Small solids in an inviscid fluid: a 1D model, theory and numerics

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joint work with
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Plan of the talk

1. Model and motivation
2. Auxiliary steps
3. Main Results
4. The frozen particle case: coupling
5. The frozen particle case: definition, uniqueness
6. The frozen particle case: numerics and existence
7. The coupled problem
MODEL AND MOTIVATION
D’Alembert paradox: a solid immersed in an inviscid fluid is not submitted to any resultant force; in other words, birds (and planes...) could not fly with such a model!

Answer 1 to the d’Alembert paradox: use viscous models of fluid-solid interaction (see e.g. M. Hillairet, for a recent review).

Answer 2 (when the Reynolds number is large): it is reasonable to neglect the viscous effects in model that governs the fluid; but we have to conserve information from the vanishing viscosity in a DRAG FORCE. The drag force takes the form of a source term which takes into account the difference between the velocity of the fluid and the velocity of the solid.
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The 1D case: the model for the interaction, via a drag force, of a point particle with a Burgers fluid writes

\[
\partial_t u + \partial_x (u^2/2) = \lambda \, D(h'(t) - u) \, \delta_0(x - h(t)),
\]
\[
 mh''(t) = \lambda \, D(u(t, h(t)) - h'(t)).
\]

here

- \( u \), the velocity of the fluid, is unknown
- \( h \), the position of the solid particle is unknown
  (then \( h' \) and \( h'' \) respectively denote its velocity and acceleration);
- the parameters are \( \lambda \) (the drag coefficient) and \( m \) (the mass of the solid particle); both are positive.
- the function \( D \) which intervenes in the drag force is an increasing odd function.

Actually, we will suppose that

- either \( D(v) = v \) (the linear case)
- or \( D(v) = v|v| \) (the quadratic case).
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...Model and motivation...

The 1D case: the model for the interaction, via a drag force, of a point particle with a Burgers fluid writes

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = \lambda \left( D\left( h'(t) - u \right) \delta_0(x - h(t)) \right),$$

$$mh''(t) = \lambda \left( D\left( u(t, h(t)) - h'(t) \right) \right).$$

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Auxiliary steps
Our study of the above coupled problem includes two auxiliary steps, that are of interest on their own. The first one is

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\begin{cases}
\partial_t u(t, x) + \partial_x (u^2/2)(t, x) = -\lambda u(t, x) \delta_0(x), & t > 0, \; x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

i.e., the particle is decoupled from the fluid and fixed at zero.

Difficulty 1: the source term is a product of distributions (\(u\) and \(\delta_0\)), which has to be carefully defined. Indeed, \(u\) can be discontinuous (and in fact, typically \(u\) is discontinuous at the particle location). To understand the meaning of the source term, the LeRoux approximation was studied in detail by Lagoutière, Seguin, Takahashi: \(\delta_0 = \partial_x H\) (\(H\): the Heavyside function) is replaced by \(\partial_x H_\varepsilon\), a smoothed version. This permits to understand what goes on at the interface; this paves the way to a well-posedness theory.

The second step is to take \(h(\cdot)\) a given path, still decoupled from the fluid, and to solve the Burgers equation with singular source term located at \(x = h(t)\). We’ll see that as soon as the first step is well understood, the second one is easy.
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Other way around, we should understand how to evolve the particle location given the fluid state at time $t$. Recall the equation (ODE) for the particle:

$$m h''(t) = \lambda (u(t, h(t)) - h'(t)).$$

Recall that $u(t, \cdot)$ has a jump at $x = h(t)$...

**Difficulty 2**: understand the equation in the Carathéodory sense? In the Filippov sense?? We will see that a nice mathematical and physical interpretation is possible:

- the ODE for $h$ can be written in a weak form that involves the values of $u(t, \cdot)$ on $\mathbb{R}$ (which is more "robust")
- the particle is driven by the lack of mass conservation in the equation for $u$; or, equivalently, the total quantity of movement $\int_{\mathbb{R}} u(t, \cdot) + mh'(t)$ is conserved.

With these auxiliary steps well understood, we can

- think of the appropriate definition of solution
- use fixed-point arguments to guarantee existence
- use time splitting algorithms (evolve the PDE and the ODE alternatively) for existence (constructive) and efficient numerics.
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MAIN RESULTS
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For the Burgers-with-Dirac-at-zero model, we apply the machinery developed for conservation laws with discontinuous flux (adapted entropies, Baiti, Jenssen and Audusse-Perthame; revisited and generalized recently by BA., Karlsen, Risebro). The outcome is:

– definition(s) of entropy solutions
– uniqueness, continuous dependence ($L^1, L^1_{loc}$ with domain of dependence) exactly as in the Kruzhkov theory

In addition, we find

– $L^\infty$ bounds, variation bounds (more delicate)
– a strikingly simple numerical method (monotone consistent finite volume scheme with a trick at the interface)
– convergence of the numerical scheme, existence.

