\partial_x\xi - \iint_{Q^r} \left(q^r(u_\varepsilon,\hat{u}_\varepsilon) - \varepsilon\partial_x\left|\varphi^r(u_\varepsilon) - \varphi^r(\hat{u}_\varepsilon)\right|\right)\partial_x\xi \leqslant 0.$$

The Kato inequality means, in the sense of distributions:

$$\partial_t\left|u_\varepsilon - \hat{u}_\varepsilon\right| + \partial_x q(\cdot;u_\varepsilon,\hat{u}_\varepsilon) \leqslant \varepsilon\partial_x\left|\lambda(\cdot;u_\varepsilon)\,\partial_x\pi(\cdot;u_\varepsilon) - \lambda(\cdot;\hat{u}_\varepsilon)\,\partial_x\pi(\cdot;\hat{u}_\varepsilon)\right|.$$

In particular $\mathcal{S}_\varepsilon$ is L^1-contractive:

$$\|u_\varepsilon(t,\cdot) - \hat{u}_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leqslant \|u_0 - \hat{u}_0\|_{L^1(\mathbb{R})}.$$

capillarity regularized problem, its well-posedness

Theorem (Cancès, Gallouët, Porretta)

There exists a solver $\mathcal{S}_\varepsilon$, i.e., a map that associates to a $[0,1]$-valued function u_0 on $\mathbb{R}$ a solution of (BL_ε) such that the Kato inequality holds: for $\xi \geqslant 0$, ξ regular,

$$-\iint_Q |u_\varepsilon - \hat{u}_\varepsilon|\, \partial_t \xi - \int_{\mathbb{R}} |u_0 - \hat{u}_0|\, \xi(0,\cdot)$$
$$- \iint_{Q^l} \left(q^l(u_\varepsilon, \hat{u}_\varepsilon) - \varepsilon \partial_x \left|\varphi^l(u_\varepsilon) - \varphi^l(\hat{u}_\varepsilon)\right|\right) \partial_x \xi$$
$$- \iint_{Q^r} \left(q^r(u_\varepsilon, \hat{u}_\varepsilon) - \varepsilon \partial_x \left|\varphi^r(u_\varepsilon) - \varphi^r(\hat{u}_\varepsilon)\right|\right) \partial_x \xi \;\leqslant\; 0.$$

The Kato inequality means, in the sense of distributions:

$$\partial_t |u_\varepsilon - \hat{u}_\varepsilon| + \partial_x q(\cdot; u_\varepsilon, \hat{u}_\varepsilon) \leqslant \varepsilon \partial_x \left|\lambda(\cdot; u_\varepsilon)\, \partial_x \pi(\cdot; u_\varepsilon) - \lambda(\cdot; \hat{u}_\varepsilon)\, \partial_x \pi(\cdot; \hat{u}_\varepsilon)\right|.$$

In particular $\mathcal{S}_\varepsilon$ is L^1-contractive:

$$\|u_\varepsilon(t,\cdot) - \hat{u}_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leqslant \|u_0 - \hat{u}_0\|_{L^1(\mathbb{R})}.$$

Stationary re-scaled equation and capillarity profiles

Let us look for solutions of (BL_ε) that are

- constant in t
- smooth in $\mathbb{R}^\pm$, with possibly a jump at $x = 0$ but with the capillary pressures connected
- solutions taking the values u^l and u^r, respectively, at $\pm\infty$, with $f^l(u^l)$ connected to ($\equiv$ equal to) $f^r(u^r)$.

NB. We can find such profiles of the form $u_\varepsilon(t,x) = U(x/\varepsilon)$!! (this is a classical technique in conservation laws: natural scaling of the equation is used !)

