A particle-in-Burgers model: theory and numerics

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joint work with

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Hyperbolic Conservation Laws
and Related Analysis with Applications

Plan of the talk

Model and motivation

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D'Alembert paradox : a solid immersed in an inviscid fluid is not submitted to any resultant force ; in other words, birds (and planes...) could not fly with a model where viscosity is neglected ! Yet, inviscid (hyperbolic !) models are ok for some fluids...

Answer 1 to the d'Alembert paradox: use viscous models of fluid-solid interaction (see e.g. M. Hillairet , for a recent review).

Answer 2 (when the Reynolds number is large): it is reasonable to neglect the viscous effects in the model that governs the fluid ; but we have to conserve information from the vanishing viscosity in a DRAG FORCE .
The drag force takes the form of a source term which takes into account the difference between the velocity of the fluid and the velocity of the solid.

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The 1D case : the Lagoutière-Seguin-Takahashi model for the interaction, via a drag force, of a point particle with a Burgers fluid writes

$$\begin{aligned} &\partial_t u + \partial_x (u^2/2) = \lambda\, D(h'(t) - u)\, \delta_0(x - h(t)), \\ &m h''(t) = \lambda\, D(u(t, h(t)) - h'(t)). \end{aligned}$$

here

- u, the velocity of the fluid, is unknown
- h, the position of the solid particle, is unknown (then h' and h'' respectively denote its velocity and acceleration);
- the parameters are λ (the drag coefficient) and m (the mass of the solid particle); both are positive.
- the function D which intervenes in the drag force is an increasing odd function.

Actually, we will suppose that

either $D(v) = v$ (the linear case)

or $D(v) = v|v|$ (the quadratic case).

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AUXILIARY STEPS

Auxiliary steps to approach the full model...

Our study of the above coupled problem includes two auxiliary steps, that are of interest on their own. The first step is

$$\begin{cases} \partial_t u(t,x) + \partial_x (u^2/2)(t,x) = -\lambda\, u(t,x)\, \delta_0(x), & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases}$$

i.e., the particle is decoupled from the fluid and fixed at zero.

Difficulty 1 : the source term has to be carefully defined. Indeed, u can be discontinuous (and in fact, typically u IS DISCONTINUOUS at the particle location).

To give an interpretation of the source term, the LeRoux approximation was studied in detail by Lagoutière, Seguin, Takahashi: $\delta_0 = \partial_x H$ (H: the Heavyside function) is replaced by $\partial_x H_\varepsilon$, a smoothed version. This permits to understand what goes on at the interface.

The second step is to take $h(\cdot)$ a given path, still decoupled from the fluid, and to solve the Burgers equation with singular source term located at $x = h(t)$. We'll see that as soon as the first step is well understood, the second one is easy.

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...Auxiliary steps to approach the full model

Other way around, we should understand how to evolve the particle location given the fluid state at time t. Recall the equation (ODE) for the particle:

$$mh''(t) = \lambda\,(u(t,h(t)) - h'(t)).$$

Recall that $u(t,\cdot)$ has a jump at $x = h(t)$...

Difficulty 2 : understand the equation in the Carathéodory sense ? In the Filippov sense ?? We will see that a nice mathematical and physical interpretation is possible:

- the particle is driven by the lack of mass conservation in the equation for u ; or, equivalently, the total quantity of movement $\int_{\mathbb{R}} u(t,\cdot) + mh'(t)$ is conserved.
- the ODE for h can be written in a weak form that involves the values of $u(t,\cdot)$ on $\mathbb{R}$ (which is more "robust")

With these auxiliary steps well understood, we can

- think of the appropriate definition of solution
- use fixed-point arguments to guarantee existence
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MAIN RESULTS

Main results: Auxiliary Problem 1 is well posed

For the Burgers-with-Dirac-at-zero model , we apply the machinery developed for conservation laws with discontinuous flux (adapted entropies, Baiti,Jenssen and Audusse,Perthame ; revisited and generalized recently by BA., Karlsen, Risebro using the notion of admissibility germ). The outcome is:

– definition(s) of entropy solutions
– uniqueness, continuous dependence (L^1, L^1_{loc} with domain of dependence) exactly as in the Kruzhkov theory

In addition, we find

– a priori L^∞ bounds and (more delicate) variation bounds
– a strikingly simple numerical method (monotone consistent finite volume scheme with a trick at the interface)
– convergence of the numerical scheme, existence .

