Time compactness tools for discretized evolution equations and applications to degenerate parabolic PDEs

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based on joint works with

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Plan of the talk

1. **Degenerate parabolic convection-diffusion problems. Structure-preserving discretization by finite volumes.**
2. **Different strategies for (time) compactness**
3. **Functional-analytic results**
4. **Direct Estimation of L^1 Time Translates**
5. **Compactness by monotonicity**

SOME DEGENERATE PARABOLIC PROBLEMS

Triply nonlinear degenerate parabolic problems...

Complex applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v),\, w = \varphi(v),$$
$$u_t + \operatorname{div}\,[\vec{G}(v) - \vec{a_0}(\nabla w)] + \psi(v) = f \;\text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), \varphi(\cdot), \psi(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{G}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian ,
where $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$ is a typical example.
Take, e.g., homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus '83; Otto '96; Bénilan, Wittbold '96 and '96; Carrillo '99; Ammar, Wittbold '03; Andreianov, Bendahmane, Karlsen, Ouaro '09
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $\varphi(\cdot)$ may be constant on intervals: parabolic-hyperbolic.

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Convergence of approximations for degenerate parabolic problems...

Finite Volumes ?? Arguments for existence proof are the same as used for convergence of numerical approximations !

Namely:

1. Construct a sequence of "approximate solutions" $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments ?) the point of this talk is to discuss this Step.
3. Prove that the accumulation point is as solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{G}(v_h) \to \vec{G}(v)$? $\vec{a}_0(\nabla\varphi(v_h)) \to \vec{a}_0(\nabla\varphi(v))$?

NB: Steps 2 and 3 are separated in "simpler" problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In "harder" problems, it may be necessary to treat simultaneously Steps 2+3 : compensated compactness, entropy-process solutions...

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Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit the features of the continuous problem that were used in the convergence proof. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different “discrete duality” schemes, examples : co-volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen... DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov... gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...
- entropy inequalities – discrete order preservation – discrete contraction for the hyperbolic-parabolic operator.

  Preserved by discretization of div $\vec{G}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin, Vovelle) + DDFV/Co-volume/... discretization of the nonlinear elliptic operator $-\mathrm{div}\vec{a}_0(\nabla\varphi(v))$ on orthogonal meshes .

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Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Yet, “structure-preserving scheme” is not always synonym of “computationally inefficient scheme” ! See the benchmark session...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce “discrete” versions of “continuous” arguments for Steps 1 – 2 – 3 .

Let us focus on “compactness in time” arguments (a part of Step 2). The message:

- the arguments for Step 2 (compactness) that work in the “continuous case” can be adapted at the discrete level
- structure preservation is especially important for arguments that combine Steps 2+3.
- I'll mainly discuss the “continuous” arguments but discrete (FV) versions of all these arguments are available by now !

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DIFFERENT STRATEGIES FOR TIME COMPACTNESS

Strategies for (time) compactness...

- "Purely" functional-analytic methods (very weak use of the PDE)
  One use only *bounds* in functional spaces;
  proofs based on estimation of time translates
  - Aubin-Lions-Dubinskii-Simon: bounds on $(u_h)_h$ and $((u_h)_t)_h$.
  - Use of fractional time derivatives: Besov-Slobodetskii bounds on u
- Direct estimation of time translates $|u(t+\Delta t)-u(t)|$ in L^1 space
  - The L^1 compactness lemma of Kruzhkov:
    similar to L^1 version of Simon Lemma, but "a bit more nonlinear"
  - Techniques for the "variational" (Alt-Luckhaus) case :
    $\left(\int_t^{t+\Delta t} Eq_h(s)\,ds\right)*(u_h(t+\Delta t)-u_h(t))$...
- "Compensated compactness" methods (strong use of the PDE)
  Use of differential relations and of structure of nonlinearities to convert weak convergence into the strong one
  - Minty trick or reduction of Young measures (used for p-laplacian)
  - Entropy-process solutions (used for hyperbolic pb.)
- "Compactness by monotonicity" (for order-preserving PDEs)
  Use of $\liminf - \limsup$ arguments; monotonicity ensures convergence.