The function U will satisfy the ε-independent ODE

$$\begin{cases} \lambda^l(U)(\pi^l(U))' = f^l(U) - F \ \text{ for } x \leqslant 0 \\ \lambda^r(U)(\pi^r(U))' = f^r(U) - F \ \text{ for } x \geqslant 0 \\ F = f^{l,r}(u^{l,r}) \ \text{ and } \ \pi^l(U(0^-)) \cap \pi^r(U(0^+)) \neq \emptyset. \end{cases}$$

Idea: when U solving the above equation exists, the "connection solution" $c(t,x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x>0\}}$ is the limit of solutions $u_\varepsilon(t,x) = U(x/\varepsilon)$ of the (BL_ε) equation. Then $c(\cdot)$ is an admissible solution of (BL) equation, according to our modelling assumption.

Stationary re-scaled equation and capillarity profiles

Let us look for solutions of (BL_ε) that are

- constant in t
- smooth in $\mathbb{R}^\pm$, with possibly a jump at $x = 0$
  but with the capillary pressures connected
- solutions taking the values u^l and u^r, respectively, at $\pm\infty$,
  with $f^l(u^l)$ connected to ($\equiv$ equal to) $f^r(u^r)$.

NB. We can find such profiles of the form $u_\varepsilon(t,x) = U(x/\varepsilon)$!!
(this is a classical technique in conservation laws: natural scaling of the equation is used !)

The function U will satisfy the ε-independent ODE

$$\begin{cases} \lambda^l(U)(\pi^l(U))' = f^l(U) - F \ \text{ for } x \leqslant 0 \\ \lambda^r(U)(\pi^r(U))' = f^r(U) - F \ \text{ for } x \geqslant 0 \\ F = f^{l,r}(u^{l,r}) \ \text{ and } \ \pi^l(U(0^-)) \cap \pi^r(U(0^+)) \neq \emptyset. \end{cases}$$

Idea: when U solving the above equation exists, the "connection solution" $c(t,x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{x>0\}}$ is the limit of solutions $u_\varepsilon(t,x) = U(x/\varepsilon)$ of the (BL_ε) equation. Then $c(\cdot)$ is an admissible solution of (BL) equation, according to our modelling assumption.

Stationary re-scaled equation and capillarity profiles

Let us look for solutions of (BL_ε) that are

- constant in t
- smooth in $\mathbb{R}^\pm$, with possibly a jump at $x = 0$
  but with the capillary pressures connected
- solutions taking the values u^l and u^r, respectively, at $\pm\infty$,
  with $f^l(u^l)$ connected to ($\equiv$ equal to) $f^r(u^r)$.

NB. We can find such profiles of the form $u_\varepsilon(t,x) = U(x/\varepsilon)$!!
(this is a classical technique in conservation laws: natural scaling of the equation is used !)

The function U will satisfy the ε-independent ODE

$$\left\{\begin{array}{l} \lambda^l(U)(\pi^l(U))' = f^l(U) - F \ \text{ for } x \leqslant 0 \\ \lambda^r(U)(\pi^r(U))' = f^r(U) - F \ \text{ for } x \geqslant 0 \\ F = f^{l,r}(u^{l,r}) \ \text{ and } \ \pi^l(U(0^-)) \cap \pi^r(U(0^+)) \neq \emptyset. \end{array}\right.$$

Idea: when U solving the above equation exists, the "connection solution" $c(t,x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x>0\}}$ is the limit of solutions $u_\varepsilon(t,x) = U(x/\varepsilon)$ of the (BL_ε) equation. Then $c(\cdot)$ is an admissible solution of (BL) equation, according to our modelling assumption.

Stationary re-scaled equation and capillarity profiles

Let us look for solutions of (BL_ε) that are

- constant in t
- smooth in $\mathbb{R}^\pm$, with possibly a jump at $x = 0$
  but with the capillary pressures connected
- solutions taking the values u^l and u^r, respectively, at $\pm\infty$,
  with $f^l(u^l)$ connected to ($\equiv$ equal to) $f^r(u^r)$.

NB. We can find such profiles of the form $u_\varepsilon(t,x) = U(x/\varepsilon)$!!
(this is a classical technique in conservation laws: natural scaling of the equation is used !)

