Dissipative coupling of scalar conservation laws across an interface: theory and applications.

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based on joint works with
K.H. Karlsen and N.N. Risebro
P. Goatin, F. Lagoutière, N. Seguin, and T. Takahashi

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Plan of the talk

GENERAL FRAMEWORK AND MODEL EQUATION

General framework and the model equation...

Consider the Cauchy problem for a scalar conservation law

$$\begin{cases} u_t + \operatorname{div} \mathfrak{f}(t,x,u) = 0, & \text{on } [0,+\infty) \times \mathbb{R}^N \\ u|_{t=0} = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

with a Carathéodory flux $\mathfrak{f}$ $\big[$ measurable in (t,x), continuous in u $\big]$.

Main Question:

Well-posedness, for an appropriate generalization of the S.N. Kruzhkov notion of entropy solutions .

Related:

– Stability with respect to perturbations

– Convergence of approximation methods : vanishing viscosity, numerical schemes,...

NB. The case of merely measurable dependence of $\mathfrak{f}$ on (t,x); few works available: Baiti&Jenssen, Audusse&Perthame; Panov.

Many techniques are restricted to the case of piecewise regular dependency on (t,x) ; and the main issue is to understand how can conservation laws be coupled across an interface .

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$$u_t + \mathfrak{f}(x,u)_x = 0,\ \mathfrak{f}(x,u) = \begin{cases} f^l(u), & x<0, \\ f^r(u), & x>0, \end{cases} = f^l(u)1\!\!1_{\{x<0\}} + f^r(u)1\!\!1_{\{x>0\}}$$

Many contributions since 1990:

T. Gimse and N.H. Risebro; S. Diehl; C. Klingenberg and Risebro; F. Kaasschietter; P. Baiti and H.K. Jenssen; J. Towers, then K.H. Karlsen, Towers and Karlsen, Risebro, Towers; R. Bürger and al.; Adimurthi and G.D. Veerappa Gowda, then Adimurthi, J. Jaffré, Gowda, and Adimurthi, S. Mishra, Gowda; D. Ostrov; N. Seguin and J. Vovelle, then F. Bachmann and Vovelle; E. Audusse and B. Perthame; E.Yu. Panov; M. Garavello, R. Natalini, B. Piccoli and A. Terracina; G.Q. Chen, N. Even and Klingenberg; J. Jimenez, then Jimenez and L.Lévi; D. Mitrovič; C. Cancès;...

Also degenerate parabolic pbs are treated (ideas: J. Carrillo'1999).

We re-use and combine several key ideas that were introduced , and construct a "general theory" for the model equation. As an outcome, we are able to treat more general cases (examples will be given).

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MAIN RESULTS AND INGREDIENTS

Restrictions, Results and Ingredients...

Restrictions :

- A scalar conservation law
- Only L^1-contractive solvers are considered
- Some structural restrictions on the fluxes $f^{l,r}$ (Lipschitz/Hölder continuity, genuine nonlinearity, range conditions may be used, but not essential)

Results :

- A definition of "L^1-dissipative germs" $\mathcal{G}$ recognized as objects governing the admissibility ; an "algebraic" study of germ properties
- Definition (s) of the associated $\mathcal{G}$-entropy solutions
- Uniqueness, comparison, L^1 contraction for $\mathcal{G}$-entropy solutions
- A general existence result, under the assumptions that
  - all Riemann problem at $x = 0$ has a $\mathcal{G}$-entropy solution
  - compactness-ensuring assumptions (conditions on $f^{l,r}$ that yield uniform L^∞ estimates on approx. solutions and either uniform localized BV estimates, or reduction of the Young measure)
- Convergence of particular approximation procedures (standard or adapted vanishing viscosity; regularization-viscosity; "Godunov-at-interface" Finite Volume scheme; "miracle" FV schemes)
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Main Ingredients :

- Strong left- and right-sided traces $\gamma^l u$, $\gamma^r u$ on the interface exist[1] (Vasseur; Panov), for L^∞ away-from-the-interface entropy sol.:

$$\forall \xi \in \mathcal{D}(0,T) \quad \lim_{h\to 0^+} \frac{1}{h}\int_{-h}^{0}\int_{0}^{T} \xi(t)\,\big|u(t,x)-(\gamma^l u)(t)\big|\,dtdx = 0.$$

- The notion of solution is entirely governed by a choice of the admissible couples $(\gamma^l u, \gamma^r u)$ of traces across the interface (Garavello&Natalini&Piccoli&Terracina , via the Riemann pb.)
- (because of the L^1-contractivity constraint) the set $\mathcal{G}$ of such admissible couples has some nice structure
- (somewhat confusing... but quite interesting !) different L^1-contractive solvers(semigroups) co-exist (Adimurthi&Mishra&V.Gowda; Bürger&Karlsen&Mishra&Risebro); moreover, there are different “physically motivated” choices of $\mathcal{G}$!

