A particle-in-Burgers model: theory and numerics

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joint work with

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Hyperbolic Conservation Laws
and Related Analysis with Applications
Plan of the talk

1. Model and motivation
2. Auxiliary steps
3. Main Results
4. The frozen particle case: coupling
5. The frozen particle case: definition, uniqueness
6. The frozen particle case: numerics and existence
7. The coupled problem
Model and motivation

Auxiliary steps

Results

$h = 0$: coupling

$h = 0$: definition, uniqueness

$h = 0$: numerics, existence

The coupled problem

MODEL AND MOTIVATION
D’Alembert paradox: a solid immersed in an inviscid fluid is not submitted to any resultant force; in other words, birds (and planes...) could not fly with a model where viscosity is neglected! Yet, inviscid (hyperbolic !) models are ok for some fluids...

Answer 1 to the d’Alembert paradox: use viscous models of fluid-solid interaction (see e.g. M. Hillairet, for a recent review).

Answer 2 (when the Reynolds number is large): it is reasonable to neglect the viscous effects in the model that governs the fluid; but we have to conserve information from the vanishing viscosity in a DRAG FORCE. The drag force takes the form of a source term which takes into account the difference between the velocity of the fluid and the velocity of the solid.
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The drag force takes the form of a source term which takes into account the difference between the velocity of the fluid and the velocity of the solid.
The 1D case: the Lagoutière-Seguin-Takahashi model for the interaction, via a drag force, of a point particle with a Burgers fluid writes

\[
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \lambda \, D(h'(t) - u) \, \delta_0(x - h(t)), \\
mh''(t) = \lambda \, D(u(t, h(t)) - h'(t)).
\]

here

- \( u \), the velocity of the fluid, is unknown
- \( h \), the position of the solid particle, is unknown
  (then \( h' \) and \( h'' \) respectively denote its velocity and acceleration);
- the parameters are \( \lambda \) (the drag coefficient) and \( m \) (the mass of the solid particle); both are positive.
- the function \( D \) which intervenes in the drag force is an increasing odd function.

Actually, we will suppose that

either \( D(\nu) = \nu \) \hspace{1cm} (the linear case)

or \( D(\nu) = \nu|\nu| \) \hspace{1cm} (the quadratic case).
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Auxiliary steps
Our study of the above coupled problem includes two auxiliary steps, that are of interest on their own. The first step is

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\begin{align*}
\partial_t u(t, x) + \partial_x \left( \frac{u^2}{2} \right)(t, x) &= -\lambda \, u(t, x) \, \delta_0(x), \quad t > 0, \; x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

i.e., the particle is decoupled from the fluid and fixed at zero.

**Difficulty 1:** the source term has to be carefully defined. Indeed, \( u \) can be discontinuous (and in fact, typically \( u \) IS DISCONTINUOUS at the particle location).

To give an interpretation of the source term, the LeRoux approximation was studied in detail by Lagoutière, Seguin, Takahashi: \( \delta_0 = \partial_x H \) (\( H \): the Heavyside function) is replaced by \( \partial_x H_\varepsilon \), a smoothed version. This permits to understand what goes on at the interface.

The second step is to take \( h(\cdot) \) a given path, still decoupled from the fluid, and to solve the Burgers equation with singular source term located at \( x = h(t) \). We'll see that as soon as the first step is well understood, the second one is easy.
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Other way around, we should understand how to evolve the particle location given the fluid state at time $t$. Recall the equation (ODE) for the particle:

$$ mh''(t) = \lambda (u(t, h(t)) - h'(t)). $$

Recall that $u(t, \cdot)$ has a jump at $x = h(t)$...

**Difficulty 2**: understand the equation in the Carathéodory sense? In the Filippov sense?? We will see that a nice mathematical and physical interpretation is possible:

- the particle is driven by the lack of mass conservation in the equation for $u$; or, equivalently, the total quantity of movement $\int_{\mathbb{R}} u(t, \cdot) + mh'(t)$ is conserved.
- the ODE for $h$ can be written in a weak form that involves the values of $u(t, \cdot)$ on $\mathbb{R}$ (which is more "robust")

With these auxiliary steps well understood, we can

- think of the appropriate definition of solution
- use fixed-point arguments to guarantee existence
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MAIN RESULTS
Main results: Auxiliary Problem 1 is well posed

For the Burgers-with-Dirac-at-zero model, we apply the machinery developed for conservation laws with discontinuous flux (adapted entropies, Baiti, Jenssen and Audusse, Perthame; revisited and generalized recently by BA., Karlsen, Risebro using the notion of admissibility germ). The outcome is:

– definition(s) of entropy solutions
– uniqueness, continuous dependence \( (L^1, \, L^1_{loc}) \) with domain of dependence) exactly as in the Kruzhkov theory

In addition, we find

– a priori \( L^\infty \) bounds and (more delicate) variation bounds
– a strikingly simple numerical method (monotone consistent finite volume scheme with a trick at the interface)
– convergence of the numerical scheme, existence.

NB: the Riemann solver at the interface was already described by Lagoutière, Seguin, Takahashi, so a Godunov scheme could be constructed; but we seek to avoid using the Riemann solver because it is intricate.
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Main results: Auxiliary Problem 2 and the full problem

Then for the Burgers-driven-by-particle model (with $x = h(t)$ GIVEN path of the particle) we deduce well-posedness rather easily. It is observed that the case of straight path, $h(t) = Vt$ with $V = \text{const}$, reduces to the Dirac-at-zero model by the simultaneous change of $u - V$ into $u$ and of $x - Vt$ into $x$. Thus, nothing new for $h(t) = Vt$. Then any $(W^2,\infty)$ path $h(\cdot)$ is approximated by piecewise affine paths; existence is established by passage to the limit. Uniqueness is straightforward from the definition of solution.

