Degenerate nonlinear parabolic-hyperbolic equations and their finite volume approximation

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based on joint work with
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Plan of the talk

INTRODUCTION TO DEGENERATE NONLINEAR CONVECTION-DIFFUSION PROBLEMS AND THEIR FINITE VOLUME APPROXIMATION

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \mathrm{div}\,[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \ \text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian, i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T) \times \partial\Omega$.

Theory:
Alt, Luckhaus ’83; Otto ’96; Bénilan, Wittbold ’96 and ‘96; Carrillo ’99; Ammar, Wittbold ’03; Andr., Bendahmane, Karlsen, Ouaro ’09

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Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of "approximate solutions" $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in "simpler" problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In "harder" problems, one has to treat simultaneously Steps 2+3 : "compensated compactness", entropy-process solutions...

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Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples : Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen... DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov... gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle) + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

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Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

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THEORETICAL FOUNDATIONS

Theoretical framework for elliptic-parabolic-hyperbolic problems...

Theoretical setting : entropy solutions + Leray-Lions framework. Key ideas:
Leray & Lions '65 – Alt & Luckhaus '83 ; Kruzhkov '69 – Carrillo '99

NB: Parallel theories and generalizations, not discussed here :

– semigroup solutions : Crandall, Bénilan, Carrillo & Wittbold
– kinetic solutions (quasilinear diffusion !) : Perthame, Chen & Perthame
– renormalized solutions : Murat & Lions, Carrillo & Wittbold, Ammar & Wittbold, Blanchard & Porretta, Bendahmane & Karlsen
– entropy (Bénilan et al.) solutions : Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez, Andreu-Vaillo & Igbida & Mazón & Toledo .

Nice features of the solution theory:

- well-posedness for L^∞ data u_0
- order-preservation : $u_0 \leqslant \hat{u}_0$ and $f \leqslant \hat{f}$ implies $u(t,\cdot) \leqslant \hat{u}(t,\cdot)$
- consequently, maximum principle : $\sup u(t,\cdot) \leqslant \sup u_0^+ + \int_0^t \sup f^+(\tau,\cdot)d\tau$
- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1}d\tau$.
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

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- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1}d\tau$.
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

Theoretical framework for parabolic-hyperbolic problems...

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Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.
An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

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Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{q}_c^{\,\pm})'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta'(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

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– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

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– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{\mathfrak{a}}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.

An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0,T;W_0^{1,p}(\Omega))$;
- for all pairs $(c,\psi) \in \mathbb{R}^\pm \times \mathcal{D}([0,T)\times\overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c,\psi) \in \mathbb{R} \times \mathcal{D}([0,T)\times\Omega)$, $\psi \geqslant 0$,

$$\int_0^T\!\!\int_\Omega \Big(\eta_c^\pm(u)\partial_t\psi + \vec{\mathfrak{q}}_c^\pm(u)\cdot\nabla\psi - k(\nabla w)\,\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot\nabla\psi\Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{\mathfrak{q}}_c^\pm)'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta'(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

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MESHES, DISCRETE OPERATORS AND THE SCHEME

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell .

Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\, \vec{a}_0(\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}} \vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}} \cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

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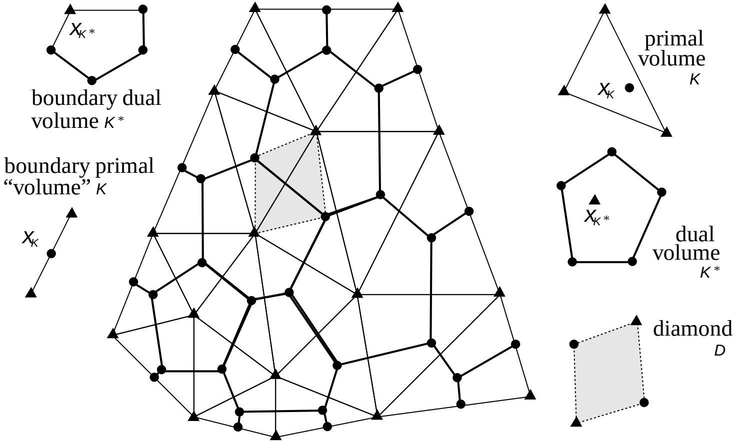
Finite volume meshes and operators...

Let us describe one meshing+operators strategy: “Discrete Duality FV”. The 2D idea is due to Hermeline and to Domelevo & Omnès . One starts with a usual mesh (called “primal”) and uses both center and vertex unknowns.



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The space of discrete functions $w^{\mathfrak{T}} = \big((w_K)_K \,;\, (w_{K^*})_{K^*} \big)$ is denoted by $\mathbb{R}^{\mathfrak{T}}$, for functions zero on the boundary we use $\mathbb{R}_0^{\mathfrak{T}}$.
The set of discrete fields $(\vec{\mathcal{F}}_D)_D$ is denoted $(\mathbb{R}^d)^{\mathfrak{D}}$.