NB: the Riemann solver at the interface was already described by Lagoutière, Seguin, Takahashi , so a Godunov scheme could be constructed; but we seek to avoid using the Riemann solver because it is intricate.

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Then for the Burgers-driven-by-particle model (with $x = h(t)$ GIVEN path of the particle) we deduce well-posedness rather easily.

It is observed that the case of straight path, $h(t) = Vt$ with $V = const$, reduces to the Dirac-at-zero model by the simultaneous change of $u - V$ into u and of $x - Vt$ into x. Thus, nothing new for $h(t) = Vt$. Then any ($W^{2,\infty}$) path $h(\cdot)$ is approximated by piecewise affine paths; existence is established by passage to the limit. Uniqueness is straightforward from the definition of solution.

For the coupled model with data u_0 and $h(0) = 0$, $h'(0) = v_0$, we get

– existence, for L^∞ data u_0

– existence, uniqueness, continuous dependence for BV solutions , for BV data u_0.

We construct a time-explicit Glimm-type scheme where particle position is updated via splitting; we get numerical results that agree with the physical phenomena that are expected for the model .

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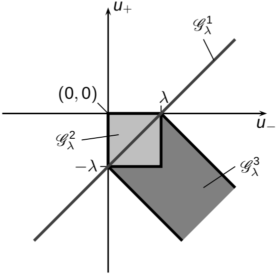
FROZEN PARTICLE

(DIRAC-AT-ZERO DRAG TERM):

UNDERSTANDING THE COUPLING

Frozen particle: understanding the coupling...

The admissibility at the interface $\{x = 0\}$ of the solution is governed by the *germ* $\mathscr{G}_\lambda$ (terminology related to the one of BA, Karlsen, Risebro):



Definition

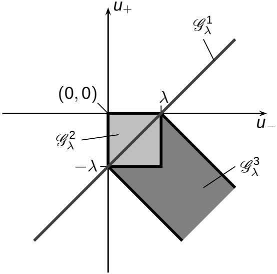
The *admissibility germ* $\mathscr{G}_\lambda \subset \mathbb{R}^2$ (or *germ*, for short) associated with the particle-at-zero problem is the union $\mathscr{G}_\lambda = \mathscr{G}^1_\lambda \cup \mathscr{G}^2_\lambda \cup \mathscr{G}^3_\lambda$, where

- $\mathscr{G}^1_\lambda = \{(a, a - \lambda), a \in \mathbb{R}\}$.
- $\mathscr{G}^2_\lambda = [0, \lambda] \times [-\lambda, 0]$.
- $\mathscr{G}^3_\lambda = \{(a, b) \in (\mathbb{R}^+ \times \mathbb{R}^-) \setminus \mathscr{G}^2_\lambda,\ -\lambda \leqslant a + b \leqslant \lambda\}$.

NB: the partition of $\mathscr{G}_\lambda$ into the three parts is dictated by the subsequent analysis, and by profiles study of LST .

Frozen particle: understanding the coupling...

The admissibility at the interface $\{x=0\}$ of the solution is governed by the *germ* $\mathscr{G}_\lambda$ (terminology related to the one of BA, Karlsen, Risebro):



Definition

The *admissibility germ* $\mathscr{G}_\lambda \subset \mathbb{R}^2$ (or *germ*, for short) associated with the particle-at-zero problem is the union $\mathscr{G}_\lambda = \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2 \cup \mathscr{G}_\lambda^3$, where

- $\mathscr{G}_\lambda^1 = \{(a, a-\lambda), a \in \mathbb{R}\}$.
- $\mathscr{G}_\lambda^2 = [0,\lambda] \times [-\lambda, 0]$.
- $\mathscr{G}_\lambda^3 = \{(a,b) \in (\mathbb{R}^+ \times \mathbb{R}^-) \setminus \mathscr{G}_\lambda^2,\ -\lambda \leqslant a+b \leqslant \lambda\}$.