The function U will satisfy the ε-independent ODE

$$\begin{cases} \lambda^l(U)(\pi^l(U))' = f^l(U) - F \ \text{ for } x \leqslant 0 \\ \lambda^r(U)(\pi^r(U))' = f^r(U) - F \ \text{ for } x \geqslant 0 \\ F = f^{l,r}(u^{l,r}) \ \text{ and } \ \pi^l(U(0^-)) \cap \pi^r(U(0^+)) \neq \emptyset. \end{cases}$$

Idea: when U solving the above equation exists, the "connection solution" $c(t,x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x>0\}}$ is the limit of solutions $u_\varepsilon(t,x) = U(x/\varepsilon)$ of the (BL_ε) equation. Then $c(\cdot)$ is an admissible solution of (BL) equation, according to our modelling assumption.

Stationary re-scaled equation and capillarity profiles

Let us look for solutions of (BL_ε) that are

- constant in t
- smooth in $\mathbb{R}^\pm$, with possibly a jump at $x = 0$
  but with the capillary pressures connected
- solutions taking the values u^l and u^r, respectively, at $\pm\infty$,
  with $f^l(u^l)$ connected to ($\equiv$ equal to) $f^r(u^r)$.

NB. We can find such profiles of the form $u_\varepsilon(t,x) = U(x/\varepsilon)$!!
(this is a classical technique in conservation laws: natural scaling of the equation is used !)

The function U will satisfy the ε-independent ODE

$$\begin{cases} \lambda^l(U)(\pi^l(U))' = f^l(U) - F \ \text{ for } x \leqslant 0 \\ \lambda^r(U)(\pi^r(U))' = f^r(U) - F \ \text{ for } x \geqslant 0 \\ F = f^{l,r}(u^{l,r}) \ \text{ and } \ \pi^l(U(0^-)) \cap \pi^r(U(0^+)) \neq \emptyset. \end{cases}$$

Idea: when U solving the above equation exists, the “connection solution” $c(t,x) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x>0\}}$ is the limit of solutions $u_\varepsilon(t,x) = U(x/\varepsilon)$ of the (BL_ε) equation. Then $c(\cdot)$ is an admissible solution of (BL) equation, according to our modelling assumption.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1] \times [0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x = -\infty$ to any state $U(0^-)$ at $x = 0^-$ provided $U^(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x < 0$, constant on $x > 0$) connecting u^l_{opt} at $x = -\infty$ to u^r_{opt} at $x = +\infty$.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1]\times[0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x=-\infty$ to any state $U(0^-)$ at $x=0^-$ provided $U^(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x<0$, constant on $x>0$) connecting u^l_{opt} at $x=-\infty$ to u^r_{opt} at $x=+\infty$.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1]\times[0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x=-\infty$ to any state $U(0^-)$ at $x=0^-$ provided $U(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x<0$, constant on $x>0$) connecting u^l_{opt} at $x=-\infty$ to u^r_{opt} at $x=+\infty$.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1]\times[0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x=-\infty$ to any state $U(0^-)$ at $x=0^-$ provided $U(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x<0$, constant on $x>0$) connecting u^l_{opt} at $x=-\infty$ to u^r_{opt} at $x=+\infty$.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1]\times[0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x=-\infty$ to any state $U(0^-)$ at $x=0^-$ provided $U(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x<0$, constant on $x>0$) connecting u^l_{opt} at $x=-\infty$ to u^r_{opt} at $x=+\infty$.

Looking for an admissible totally non-Lax connection.

We have seen that

– The set $\mathcal{L}_\pi$ of (u^l, u^r) with connected capillary pressures is an increasing (maximal monotone) graph in the unit square.

– Totally non-Lax connections form a decreasing curve $\mathcal{L}_f$ in part of $[0,1] \times [0,1]$ with endpoints (u^l_{opt}, u^r_{opt}) and (u^l_{bar}, u^r_{bar});
– the endpoint (u^l_{bar}, u^r_{bar}) is on the boundary of the unit square.