NB: the Riemann solver at the interface was already described by LST, so a Godunov scheme could be constructed; but we seek to avoid using the Riemann solver because it is intricate.
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...Main results...

For the Burgers-driven-by-particle model (with $x = h(t)$ GIVEN path of the particle) we get well-posedness rather easily.

It is observed that the case of straight path, $h(t) = Vt$ with $V = \text{const}$, reduces to the Dirac-at-zero model by the simultaneous change of $u - V$ into $u$ and of $x - Vt$ into $x$. Thus, nothing new for $h(t) = Vt$. Then any $(W^2, \infty)$ path $h(\cdot)$ is approximated by piecewise affine paths; existence is established by passage to the limit. Uniqueness is straightforward from the definition of solution.

For the coupled model with data $u_0$ and $h(0) = 0$, $h'(0) = v_0$, we get

– existence, for $L^\infty$ data $u_0$
– existence, uniqueness, continuous dependence for $BV$ solutions, for $BV$ data $u_0$.

We construct a time-explicit Glimm-type scheme where particle position is updated via splitting; we get numerical results that agree with the physical phenomena that are expected for the model. One example is “drafting-kissing-tumbling” phenomenon observed in presence of two particles.
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FROZEN PARTICLE
(DIRAC-AT-ZERO DRAG TERM):
UNDERSTANDING THE COUPLING
Frozen particle: understanding the coupling...

The admissibility at the interface \( \{ x = 0 \} \) of the solution is governed by the germ \( \mathcal{G}_\lambda \) (terminology related to the one of BA, Karlsen, Risebro ’10):

\[
\lambda 
\]

**Definition**

The admissibility germ \( \mathcal{G}_\lambda \subset \mathbb{R}^2 \) (or germ, for short) associated with the particle-at-zero problem is the union \( \mathcal{G}_\lambda = \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2 \cup \mathcal{G}_\lambda^3 \), where

- \( \mathcal{G}_\lambda^1 = \{(a, a - \lambda), a \in \mathbb{R}\} \).
- \( \mathcal{G}_\lambda^2 = [0, \lambda] \times [-\lambda, 0] \).
- \( \mathcal{G}_\lambda^3 = \{(a, b) \in (\mathbb{R}^+ \times \mathbb{R}^{-}) \setminus \mathcal{G}_\lambda^2, -\lambda \leq a + b \leq \lambda\} \).

NB: the partition of \( \mathcal{G}_\lambda \) into the three parts is dictated by the subsequent analysis.
Frozen particle: understanding the coupling...

The admissibility at the interface \( \{ x = 0 \} \) of the solution is governed by the \textit{germ} \( \mathcal{G}_\lambda \) (terminology related to the one of BA, Karlsen, Risebro ’10):

\[
\begin{align*}
\mathcal{G}_\lambda &= \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2 \cup \mathcal{G}_\lambda^3, \\
\mathcal{G}_\lambda^1 &= \{(a, a - \lambda), a \in \mathbb{R} \}, \\
\mathcal{G}_\lambda^2 &= [0, \lambda] \times [-\lambda, 0], \\
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...Frozen particle: understanding the coupling...

**Explanation:** the Burgers equation with Dirac-at-zero drag term is equivalent to
\[ \partial_t u + \partial_x (u^2/2) = -\lambda u \partial_x H. \]

We introduce \( H_\varepsilon \in C^1(\mathbb{R}) \) a non-decreasing function such that \( H_\varepsilon(x) = H(x) \) when \( |x| \geq \varepsilon \). Since we are interested in understanding the behavior of the solution through the stationary interface \( \{x = 0\} \), we can study only stationary solutions. We then obtain the regularized equation for \( U_\varepsilon(x) = u(t, x) \) in the strip \( -\varepsilon < x < \varepsilon \):
\[
(U_\varepsilon^2/2)'(x) + \lambda U_\varepsilon(x) \partial_x H_\varepsilon(x) = 0.
\]

**Proposition (Lagoutière, Seguin, Takahashi '08)**

Independently from the choice of \( H_\varepsilon \), there exists a solution to the above ODE with \( U_\varepsilon(-\varepsilon) = c_- \) and \( U_\varepsilon(+\varepsilon) = c_+ \) if and only if \( (c_-, c_+) \in G_\lambda \).

The modelling assumption we make is the following:

the traces \( \gamma_- u \) and \( \gamma_+ u \) at \( \{x = 0\} \) of a solution \( u \) of the Burgers equation on \( \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \) are compatible if and only if there exists a solution to above ODE such that \( U_\varepsilon(-\varepsilon) = \gamma_- u \), \( U_\varepsilon(\varepsilon) = \gamma_+ u \).