For the coupled model with data $u_0$ and $h(0) = 0$, $h'(0) = v_0$, we get

- existence, for $L^\infty$ data $u_0$
- existence, uniqueness, continuous dependence for $BV$ solutions, for $BV$ data $u_0$.

We construct a time-explicit Glimm-type scheme where particle position is updated via splitting; we get numerical results that agree with the physical phenomena that are expected for the model.
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FROZEN PARTICLE

(DIRAC-AT-ZERO DRAG TERM):
UNDERSTANDING THE COUPLING
Frozen particle: understanding the coupling...

The admissibility at the interface \( \{ x = 0 \} \) of the solution is governed by the germ \( G_\lambda \) (terminology related to the one of BA, Karlsen, Risebro):

**Definition**

The *admissibility germ* \( G_\lambda \subset \mathbb{R}^2 \) (or *germ*, for short) associated with the particle-at-zero problem is the union \( G_\lambda = G_\lambda^1 \cup G_\lambda^2 \cup G_\lambda^3 \), where

- \( G_\lambda^1 = \{ (a, a - \lambda), a \in \mathbb{R} \} \).
- \( G_\lambda^2 = [0, \lambda] \times [-\lambda, 0] \).
- \( G_\lambda^3 = \{ (a, b) \in (\mathbb{R}^+ \times \mathbb{R}^-) \setminus G_\lambda^2, -\lambda \leq a + b \leq \lambda \} \).

NB: the partition of \( G_\lambda \) into the three parts is dictated by the subsequent analysis, and by profiles study of LST.
Frozen particle: understanding the coupling...

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Explaination: the Burgers equation with Dirac-at-zero drag term is equivalent to
\[ \partial_t u + \partial_x (u^2/2) = -\lambda u \partial_x H. \]
We introduce \( H_\varepsilon \in C^1(\mathbb{R}) \) a non-decreasing function such that \( H_\varepsilon(x) = H(x) \) when \( |x| \geq \varepsilon \). Since we are interested in understanding the behavior of the solution through the stationary interface \( \{x = 0\} \), we can study only stationary solutions. We then obtain the regularized equation for \( U_\varepsilon(x) = u(t, x) \) in the strip \(-\varepsilon < x < \varepsilon\):
\[ (U_\varepsilon^2/2)'(x) + \lambda U_\varepsilon(x) \partial_x H_\varepsilon(x) = 0. \]

Proposition (Lagoutière, Seguin, Takahashi '08)

Independently from the choice of \( H_\varepsilon \), there exists a solution to the above ODE with \( U_\varepsilon(-\varepsilon) = c_- \) and \( U_\varepsilon(\varepsilon) = c_+ \) if and only if \((c_-, c_+) \in \mathcal{G}_\lambda\).

The modelling assumption we make is the following:
the traces \( \gamma_- u \) and \( \gamma_+ u \) at \( \{x = 0\} \) of a solution \( u \) of the Burgers equation on \( \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \) are compatible if and only if there exists a solution to above ODE such that \( U_\varepsilon(-\varepsilon) = \gamma_- u, U_\varepsilon(\varepsilon) = \gamma_+ u \).

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**Proposition (Lagoutière, Seguin, Takahashi '08)**

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...Frozen particle: understanding the coupling...

Now, the dissipativity properties of the interface coupling are encoded in the germ $\mathcal{G}_\lambda$. Indeed, define $\Xi : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$
\Xi^\pm((u_-, u_+), (v_-, v_+)) = \Phi^\pm(u_-, v_-) - \Phi^\pm(u_+, v_+)
$$

where $\Phi^\pm$ are the so-called semi-Kruzhkov entropy fluxes for Burgers eqn:

$$
\Phi^\pm(u, v) = \text{sgn}^\pm(u - v)(u^2 - v^2)/2.
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Splitting the germ $\mathcal{G}_\lambda$ into three subsets, we have

**Proposition (dissipativity and maximality of $\mathcal{G}_\lambda$)**

The following properties hold:

(i) **(dissipativity)** \( \forall (u_-, u_+), (v_-, v_+) \in \mathcal{G}_\lambda, \)

\[ \Xi^\pm((u_-, u_+), (v_-, v_+)) \geq 0. \]

(ii) **(maximality + ...)** If a pair \((u_-, u_+) \in \mathbb{R}^2\) verifies:

\[ \forall (v_-, v_+) \in \mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda \quad \Xi((u_-, u_+), (v_-, v_+)) \geq 0, \]

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One can prove the proposition directly, by a tedious case study... but...
A “better” (indirect) proof comes from the general theory from AKR. First, property (i) is actually equivalent to the “Kato inequality” ($\Leftrightarrow$ $L^1$-dissipativity)

$$-\int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u-v)^+ \partial_t \varphi + \Phi^+(u, v) \partial_x \varphi) \leq 0 \quad \forall \varphi \in D(Q), \varphi \geq 0.$$  

for the solutions

$$u(t, x) := u_- 1_{\{x<0\}} + u_+ 1_{\{x>0\}}, \quad v(t, x) := v_- 1_{\{x<0\}} + v_+ 1_{\{x>0\}}$$

of our equation; and the Kato inequality comes by passage to the limit from the LeRoux approximation case:

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\lambda (u^\varepsilon - v^\varepsilon)^+(\partial_x H_\varepsilon) \varphi - (u^\varepsilon - v^\varepsilon)^+ \partial_t \varphi - \Phi^+(u^\varepsilon, v^\varepsilon) \partial_x \varphi) \leq 0.$$  

Further, property (ii) means that “$G^1_\lambda \cup G^2_\lambda$ is a definite germ of which $G_\lambda$ is the unique maximal extension”. This follows (with some work) from the fact that $G_\lambda$ is a complete germ ($\Leftrightarrow$ the germ allows to solve every Riemann problem).
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FROZEN PARTICLE (DIRAC-AT-ZERO DRAG TERM):
DEFINITION, UNIQUENESS
Frozen particle: definition(s)...

First, let us describe some elementary solutions of this problem: these are the stationary piecewise constant functions $c$:

$$c(t, x) = c_- 1_{\{x < 0\}} + c_+ 1_{\{x > 0\}} = \begin{cases} c_- & \text{if } x < 0, \\ c_+ & \text{if } x > 0, \end{cases} \quad (c_-, c_+) \in \mathcal{G}_\lambda.$$  

They play the role of the constants in the standard Kruzhkov entropy formulation. With the idea of adapted Kruzhkov entropies, we set up

**Definition (entropy solution)**

Let $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said to be an entropy solution of the “particle-at-zero” problem if for all function $c$ defined above with $(c_-, c_+) \in \mathcal{G}_\lambda$,

$$\forall \varphi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}), \varphi \geq 0 \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[ |u - c(x)| \partial_t \varphi + \Phi(u, c(x)) \partial_x \varphi \right] dx \, dt$$

$$+ \int_{\mathbb{R}} |u_0 - c(x)| \varphi(0, x) dx \geq 0.$$
Let us provide alternative characterizations of entropy solutions:

**Proposition (equivalent definitions)**

A function \( u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) is an entropy solution if and only if it satisfies any of the following assertions:

A. The function \( u \) verifies the adapted entropy inequalities with \( (c_-, c_+) \in \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2 \).

B. The function \( u \) verifies the Kruzhkov entropy inequalities for all nonnegative test function \( \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}) \) such that \( \varphi|_{x=0} = 0 \), moreover, for a. e. \( t > 0 \) \((\gamma_- u)(t), (\gamma_+ u)(t)\) \( \in \mathcal{G}_\lambda \).

D. There exists \( C = C(\lambda, \|u\|_\infty, c_\pm) \) such that the function \( u \) verifies

\[
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+ \int_{\mathbb{R}} |u_0 - c(x)| \varphi(0, x) \, dx \geq -C(\varphi) \text{dist} \left( (c_-, c_+), \mathcal{G}_\lambda \right)
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**for all** \((c_-, c_+) \in \mathbb{R} \times \mathbb{R}\).
...Frozen particle: uniqueness, comparison, $L^1$ contraction.

**Theorem ($L^1$ contraction+comparison, analogous to Kruzhkov theory)**

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

$$
\text{for a.e. } t > 0 \quad \int_R^R (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x) \, dx
$$

where $L = \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$. Consequently, if $(u_0 - v_0)^+ \in L^1(\mathbb{R})$, we have

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$$

In particular, for all $u_0 \in L^\infty(\mathbb{R})$, there exists at most one solution and the map $S(t) : u_0 \mapsto u(t, \cdot)$ on its domain is an order-preserving $L^1$ contraction.

The proof is straightforward using

– the Kato inequality away from the interface (standard Kruzhkov)

– the characterization $B$. (“with traces”) of entropy solutions

– and the dissipativity of $G_\lambda$. 
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where \(L = \max\{\|u\|_\infty, \|v\|_\infty\}\). Consequently, if \((u_0 - v_0)^+ \in L^1(\mathbb{R})\), we have

\[
\text{for a.e. } t > 0 \quad \int_{\mathbb{R}} (u - v)^+(t, x) \, dx \leq \int_{\mathbb{R}} (u_0 - v_0)^+(x) \, dx.
\]

In particular, for all \(u_0 \in L^\infty(\mathbb{R})\), there exists at most one solution and the map \(S(t) : u_0 \mapsto u(t, \cdot)\) on its domain is an order-preserving \(L^1\) contraction.

The proof is straightforward using
– the Kato inequality away from the interface (standard Kruzhkov)
– the characterization \(B\). (“with traces”) of entropy solutions
– and the dissipativity of \(G_\lambda\).
Theorem (L¹ contraction+comparison, analogous to Kruzhkov theory)

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

$$ \text{for a.e. } t > 0 \quad \int_R^R (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x) \, dx $$

where $L = \max\{\|u\|_\infty, \|v\|_\infty\}$. Consequently, if $(u_0 - v_0)^+ \in L^1(\mathbb{R})$, we have

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...Frozen particle: uniqueness, comparison, $L^1$ contraction.

