Attractors of reaction-diffusion systems via a preconditioning technique

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based on joint work with
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Plan of the talk

CONCRETE EXAMPLES AND GOAL OF THE STUDY

Concrete examples to be treated

The motivation comes from the concrete example of 3×3 "facilitated diffusion" system with different boundary conditions (BC) modelling the blood oxigenation reaction $Hb + O_2 \rightleftharpoons HbO_2$, or of 5×5 system modelling coupled reactions $Hb + O_2 \rightleftharpoons HBO_2$, $Hb + CO_2 \rightleftharpoons HbCO_2$.

Let Ω be a bounded, smooth enough domain of $\mathbb{R}^n$. Consider

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = u_3 - u_1 u_2 \\ \partial_t u_2 - d_2 \Delta u_2 = u_3 - u_1 u_2 \\ \partial_t u_3 - d_3 \Delta u_3 = u_1 u_2 - u_3, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -K_1 u_1 u_5 + K_2 u_2 \\ \partial_t u_2 - d_2 \Delta u_2 = K_1 u_1 u_5 - K_2 u_2 \\ \partial_t u_3 - d_3 \Delta u_3 = -K_3 u_3 u_5 + K_4 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = K_3 u_3 u_5 - K_4 u_4 \\ \partial_t u_5 - d_5 \Delta u_5 = (-K_1 u_1 u_5 + K_2 u_2) + (-K_3 u_3 u_5 + K_4 u_4) \end{cases} \tag{2}$$

with non-homogeneous BC of the following general form:

$$\lambda_i \partial_n u_i + (1 - \lambda_i) u_i = \alpha_i \quad \text{on } \partial\Omega, \quad \alpha_i \geqslant 0, \quad i = 1..3 \text{ or } i = 1..5. \tag{3}$$

Here $0 \leqslant \lambda_i \leqslant 1$, and: $\lambda_i \neq \lambda$ for all i is a difficulty.

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Issues of interest. Estimates and (E.A.T) estimates.

- local in time existence of solutions : standard, one writes a fixed-point formulation in L^∞ from the Duhamel formula (solution := mild solution $\equiv$ strong solution in L^2 $\equiv$ ...)
- global existence requires uniform L^∞ bounds:

$$\|U(t)\|_\infty \leqslant \Psi(\|U_0\|_\infty)$$

where Ψ denotes a generic non-decreasing function on $\mathbb{R}^+$

- existence of a maximal attractor in L^∞ relies upon:

— compactness of the linear semigroups $e^{-td_i\Delta}$

— a bounded absorbing set that is obtained via "estimates of attractor type" (E.A.T.) in L^∞:

$$\|U(t)\|_\infty \leqslant \Phi(\|U_0\|_\infty\, ,\, t)$$

where for all t, $\Psi(\cdot, t)$ is non-decreasing and

$$\sup_{r>0} \lim_{t\to\infty} \Phi(r,t) \leqslant const\ .$$

NB: without loss of generality, $\Phi(r, \cdot)$ can be assumed non-increasing.

Additional dependence is in subscripts. With this notation, we have e.g.

$\sup_{t\in\mathbb{R}} \Phi(r,t) = \Psi(r)$, $\qquad C + e^{-\delta t}\Psi_p(r) + \Phi(r,t) = \Phi_{\delta,p}(r,t)$,

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MAIN RESULTS. ABSTRACT FRAMEWORK AND ASSUMPTIONS

Results for the concrete systems above...

Assume the following :

- $U^0 = (u_1^0, u_2^0, u_3^0)$ in $\left((L^\infty)^+\right)^3$
- $\alpha_i \in L^\infty(\partial\Omega)$ are $\geqslant 0$ and belong to the *ad hoc* trace space
- Assume one of the three following situations occurs:

either $\lambda_i \in (0,1)$, $i = 1..3$, or $\lambda_1 = \lambda_2 = \lambda_3 = 0$,
or $\lambda_i \in [0,1)$ with $\alpha_i = 0$ for i such that $\lambda_i = 0$.

[$\lambda_i = 0$: Dirichlet ; $\lambda_i = 1$: Neumann ; $0 < \lambda_i < 1$: Robin]

Results :

- global existence of solutions in L^∞ ($\Rightarrow$ there exists a nonlinear semigroup $\{S(t)\}_{t\geqslant 0}$ on $((L^\infty(\Omega))^+)^3$ of solutions of the system)
- existence of a maximal attractor in L^∞ :
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$$\forall\, r > 0 \ \lim_{t\to\infty} \sup_{U^0 \in ((L^\infty(\Omega))^+)^3,\, \|U^0\|_\infty \leqslant r} \mathrm{dist}(S(t)U^0, \mathcal{M}) = 0.$$

- if Neumann BC also allowed : only global existence is proved.