NB: the partition of $\mathscr{G}_\lambda$ into the three parts is dictated by the subsequent analysis, and by profiles study of LST .

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Explaination : the Burgers equation with Dirac-at-zero drag term is equivalent to

$$\partial_t u + \partial_x (u^2/2) = -\lambda u \partial_x H.$$

We introduce $H_\varepsilon \in C^1(\mathbb{R})$ a non-decreasing function such that $H_\varepsilon(x) = H(x)$ when $|x| \geqslant \varepsilon$. Since we are interested in understanding the behavior of the solution through the stationary interface $\{x=0\}$, we can study only stationary solutions. We then obtain the regularized equation for $U_\varepsilon(x) = u(t,x)$ in the strip $-\varepsilon < x < \varepsilon$:

$$(U_\varepsilon^2/2)'(x) + \lambda U_\varepsilon(x) \partial_x H_\varepsilon(x) = 0.$$

Proposition (Lagoutière, Seguin, Takahashi '08)

Independently from the choice of H_ε, there exists a solution to the above ODE with $U_\varepsilon(-\varepsilon) = c_-$ and $U_\varepsilon(+\varepsilon) = c_+$ if and only if $(c_-, c_+) \in \mathcal{G}_\lambda$.

The modelling assumption we make is the following :

the traces $\gamma_- u$ and $\gamma_+ u$ at $\{x=0\}$ of a solution u of the Burgers equation on $\mathbb{R}^+ \times (\mathbb{R} \backslash \{0\})$ are compatible if and only if there exists a solution to above ODE such that $U_\varepsilon(-\varepsilon) = \gamma_- u$, $U_\varepsilon(\varepsilon) = \gamma_+ u$.

Thus the germ $\mathcal{G}_\lambda$ is the set of couples $(\gamma_- u, \gamma_+ u)$ of possible traces at $\{x=0\}$ (for a.e. $t>0$) of the admissible solutions.

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Now, the dissipativity properties of the interface coupling are encoded in the germ $\mathscr{G}_\lambda$. Indeed, define $\Xi \colon \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\Xi^\pm((u_-,u_+),(v_-,v_+)) = \Phi^\pm(u_-,v_-) - \Phi^\pm(u_+,v_+)$$

where $\Phi^\pm$ are the so-called semi-Kruzhkov entropy fluxes for Burgers eqn:

$$\Phi^\pm(u,v) = \operatorname{sgn}^\pm(u-v)(u^2-v^2)/2.$$

Splitting the germ $\mathscr{G}_\lambda$ into three subsets, we have

Proposition (dissipativity and maximality of $\mathscr{G}_\lambda$)

The following properties hold:

(i) *(dissipativity)* $\forall (u_-,u_+),(v_-,v_+) \in \mathscr{G}_\lambda$,

$$\Xi^\pm((u_-,u_+),(v_-,v_+)) \geqslant 0.$$

(ii) *(maximality + ...)* *If a pair* $(u_-,u_+) \in \mathbb{R}^2$ *verifies:*

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A "better" (indirect) proof comes from the general theory from AKR .