**Theorem (L$^1$ contraction+comparison, analogous to Kruzhkov theory)**

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

$$
\text{for a.e. } t > 0 \quad \int_{R}^{R+Lt} (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R} (u_0 - v_0)^+(x) \, dx
$$

where $L = \max\{\|u\|_\infty, \|v\|_\infty\}$. Consequently, if $(u_0 - v_0)^+ \in L^1(\mathbb{R})$, we have

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Let \( u_0 \) and \( v_0 \) be two initial data in \( L^\infty(\mathbb{R}) \) and let \( u \) and \( v \) be the associated entropy solutions. Then for all \( R > 0 \),

\[
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where \( L = \max\{\|u\|_\infty, \|v\|_\infty\} \). Consequently, if \( (u_0 - v_0)^+ \in L^1(\mathbb{R}) \), we have

\[
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\]

In particular, for all \( u_0 \in L^\infty(\mathbb{R}) \), there exists at most one solution and the map \( S(t) : u_0 \mapsto u(t, \cdot) \) on its domain is an order-preserving \( L^1 \) contraction.

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**Theorem ($L^1$ contraction+comparison, analogous to Kruzhkov theory)**

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

$$
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where $L = \max\{\|u\|_\infty, \|v\|_\infty\}$. Consequently, if $(u_0 - v_0)^+ \in L^1(\mathbb{R})$, we have

$$
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FROZEN PARTICLE
(DIRAC-AT-ZERO DRAG TERM):
NUMERICAL SCHEME AND EXISTENCE
Frozen particle: a finite volume scheme...

We use a well-balanced finite volume scheme, preserving exactly (some of) the stationary sols $u(t, x) := u_- 1_{\{x<0\}} + u_+ 1_{\{x>0\}}$.

Usual schemes are determined by a numerical flux $g(\cdot, \cdot)$:

- $g$ locally Lipschitz;
- $g(u, u) = \frac{u^2}{2}$ (consistency);
- $g(\cdot, b)$ is $\nearrow$, $g(a, \cdot)$ is $\searrow$ (monotonicity).

We only modify $g(\cdot, \cdot)$ at the interface $x = 0$ (between $x_0$ and $x_1$):

$$g^-_\lambda (a, b) = g(a, b + \lambda) \quad \text{and} \quad g^+_\lambda (a, b) = g(a - \lambda, b).$$

Idea: $g^\pm_\lambda$ “only see” the $G^1_\lambda$ part of the germ!

Then the scheme writes

$$\forall i \neq 0, 1 \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n));$$

$$i = 0 : \quad u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x} (g^-_{\lambda}(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n));$$

$$i = 1 : \quad u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x} (g(u_1^n, u_2^n) - g^+_\lambda(u_0^n, u_1^n)).$$

Numerical solution:

$$u_\Delta(t, x) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} u_i^n \mathbbm{1}_{(n\Delta t,(n+1)\Delta t)}(t) \mathbbm{1}_{(x_{i-1/2}, x_{i+1/2})}(x).$$
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Numerical solution:

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Idea: $g^\pm_\lambda$ “only see” the $G_\lambda^1$ part of the germ!

Then the scheme writes

$$
\forall i \neq 0, 1 \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n));
$$

$$
i = 0 : \quad u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x} (g^-(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n));
$$

$$
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$$

Numerical solution:

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We only modify \( g(\cdot, \cdot) \) at the interface \( x = 0 \) (between \( x_0 \) and \( x_1 \)):

\[
\begin{align*}
g^-_\lambda(a, b) &= g(a, b + \lambda) \quad \text{and} \quad g^+_\lambda(a, b) = g(a - \lambda, b).
\end{align*}
\]

Idea: \( g^+_\lambda \) "only see" the \( G^1_\lambda \) part of the germ!

Then the scheme writes

\[
\begin{align*}
\forall i \neq 0, 1 & \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)); \\
i = 0 & \quad u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x} (g^-_\lambda(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n)); \\
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\end{align*}
\]

Numerical solution:

\[
u_\Delta(t, x) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} u_i^n \mathbb{1}_{(n\Delta t, (n+1)\Delta t)}(t) \mathbb{1}_{(x_{i-1/2}, x_{i+1/2})}(x).
\]
Properties of the scheme...

Under the CFL condition: $2M\Delta t \leq \Delta x$, ($M$ being the Lipschitz constant of the numerical flux $g$ on the \textit{ad hoc} interval of values of $(u^n_i)_{n,i}$, the scheme writes

$$\forall i \in \mathbb{Z} \quad u^{n+1}_i = H_i(u^n_{i-1}, u^n_i, u^n_{i+1}),$$

where functions $H_i$ are monotone in each of the three arguments.

NB: since $\cdot \mapsto \cdot \pm \lambda$ are functions, monotonicity OK also for $i = 0, 1$.

**Lemma (L\textsuperscript{∞} bound — choice of $M$ in the CFL condition)**

Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$

$$\min\{\text{ess inf } u_0 - \lambda, \text{ ess inf } u_0\} \leq u^n_i \leq \max\{\text{ess sup } u_0, \text{ ess sup } u_0 + \lambda\}.$$  

**Proposition (the scheme is (partially) well-balanced)**

(i) The initial datum $v_0(\cdot) = c(\cdot) = c_- 1_{\{x < 0\}} + c_+ 1_{\{x > 0\}}$, $(c_-, c_+) \in \mathcal{G}^1_\lambda$, is exactly preserved in the evolution by the scheme.

(ii) Let $v_\Delta$ be the solution of the numerical scheme with the initial datum $v_0(\cdot) = c(\cdot) = c_- 1_{\{x < 0\}} + c_+ 1_{\{x > 0\}}$, $(c_-, c_+) \in \mathcal{G}^2_\lambda$. Then $v_\Delta$ converge to $c$ \textit{in} $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ as $\Delta x \to 0$. 
Properties of the scheme...