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FUNCTIONAL-ANALYTIC RESULTS

J. Simon lemma and use of fractional derivatives

The paper of J. Simon '87 (*Ann. Mat. Pura ed Appl.*)
"Compact sets in the space $L^p(0, T; B)$" is the main reference.

It contains a "metatheorem" with the following conditions:

– "space compactness" ensured by a bound in $L^p(0, T; X)$ with $X \Subset B$ (compact embedding)
– "time compactness" ensured by a bound on time translates $u(t+\Delta t)-u(t)$

Then several sufficient conditions are given. Mention two of them:

- "Aubin-Lions-Simon lemma": ask $((u_h)_t)_h$ bdd in a "very weak" space.
  Example: $(u_h)_h$ bdd in $L^1(0, T; W_0^{1,1}(\Omega))$
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DIRECT ESTIMATION OF L^1 TIME TRANSLATES

S.N. Kruzhkov lemma

Reference: Kruzhkov '69 (*Math. Zametki / Math. Notes*)
"Results on the nature of the continuity of solutions of parab. eqns..."

Lemma (actually this is an L^1_{loc} result)

Assume the families of functions $(u_h)_h, (F_h^\alpha)_{h,\alpha}$ are bounded in $L^1(Q)$ and satisfy

$$\partial/\partial t\, u_h = \sum_{|\alpha|\leqslant m} D^\alpha F_h^\alpha \quad \text{in } \mathcal{D}'(Q).$$

Assume that u_h can be extended outside Q, and one has

$$\iint_Q |u_h(t,x+\delta)-u_h(t,x)|\,dxdt \leqslant \omega(\delta), \quad \text{with } \lim_{\delta\to 0}\omega(\delta)=0,$$

with $\omega(\cdot)$ independent of h. Then $(u_h)_h$ is relatively compact in $L^1(Q)$.

Differences with the Simon lemma:

– in Kruzhkov Lemma, space compactness is most general
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"Compacité est toujours plus simple dans L^1"

Th. Gallouët

A discrete version of the Kruzhkov lemma

– adaptation in FV setting (for the value $m = 1$: relevant for divergence problems $u_t + \text{div}\,[...] = ...$) is quite straightforward, because "no h-dependent spaces are used": all discrete objects naturally live in L^1 !
Andr., Bendahmane, Ruiz Baier '11 ; Andr., Bendahmane, Hubert

Assume we are given a family of meshes with sizes $h \to 0$,
an initial condition b_h^0 ,
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$$\text{for } n \in [1, N_h + 1], \quad \frac{b(v_h^n) - b(v_h^{n-1})}{\delta_h} = \text{div}_h\,[\vec{F}_h^n] + f_h^n,$$

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L^1 time translates in variational context

The technique of Kruzhkov lemma applies to the elliptic-parabolic problem

$$b(v)_t + \text{div}\,[\vec{G}(v) - \vec{a_0}(\nabla v)] + \psi(v) = f$$

under the "structure condition" $G(v) = \tilde{G}(b(v))$ ($\Rightarrow$ compactness of $(b(v_h))_h$)

How to cope with the other cases ? For parabolic-hyperbolic problem:

$$v_t + \text{div}\,[\vec{G}(v) - \vec{a_0}(\nabla\varphi(v))] + \psi(v) = f$$

Reference: Alt, Luckhaus '83 (*Math. Z.*)
"Quasilinear elliptic-parabolic differential equations"

Works also on parabolic-hyperbolic case !
Technique: take $\varphi(v)(t{+}\Delta t) - \varphi(v)(t)$ in the test function in the equation integrated from t to $t{+}\Delta t$. Use Sobolev bounds + Fubini $\Rightarrow$

$$\int_0^{T-\delta}\int_\Omega (v_h(t{+}\delta) - v_h(t))\,(\varphi(v_h)(t{+}\delta) - \varphi(v_h)(t)) \leqslant \; Const\;\delta.$$

Easily adapted to finite volume context:
Eymard, Gallouët, Herbin '00 for $\varphi^{-1}(\cdot)$ Lipschitz
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“COMPACTNESS FROM MONOTONICITY”

Elliptic-parabolic case without the structure condition

Come back to the elliptic-parabolic problem

$$b(v)_t + \text{div}\,[\vec{G}(v) - \vec{a}_0(\nabla v)] + \psi(v) = f$$

but without the structure condition: assume $G \circ b^{-1}$ is not a function !