Thus the germ \( G_\lambda \) is the set of couples \( (\gamma_- u, \gamma_+ u) \) of possible traces at \( \{x = 0\} \) (for a.e. \( t > 0 \)) of the admissible solutions.
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...Frozen particle: understanding the coupling...

Now, the dissipativity properties of the interface coupling are encoded in the germ $G_{\lambda}$. Indeed, define $\Xi : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\Xi^\pm((u-, u+), (v-, v+)) = \Phi^\pm(u-, v-) - \Phi^\pm(u+, v+)$$

where $\Phi^\pm$ are the so-called semi-Kruzhkov entropy fluxes for Burgers eqn:

$$\Phi^\pm(u, v) = \text{sgn}^\pm(u - v)(u^2 - v^2)/2.$$

Splitting the germ $G_{\lambda}$ into three subsets, we have

**Proposition (dissipativity and maximality of $G_{\lambda}$)**

The following properties hold:

(i) (dissipativity) $\forall (u-, u+), (v-, v+) \in G_{\lambda},$

$$\Xi^\pm((u-, u+), (v-, v+)) \geq 0.$$

(ii) (maximality + ...) If a pair $(u-, u+) \in \mathbb{R}^2$ verifies:

$$\forall (v-, v+) \in G_{\lambda}^1 \cup G_{\lambda}^2 \quad \Xi((u-, u+), (v-, v+)) \geq 0,$$

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A “better” (indirect) proof comes from the general theory from AKR. First, property (i) is actually equivalent to the “Kato inequality” \((\L^1\)-dissipativity)

\[
-\int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u - v)^+ \partial_t \varphi + \Phi^+(u, v) \partial_x \varphi) \leq 0 \quad \forall \varphi \in D(Q), \varphi \geq 0.
\]

for the solutions

\[
u(t, x) := u_- 1_{\{x < 0\}} + u_+ 1_{\{x > 0\}}, \quad v(t, x) := v_- 1_{\{x < 0\}} + v_+ 1_{\{x > 0\}}
\]

of our equation; and the Kato inequality comes by passage to the limit from the LeRoux approximation case:

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\lambda(u^\varepsilon - v^\varepsilon)^+(\partial_x H^\varepsilon) \varphi - (u^\varepsilon - v^\varepsilon)^+ \partial_t \varphi - \Phi^+(u^\varepsilon, v^\varepsilon) \partial_x \varphi) \leq 0.
\]

Further, property (ii) means that \(G^1_\lambda \cup G^2_\lambda\) is a definite germ of which \(G_\lambda\) is the unique maximal extension”. This follows (with some work) from the fact that \(G_\lambda\) is a complete germ \((\L_0\) the germ allows to solve every Riemann problem).
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\[-\int_\mathbb{R}^+ \int_\mathbb{R} ((u-v)^+ \partial_t \varphi + \Phi^+ (u, v) \partial_x \varphi) \leq 0 \ \forall \varphi \in \mathcal{D}(Q), \varphi \geq 0.\]

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\[u(t, x) := u_- 1_{\{x<0\}} + u_+ 1_{\{x>0\}}, \quad v(t, x) := v_- 1_{\{x<0\}} + v_+ 1_{\{x>0\}}\]

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Further, property (ii) means that “\(\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda\) is a definite germ of which \(\mathcal{G}_\lambda\) is the unique maximal extension”. This follows (with some work) from the fact that \(\mathcal{G}_\lambda\) is a complete germ \((\Leftrightarrow \text{the germ allows to solve every Riemann problem})\).
FROZEN PARTICLE
(DIRAC-AT-ZERO DRAG TERM):
DEFINITION, UNIQUENESS
Frozen particle: definition(s)...

First, let us describe some elementary solutions of this problem: these are the stationary piecewise constant functions \( c \):

\[
c(t, x) = c_1 \mathbb{1}_{\{x<0\}} + c_2 \mathbb{1}_{\{x>0\}} = \begin{cases} c_- & \text{if } x < 0, \\ c_+ & \text{if } x > 0, \end{cases}
\]

\((c_-, c_+) \in G_\lambda\).

They play the role of the constants in the standard Kruzhkov entropy formulation. With the idea of adapted Kruzhkov entropies, we set up

**Definition (entropy solution)**

Let \( u_0 \in L^\infty(\mathbb{R}) \). A function \( u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) is said to be an entropy solution of the “particle-at-zero” problem if for all function \( c \) defined above with \((c_-, c_+) \in G_\lambda\),

\[
\forall \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}), \varphi \geq 0 \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}} [\lvert u - c(x) \rvert \partial_t \varphi + \Phi(u, c(x)) \partial_x \varphi] \, dx \, dt
\]

\[
+ \int_{\mathbb{R}} \lvert u_0 - c(x) \rvert \varphi(0, x) \, dx \geq 0.
\]
Let us provide alternative characterizations of entropy solutions:

**Proposition (equivalent definitions)**

A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution if and only if it satisfies any of the following assertions:

A. The function $u$ verifies the adapted entropy inequalities with $(c_-, c_+) \in G^1_\lambda \cup G^2_\lambda$.

B. The function $u$ verifies the Kruzhkov entropy inequalities for all nonnegative test function $\varphi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R})$ such that $\varphi|_{x=0} = 0$, moreover, for a.e. $t > 0$ $( (\gamma_- u)(t), (\gamma_+ u)(t) ) \in G_\lambda$.