Under the CFL condition: $2M\Delta t \leq \Delta x$, ($M$ being the Lipschitz constant of the numerical flux $g$ on the \textit{ad hoc} interval of values of $(u^n_i)_n,i$, the scheme writes

$$\forall i \in \mathbb{Z} \quad u^{n+1}_i = H_i(u^n_{i-1}, u^n_i, u^n_{i+1}),$$

where functions $H_i$ are monotone $\uparrow$ in each of the three arguments.

NB: since $\cdot \mapsto \cdot \pm \lambda$ are $\uparrow$ functions, monotonicity OK also for $i = 0, 1$.

Lemma (L$^\infty$ bound — choice of $M$ in the CFL condition)

Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}, i \in \mathbb{Z}$

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(i) The initial datum $v_0(\cdot) = c(\cdot) = c_- \mathbb{1}_{\{x<0\}} + c_+ \mathbb{1}_{\{x>0\}}$, $(c_-, c_+) \in \mathcal{G}^1_\lambda$, is exactly preserved in the evolution by the scheme.

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Under the CFL condition: $2M\Delta t \leq \Delta x$, ($M$ being the Lipschitz constant of the numerical flux $g$ on the ad hoc interval of values of $(u_i^n)_{n,i}$, the scheme writes

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where functions $H_i$ are monotone $\nearrow$ in each of the three arguments. NB: since $\cdot \mapsto \cdot \pm \lambda$ are $\nearrow$ functions, monotonicity OK also for $i = 0, 1$.

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*Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$*

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$$
\forall i \in \mathbb{Z} \quad u^{n+1}_i = H_i(u^n_{i-1}, u^n_i, u^n_{i+1}),
$$

where functions $H_i$ are monotone $\leadsto$ in each of the three arguments.

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**Lemma (L∞ bound — choice of $M$ in the CFL condition)**

Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$

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\min\{\text{ess inf } u_0 - \lambda, \text{ess inf } u_0\} \leq u^n_i \leq \max\{\text{ess sup } u_0, \text{ess sup } u_0 + \lambda\}.
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**Proposition (the scheme is (partially) well-balanced)**

(i) The initial datum $v_0(\cdot) = c(\cdot) = c_- \mathbf{1}_{\{x<0\}} + c_+ \mathbf{1}_{\{x>0\}},$

$(c_-, c_+) \in \mathcal{G}_\lambda^1$, is exactly preserved in the evolution by the scheme.

(ii) Let $v_\Delta$ be the solution of the numerical scheme with the initial datum $v_0(\cdot) = c(\cdot) = c_- \mathbf{1}_{\{x<0\}} + c_+ \mathbf{1}_{\{x>0\}},$

$(c_-, c_+) \in \mathcal{G}_\lambda^2$. Then $v_\Delta$ converge to $c$ in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ as $\Delta x \to 0$. 
In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

$$(H) \quad \partial_a (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0, \quad \partial_b (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0;$$

**Lemma (\textit{BV}\textsubscript{loc} bound, Bürger, García, Karlsen, Towers)**

Let $T > 0$ and $A > 0$. Assume that $u_0 \in \text{BV}(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption $(H)$, we have

$$\|u(\cdot, \cdot)\|_{\text{BV}([0, T] \times \mathbb{R} \setminus (-A, A))} \leq \frac{1}{A} \text{Const}(T, \|u_0\|_{L^\infty}, \|u_0\|_{\text{BV}(\mathbb{R})}, \lambda).$$

– estimate in $\text{BV}(0, T; L^1(\mathbb{R}))$ (i.e., time $\text{BV}$ estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + $(H)$)
– mean-value theorem: for some $r \in (0, A)$, $\|u(\cdot, \pm r)\|_{\text{BV}(0, T)} \leq \text{const}/A$
– look at our solution as solution to a Cauchy-Dirichlet problem with $\text{BV}$ initial and boundary data.

**Proposition (approximate Kato inequality)**

Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in \mathcal{D}([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

$$- \int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u_\Delta - v_\Delta)^+ \partial_t \varphi + \Phi^+(u_\Delta, v_\Delta) \partial_x \varphi) \leq \text{Rem}(\Delta x, \varphi).$$

Arguments: monotonicity of the scheme, consistency, the $\text{BV}\textsubscript{loc}$ in space bound
...Properties of the scheme...

In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

(H) $\partial_a(\partial_ag(a, b) + \partial_bg(a, b)) \geq 0, \quad \partial_b(\partial_ag(a, b) + \partial_bg(a, b)) \geq 0$;

**Lemma ( $BV_{loc}$ bound, Bürger, García, Karlsen, Towers )**

Let $T > 0$ and $A > 0$. Assume that $u_0 \in BV(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption (H), we have

$$
\| u_\Delta(\cdot, \cdot) \|_{BV([0, T] \times \mathbb{R} \setminus (-A, A))} \leq \frac{1}{A} Const(T, \| u_0 \|_{L^\infty}, \| u_0 \|_{BV(\mathbb{R})}, \lambda).
$$

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– mean-value theorem: for some $r \in (0, A)$, $\| u(\cdot, \pm r) \|_{BV(0, T)} \leq const/A$
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Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in D([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

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$$

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**Proposition (approximate Kato inequality)**

Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in \mathcal{D}([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

$$- \int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u_\Delta - v_\Delta)^+ \partial_t \varphi + \Phi^+(u_\Delta, v_\Delta) \partial_x \varphi) \leq \operatorname{Rem}(\Delta x, \varphi).$$