- If there exists $(u^l_\pi, u^r_\pi) \in \mathcal{L}_\pi \cap \mathcal{L}_f$ (the intersection point is necessarily unique), then $U(x) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$ is a viscosity profile !
- Fact (simple): if $\mathcal{L}_f \cap \mathcal{L}_\pi = \emptyset$, a viscosity profile exists for (u^l_{opt}, u^r_{opt}) ! We then set $u^{l,r}_\pi := u^{l,r}_{opt}$ for this case .

$\Longrightarrow$ in all cases, (u^l_π, u^r_π) is an admissible totally non-Lax connection.

Second case, details:

consider, e.g., the case where the optimal connection level F is the maximum value of f^l (otherwise, it is the maximum value of f^r). Then we construct a solution of the ODE of the previous slide, by connecting the state u^l_{opt} at $x = -\infty$ to any state $U(0^-)$ at $x = 0^-$ provided $U^(0^-) < u^l_{opt}$. One of the couples $(U(0^-), u^r_{opt})$ belongs to the graph $\mathcal{L}_\pi$, thus we get a profile U (decreasing for $x < 0$, constant on $x > 0$) connecting u^l_{opt} at $x = -\infty$ to u^r_{opt} at $x = +\infty$.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) *are entropy solutions on* $\{x<0\}$ *and on* $\{x>0\}$ *of* (BL).
(ii) The resulting limit solver $\mathcal{S}$ *for* (BL) *is* L^1*-contractive.*
(iii) The solvers $\mathcal{S}_\varepsilon$ *contain the solution* $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S contains the admissible connection $c_\pi(t,x)=u^l_\pi 1\!\!1_{\{x<0\}}+u^r_\pi 1\!\!1_{x>0\}}$ *obtained by intersecting* $\mathcal{L}_\pi$ *with* $\mathcal{L}_f$ *(with the optimal connection chosen if* $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$*);*
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$*, we inherit one particular Kato inequality:*

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\,q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the “usine à gaz” of the general theory of conservation laws with discontinuous flux.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) *are entropy solutions on* $\{x<0\}$ *and on* $\{x>0\}$ *of* (BL)*.*
(ii) The resulting limit solver $\mathcal{S}$ *for* (BL) *is* L^1*-contractive.*
(iii) The solvers $\mathcal{S}_\varepsilon$ *contain the solution* $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S contains the admissible connection $c_\pi(t,x)=u_\pi^l 1\!\!1_{\{x<0\}}+u_\pi^r 1\!\!1_{x>0\}}$ *obtained by intersecting* $\mathcal{L}_\pi$ *with* $\mathcal{L}_f$ *(with the optimal connection chosen if* $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$*);*
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$*, we inherit one particular Kato inequality:*

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\,q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the “usine à gaz” of the general theory of conservation laws with discontinuous flux.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) *are entropy solutions on* $\{x<0\}$ *and on* $\{x>0\}$ *of* (BL).
(ii) The resulting limit solver $\mathcal{S}$ *for* (BL) *is* L^1*-contractive.*
(iii) The solvers $\mathcal{S}_\varepsilon$ *contain the solution* $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S contains the admissible connection $c_\pi(t,x)=u^l_\pi 1\!\!1_{\{x<0\}}+u^r_\pi 1\!\!1_{x>0\}}$ *obtained by intersecting* $\mathcal{L}_\pi$ *with* $\mathcal{L}_f$ *(with the optimal connection chosen if* $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$*);*
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$, *we inherit one particular Kato inequality:*

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\,q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the "usine à gaz" of the general theory of conservation laws with discontinuous flux.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) *are entropy solutions on* $\{x<0\}$ *and on* $\{x>0\}$ *of* (BL).
(ii) The resulting limit solver $\mathcal{S}$ *for* (BL) *is* L^1*-contractive.*
(iii) The solvers $\mathcal{S}_\varepsilon$ *contain the solution* $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S contains the admissible connection $c_\pi(t,x)=u^l_\pi 1\!\!1_{\{x<0\}}+u^r_\pi 1\!\!1_{x>0\}}$ *obtained by intersecting* $\mathcal{L}_\pi$ *with* $\mathcal{L}_f$ *(with the optimal connection chosen if* $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$*);*
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$*, we inherit one particular Kato inequality:*