D. There exists $C = C(\lambda, \|u\|_{L^\infty}, c_{\pm})$ such that the function $u$ verifies

$$\forall \varphi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}), \varphi \geq 0 \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}} [ |u - c(x)| \partial_t \varphi + \Phi(u, c(x)) \partial_x \varphi ] \, dx \, dt$$

$$+ \int_{\mathbb{R}} |u_0 - c(x)| \varphi(0, x) \, dx \geq -C \text{dist} \left( (c_-, c_+), G_\lambda \right)$$

for all $(c_-, c_+) \in \mathbb{R} \times \mathbb{R}$.
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\[+ \int_{\mathbb{R}} |u_0 - c(x)| \, \varphi(0, x) \, dx \geq -C \text{dist} \left( (c_-, c_+), \mathcal{G}_\lambda \right) \]

**for all** \((c_-, c_+) \in \mathbb{R} \times \mathbb{R} \).
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**D.** There exists $C = C(\lambda, \|u\|_{\infty}, c_{\pm})$ such that the function $u$ verifies

\[
\forall \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}), \, \varphi \geq 0 \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[ |u - c(x)| \, \partial_t \varphi + \Phi(u, c(x)) \, \partial_x \varphi \right] \, dx \, dt \\
+ \int_{\mathbb{R}} |u_0 - c(x)| \, \varphi(0, x) \, dx \geq -Cd_{\text{dist}}((c_-, c_+), \mathcal{G}_\lambda)
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for all $(c^-, c^+) \in \mathbb{R} \times \mathbb{R}$. 

...Frozen particle: definition(s)...

Theorem ($L^1$ contraction+comparison, analogous to Kruzhkov theory)

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

\[
\text{for a.e. } t > 0 \quad \int_R^R (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x) \, dx
\]

where $L = \max\{\|u\|_\infty, \|v\|_\infty\}$. Consequently, if $(u_0 - v_0)^+ \in L^1(\mathbb{R})$, we have

\[
\text{for a.e. } t > 0 \quad \int_{\mathbb{R}} (u - v)^+(t, x) \, dx \leq \int_{\mathbb{R}} (u_0 - v_0)^+(x) \, dx.
\]

In particular, for all $u_0 \in L^\infty(\mathbb{R})$, there exists at most one solution and the map $S(t) : u_0 \mapsto u(t, \cdot)$ on its domain is an order-preserving $L^1$ contraction.

The proof is straightforward using

– the Kato inequality away from the interface (standard Kruzhkov)
– the characterization $B$. ("with traces") of entropy solutions
– and the dissipativity of $\mathcal{G}_\lambda$. 
...Frozen particle: uniqueness, comparison, $L^1$ contraction.

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\text{for a.e. } t > 0 \quad \int_{-R-Lt}^{R+Lt} (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x) \, dx
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The proof is straightforward using

– the Kato inequality away from the interface (standard Kruzhkov)
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**Theorem (L$^1$ contraction+comparison, analogous to Kruzhkov theory)**

Let $u_0$ and $v_0$ be two initial data in $L^\infty(\mathbb{R})$ and let $u$ and $v$ be the associated entropy solutions. Then for all $R > 0$,

$$\text{for a.e. } t > 0 \quad \int_R^R (u - v)^+(t, x) \, dx \leq \int_{-R-Lt}^{R+Lt} (u_0 - v_0)^+(x) \, dx$$

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FROZEN PARTICLE

(DIRAC-AT-ZERO DRAG TERM):

NUMERICAL SCHEME AND EXISTENCE
We use a well-balanced finite volume scheme, preserving exactly (some of) the stationary sols \( u(t, x) := u_- \mathbb{1}_{\{x < 0\}} + u_+ \mathbb{1}_{\{x > 0\}} \).

Usual schemes are determined by a numerical flux \( g(\cdot, \cdot) : \)

\(- g \) locally Lipschitz;
\(- g(u, u) = \frac{u^2}{2} \) (consistency);
\(- g(\cdot, b) \) is \( \nearrow \), \( g(a, \cdot) \) is \( \searrow \) (monotonicity).

We only modify \( g(\cdot, \cdot) \) at the interface \( x = 0 \) (between \( x_0 \) and \( x_1 \)):

\[ g^-_\lambda (a, b) = g(a, b + \lambda) \quad \text{and} \quad g^+\lambda (a, b) = g(a - \lambda, b). \]

Idea: \( g^\pm_\lambda \) “only see” the \( \mathcal{G}^1_\lambda \) part of the germ!