Kruzhkov lemma gives compactness estimates on $b(v)$, but it is not enough to pass to the limit in $G(v)$... time oscillations cannot be precluded. Way out ? Monotonicity arguments : Ammar, Wittbold '03

The zero-order term $\psi(v)$ brings translation estimates and $L^1_{loc}((0,T])$ compactness of $(v_h)_h$ if $\psi(\cdot)$ is strictly monotone.
And if not ?
Ammar, Wittbold and Zimmermann '11 use penalization e.g. by $\psi^\varepsilon(v) = \varepsilon(\arctan v \mp \frac{\pi}{2}\text{sign}\,\varepsilon)$ which allows to get time compactness.

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Elliptic-parabolic case without the structure condition

Come back to the elliptic-parabolic problem

$$b(v)_t + \operatorname{div}\,[\vec{G}(v) - \vec{a}_0(\nabla v)] + \psi(v) = f$$

but without the structure condition: assume $G \circ b^{-1}$ is not a function !

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Application to an elliptic-parabolic case without the structure condition

Come back to the elliptic-parabolic problem

$$b(v)_t + \text{div}\,[\vec{G}(v) - \vec{a}_0(\nabla v)] + \psi(v) = f$$

without the structure condition.

The assumptions of the above convergence method are fulfilled if the finite volume scheme is order-preserving (Crandall-Tartar Lemma). Ok for monotone two-points' flux FV discretization of the convection term. Reference : Andreianov, Wittbold '11 .

Translations or Kruzhkov lemma give translation estimates on $b(v_h^\varepsilon)$.

For each ε, in presence of strictly increasing penalization term $\psi^\varepsilon(v_h^\varepsilon)$ we derive translation estimates on $\psi^\varepsilon(v_h^\varepsilon)$ and thus on $(v_h^\varepsilon)_h$ (we need a subtler version of the direct L^1 translation approach).

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Come back to the elliptic-parabolic problem

$$b(v)_t + \text{div}\,[\vec{G}(v) - \vec{a}_0(\nabla v)] + \psi(v) = f$$

without the structure condition.

The assumptions of the above convergence method are fulfilled if the finite volume scheme is order-preserving (Crandall-Tartar Lemma). Ok for monotone two-points' flux FV discretization of the convection term. Reference : Andreianov, Wittbold '11 .

Translations or Kruzhkov lemma give translation estimates on $b(v_h^\varepsilon)$.

For each ε, in presence of strictly increasing penalization term $\psi^\varepsilon(v_h^\varepsilon)$ we derive translation estimates on $\psi^\varepsilon(v_h^\varepsilon)$ and thus on $(v_h^\varepsilon)_h$ (we need a subtler version of the direct L^1 translation approach).

Then the order-preserving feature of the scheme permits us to apply the argument of the previous slide and get convergence of $(v_h)_h$ to the unique solution of the problem "without the structure condition".

Conclusions and references

• Different time compactness methods generalize well to sequences created by finite volume approximation in space with Euler in time scheme.

• Aubin-Lions-Simon lemma : Th. Gallouët, and J. C. Latché '11
"Compactness of discrete approximate solutions to parabolic PDEs..."
Comm.Pure Appl. Anal.
+ *http://www.cmi.univ-mrs.fr/ gallouet/*, "Heraklion Sept. 2010" talk

• Fractional time derivatives : E. Emmrich and M. Thalhammer '11
J. Differential Equations 251 (2011), pp. 82-118

• Kruzhkov lemma : Kruzhkov '69 , easy to get discrete versions !
B.A., M. Bendahmane, R. Ruiz Baier '11 , M^3AS (general schemes)
+ B.A., M. Bendahmane, F. Hubert , HAL preprint (DDFV schemes)

• Direct translation estimates : Eymard, Gallouët, Herbin '00,.......
· L^1 variant: B.A., M. Bendahmane, K.H. Karlsen '10 J. Hyperbolic Diff. Eq.
· compactness from zero-order term: B.A., P. Wittbold , to appear on HAL

• Compactness from monotonicity : math. folklore ?
Requires severe restrictions on the discretization (order-preservation...).
B.A., P. Wittbold , and also B.A. , FVCA6 proceeding.

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Thank you — Děkuji — Merci

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