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\, q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the “usine à gaz” of the general theory of conservation laws with discontinuous flux.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) are entropy solutions on $\{x<0\}$ and on $\{x>0\}$ of (BL).
(ii) The resulting limit solver $\mathcal{S}$ for (BL) is L^1-contractive.
(iii) The solvers $\mathcal{S}_\varepsilon$ contain the solution $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S contains the admissible connection $c_\pi(t,x)=u^l_\pi 1\!\!1_{\{x<0\}}+u^r_\pi 1\!\!1_{x>0\}}$ obtained by intersecting $\mathcal{L}_\pi$ with $\mathcal{L}_f$ (with the optimal connection chosen if $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$);
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$, we inherit one particular Kato inequality:

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\,q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the “usine à gaz” of the general theory of conservation laws with discontinuous flux.

Physically motivated solutions to BL equation in fractured medium

Let us recapitulate the key elements of our analysis:

Proposition

(i) Limits (a.e. on Q) of solutions of (BL_ε) *are entropy solutions on* $\{x<0\}$ *and on* $\{x>0\}$ *of* (BL).
(ii) The resulting limit solver $\mathcal{S}$ *for* (BL) *is* L^1*-contractive.*
(iii) The solvers $\mathcal{S}_\varepsilon$ *contain the solution* $u_\varepsilon(t,x)=U(x/\varepsilon)$
(iv) Consequently, the limit solver S *contains the admissible connection* $c_\pi(t,x)=u^l_\pi 1\!\!1_{\{x<0\}}+u^r_\pi 1\!\!1_{x>0\}}$ *obtained by intersecting* $\mathcal{L}_\pi$ *with* $\mathcal{L}_f$ *(with the optimal connection chosen if* $\mathcal{L}_\pi\cap\mathcal{L}_f=\emptyset$*);*
(v) and for $u=\lim_{\varepsilon\to 0}u_\varepsilon$, *we inherit one particular Kato inequality:*

$$\partial_t\,|u-c_\pi(\cdot)|+\partial_x\,q(\cdot;u,c_\pi(\cdot))\leqslant 0.$$

These facts will be now inserted in the “usine à gaz” of the general theory of conservation laws with discontinuous flux.

MAIN RESULTS

Notion of entropy solution...

Definition (Bürger, Karlsen, Towers)

Let (u^l_π, u^r_π) be a totally non-Lax connection .

A $\mathcal{G}_\pi$-entropy solution to (BL) equation is a $[0,1]$-valued function u which takes value u_0 at $t=0$ (in the sense of traces) and satisfies

- Kruzhkov entropy inequalities away from the interface hold:

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \partial_x q^{l,r}(u,k) \leqslant 0 \;\text{ in } Q^{l,r}$$

- The global "connection-adapted entropy" inequality ($\equiv$ the Kato inequality wrt $c_\pi(\cdot) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$) holds:

$$\partial_t |u - c_\pi(\cdot)| + \partial_x q(\cdot; u, c_\pi(\cdot)) \leqslant 0 \;\text{ in } Q.$$

NB The reason for which only one adapted global entropy inequality is needed lies in the particular configuration of Buckley-Leverett fluxes $f^{l,r}$ (bell-shaped fluxes).

Notion of entropy solution...

Definition (Bürger, Karlsen, Towers)

Let (u^l_π, u^r_π) be a totally non-Lax connection .
A $\mathcal{G}_\pi$-entropy solution to (BL) equation is a $[0,1]$-valued function u which takes value u_0 at $t=0$ (in the sense of traces) and satisfies

- Kruzhkov entropy inequalities away from the interface hold:

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \partial_x q^{l,r}(u,k) \leqslant 0 \;\; \text{in } Q^{l,r}$$

- The global "connection-adapted entropy" inequality ($\equiv$ the Kato inequality wrt $c_\pi(\cdot) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$) holds:

$$\partial_t |u - c_\pi(\cdot)| + \partial_x q(\cdot; u, c_\pi(\cdot)) \leqslant 0 \;\; \text{in } Q.$$

NB The reason for which only one adapted global entropy inequality is needed lies in the particular configuration of Buckley-Leverett fluxes $f^{l,r}$ (bell-shaped fluxes).