Arguments: monotonicity of the scheme, consistency, the $BV_{\text{loc}}$ in space bound
In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

$$(H) \quad \partial_a (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0, \quad \partial_b (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0;$$

**Lemma ( $BV_{loc}$ bound, Bürger, García, Karlsen, Towers )**

Let $T > 0$ and $A > 0$. Assume that $u_0 \in BV(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption $$(H),$$ we have

$$\|u_\Delta (\cdot, \cdot)\|_{BV([0, T] \times \mathbb{R} \setminus (-A, A))} \leq \frac{1}{A} \text{Const}(T, \|u_0\|_{L^\infty}, \|u_0\|_{BV(\mathbb{R})}, \lambda).$$

- estimate in $BV(0, T; L^1(\mathbb{R}))$ (i.e., time $BV$ estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + $$(H))$
- mean-value theorem: for some $r \in (0, A)$, $\|u(\cdot, \pm r)\|_{BV(0, T)} \leq \text{const}/A$
- look at our solution as solution to a Cauchy-Dirichlet problem with $BV$ initial and boundary data.

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Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in \mathcal{D}([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

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Arguments: monotonicity of the scheme, consistency, the $BV_{loc}$ in space bound
In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

(H) $\partial_a(\partial_a g(a, b) + \partial_b g(a, b)) \geq 0$, $\partial_b(\partial_a g(a, b) + \partial_b g(a, b)) \geq 0$;

Lemma (BV$_{\text{loc}}$ bound, Bürger, García, Karlsen, Towers)

Let $T > 0$ and $A > 0$. Assume that $u_0 \in BV(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption (H), we have

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Let $u_{\Delta}, v_{\Delta}$ be solutions of the scheme. Let $\varphi \in D([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

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Arguments: monotonicity of the scheme, consistency, the BV$_{\text{loc}}$ in space bound.
...Properties of the scheme...

In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

\( (H) \quad \partial_a (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0, \quad \partial_b (\partial_a g(a, b) + \partial_b g(a, b)) \geq 0; \)

**Lemma (BV\textsubscript{loc} bound, Bürger, García, Karlsen, Towers)**

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\[
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- estimate in $BV(0, T; L^1(\mathbb{R}))$ (i.e., time $BV$ estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + $(H)$)
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Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in D([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

\[ - \int_{\mathbb{R}^+} \int_{\mathbb{R}} (u_\Delta - v_\Delta)^+ \partial_t \varphi + \Phi^+(u_\Delta, v_\Delta) \partial_x \varphi \right) \leq Rem(\Delta x, \varphi). \]

Arguments: monotonicity of the scheme, consistency, the $BV_{\text{loc}}$ in space bound
Theorem (convergence of the scheme; existence of solutions)

Assume $u_0 \in L^\infty(\mathbb{R})$. Then, under the CFL condition and assumption $(H)$, the numerical scheme converges in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ to the unique entropy solution to “Burgers with particle-at-zero” problem when $\Delta x$ tends to 0.

In particular, the problem is well-posed, for $L^\infty$ data and $L^1_{\text{loc}}$ topology.

Proof.

- First assume that $u_0 \in BV(\mathbb{R})$.
  - $BV_{\text{loc}}$ bounds yield compactness: we get $u$ an accumulation point of $(u_\Delta)_\Delta$;
  - well-balance property for $(c_-, c_+) \in G^1_\lambda \cup G^2_\lambda$ yields enough explicit stationary solutions $v_\Delta$ to the scheme (at least, at the limit $\Delta x \to 0$);
  - using the approximate Kato inequalities on $u_\Delta$ and the above special solutions $v_\Delta$, at the limit we get Kato inequalities... but, these are precisely the adapted entropy inequalities!!
  - then $u$ is (the unique) entropy solution (use caract. A. of entropy sols).

- For the general case $u_0 \in L^\infty(\mathbb{R})$, localize using finite speed of propagation; approximate $u_0$ by $BV(\mathbb{R}) \cap L^1(\mathbb{R})$ functions $(u^n_0)_n$. Use discrete $L^1$ contraction and the result of the $BV$ case.
**Theorem (convergence of the scheme; existence of solutions)**

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (\( H \)), the numerical scheme converges in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

In particular, the problem is well-posed, for \( L^\infty \) data and \( L^1_{loc} \) topology.

**Proof.**

- First assume that \( u_0 \in BV(\mathbb{R}) \).
  - \( BV_{loc} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_\Delta)_\Delta \);
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- For the general case \( u_0 \in L^\infty(\mathbb{R}) \), localize using finite speed of propagation; approximate \( u_0 \) by \( BV(\mathbb{R}) \cap L^1(\mathbb{R}) \) functions \( (u_0^n)_n \). Use discrete \( L^1 \) contraction and the result of the \( BV \) case.
Convergence; existence of entropy solutions.

**Theorem (convergence of the scheme; existence of solutions)**

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (H), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

In particular, the problem is well-posed, for \( L^\infty \) data and \( L^1_{\text{loc}} \) topology.