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Results for the concrete systems above...

Assume the following :

- $U^0 = (u_1^0, u_2^0, u_3^0)$ in $\left((L^\infty)^+\right)^3$
- $\alpha_i \in L^\infty(\partial\Omega)$ are $\geqslant 0$ and belong to the *ad hoc* trace space
- Assume one of the three following situations occurs:

either $\lambda_i \in (0,1),\ i = 1..3$, or $\lambda_1 = \lambda_2 = \lambda_3 = 0$,
or $\lambda_i \in [0,1)$ with $\alpha_i = 0$ for i such that $\lambda_i = 0$.

[$\lambda_i = 0$: Dirichlet ; $\lambda_i = 1$: Neumann ; $0 < \lambda_i < 1$: Robin]

Results :

- global existence of solutions in L^∞ ($\Rightarrow$ there exists a nonlinear semigroup $\{S(t)\}_{t\geqslant 0}$ on $((L^\infty(\Omega))^+)^3$ of solutions of the system)
- existence of a maximal attractor in L^∞ :
  $\exists\mathcal{M}$ a compact set that is invariant for the semigroup $\{S(t)\}_{t\geqslant 0}$ on $((L^\infty(\Omega))^+)^3$ and satisfies

$$\forall\, r > 0\ \lim_{t\to\infty} \sup_{U^0\in((L^\infty(\Omega))^+)^3,\ \|U^0\|_\infty\leqslant r} \operatorname{dist}\left(S(t)U^0, \mathcal{M}\right) = 0.$$

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Abstract framework and assumptions...

Following Bénilan and Labani, we recast the 3×3 problem under the abstract form :

$$(S) \qquad \begin{cases} \frac{d}{dt} u_i + A_i(u_i - \bar{\alpha}_i) = f_i(u_1, u_2, u_3), \\ u_i(0) = u_i^0, \quad i = 1..3, \end{cases}$$

where for $i = 1..3$,

- $(-A_i)$ is the infinitesimal generator of an analytic exponentially stable semigroup of positive linear operators e^{-tA_i} on $L^2(\Omega)$;
- we assume that these semigroups are L^p-nonexpansive ;
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Further, in (S) we assume

$$\bar{\alpha}_i \in (L^\infty(\Omega))^+ \text{ with } e^{-tA_i}\bar{\alpha}_i \leqslant \bar{\alpha}_i, \quad i = 1..3$$

To get from the concrete system to (S) one takes for $\bar{\alpha}_i$ the solution of the appropriately defined elliptic problem with BC given by α_i :

$$\begin{cases} -d_i \Delta \bar{\alpha}_i = 0 & \text{in } \Omega \\ \lambda_i \partial_n \bar{\alpha}_i + (1 - \lambda_i)\bar{\alpha}_i = \alpha_i & \text{on } \partial\Omega. \end{cases}$$

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["preservation of the positive cone"]

$$f_1(0, u_2, u_3) \geqslant 0, \quad f_2(u_1, 0, u_3) \geqslant 0, \quad f_3(u_1, u_2, 0) \geqslant 0; \tag{4}$$

["some amount of compensation"]

$$f_1(u_1, u_2, u_3) + f_3(u_1, u_2, u_3) \leqslant 0; \tag{5}$$

["linear growth in u_3 for «good components» u_1 and u_2"]

$$\exists a \geqslant 0 \quad f_1(u_1, u_2, u_3) \leqslant a(1 + u_3), \quad f_2(u_1, u_2, u_3) \leqslant a(1 + u_3); \tag{6}$$

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$$\exists b \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0 \quad f_3(u_1, u_2, u_3) \leqslant b(1 + u_1^\beta + u_2^\gamma). \tag{7}$$

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Preconditioning operator...