Notion of entropy solution...

Definition (Bürger, Karlsen, Towers)

Let (u^l_π, u^r_π) be a totally non-Lax connection .
A $\mathcal{G}_\pi$-entropy solution to (BL) equation is a $[0,1]$-valued function u which takes value u_0 at $t=0$ (in the sense of traces) and satisfies

- Kruzhkov entropy inequalities away from the interface hold:

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \partial_x q^{l,r}(u,k) \leqslant 0 \;\; \text{in } Q^{l,r}$$

- The global “connection-adapted entropy” inequality ($\equiv$ the Kato inequality wrt $c_\pi(\cdot) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$) holds:

$$\partial_t |u - c_\pi(\cdot)| + \partial_x q(\cdot; u, c_\pi(\cdot)) \leqslant 0 \;\; \text{in } Q.$$

NB The reason for which only one adapted global entropy inequality is needed lies in the particular configuration of Buckley-Leverett fluxes $f^{l,r}$ (bell-shaped fluxes).

Notion of entropy solution...

Definition (Bürger, Karlsen, Towers)

Let (u^l_π, u^r_π) be a totally non-Lax connection .
A $\mathcal{G}_\pi$-entropy solution to (BL) equation is a $[0,1]$-valued function u which takes value u_0 at $t=0$ (in the sense of traces) and satisfies

- Kruzhkov entropy inequalities away from the interface hold:

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \partial_x q^{l,r}(u,k) \leqslant 0 \;\; \text{in } Q^{l,r}$$

- The global "connection-adapted entropy" inequality ($\equiv$ the Kato inequality wrt $c_\pi(\cdot) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$) holds:

$$\partial_t |u - c_\pi(\cdot)| + \partial_x q(\cdot; u, c_\pi(\cdot)) \leqslant 0 \;\; \text{in } Q.$$

NB The reason for which only one adapted global entropy inequality is needed lies in the particular configuration of Buckley-Leverett fluxes $f^{l,r}$ (bell-shaped fluxes).

Notion of entropy solution...

Definition (Bürger, Karlsen, Towers)

Let (u^l_π, u^r_π) be a totally non-Lax connection .
A $\mathcal{G}_\pi$-entropy solution to (BL) equation is a $[0,1]$-valued function u which takes value u_0 at $t=0$ (in the sense of traces) and satisfies

- Kruzhkov entropy inequalities away from the interface hold:

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \partial_x q^{l,r}(u,k) \leqslant 0 \;\; \text{in } Q^{l,r}$$

- The global "connection-adapted entropy" inequality ($\equiv$ the Kato inequality wrt $c_\pi(\cdot) = u^l_\pi 1\!\!1_{\{x<0\}} + u^r_\pi 1\!\!1_{\{x<0\}}$) holds:

$$\partial_t |u - c_\pi(\cdot)| + \partial_x q(\cdot; u, c_\pi(\cdot)) \leqslant 0 \;\; \text{in } Q.$$

NB The reason for which only one adapted global entropy inequality is needed lies in the particular configuration of Buckley-Leverett fluxes $f^{l,r}$ (bell-shaped fluxes).

...Notion of entropy solution.

To understand why the definition works, we will invoke the "general theory" (A., Karlsen, Risebro). According to this theory,

– a "good definition" prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$

$$= \{(u^l_\pi, u^r_\pi)\}$$

$$\cup\ \Big\{\text{all Lax and "half-Lax" connections at levels } \leqslant\ F_\pi := f^{l,r}(u^{l,r}_\pi)\Big\}$$

In terms of the "general theory": $\mathcal{G}_\pi$ is maximal, complete L^1D germ
(⇔ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite L^1D germ
with complete maximal extension $\mathcal{G}_\pi$
(⇔ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the "general theory" (A., Karlsen, Risebro). According to this theory,