First, property (i) is actually equivalent to the "Kato inequality" ($\Leftrightarrow$ L^1-dissipativity)

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big((u-v)^+\,\partial_t\varphi+\Phi^+(u,v)\,\partial_x\varphi\Big)\leqslant 0\quad\forall\varphi\in\mathcal{D}(Q),\ \varphi\geqslant 0.$$

for the solutions

$$u(t,x):=u_-1\!\!1_{\{x<0\}}+u_+1\!\!1_{\{x>0\}},\quad v(t,x):=v_-1\!\!1_{\{x<0\}}+v_+1\!\!1_{\{x>0\}}$$

of our equation; and the Kato inequality comes by passage to the limit from the LeRoux approximation case :

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big(\lambda(u^\varepsilon-v^\varepsilon)^+(\partial_x H_\varepsilon)\,\varphi-(u^\varepsilon-v^\varepsilon)^+\partial_t\varphi-\Phi^+(u^\varepsilon,v^\varepsilon)\partial_x\varphi\Big)\leqslant 0.$$

Further, property (ii) means that "$\mathscr{G}^1_\lambda\cup\mathscr{G}^2_\lambda$ is a definite germ of which $\mathscr{G}_\lambda$ is the unique maximal extension". This follows (with some work) from the fact that $\mathscr{G}_\lambda$ is a complete germ ($\Leftrightarrow$ the germ allows to solve every Riemann problem).

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FROZEN PARTICLE (DIRAC-AT-ZERO DRAG TERM): DEFINITION, UNIQUENESS

Frozen particle: definition(s)...

First, let us describe some elementary solutions of this problem: these are the stationary piecewise constant functions c:

$$c(t,x) = c_- 1\!\!1_{\{x<0\}} + c_+ 1\!\!1_{\{x>0\}} = \begin{cases} c_- & \text{if } x<0, \\ c_+ & \text{if } x>0, \end{cases} \quad (c_-,c_+) \in \mathcal{G}_\lambda.$$

They play the role of the constants in the standard Kruzhkov entropy formulation. With the idea of adapted Kruzhkov entropies , we set up

Definition (entropy solution)

Let $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said to be an entropy solution of the "particle-at-zero" problem if for all function c defined above with $(c_-,c_+) \in \mathcal{G}_\lambda$,

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}),\ \varphi \geqslant 0 \quad \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left[|u - c(x)|\, \partial_t \varphi + \Phi(u, c(x))\, \partial_x \varphi \right] dx\, dt$$
$$+ \int_{\mathbb{R}} |u_0 - c(x)|\, \varphi(0,x)\, dx \geqslant 0.$$

...Frozen particle: definition(s)...

Let us provide alternative characterizations of entropy solutions:

Proposition (equivalent definitions)

A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution if and only if it satisfies any of the following assertions:

A. *The function u verifies the adapted entropy inequalities with $(c_-, c_+) \in \mathscr{G}_\lambda^1 \cup \mathscr{G}_\lambda^2$.*

B. *The function u verifies the Kruzhkov entropy inequalities for all nonnegative test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ such that $\varphi|_{x=0} = 0$, moreover,*

$$\text{for a. e. } t > 0 \quad \left((\gamma_- u)(t), (\gamma_+ u)(t) \right) \in \mathscr{G}_\lambda.$$

D. *There exists $C = C(\lambda, \|u\|_\infty, c_\pm)$ such that the function u verifies*

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$$+ \int_{\mathbb{R}} |u_0 - c(x)|\, \varphi(0, x)\, dx \geqslant -C(\varphi)\mathrm{dist}\left((c_-, c_+), \mathscr{G}_\lambda \right)$$

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...Frozen particle: uniqueness, comparison, L^1 contraction.

Theorem ($\mathbf{L}^1$ contraction+comparison, analogous to Kruzhkov theory)

Let u_0 and v_0 be two initial data in $\mathbf{L}^\infty(\mathbb{R})$ and let u and v be the associated entropy solutions. Then for all $R > 0$,

$$\text{for a.e. } t > 0 \quad \int_R^R (u-v)^+(t,x)\,dx \leqslant \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x)\,dx$$

where $L = \max\{\|u\|_\infty, \|v\|_\infty\}$. Consequently, if $(u_0 - v_0)^+ \in \mathbf{L}^1(\mathbb{R})$, we have

$$\text{for a.e. } t > 0 \quad \int_{\mathbb{R}} (u-v)^+(t,x)\,dx \leqslant \int_{\mathbb{R}} (u_0 - v_0)^+(x)\,dx.$$

In particular, for all $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, there exists at most one solution and the map $S(t) : u_0 \mapsto u(t,\cdot)$ on its domain is an order-preserving $\mathbf{L}^1$ contraction.