Finally, assume there exists a "preconditioning operator" B on $L^2(\Omega)$ satisfying the same requirements as those imposed on A_i, $i = 1..3$

(infinitesimal generator of a positive, analytic, exponentially stable semigroup on L^2, non-expansive in all L^p spaces, hypercontractive) ; and such that, for $A = A_i$, $i = 1..3$, the two properties hold:

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$$\begin{cases} (I - B^{-1}A) \leqslant 0 \quad \text{in the sense that} \\ \text{for all } u \in D(A) \cap L^\infty(\Omega),\ u \geqslant 0,\ \text{one has } u \leqslant B^{-1}A\, u \end{cases}$$

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Idea of the preconditioning:

roughly speaking, the study of solutions of $\left(\frac{d}{dt}+A\right)(u-\bar{\alpha})=f$ is reduced to the study of solutions to

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If $\bar{\alpha}_i=0$, the negativity of $(I-B^{-1}A)$ permits to upper bound the rhs. In general, also some bound for control of $B^{-1}A\,(u-\bar{\alpha})^-$ is needed.

In practice, when does a preconditioner operator exist ?

Proposition

Let A be the operator associated with $-d\Delta$ on Ω with the BC $\lambda\partial_n u+(1-\lambda)u=0$ on $\partial\Omega$ with parameter $\lambda\in[0,1)$.
Let B be another such operator with parameters e and μ. Then

- $(I-B^{-1}A)\leqslant 0$ *if* $0<e\leqslant d$ *and* $\lambda\leqslant\mu<1$
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Proof: maximum principle for (i); duality + Calderón-Zygmund for (ii).

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MAIN INGREDIENTS

Properties and arguments used in the proof...

Definition

We say that A is an operator of class $\mathcal{A}$ if the following holds:

- $-A$ is the inf. generator of an analytic semigroup e^{-tA} on $L^2(\Omega)$
- the semigr. e^{-tA} is positive , in the sense $e^{-tA}u \geqslant 0$ for $u \geqslant 0$;
- e^{-tA} is non-expansive on all spaces $L^p(\Omega)$, i.e., for all $t > 0$,

$$\forall p \in [1, +\infty] \quad \|e^{-tA}u\|_p \leqslant \|u\|_p \text{ for } u \in L^\infty(\Omega);$$

- e^{-tA} is exponentially stable on $L^2(\Omega)$, i.e. there exists $\omega > 0$ st

$$\text{for all } t > 0 \quad \|e^{-tA}\|_{\mathcal{L}(L^2)} \leqslant e^{-\omega t};$$

- e^{-tA} is hypercontractive , i.e., there exist $\sigma > 0$ and $c > 0$ such that

$$\|e^{-tA}u\|_{L^\infty} \leqslant \frac{c}{t^\sigma}\|u\|_{L^1}.$$

Remark: The different Laplace operators A_i in our concrete examples are of class $\mathcal{A}$, provided $\lambda_i < 1$, $i = 1..3$ (Dirichlet or Robin BC case) . If $\lambda_i = 1$ (Neumann BC case), then for all $c > 0$, $(A_i + cI)$ is of class $\mathcal{A}$.

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Finally, recall the following maximal regularity statement (Lamberton).

Theorem

Assume that $-A$ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$, non-expansive in L^p. Let $p \in (1, +\infty)$. Then the unique mild solution of the evolution problem

$$\frac{d}{dt}u + Au = f \in L^p_{loc}([0, +\infty) \times \Omega), \quad u(0) = 0$$

verifies the equation in the strong sense in $L^p(\Omega)$. Namely, $u \in W_0^{1,p}([0, \infty); L^p(\Omega))$, both $\frac{d}{dt}u$ and Au belong to $L^p(\Omega)$, and the equality holds in $L^p(\Omega)$ for a.e. $t > 0$.
Moreover, there exists $C_p > 0$ such that for all $T > 0$, the maximal regularity estimate holds:

$$\|\frac{d}{dt}u\|_{L^p([0,T]\times\Omega)} + \|Au\|_{L^p([0,T]\times\Omega)} \leqslant C_p \|f\|_{L^p([0,T]\times\Omega)}.$$

STEPS OF THE PROOF

Step 1...

The proof of global existence + (E.A.T) consists in four Steps.

Step 1 The following (E.A.T) hold:

$$\forall i,j = 1..3 \ \forall p \in [1,+\infty) \ \forall t < T_{max}$$
$$\|A_j^{-1} u_i(t)\|_{L^p(\Omega)} \leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t);$$

$$\forall i = 1..3 \ \forall \delta > 0 \ \forall t \geqslant \delta \ \forall \tau < T_{max}$$
$$\|e^{-tA_i} u_i(\tau)\|_{L^\infty(\Omega)} \leqslant \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau).$$