– a "good definition" prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\begin{aligned} \mathcal{G}_\pi = & \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \} \\ & = \{(u^l_\pi, u^r_\pi)\} \\ & \quad \cup \left\{\text{all Lax and "half-Lax" connections at levels } \leqslant F_\pi := f^{l,r}(u^{l,r}_\pi)\right\} \end{aligned}$$

In terms of the "general theory": $\mathcal{G}_\pi$ is maximal, complete L^1D germ

($\Leftrightarrow$ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite L^1D germ with complete maximal extension $\mathcal{G}_\pi$

($\Leftrightarrow$ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the "general theory" (A., Karlsen, Risebro). According to this theory,

– a "good definition" prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$

$$= \{(u^l_\pi, u^r_\pi)\}$$

$$\cup\ \{\text{all Lax and "half-Lax" connections at levels} \leqslant F_\pi := f^{l,r}(u^{l,r}_\pi)\}$$

In terms of the "general theory": $\mathcal{G}_\pi$ is maximal, complete L^1D germ

($\Leftrightarrow$ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite L^1D germ
with complete maximal extension $\mathcal{G}_\pi$

($\Leftrightarrow$ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the “general theory” (A., Karlsen, Risebro). According to this theory,

– a “good definition” prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$
$$= \{(u^l_\pi, u^r_\pi)\}$$
$$\cup \left\{\text{all Lax and “half-Lax” connections at levels} \leqslant F_\pi := f^{l,r}(u^{l,r}_\pi)\right\}$$

In terms of the “general theory”: $\mathcal{G}_\pi$ is maximal, complete L^1D germ

($\Leftrightarrow$ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite L^1D germ
with complete maximal extension $\mathcal{G}_\pi$

($\Leftrightarrow$ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the "general theory" (A., Karlsen, Risebro). According to this theory,

– a "good definition" prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$

$$= \{(u^l_\pi, u^r_\pi)\}$$

$$\cup \left\{\text{all Lax and "half-Lax" connections at levels} \leqslant F_\pi := f^{l,r}(u^{l,r}_\pi)\right\}$$

In terms of the "general theory": $\mathcal{G}_\pi$ is maximal, complete $L^1 D$ germ

($\Leftrightarrow$ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite $L^1 D$ germ
with complete maximal extension $\mathcal{G}_\pi$

($\Leftrightarrow$ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the “general theory” (A., Karlsen, Risebro). According to this theory,

– a “good definition” prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$

$$= \{(u^l_\pi, u^r_\pi)\}$$

$$\cup \left\{\text{all Lax and “half-Lax” connections at levels} \leqslant F_\pi := f^{l,r}(u^{l,r}_\pi)\right\}$$

In terms of the “general theory”: $\mathcal{G}_\pi$ is maximal, complete $L^1 D$ germ

($\Leftrightarrow$ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite $L^1 D$ germ
with complete maximal extension $\mathcal{G}_\pi$

($\Leftrightarrow$ everything is determined by $\mathcal{G}^0_\pi$.)

...Notion of entropy solution.

To understand why the definition works, we will invoke the "general theory" (A., Karlsen, Risebro). According to this theory,

– a "good definition" prescribes all possible couples of traces (u^l, u^r) at the interface; the notation $\mathcal{G}_\pi$ refers to the set of all these couples;

– actually, the adapted global entropy inequalities must hold wrt to all the connections $c(\cdot) = u^l 1\!\!1_{\{x<0\}} + u^r 1\!\!1_{\{x<0\}}$ with $(u^l, u^r) \in \mathcal{G}_\pi$;

– in the Buckley-Leverett context, they hold for all $(u^l, u^r) \in \mathcal{G}_\pi$ as soon as they hold for the totally non-Lax connection (u^l_π, u^r_π).

$$\mathcal{G}_\pi = \{\text{the set of all admissible trace couples } (\gamma^l u, \gamma^r u) \text{ at } \{x=0\}\ \}$$

$$= \{(u^l_\pi, u^r_\pi)\}$$

$$\cup \left\{\text{all Lax and "half-Lax" connections at levels } \leqslant\ F_\pi := f^{l,r}(u^{l,r}_\pi)\right\}$$

In terms of the "general theory": $\mathcal{G}_\pi$ is maximal, complete $L^1 D$ germ
(⇔ the theory works well !)

and $\mathcal{G}^0_\pi := \{(u^l_\pi, u^r_\pi)\}$ is a definite $L^1 D$ germ
with complete maximal extension $\mathcal{G}_\pi$
(⇔ everything is determined by $\mathcal{G}^0_\pi$.)