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[1] well, that’s false unless $f^{l,r}$ are genuinely nonlinear.
Yet “traces that are needed” always exist (Panov; A.&Sbihi). More exactly, for $f=f^{l,r}$, introduce the “variation fct” V_f of f by $V_f(u)=\int_0^u |f'(z)|dz$ (singular mapping: Temple). Then Panov shows that $V_f(u)$ has traces; then one remarks that the fluxes $f(z), q(z,k)$ are cont. fcts of $V_f(z)$ and therefore have strong traces.

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Main Ingredients (cont^{d}) :

- The fundamental idea of Kruzhkov , seen from the viewpoint of the nonlinear semigroup theory (Crandall; Bénilan):
  - a set of "elementary" admissible solutions is selected (denoted u_{el})
  - the L^1 contraction property with respect to these pre-selected solutions is postulated: $\|u-u_{el}\|_{L^1(R^N)}(t) \leqslant \|u-u_{el}\|_{L^1(R^N)}(0)$
  - (since the elementary solutions are not in L^1...) the L^1 contraction is localized: this is called the Kato inequality:

$$\forall \xi \in \mathcal{D}(\mathbb{R}) \quad \int_{\mathbb{R}} |u-u_{el}|(t)\,\xi + \int_0^t\int_{\mathbb{R}} q(x;u,u_{el})\cdot\nabla\xi \leqslant \int_{\mathbb{R}} |u-u_{el}|(0)\,\xi$$

  - from the Kato ineq. between u and every elementary solution u_{el}, the Kato inequality between u and another solution $\hat{u}$ is deduced!

  This leads to "adapted entropies" $\eta(x;u) := |u-u_{el}(x)|$ (first used by Baiti&Jenssen ; popularized by Audusse&Perthame)

- Either relating the choice of the support of the test function to the choice of the entropy (Carrillo); or incorporating "error terms" into the "too general" entropy inequalities (Otto; Towers, Karlsen&Risebro&Towers,...)

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  - a set of "elementary" admissible solutions is selected (denoted u_{el})
  - the L^1 contraction property with respect to these pre-selected solutions is postulated: $\|u - u_{el}\|_{L^1(R^N)}(t) \leqslant \|u - u_{el}\|_{L^1(R^N)}(0)$
  - (since the elementary solutions are not in L^1...) the L^1 contraction is localized: this is called the Kato inequality:

$$\forall \xi \in \mathcal{D}(\mathbb{R}) \quad \int_{\mathbb{R}} |u - u_{el}|(t)\, \xi + \int_0^t \int_{\mathbb{R}} q(x; u, u_{el}) \cdot \nabla \xi \leqslant \int_{\mathbb{R}} |u - u_{el}|(0)\, \xi$$

  - from the Kato ineq. between u and every elementary solution u_{el}, the Kato inequality between u and another solution $\hat{u}$ is deduced!

This leads to "adapted entropies" $\eta(x; u) := |u - u_{el}(x)|$ (first used by Baiti&Jenssen ; popularized by Audusse&Perthame)

- Either relating the choice of the support of the test function to the choice of the entropy (Carrillo); or incorporating "error terms" into the "too general" entropy inequalities (Otto; Towers, Karlsen&Risebro&Towers,...)

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GERMS AND $\mathcal{G}$-ENTROPY SOLUTIONS. UNIQUENESS.

L^1-dissipative germs and $\mathcal{G}$-entropy solutions; uniqueness...

Set $q^{l,r}(.,k) := sign(.-k)\big(f^{l,r}(.)-f^{l,r}(k)\big)$ (Kruzhkov entropy-fluxes for $f^{l,r}$).

Definition (Germs; maximal, definite, closed, and complete germs)

- Any set $\mathcal{G}$ of couples $(c^l,c^r)\in\mathbb{R}\times\mathbb{R}$ satisfying the Rankine-Hugoniot relation $f^l(c^l)=f^r(c^r)$ and the L^1-dissipativity relation

  $$(L^1D)\qquad \forall (c^l,c^r),(b^l,b^r)\in\mathcal{G},\quad q^l(c^l,b^l)\geqslant q^r(c^r,b^r).$$

  is called L^1D admissibility germ (an L^1D-germ, for short) associated with the couple of fluxes (f^l,f^r).
- We say that $\mathcal{G}'$ is an extension of an L^1D-germ $\mathcal{G}$ if $\mathcal{G}\subset\mathcal{G}'$ and $\mathcal{G}'$ still satisfies the L^1-dissipation property (L^1D) and the Rankine-Hugoniot condition.
- L^1D-germ $\mathcal{G}$ is called maximal, if it does not admit a nontrivial extension.
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$$(L^1 D) \qquad \forall (c^l, c^r), (b^l, b^r) \in \mathcal{G}, \quad q^l(c^l, b^l) \geqslant q^r(c^r, b^r).$$

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In relation with definite and maximal germs, consider one more definition.