**Proof.**

- First assume that \( u_0 \in BV(\mathbb{R}) \).
  - \( BV_{\text{loc}} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_{\Delta})_{\Delta} \);
  - well-balance property for \( (c_-, c_+) \in G^1_\lambda \cup G^2_\lambda \) yields enough explicit stationary solutions \( v_{\Delta} \) to the scheme (at least, at the limit \( \Delta x \to 0 \));
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- For the general case \( u_0 \in L^\infty(\mathbb{R}) \), localize using finite speed of propagation; approximate \( u_0 \) by \( BV(\mathbb{R}) \cap L^1(\mathbb{R}) \) functions \( (u_{0n})_n \) Use discrete \( L^1 \) contraction and the result of the \( BV \) case.
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Proof.

• First assume that \( u_0 \in BV(\mathbb{R}) \).
  – \( BV_{loc} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_\Delta)_\Delta \);
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The coupled problem

Convergence; existence of entropy solutions.

**Theorem (convergence of the scheme; existence of solutions)**

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (H), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

In particular, the problem is well-posed, for \( L^\infty \) data and \( L^1_{\text{loc}} \) topology.

**Proof.**

- First assume that \( u_0 \in BV(\mathbb{R}) \).
  - \( BV_{\text{loc}} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_\Delta)_\Delta \);
  - well-balance property for \((c_-, c_+) \in G^1_\lambda \cup G^2_\lambda \) yields enough explicit stationary solutions \( v_\Delta \) to the scheme (at least, at the limit \( \Delta x \to 0 \));
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Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (H), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

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Convergence; existence of entropy solutions.

Theorem (convergence of the scheme; existence of solutions)

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption \( (H) \), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

In particular, the problem is well-posed, for \( L^\infty \) data and \( L^1_{\text{loc}} \) topology.

Proof.

- First assume that \( u_0 \in BV(\mathbb{R}) \).
  - \( BV_{\text{loc}} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_\Delta)_\Delta \);
  - well-balance property for \( (c_-, c_+) \in G^1_\lambda \cup G^2_\lambda \) yields enough explicit stationary solutions \( \nu_\Delta \) to the scheme (at least, at the limit \( \Delta x \to 0 \));
  - using the approximate Kato inequalities on \( u_\Delta \) and the above special solutions \( \nu_\Delta \), at the limit we get Kato inequalities...
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Coupled Problem
Case $h = h(t)$: existence, uniqueness

Theorem (well-posedness for moving but decoupled particle)

Given $h(\cdot)$ a $C^1$ path, there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda (u - h'(t))\delta_0(x - h(t))$; (localized) $L^1$ contraction property holds.

Definition:

- the germ $G_\lambda$ changes into $(h'(t), h'(t)) + G_\lambda$;
- versions B. ("with traces") and D. ("adapted entropy inequalities with remainder term") permit to define entropy solutions.

Arguments:

- for uniqueness, comparison, $L^1$ contraction: same technique;
- for existence: use characterization D. (it is stable by passage to the limit!);
- approximate $h(\cdot)$ by a family $(h_n)_n$ of piecewise affine paths;
- construction of solutions for "particle at $h_n$" is straightforward: $h'_n$ being piecewise constant, one changes variables to reduce to the "drag force-at-zero" case. Procedure restarted at each time where $h'_n$ jumps.
- because $h'_n \rightarrow h'$, the associated germs converge; thus we pass to the limit in characterization D.
Case $h = h(t)$: existence, uniqueness

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Theorem (well-posedness for moving but decoupled particle)

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Case $h = h(t)$: existence, uniqueness

Theorem (well-posedness for moving but decoupled particle)

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Case $h = h(t)$: existence, uniqueness

**Theorem (well-posedness for moving but decoupled particle)**

*Given* $h(\cdot)$ *a* $C^1$ *path*, there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda (u - h'(t))\delta_0(x - h(t))$; (localized) $L^1$ contraction property holds.

**Definition**:

– the germ $\mathcal{G}_\lambda$ changes into $(h'(t), h'(t)) + \mathcal{G}_\lambda$;
– versions **B.** (“with traces”) and **D.** (“adapted entropy inequalities with remainder term”) permit to define entropy solutions.

**Arguments**:

– for uniqueness, comparison, $L^1$ contraction: same technique;
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– construction of solutions for “particle at $h_n$” is straightforward: $h'_n$ being piecewise constant, one changes variables to reduce to the “drag force-at-zero” case. Procedure restarted at each time where $h'_n$ jumps.
– because $h'_n \to h'$, the associated germs converge; thus we pass to the limit in characterization **D.**
Case $h = h(t)$: continuous dependence on $h(\cdot)$

Uniqueness proof for the coupled problem relies on a Gronwall inequality, which in turn relies on a Lipschitz dependence estimate for the map $h(\cdot) \mapsto u(\cdot, \cdot)$.

**Theorem (dependence of $u$ on the path $h(\cdot)$)**

Assume $u, \hat{u}$ are entropy solutions corresponding to the particles located at $h(\cdot), \hat{h}(\cdot)$, respectively, with $h(0) = \hat{h}(0) = 0$ and same initial datum $u_0$. Assume $\hat{u} \in L^\infty(0, T; BV(\mathbb{R}))$. Then for a.e. $t \in (0, T)$,

$$
\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^1(\mathbb{R})} \leq C(\|u\|_\infty, \|\hat{u}\|_\infty, \|\hat{u}\|_{BV, \lambda}) \int_0^t |h'(s) - \hat{h}'(s)| \, ds.
$$

Arguments:

– change of variables $y = x - h(t)$, resp. $x - h'(t)$. Two eqns, both with singularity at zero, come out, with different fluxes of the kind $u \mapsto \frac{u^2}{2} - h'(t)u$.
– use the techniques of dependence of entropy solutions on the flux function ($BV$ regularity needed!): Kuznetsov, Bouchut-Perthame, Karlsen-Risebro...: the $C^1$ norm of the difference of the fluxes pops up, which yields $|h' - \hat{h}'|$.
– use Lipschitz dependence of the germ on $h'$ to describe additional (small) “non-dissipation” term coming from the interface.
Case $h = h(t)$: continuous dependence en $h(\cdot)$

Uniqueness proof for the coupled problem relies on a Gronwall inequality, which in turn relies on a Lipschitz dependence estimate for the map $h(\cdot) \mapsto u(\cdot, \cdot)$.