Well-posedness.

Theorem (A.,Karlsen,Risebro + A., Goatin, Seguin+A., Cancès)

(i) For every configuration of fluxes $f^{l,r}$ and of capillary pressures $\pi^{l,r}$ there exists a unique totally non-Lax connection (u^l_π, u^r_π) that can be obtained as the limit of capillarity regularized Buckley-Leverett equation.

(ii) Assume the fluxes $f^{l,r}$ are either Lipshitz, or genuinely nonlinear. A connection (u^l_π, u^r_π) being fixed, for every initial datum there exists a unique $\mathcal{G}_\pi$-entropy solution of the Buckley-Leverett equation . These solutions form the unique L^1-dissipative solver $\mathcal{S}$ obtained as limit of the L^1-dissipative solvers $\mathcal{S}_\varepsilon$ of the capillarity-regularized equation .

(iii) Consider a monotone finite volume numerical scheme (in the spirit of Eymard, Gallouët, Herbin) for (BL). Assume that it is "well-balanced" wrt to the connection (u^l_π, u^r_π). Then converges to the unique $\mathcal{G}_\pi$-entropy solution of (BL).

Well-posedness.

Theorem (A.,Karlsen,Risebro + A., Goatin, Seguin+A., Cancès)

(i) For every configuration of fluxes $f^{l,r}$ and of capillary pressures $\pi^{l,r}$ there exists a unique totally non-Lax connection (u^l_π, u^r_π) that can be obtained as the limit of capillarity regularized Buckley-Leverett equation.

(ii) Assume the fluxes $f^{l,r}$ are either Lipshitz, or genuinely nonlinear. A connection (u^l_π, u^r_π) being fixed, for every initial datum there exists a unique $\mathcal{G}_\pi$-entropy solution of the Buckley-Leverett equation . These solutions form the unique L^1-dissipative solver $\mathcal{S}$ obtained as limit of the L^1-dissipative solvers $\mathcal{S}_\varepsilon$ of the capillarity-regularized equation .

(iii) Consider a monotone finite volume numerical scheme (in the spirit of Eymard, Gallouët, Herbin) for (BL). Assume that it is "well-balanced" wrt to the connection (u^l_π, u^r_π). Then converges to the unique $\mathcal{G}_\pi$-entropy solution of (BL).

Well-posedness.

Theorem (A.,Karlsen,Risebro + A., Goatin, Seguin+A., Cancès)

(i) For every configuration of fluxes $f^{l,r}$ and of capillary pressures $\pi^{l,r}$ there exists a unique totally non-Lax connection (u^l_π, u^r_π) that can be obtained as the limit of capillarity regularized Buckley-Leverett equation.

(ii) Assume the fluxes $f^{l,r}$ are either Lipshitz, or genuinely nonlinear. A connection (u^l_π, u^r_π) being fixed, for every initial datum there exists a unique $\mathcal{G}_\pi$-entropy solution of the Buckley-Leverett equation . These solutions form the unique L^1-dissipative solver $\mathcal{S}$ obtained as limit of the L^1-dissipative solvers $\mathcal{S}_\varepsilon$ of the capillarity-regularized equation .

(iii) Consider a monotone finite volume numerical scheme (in the spirit of Eymard, Gallouët, Herbin) for (BL). Assume that it is "well-balanced" wrt to the connection (u^l_π, u^r_π). Then converges to the unique $\mathcal{G}_\pi$-entropy solution of (BL).

Choukrane — Merci — Gracias — Thank you !!!

شكرا