The proof is straightforward using

– the Kato inequality away from the interface (standard Kruzhkov)
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Theorem (L^1 contraction+comparison, analogous to Kruzhkov theory)

Let u_0 and v_0 be two initial data in $\mathbf{L}^\infty(\mathbb{R})$ and let u and v be the associated entropy solutions. Then for all $R>0$,

$$\text{for a.e. } t>0 \quad \int_R^R (u-v)^+(t,x)\,dx \leqslant \int_{-R-Lt}^{R+Lt} (u_0-v_0)^+(x)\,dx$$

where $L=\max\{\|u\|_\infty,\|v\|_\infty\}$. Consequently, if $(u_0-v_0)^+\in\mathbf{L}^1(\mathbb{R})$, we have

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FROZEN PARTICLE

(DIRAC-AT-ZERO DRAG TERM):

NUMERICAL SCHEME AND EXISTENCE

Frozen particle: a finite volume scheme...

We use a well-balanced finite volume scheme , preserving exactly (some of) the stationary sols $u(t,x) := u_- 1\!\!1_{\{x<0\}} + u_+ 1\!\!1_{\{x>0\}}$.
Usual schemes are determined by a numerical flux $g(\cdot,\cdot)$:

– g locally Lipschitz;
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Idea : $g^\pm_\lambda$ "only see" the $\mathscr{G}^1_\lambda$ part of the germ !

Then the scheme writes

$$\begin{aligned} \forall i \neq 0,1 \qquad & u_i^{n+1} = u_i^n - \tfrac{\Delta t}{\Delta x}\big(g(u_i^n,u_{i+1}^n) - g(u_{i-1}^n,u_i^n)\big);\\ i=0: \qquad & u_0^{n+1} = u_0^n - \tfrac{\Delta t}{\Delta x}\big(g^-_\lambda(u_0^n,u_1^n) - g(u_{-1}^n,u_0^n)\big);\\ i=1: \qquad & u_1^{n+1} = u_1^n - \tfrac{\Delta t}{\Delta x}\big(g(u_1^n,u_2^n) - g^+_\lambda(u_0^n,u_1^n)\big). \end{aligned}$$

Numerical solution:

$$u_\Delta(t,x) = \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{Z}} u_i^n \, 1\!\!1_{(n\Delta t,(n+1)\Delta t)}(t)\, 1\!\!1_{(x_{i-1/2},x_{i+1/2})}(x).$$

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Under the CFL condition: $2M\Delta t \leqslant \Delta x$, ($M$ being the Lipschitz constant of the numerical flux g on the *ad hoc* interval of values of $(u_i^n)_{n,i}$, the scheme writes

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Lemma ($\mathbf{L}^\infty$ bound — choice of M in the CFL condition)

Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$

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Proposition (the scheme is (partially) well-balanced)

(i) The initial datum $v_0(\cdot) = c(\cdot) = c_- 1\!\!1_{\{x<0\}} + c_+ 1\!\!1_{\{x>0\}}$, $(c_-, c_+) \in \mathcal{G}_\lambda^1$, is exactly preserved in the evolution by the scheme .