Then the scheme writes

\[ \forall i \neq 0, 1 \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)); \]
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Numerical solution:

\[ u_\Delta (t, x) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} u_i^n \mathbb{1}_{(n \Delta t, (n+1) \Delta t)}(t) \mathbb{1}_{(x_{i-1/2}, x_{i+1/2})}(x). \]
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\]
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\]
\[
i = 1 : \quad u^{n+1}_1 = u^n_1 - \frac{\Delta t}{\Delta x}(g(u^n_1, u^n_2) - g^+_\lambda(u^n_0, u^n_1)).
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\]
Properties of the scheme...

Under the CFL condition: $2M\Delta t \leq \Delta x$, ($M$ being the Lipschitz constant of the numerical flux $g$ on the \textit{ad hoc} interval of values of $(u^n_i)_{n,i}$, the scheme writes

$$\forall i \in \mathbb{Z} \quad u^{n+1}_i = H_i(u^n_{i-1}, u^n_i, u^n_{i+1}),$$

where functions $H_i$ are monotone ↗ in each of the three arguments. NB: since $\cdot \mapsto \cdot \pm \lambda$ are ↗ functions, monotonicity OK also for $i = 0, 1$.

**Lemma (L$^\infty$ bound — choice of $M$ in the CFL condition)**

Under the CFL condition, the scheme satisfies for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$

$$\min\{\text{ess inf } u_0 - \lambda, \text{ess inf } u_0\} \leq u^n_i \leq \max\{\text{ess sup } u_0, \text{ess sup } u_0 + \lambda\}.$$  

**Proposition (the scheme is (partially) well-balanced)**

(i) The initial datum $v_0(\cdot) = c(\cdot) = c_- 1_{\{x<0\}} + c_+ 1_{\{x>0\}}$, $(c_-, c_+) \in \mathcal{G}^1_\lambda$, is exactly preserved in the evolution by the scheme.

(ii) Let $v_\Delta$ be the solution of the numerical scheme with the initial datum $v_0(\cdot) = c(\cdot) = c_- 1_{\{x<0\}} + c_+ 1_{\{x>0\}}$, $(c_-, c_+) \in \mathcal{G}^2_\lambda$. Then $v_\Delta$ converge to $c$ in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ as $\Delta x \to 0$. 
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In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

\((H)\) \quad \partial_a(\partial_a g(a, b) + \partial_b g(a, b)) \geq 0, \quad \partial_b(\partial_a g(a, b) + \partial_b g(a, b)) \geq 0;

**Lemma (BV_{loc} bound, Bürger, García, Karlsen, Towers)**

Let $T > 0$ and $A > 0$. Assume that $u_0 \in BV(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption \((H)\), we have

$$
\|u_\Delta (\cdot, \cdot)\|_{BV([0, T] \times \mathbb{R} \setminus (-A, A))} \leq \frac{1}{A} Const(T, \|u_0\|_{L^\infty}, \|u_0\|_{BV(\mathbb{R})}, \lambda).
$$

- estimate in $BV(0, T; L^1(\mathbb{R}))$ (i.e., time $BV$ estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + \((H)\))
- mean-value theorem: for some $r \in (0, A)$, $\|u(\cdot, \pm r)\|_{BV(0, T)} \leq const/A$
- look at our solution as solution to a Cauchy-Dirichlet problem with $BV$ initial and boundary data.

**Proposition (approximate Kato inequality)**

Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in D([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

$$
- \int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u_\Delta - v_\Delta)^+ \partial_t \varphi + \Phi^+(u_\Delta, v_\Delta) \partial_x \varphi) \leq Rem(\Delta x, \varphi).
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Arguments: monotonicity of the scheme, consistency, $BV$ in space bound
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Let $T > 0$ and $A > 0$. Assume that $u_0 \in BV(\mathbb{R})$ and $\Delta x$ is small enough. Then, under the CFL condition and assumption (H), we have

\[ \|u(\cdot, \cdot)\|_{BV([0, T] \times \mathbb{R} \setminus (-A, A))} \leq \frac{1}{A} Const(T, \|u_0\|_{L^\infty}, \|u_0\|_{BV(\mathbb{R})}, \lambda). \]

- estimate in $BV(0, T; L^1(\mathbb{R}))$ (i.e., time $BV$ estimate in the mean) from translation invariance + contraction (use Crandall-Tartar lemma + (H))
- mean-value theorem: for some $r \in (0, A)$, $\|u(\cdot, \pm r)\|_{BV(0, T)} \leq \text{const}/A$
- look at our solution as solution to a Cauchy-Dirichlet problem with $BV$ initial and boundary data.