**Theorem (dependence of $u$ on the path $h(\cdot)$)**

Assume $u, \hat{u}$ are entropy solutions corresponding to the particles located at $h(\cdot), \hat{h}(\cdot)$, respectively, with $h(0) = 0 = \hat{h}(0)$ and same initial datum $u_0$. Assume $\hat{u} \in L^\infty(0, T; BV(\mathbb{R}))$. Then for a.e. $t \in (0, T)$,

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Arguments:

- change of variables $y = x - h(t)$, resp. $x - h'(t)$. Two eqns, both with singularity at zero, come out, with different fluxes of the kind $u \mapsto \frac{u^2}{2} - h'(t)u$.
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Case $h = h(t)$: continuous dependence on $h(\cdot)$, $L^\infty$ and BV stability

**Proposition (BV estimate)**

The solution constructed for the $h = 0$ case obeys

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \downarrow \text{ for all } t > 0$$

(at $t = 0$ the variation may increase by a const. depending on $\|u_0\|_\infty, G_\lambda$).

The solution constructed for the fixed-$h(\cdot)$ case obeys the BV estimate

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})} + \text{const}(\lambda, \|u_0\|_\infty) + 2 \int_0^t |h''(s)| \, ds.$$  

Argument: (re)-construct solutions by wave-front tracking algorithm (Dafermos, Holden-Risebro, Bressan et al.) (better control of interactions).

**Lemma (L$^\infty$ bounds)**

We get a uniform $L^\infty$ bound on ad hoc sequences of $h'(\cdot)$ and $u(\cdot, \cdot)$.

To be precise: if we look at solutions to the coupled problem, we get

$$\max\{\|u\|_\infty, \|h'\|_\infty\} \leq \max\{\|u_0\|_\infty, |h'(0)|\}.$$  

For solutions appearing in the fixed-point or splitting arguments, we get somewhat weaker bounds.
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Case of $u$ frozen: evolving $h = h(\cdot)$

**Proposition (modelling/“traces” interpretation of the ODE on $h(\cdot)$ )**

*For every drag force, the ODE in the coupled problem writes*

$$mh''(t) = \left(\left(\frac{u_-}{2} - h'(t)u_\cdot\right) - \left(\frac{u_+}{2} - h'(t)u_\cdot\right)\right).$$

Notice that the right-hand side above is expressed as the difference of the normal components of the 2D-field $(u, u^2/2)$ on the curve $\{x = h(t)\}$ from the left and from the right. Combining this observation with the Green-Gauss formula, we get the following weak formulation of the ODE:

**Lemma (second interpretation of the ODE on $h(\cdot)$ )**

*Let $u$ be a weak solution of the PDE on $\{x \neq h(t)\}$; let $h \in W^{2,\infty}(0, T)$. Then $h(\cdot)$ verifies the ODE if and only if for all $\xi \in D([0, T]),$ for all $\psi \in D(\mathbb{R})$ such that $\psi \equiv 1$ on the set $\{x \in \mathbb{R} : \exists t \in [0, T] \text{ such that } h(t) = x\},$ there holds*

$$-m \int_0^T h'(t)\xi'(t)dt = mh'(0)\xi(0) + \int_0^T \int_{\mathbb{R}} \left[u_\psi \xi_t + \frac{u^2}{2} \xi \psi_x\right] + \int_{\mathbb{R}} u_0 \psi \xi(0).$$
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Coupled problem: existence, uniqueness of $BV$ solutions / existence of $L^\infty$ solutions

The above ingredients can be used in several ways:

- **In a fixed-point argument** $h(\cdot) \mapsto u(\cdot, \cdot) \mapsto h(\cdot)$
  (compactness: work in $C^1([0, T])$, exploit a $W^{2,\infty}(0, T)$ bound on $h(\cdot)$)

- **In a time splitting algorithm** (alternatively evolving $u$ and $h$ on small time intervals):
  - $u$ updated from $h$ using the theory of entropy solutions for $h$ frozen;
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- **In a numerical scheme** (same time splitting + approximation in space of the conservation law); an interesting possibility is the random-choice algorithm (Glimm), in order not to adapt the space meshing to the particle location.

**Theorem (Main result)**

For all BV datum $u_0$ and given $h(0), h'(0)$, there exists a unique entropy solution to the coupled problem.

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**Coupled problem: a well-balanced random-choice numerical scheme**

**Figure:** Representation of the algorithm based on the well-balanced scheme.
**Model and motivation**

**Auxiliary steps**

- $h = 0$: coupling
- $h = 0$: definition, uniqueness
- $h = 0$: numerics, existence

**Results**

The coupled problem

**Numerics: drafting-kissing-tumbling**

**Figure:** Trajectories of two particles

![Figure: Trajectories of two particles](image-url)
THANK YOU!