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(H) $\partial_a(\partial_a g(a,b)+\partial_b g(a,b)) \geqslant 0, \quad \partial_b(\partial_a g(a,b)+\partial_b g(a,b)) \geqslant 0;$

Lemma (BV_{loc} bound, Bürger, García, Karlsen, Towers)

Let $T>0$ and $A>0$. Assume that $u_0 \in BV(\mathbb{R})$ and Δx is small enough. Then, under the CFL condition and assumption **(H)**, *we have*

$$\|u_\Delta(\cdot,\cdot)\|_{BV([0,T]\times\mathbb{R}\setminus(-A,A))} \leqslant \frac{1}{A} Const(T, \|u_0\|_{L^\infty}, \|u_0\|_{BV(\mathbb{R})}, \lambda).$$

– estimate in $BV(0,T;L^1(\mathbb{R}))$ (i.e., time BV estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + **(H)**)

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– look at our solution as solution to a Cauchy-Dirichlet problem with BV initial and boundary data.

Proposition (approximate Kato inequality)

Let u_Δ, v_Δ be solutions of the scheme. Let $\varphi \in \mathcal{D}([0,T)\times\mathbb{R})$, $\varphi \geqslant 0$. Then

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left((u_\Delta - v_\Delta)^+ \partial_t\varphi + \Phi^+(u_\Delta, v_\Delta)\,\partial_x\varphi\right) \leqslant Rem(\Delta x, \varphi).$$

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Proposition (approximate Kato inequality)

Let u_Δ, v_Δ be solutions of the scheme. Let $\varphi\in\mathcal{D}([0,T)\times\mathbb{R})$, $\varphi\geqslant 0$. Then

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}}\left((u_\Delta-v_\Delta)^+\,\partial_t\varphi+\Phi^+(u_\Delta,v_\Delta)\,\partial_x\varphi\right)\leqslant Rem(\Delta x,\varphi).$$

Arguments: monotonicity of the scheme, consistency , the BV_{loc} in space bound

...Properties of the scheme...

In what follows, we need a technical hypothesis (dissipativity at $x=0$):

(H) $\partial_a(\partial_a g(a,b)+\partial_b g(a,b))\geqslant 0,\quad \partial_b(\partial_a g(a,b)+\partial_b g(a,b))\geqslant 0;$

Lemma (BV_{loc} bound, Bürger, García, Karlsen, Towers)

Let $T>0$ and $A>0$. Assume that $u_0\in BV(\mathbb{R})$ and Δx is small enough. Then, under the CFL condition and assumption **(H)***, we have*

$$\|u_\Delta(\cdot,\cdot)\|_{BV([0,T]\times\mathbb{R}\setminus(-A,A))}\leqslant \frac{1}{A}Const(T,\|u_0\|_{L^\infty},\|u_0\|_{BV(\mathbb{R})},\lambda).$$

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In particular, the problem is well-posed, for L^∞ data and L^1_{loc} topology.

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- For the general case $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, localize using finite speed of propagation; approximate u_0 by $BV(\mathbb{R}) \cap L^1(\mathbb{R})$ functions $(u_0^n)_n$ Use discrete L^1 contraction and the result of the BV case. □

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Coupled Problem

Case $h = h(t)$: existence, uniqueness

Theorem (well-posedness for moving but decoupled particle)

Given $h(\cdot)$ a C^1 path, there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda(u - h'(t))\delta_0(x - h(t))$; (localized) L^1 contraction property holds.

Definition :

– the germ $\mathscr{G}_\lambda$ changes into $(h'(t), h'(t)) + \mathscr{G}_\lambda$;
– versions **B.** ("with traces") and **D.** ("adapted entropy inequalities with remainder term") permit to define entropy solutions.

Arguments :

– for uniqueness, comparison, L^1 contraction: same technique;
– for existence: use characterization **D.** (it is stable by passage to the limit!) ;
– approximate $h(\cdot)$ by a family $(h_n)_n$ of piecewise affine paths
– construction of solutions for "particle at h_n" is straightforward: h_n' being piecewise constant, one changes variables to reduce to the "drag force-at-zero" case. Procedure restarted at each time where h_n' jumps.
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Case $h=h(t)$: continuous dependence en $h(\cdot)$

Uniqueness proof for the coupled problem relies on a Gronwall inequality, which in turn relies on a Lipschitz dependence estimate for the map $h(\cdot)\mapsto u(\cdot,\cdot)$.