Principle: apply $(\frac{d}{dt} + B)$ to

$$w(t) := B^{-1}((u_1 - \bar{\alpha}_1) + (u_3 - \bar{\alpha}_3)).$$

Because (S) is verified in the strong sense for $t < T_{max}$, we get

$$\begin{cases} \frac{d}{dt} w + Bw = B^{-1}(f_1 + f_3) + \sum_{i=1,3} (I - B^{-1}A_i)(u_i - \bar{\alpha}_i) \\ w(0) = B^{-1}((u_1^0 - \bar{\alpha}_1) + (u_3^0 - \bar{\alpha}_3)) \end{cases}$$

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From the compensation assumption on $f_1 + f_3$ and the assumptions on the preconditioning operator B, we infer

$$\frac{d}{dt}w + Bw \leqslant g(t), \quad \|g(t)\|_{L^q(\Omega)} \leqslant C_q \text{ for all } q < +\infty,$$

(in the strong sense); by the Duhamel formula and the positivity of e^{-tB},

$$w(t) \leqslant e^{-tB}w(0) + \int_0^t e^{-(t-s)B}g(s)\,ds, \quad \|g(t)\|_{L^q(\Omega)} \leqslant C_q.$$

The first term admits an (E.A.T.) (boundedness in terms of $\|U^0\|_\infty$ and exponential decay to zero in all L^p, $p < \infty$).

For p large , we have $\frac{\sigma}{p} < 1$ and take $q = p/2$. Then

$$\begin{aligned}\|w^+(t)\|_{L^p(\Omega)} &\leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t) + C\int_0^t e^{\lambda_p(t-s)}(t-s)^{-\frac{\sigma}{p}}\|g(s)\|_{L^{p/2}(\Omega)}\,ds \\ &\leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t) + C_p\int_0^t e^{-\lambda_p z}z^{-\frac{\sigma}{p}}\,dz \\ &\leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t) + C_p.\end{aligned}$$

This yields the (E.A.T.) $\|w^+(t)\|_{L^p(\Omega)} \leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t)$.
The negative part is bounded by $B^{-1}(\bar{\alpha}_1 + \bar{\alpha}_2)$, whence the $(E.A.T.)$ estimate on $w(t)$ and then on $B^{-1}u_1(t)$ and $B^{-1}u_2(t)$ in L^p.

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This yields the (E.A.T.) $\|w^+(t)\|_{L^p(\Omega)} \leqslant \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t)$.
The negative part is bounded by $B^{-1}(\bar{\alpha}_1 + \bar{\alpha}_2)$, whence the $(E.A.T.)$ estimate on $w(t)$ and then on $B^{-1}u_1(t)$ and $B^{-1}u_2(t)$ in L^p.

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From the compensation assumption on $f_1 + f_3$ and the assumptions on the preconditioning operator B, we infer

$$\frac{d}{dt} w + Bw \leqslant g(t), \quad \|g(t)\|_{L^q(\Omega)} \leqslant C_q \text{ for all } q < +\infty,$$

(in the strong sense); by the Duhamel formula and the positivity of e^{-tB},

$$w(t) \leqslant e^{-tB} w(0) + \int_0^t e^{-(t-s)B} g(s)\, ds, \quad \|g(t)\|_{L^q(\Omega)} \leqslant C_q.$$

The first term admits an (E.A.T.) (boundedness in terms of $\|U^0\|_\infty$ and exponential decay to zero in all L^p, $p < \infty$).

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It remains to bound $B^{-1}u_2$ (then we deduce bounds on $A_j^{-1}u_2$ as above). We simply apply $(\frac{d}{dt} + B)$ to $B^{-1}(u_2 - \bar{\alpha}_2)$ and argue as above.

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Finally, we can estimate $e^{-tA_i}u_i(\tau)$ writing $(A_ie^{-tA_i})A_i^{-1}$:
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Step 2.

Step 2 The following (E.A.T) of «good components» hold:

$$\forall i=1,2\ \forall p\in[1,+\infty)\ \forall\delta>0\ \forall\tau<T_{max}-2\delta$$
$$\|u_i(\tau+\cdot)\|_{L^p((\delta,2\delta)\times\Omega)}\leqslant\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)},\ \tau).$$

Moreover, if $T_{max}>2\delta$, then

$$\forall i=1,2\ \forall p\in[1,+\infty)\ \forall\delta>0\ \forall t\leqslant 2\delta$$
$$\|u_i(\cdot)\|_{L^p((0,t)\times\Omega)}\leqslant\Psi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}).$$

Indeed, because we already have a bound on $e^{-tA_1}u_1(\tau)$, we can fix the initial time at τ (i.e., consider $u_1(\cdot+\tau)$), assume the initial datum to be zero, and write:

$$(\frac{d}{dt}+A_1)[A_1^{-1}u_1(\cdot+\tau)]=A_1^{-1}f_1(U(\cdot+\tau)\leqslant A_1^{-1}a(1+u_3)(\cdot+\tau).$$

(we applied A_1^{-1} to each term).The rhs is L^p bounded because of the previous bound on $A_1^{-1}u_3$; but then the maximum regularity yields an $L^p((\delta,2\delta)\times\Omega)$ bound on $A_1[A_1^{-1}u_1(\cdot+\tau)]\equiv u_1(\cdot+\tau)$.