**Proposition (approximate Kato inequality)**

Let $u_\Delta, v_\Delta$ be solutions of the scheme. Let $\varphi \in D([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Then

\[ - \int_{\mathbb{R}^+} \int_{\mathbb{R}} ((u_\Delta - v_\Delta)^+ \partial_t \varphi + \Phi^+(u_\Delta, v_\Delta) \partial_x \varphi) \leq Rem(\Delta x, \varphi). \]

Arguments: monotonicity of the scheme, consistency, $BV$ in space bound
In what follows, we need a technical hypothesis (dissipativity at $x = 0$):

\[ (H) \quad \partial_a(\partial_ag(a, b) + \partial_bg(a, b)) \geq 0, \quad \partial_b(\partial_ag(a, b) + \partial_bg(a, b)) \geq 0; \]

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Arguments: monotonicity of the scheme, consistency, $BV$ in space bound.
Convergence; existence of entropy solutions.

Theorem (convergence of the scheme; existence of solutions)

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (\( H \)), the numerical scheme converges in \( L_{\text{loc}}^1(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

In particular, the problem is well-posed, for \( L^\infty \) data and \( L_{\text{loc}}^1 \) topology.

Proof.

- First assume that \( u_0 \in BV(\mathbb{R}) \).
  - \( BV_{\text{loc}} \) bounds yield compactness: we get \( u \) an accumulation point of \( (u_\Delta)_\Delta \);  
  - well-balance property for \((c_-, c_+) \in G^1_\lambda \cup G^2_\lambda \) yields enough explicit stationary solutions \( v_\Delta \) to the scheme (at least, at the limit \( \Delta x \to 0 \));  
  - using the approximate Kato inequalities on \( u_\Delta \) and the above special solutions \( v_\Delta \), at the limit we get Kato inequalities... but, these are precisely the adapted entropy inequalities!!  
  - then \( u \) is (the unique) entropy solution (use caract. A. of entropy sols).
- For the general case \( u_0 \in L^\infty(\mathbb{R}) \), approximate \( u_0 \) by two a.e. convergent sequences of \( BV(\mathbb{R}) \) functions \( (u_0^n)_n \) (↗) and \( (\overline{u}_0^n)_n \) (↘) such that \( u_0^n \leq u_0 \leq \overline{u}_0^n \). Use comparison arguments and the result of the \( BV \) case.
Convergence; existence of entropy solutions.

**Theorem (convergence of the scheme; existence of solutions)**

Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption (\( H \)), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to "Burgers with particle-at-zero" problem when \( \Delta x \) tends to 0.

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The coupled problem

Convergence; existence of entropy solutions.

Theorem (convergence of the scheme; existence of solutions)

Assume $u_0 \in L^\infty(\mathbb{R})$. Then, under the CFL condition and assumption $(H)$, the numerical scheme converges in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ to the unique entropy solution to “Burgers with particle-at-zero” problem when $\Delta x$ tends to 0.

In particular, the problem is well-posed, for $L^\infty$ data and $L^1_{\text{loc}}$ topology.

Proof.

• First assume that $u_0 \in BV(\mathbb{R})$.
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Convergence; existence of entropy solutions.

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Assume \( u_0 \in L^\infty(\mathbb{R}) \). Then, under the CFL condition and assumption \( (H) \), the numerical scheme converges in \( L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) to the unique entropy solution to “Burgers with particle-at-zero” problem when \( \Delta x \) tends to 0.

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**Proof.**

- First assume that \( u_0 \in BV(\mathbb{R}) \).
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**Theorem (convergence of the scheme; existence of solutions)**

Assume $u_0 \in L^\infty(\mathbb{R})$. Then, under the CFL condition and assumption $(H)$, the numerical scheme converges in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ to the unique entropy solution to “Burgers with particle-at-zero” problem when $\Delta x$ tends to 0.

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Convergence; existence of entropy solutions.

Theorem (convergence of the scheme; existence of solutions)

Assume $u_0 \in L^\infty(\mathbb{R})$. Then, under the CFL condition and assumption ($H$), the numerical scheme converges in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ to the unique entropy solution to “Burgers with particle-at-zero” problem when $\Delta x$ tends to 0.

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In particular, the problem is well-posed, for \( L^\infty \) data and \( L^1_{\text{loc}} \) topology.

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Theorem (convergence of the scheme; existence of solutions)

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COUPLED PROBLEM
Case $h = h(t)$: existence, uniqueness

**Theorem (well-posedness for moving but decoupled particle)**

*Given $h(\cdot)$ a $C^1$ path, there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda (u - h'(t))\delta_0(x - h(t))$; (localized) $L^1$ contraction property holds.*

**Definition:**

- the germ $G_\lambda$ changes into $(h'(t), h'(t)) + G_\lambda$;
- versions B. (“with traces”) and D. (“adapted entropy inequalities with remainder term”) permit to define entropy solutions.