Theorem (dependence of u on the path $h(\cdot)$)

Assume $u,\hat u$ are entropy solutions corresponding to the particles located at $h(\cdot),\hat h(\cdot)$, respectively, with $h(0)=0=\hat h(0)$ and same initial datum u_0. Assume $\hat u\in L^\infty(0,T;BV(\mathbb{R}))$. Then for a.e. $t\in(0,T)$,

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– use the techniques of dependence of entropy solutions on the flux function (BV regularity needed!): Kuznetsov, Bouchut-Perthame, Karlsen-Risebro... : the C^1 norm of the difference of the fluxes pops up, which yields $|h'-\hat h'|$

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Theorem (dependence of u on the path $h(\cdot)$)

Assume $u,\hat{u}$ are entropy solutions corresponding to the particles located at $h(\cdot),\hat{h}(\cdot)$, respectively, with $h(0)=0=\hat{h}(0)$ and same initial datum u_0. Assume $\hat{u}\in L^\infty(0,T;BV(\mathbb{R}))$. Then for a.e. $t\in(0,T)$,

$$\|u(t,\cdot)-\hat{u}(t,\cdot)\|_{L^1(\mathbb{R})}\leqslant C(\|u\|_\infty,\|\hat{u}\|_\infty,\|\hat{u}\|_{BV},\lambda)\int_0^t|h'(s)-\hat{h}'(s)|\,ds.$$

Arguments:

– change of variables $y=x-h(t)$, resp. $x-h'(t)$. Two eqns, both with singularity at zero, come out, with different fluxes of the kind $u\mapsto\frac{u^2}{2}-h'(t)u$.
– use the techniques of dependence of entropy solutions on the flux function (BV regularity needed!): Kuznetsov, Bouchut-Perthame, Karlsen-Risebro... : the C^1 norm of the difference of the fluxes pops up, which yields $|h'-\hat{h}'|$
– use Lipschitz dependence of the germ on h' to describe additional (small) “non-dissipation” term coming from the interface.

Case $h=h(t)$: continuous dependence en $h(\cdot)$, L^∞ and BV stability

Proposition (BV estimate)

The solution constructed for the $h=0$ case obeys

$$\|u(t,\cdot)\|_{BV(\mathbb{R})} \searrow \quad \textit{for all} \quad t>0$$

(at $t=0$ the variation may increase by a const. depending on $\|u_0\|_\infty, \mathscr{G}_\lambda$).

The solution constructed for the fixed-$h(\cdot)$ case obeys the BV estimate

$$\|u(t,\cdot)\|_{BV(\mathbb{R})} \leqslant \|u_0\|_{BV(\mathbb{R})} + const(\lambda, \|u_0\|_\infty) + 2\int_0^t |h''(s)|\, ds.$$

Argument: (re)-construct solutions by wave-front tracking algorithm (Dafermos, Holden-Risebro, Bressan et al.) (better control of interactions).

Lemma (L^∞ bounds)

We get a uniform $\mathbf{L}^\infty$ bound on ad hoc sequences of $h'(\cdot)$ and $u(\cdot,\cdot)$.

To be precise: if we look at solutions to the coupled problem, we get $\max\{\|u\|_\infty, \|h'\|_\infty\} \leqslant \max\{\|u_0\|_\infty, |h'(0)|\}$.
For solutions appearing in the fixed-point or splitting arguments, we get somewhat weaker bounds.