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$$\forall i=1..3 \ \ \forall\delta>0 \ \ \forall\tau\in[2\delta, T_{max}) \quad \|u_i(\tau)\|_{L^\infty(\Omega)} \leqslant \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau).$$

We first estimate the «bad component» u_3; at the very end, with this estimate in hand one easily bounds u_1 and u_2.

Again, we can shift the initial moment (now at $t=\tau-\delta$) and drop the IC. Then

$$(\frac{d}{dt}+A_3)u_3(\cdot+\tau-\delta)=g_\tau, \quad \tilde{w}(0)=0,$$

with $g_\tau(\cdot):=f_3(U(\cdot+\tau-\delta)) \leqslant b\,(1+|u_1|^\beta(\cdot+\tau-\delta)+|u_2|^\gamma(\cdot+\tau-\delta))$.

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in this way, for p large enough (i.e., for $\sigma\frac{p'}{p}<1$) we get at $t=\delta$:

$$\begin{aligned}\|u_3(\delta+\tau-\delta)\|_{L^\infty(\Omega)} &\leqslant \int_0^\delta C_p(\delta-s)^{-\frac{\sigma}{p}}\|g_\tau^+(s)\|_{L^p(\Omega)}\,ds\\ &\leqslant C_p\left(\int_0^\delta((\delta-s)^{-\frac{\sigma}{p}})^{p'}\right)^{1/p'}\|g_\tau^+\|_{L^p((0,\delta)\times\Omega)}\\ &\leqslant C_{\delta,p}\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)},\tau-\delta)=\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)},\tau).\end{aligned}$$

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$$\begin{aligned}\|u_3(\delta+\tau-\delta)\|_{L^\infty(\Omega)} &\leqslant \int_0^\delta C_p(\delta-s)^{-\frac{\sigma}{p}}\|g_\tau^+(s)\|_{L^p(\Omega)}\,ds\\ &\leqslant C_p\left(\int_0^\delta((\delta-s)^{-\frac{\sigma}{p}})^{p'}\right)^{1/p'}\|g_\tau^+\|_{L^p((0,\delta)\times\Omega)}\\ &\leqslant C_{\delta,p}\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau-\delta)=\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau).\end{aligned}$$

Attractor.

Denote by $\mathcal{E}$ an absorbing set obtained from the (E.A.T).
By the general result (see Bénilan and Labani; cf. Temam), under the additional assumption of asymptotic compactness of the solution semigroup,

$$\mathcal{M} = \bigcap_{t \geqslant 0} \overline{\bigcup_{\delta > 0} S(t+\delta)\mathcal{E}} \qquad \text{is the maximal attractor in } L^\infty.$$

[maximal attractor = compact invariant set
that attracts the images of bounded sets as $t \to \infty$]

Indeed, it is not difficult to show that $\mathcal{M}$ is invariant for $S(t)$ and

$$\forall\, r > 0 \ \lim_{t\to\infty} \sup_{U^0 \in ((L^\infty(\Omega))^+)^3,\, \|U^0\|_\infty \leqslant r} \operatorname{dist}\big(S(t)U^0, \mathcal{M}\big) = 0.$$

By construction, $\mathcal{M}$ is bounded and closed. Because $S(t)\mathcal{M} = \mathcal{M}$, the compactness of $S(t)$ is enough to infer the compactness of $\mathcal{M}$.

Thus it remains to deduce compactness of the nonlinear semigroup $(S(t))_{t\geqslant 0}$ from the compactness of the linear semigroups e^{-tA_i}. The only delicate point is to bypass the continuity of the semigroup in L^∞: indeed, $\{S(t)\}_{t\geqslant 0}$ is not continuous in the topology $(L^\infty(\Omega))^3$, thus the L^2-continuity and the $L^2 - L^\infty$ regularizing effect are used instead.

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Oufff !!!

MERCI !