**Arguments:**

- for uniqueness, comparison, $L^1$ contraction: same technique;
- for existence: use characterization D. (it is stable by passage to the limit!) ;
- approximate $h(\cdot)$ by a family $(h_n)_n$ of piecewise affine paths
- construction of solutions for “particle at $h_n$” is straightforward: $h'_n$ being piecewise constant, one changes variables to reduce to the “drag force-at-zero” case. Procedure restarted at each time where $h'_n$ jumps.
- because $h'_n \to h'$, the associated germs converge; thus we pass to the limit in characterization D.
Case $h = h(t)$: existence, uniqueness

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– versions **B.** (“with traces”) and **D.** (“adapted entropy inequalities with remainder term”) permit to define entropy solutions.

**Arguments:**

– for uniqueness, comparison, $L^1$ contraction: same technique;
– for existence: use characterization **D.** (it is stable by passage to the limit!);
– approximate $h(\cdot)$ by a family $(h_n)_n$ of piecewise affine paths
– construction of solutions for “particle at $h_n$” is straightforward: $h_n'$ being piecewise constant, one changes variables to reduce to the “drag force-at-zero” case. Procedure restarted at each time where $h_n'$ jumps.
– because $h_n' \to h'$, the associated germs converge; thus we pass to the limit in characterization **D..**
Case $h = h(t)$: existence, uniqueness

**Theorem (well-posedness for moving but decoupled particle)**

*Given* $h(\cdot)$ *a $C^1$ path*, there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda (u - h'(t))\delta_0(x - h(t))$; *(localized)* $L^1$ contraction property holds.

**Definition**:
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Case \( h = h(t) \): existence, uniqueness

**Theorem (well-posedness for moving but decoupled particle)**

*Given* \( h(\cdot) \) *a \( C^1 \) path,* *there exists a unique entropy solution to the Burgers equation with singular drag term* \(-\lambda (u - h'(t))\delta_0(x - h(t))\); *(localized) \( L^1 \) contraction property holds.*

**Definition:**

- the germ \( \mathcal{G}_\lambda \) changes into \((h'(t), h'(t)) + \mathcal{G}_\lambda\);
- versions \( \text{B.} \) ("with traces") and \( \text{D.} \) ("adapted entropy inequalities with remainder term") permit to define entropy solutions.

**Arguments:**

- for uniqueness, comparison, \( L^1 \) contraction: same technique;
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- because \( h'_n \to h' \), the associated germs converge; thus we pass to the limit in characterization \( \text{D.} \).
Case $h = h(t)$: existence, uniqueness

**Theorem (well-posedness for moving but decoupled particle)**

*Given* $h(\cdot)$ *a* $C^1$ *path,* there exists a unique entropy solution to the Burgers equation with singular drag term $-\lambda (u - h'(t))\delta_0(x - h(t));$ (localized) $L^1$ contraction property holds.

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- construction of solutions for "particle at $h_n$" is straightforward: $h'_n$ being piecewise constant, one changes variables to reduce to the "drag force-at-zero" case. Procedure restarted at each time where $h'_n$ jumps.
- because $h'_n \rightarrow h'$, the associated germs converge; thus we pass to the limit in characterization D.
Case $h = h(t)$: continuous dependence en $h(\cdot)$

Uniqueness proof for the coupled problem relies on a Gronwall inequality, which in turn relies on a Lipschitz dependence estimate for the map $h(\cdot) \mapsto u(\cdot, \cdot)$.

**Theorem (dependence of $u$ on the path $h(\cdot)$)**

Assume $u$, $\hat{u}$ are entropy solutions corresponding to the particles located at $h(\cdot)$, $\hat{h}(\cdot)$, respectively, with $h(0) = 0 = h'(0)$ and same initial datum $u_0$.

Assume $\hat{u} \in L^\infty(0, T; BV(\mathbb{R}))$. Then for a.e. $t \in (0, T)$,

$$
\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^1(\mathbb{R})} \leq C(\|u\|_{\infty}, \|\hat{u}\|_{\infty}, \|\hat{u}\|_{BV, \lambda}) \int_0^t |h'(s) - \hat{h}'(s)| \, ds.
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Arguments:

– change of variables $y = x - h(t)$, resp. $x - h'(t)$. Two eqns, both with singularity at zero, come out, with different fluxes of the kind $u \mapsto \frac{u^2}{2} - h'(t)u$.

– use the techniques of dependence of entropy solutions on the flux function ($BV$ regularity needed!): Kuznetsov, Bouchut-Perthame, Karlsen-Risebro...: the $C^1$ norm of the difference of the fluxes pops up, which yields $|h' - \hat{h}'|$.

– use Lipschitz dependence of the germ on $h'$ to describe additional (small) “non-dissipation” term coming from the interface.
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**Case** $h = h(t)$: continuous dependence en $h(\cdot)$, $L^\infty$ and $BV$ stability

**Proposition (BV estimate)**

The solution constructed for the $h = 0$ case obeys

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \downarrow \text{ for all } t > 0$$

(at $t = 0$ the variation may increase by a const. depending on $\|u_0\|_\infty, G_\lambda$).