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Case of u frozen: evolving $h=h(\cdot)$

Proposition (modelling/"traces" interpretation of the ODE on $h(\cdot)$)

For every drag force, the ODE in the coupled problem writes

$$mh''(t)=\Big((u_-)^2/2-h'(t)u_-\Big) \quad -\Big((u_+)^2/2-h'(t)u_+\Big).$$

Notice that the right-hand side above is expressed as the difference of the normal components of the 2D-field $(u,u^2/2)$ on the curve $\{x=h(t)\}$ from the left and from the right . Combining this observation with the Green-Gauss formula, we get the following weak formulation of the ODE:

Lemma (second interpretation of the ODE on $h(\cdot)$)

Let u be a weak solution of the PDE on $\{x\neq h(t)\}$; let $h\in W^{2,\infty}(0,T)$. Then $h(\cdot)$ verifies the ODE if and only if for all $\xi\in\mathcal{D}([0,T))$, for all $\psi\in\mathcal{D}(\mathbb{R})$ such that $\psi\equiv 1$ on the set $\{x\in\mathbb{R}\ :\ \exists t\in[0,T]$ such that $h(t)=x\}$, there holds

$$-m\int_0^T h'(t)\xi'(t)dt=mh'(0)\xi(0)+\int_0^T\int_{\mathbb{R}}\Big[u\psi\xi_t+\frac{u^2}{2}\xi\psi_x\Big]+\int_{\mathbb{R}}u_0\psi\xi(0).$$

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Coupled problem: existence, uniqueness of BV solutions / existence of $\mathbf{L}^\infty$ solutions

The above ingredients can be used in several ways:

– In a fixed-point argument $h(\cdot) \mapsto u(\cdot,\cdot) \mapsto h(\cdot)$
(compactness: work in $C^1([0,T])$, exploit a $W^{2,\infty}(0,T)$ bound on $h(\cdot)$)

– In a time splitting algorithm (alternatively evolving u and h on small time intervals):

- u updated from h using the theory of entropy solutions for h frozen;
- h updated from u using the above weak formulation of the ODE.

– In a numerical scheme (same time splitting + approximation in space of the conservation law); an interesting possibility is the random-choice algorithm (Glimm), in order not to adapt the space meshing to the particle location.

Theorem (Main result)

For all BV datum u_0 and given $h(0), h'(0)$, there exists a unique entropy solution to the coupled problem.
For all $\mathbf{L}^\infty$ datum u_0 and given $h(0), h'(0)$, there exists an entropy solution to the coupled problem.

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Coupled problem: a well-balanced random-choice numerical scheme

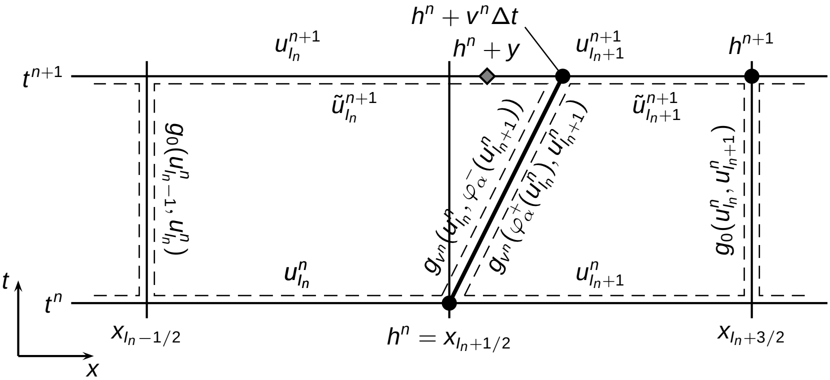

Figure: Representation of the algorithm based on the well-balanced scheme.

Numerics: drafting-kissing-tumbling

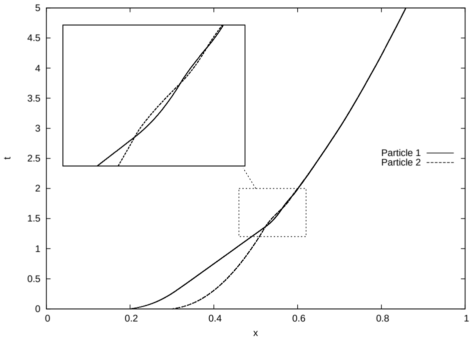

Figure: Trajectories of two particles

Thx !!!

THANK YOU !