On spaces $\mathbb{R}^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$, we introduce scalar products

$$[\![w^{\mathfrak{T}}, v^{\mathfrak{T}}]\!] = \frac{1}{d} \sum_{K \in \mathfrak{M}} m_K \, w_K v_K + \frac{d-1}{d} \sum_{K^* \in \mathfrak{M}^*} m_{K^*} \, w_{K^*} v_{K^*}$$

and

$$\Big\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \Big\} = \sum_{D \in \mathfrak{D}} m_D \, \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D;$$

The discrete divergence operator is the usual Finite Volumes' one: we apply the Green-Gauss formula in each primal cell K and in each dual cell K^*:

$$\operatorname{div}^{\mathfrak{T}} : (R^d)^{\mathfrak{D}} \longrightarrow R^{\mathfrak{T}}, \quad \text{with e.g. } (\operatorname{div}^{\mathfrak{T}})_K \vec{\mathcal{F}} := \sum_{D \in \mathfrak{D}} \int_{\partial K \cap D} \vec{\mathcal{F}}_D \cdot \nu_K .$$

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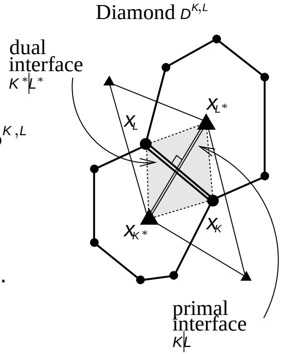

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We follow the last one, called “3D CeVe-DDFV”.
The DDFV schemes enjoy discrete duality:

Proposition (discrete duality)

For $v^{\mathfrak{T}} \in \mathbb{R}_0^{\mathfrak{T}}$ and $\vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^d)^{\mathfrak{D}}$, $\left[\!\left[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\, v^{\mathfrak{T}} \right]\!\right] = \left\{\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}},\, \nabla^{\mathfrak{T}} v^{\mathfrak{T}} \right\}\!\right\}.$

With this property, all “variational” techniques can be used at the discrete level ! The discretization of the Leray-Lions diffusion operator $-\operatorname{div}\vec{a}_0(\nabla\cdot)$ by $-\operatorname{div}^{\mathfrak{T}}\vec{a}_0(\nabla^{\mathfrak{T}}\cdot)$ preserves the key features of the continuous operator: coercivity, monotonicity, growth, existence of a potential...

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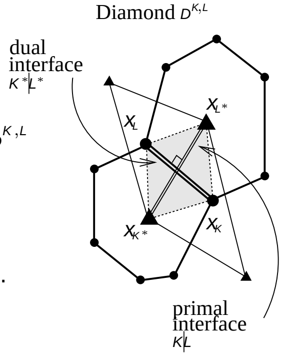

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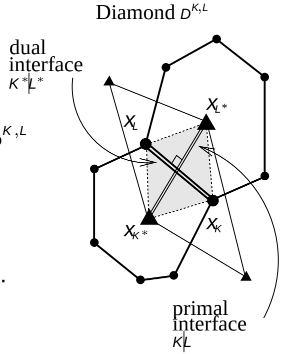

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Finite volume scheme for the problem...

In addition, we take a usual two-point monotone consistent flux approximation to produce a discrete operator $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$ which approximates the convection operator $\mathrm{div}\,\vec{F}(\cdot)$.

With the notation introduced above, our discretization writes:

find a discrete function $u^{\mathfrak{T},\Delta t}$ satisfying for $n = 1, \ldots, N = T/\Delta t$ the equations

$$\left|\begin{array}{l} \dfrac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} + (\mathrm{div}_c\vec{F})^{\mathfrak{T}}[u^{\mathfrak{T},n}] - \mathrm{div}^{\mathfrak{T}}[\vec{a}_0(\nabla^{\mathfrak{T}} w^{\mathfrak{T},n})] = 0, \\ w^{\mathfrak{T},n} = A(u^{\mathfrak{T},n}), \end{array}\right.$$

together with the boundary and initial conditions

$$\text{for all } n = 1, \ldots, N, \qquad \begin{cases} u_K^n = 0 & \text{for all } \kappa \text{ near } \partial\Omega \\ u_{K^*}^n = 0 & \text{for all } \kappa^* \text{ near } \partial\Omega; \end{cases}$$

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DISCRETE CALCULUS TOOLS AND CONVERGENCE ANALYSIS

Discrete calculus tools...

Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

"Variational" arguments: take $w^{\mathfrak{T}}$ for test function, get

- Energy estimates
  ($\Longrightarrow$ Existence + Weak L^p compactness for gradients $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$
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Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

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- Discrete entropy inequalities
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We already know that the discrete convection operator with monotone flux leads to discrete entropy inequalities with remainder terms controlled by the "weak BV" estimate , Eymard, Gallouët, Herbin .

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Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
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Thank you — Merci — Danke

Merci !