The solution constructed for the fixed-$h(\cdot)$ case obeys the BV estimate

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})} + \text{const}(\lambda, \|u_0\|_\infty) + 2 \int_0^t |h''(s)| \, ds.$$ 

Argument: (re)-construct solutions by wave-front tracking algorithm (Dafermos, Holden-Risebro, Bressan et al.) (better control of interactions).

**Lemma (L^\infty bounds)**

We get a uniform $L^\infty$ bound on ad hoc sequences of $h'(\cdot)$ and $u(\cdot, \cdot)$.

To be precise: if we look at solutions to the coupled problem, we get

$$\max\{\|u\|_\infty, \|h'\|_\infty\} \leq \max\{\|u_0\|_\infty, |h'(0)|\}.$$ 

For solutions appearing in the fixed-point or splitting arguments, we get somewhat weaker bounds.
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For solutions appearing in the fixed-point or splitting arguments, we get somewhat weaker bounds.
Case of \( u \) frozen: evolving \( h = h(\cdot) \)

**Proposition (modelling/“traces” interpretation of the ODE on \( h(\cdot) \))**

For every drag force, the ODE in the coupled problem writes

\[
 mh''(t) = \left( (u_-)^2 / 2 - h'(t)u_- \right) - \left( (u_+)^2 / 2 - h'(t)u_+ \right).
\]

Notice that the right-hand side above is expressed as the difference of the normal components of the 2D-field \((u, u^2/2)\) on the curve \(\{x = h(t)\}\) from the left and from the right. Combining this observation with the Green-Gauss formula, we get the following weak formulation of the ODE:

**Lemma (second interpretation of the ODE on \( h(\cdot) \))**

Let \( u \) be a weak solution of the PDE on \(\{x \neq h(t)\}\); let \( h \in W^{2,\infty}(0, T) \). Then \( h(\cdot) \) verifies the ODE if and only if for all \( \xi \in D([0, T]) \), for all \( \psi \in D(\mathbb{R}) \) such that \( \psi \equiv 1 \) on the set \( \{x \in \mathbb{R} : \exists t \in [0, T] \text{ such that } h(t) = x\} \), there holds

\[
 -m \int_0^T h'(t)\xi'(t)dt = mh'(0)\xi(0) + \int_0^T \int_{\mathbb{R}} \left[ u\psi\xi_t + \frac{u^2}{2}\xi\psi_x \right] + \int_{\mathbb{R}} u_0\psi\xi(0).
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Case of $u$ frozen: evolving $h = h(\cdot)$

Proposition (modelling/“traces” interpretation of the ODE on $h(\cdot)$ )

For every drag force, the ODE in the coupled problem writes

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Lemma (second interpretation of the ODE on $h(\cdot)$ )

Let $u$ be a weak solution of the PDE on $\{x \neq h(t)\}$; let $h \in W^{2,\infty}(0, T)$. Then $h(\cdot)$ verifies the ODE if and only if for all $\xi \in \mathcal{D}([0, T])$, for all $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi \equiv 1$ on the set $\{x \in \mathbb{R} : \exists t \in [0, T] \text{ such that } h(t) = x\}$, there holds

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Coupled problem: existence, uniqueness of \( BV \) solutions / existence of \( L^\infty \) solutions

The above ingredients can be used in several ways:

- In a fixed-point argument \( h(\cdot) \leftrightarrow u(\cdot, \cdot) \leftrightarrow h(\cdot) \)
  
  (compactness: work in \( C^1([0, T]) \) and enjoy a \( W^{2, \infty}(0, T) \) bound on \( h(\cdot) \))

- In a time splitting algorithm (alternatively evolving \( u \) and \( h \) on small time intervals):
  
  - \( u \) updated from \( h \) using the theory of entropy solutions for \( h \) frozen;
  
  - \( h \) updated from \( u \) using the above weak formulation of the ODE.

- In a numerical scheme (same time splitting + approximation in space of the conservation law); an interesting possibility is the random-choice algorithm (Glimm), in order not to adapt the space meshing to the particle location.

**Theorem (Main result)**

For all \( BV \) datum \( u_0 \) and given \( h(0), h'(0) \), there exists a unique entropy solution to the coupled problem.

For all \( L^\infty \) datum \( u_0 \) and given \( h(0), h'(0) \), there exists an entropy solution to the coupled problem.
The coupled problem: existence, uniqueness of BV solutions / existence of $L^\infty$ solutions

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Coupled problem: a well-balanced random-choice numerical scheme

Figure: Representation of the algorithm based on the well-balanced scheme.
Numerics: drafting-kissing-tumbling

Figure: Trajectories of two particles
Oufff !!!

DANKE SCHÖN !