Definition (dual of a germ)

Let $\mathcal{G}$ be a subset of $\mathbb{R} \times \mathbb{R}$. The dual of $\mathcal{G}$ is the set

$$\mathcal{G}^* := \left\{ \left(b^l, b^r\right) \in \mathbb{R} \times \mathbb{R} \;\middle|\; \begin{array}{l} f^l\left(b^l\right) = f^r\left(b^r\right) \quad \text{and} \\ \forall \left(c^l, c^r\right) \in \mathcal{G} \quad q^l\left(c^l, b^l\right) \geq q^r\left(c^r, b^r\right) \end{array} \right\}.$$

Then we establish a series of properties of the following kind:

Proposition (dual germ, maximality and definiteness)

Let $\mathcal{G}$ be a subset of $\mathbb{R} \times \mathbb{R}$; let $\mathcal{G}^*$ be the dual of $\mathcal{G}$ defined above.

- *One has $\mathcal{G} \subset \mathcal{G}^*$ if and only if $\mathcal{G}$ is an L^1D-germ.*
- *Assume $\mathcal{G}$ is an L^1D-germ. Then $\mathcal{G}^*$ is the union of all extensions of $\mathcal{G}$.*
  In particular, if $\mathcal{G}$ is a definite germ, then $\mathcal{G}^*$ is the unique max. extension of $\mathcal{G}$.
- *One has $\mathcal{G}^* = \mathcal{G}$ if and only if $\mathcal{G}$ is a maximal germ.*
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Heuristically:

– Complete germs are good for existence
– Maximal germs are good for uniqueness
– Yet a definite germ $\mathcal{G}$ determines completely its unique maximal extension $\mathcal{G}^*$; therefore our main object is a definite germ $\mathcal{G}$ of which the dual $\mathcal{G}^*$ is complete

Definition (with traces)

Given $f^{l,r} \in C(\mathbb{R}, \mathbb{R})$, let $\mathcal{G}$ be a definite germ.
A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is called a $\mathcal{G}$-entropy solution of

$$u_t + \big(\mathfrak{f}(x; u)\big)_x = 0, \quad u|_{t=0} = u_0$$

with flux $\mathfrak{f}(x, .)$ given by $f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$, if

(i) the restriction of u on $\{x > 0\}$ (resp., on $\{x < 0\}$) is a Kruzhkov entropy solution of the pb. with flux f^r (resp., with flux f^l);

(ii) for a.e. $t > 0$, the couple of strong traces $\big((\gamma^l u)(t), (\gamma^r u)(t)\big)$ of u on the interface $\{x = 0\}$ belongs to the dual germ $\mathcal{G}^*$.

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The above def. is perfectly suited to get uniqueness of a $\mathcal{G}$-entropy solution:

Theorem (uniqueness, comparison, L^1 contraction)

Assume that $\mathcal{G}^*$ is an L^1D germ.
If u and $\hat{u}$ are two $\mathcal{G}$-entropy solutions of the model problem, then the comparison principle and the L^1-contractivity property hold .
In particular, for each L^∞ datum u_0 there exists at most one $\mathcal{G}$-entropy solution of the Cauchy problem .

Proof.

We only prove the Kato inequality; the L^1-contractivity, comparison and uniqueness will follow by the usual Kruzhkov choice of test functions.

First, the standard doubling of variables away from the boundary yields the Kato inequality with test functions $\xi\xi_h$, where ξ_h is a cut-off of the interface.

Letting $h \to 0$ (thus $\xi_h \to 1$) and using the existence of strong traces, we get the additional term $q^r(\gamma^r u, \gamma^r \hat{u}) - q^l(\gamma^l u, \gamma^l \hat{u})$.

Yet the sign of this term is built-in into the definition of L^1D-germ !
This term is $\leqslant 0$ since $(\gamma^l u, \gamma^r u), (\gamma^l \hat{u}, \gamma^r \hat{u}) \in \mathcal{G}^*$ and $\mathcal{G}^*$ is L^1-dissipative. □

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Yet the sign of this term is built-in into the definition of L^1D-germ !
This term is $\leqslant 0$ since $(\gamma^l u, \gamma^r u), (\gamma^l \hat{u}, \gamma^r \hat{u}) \in \mathcal{G}^*$ and $\mathcal{G}^*$ is L^1-dissipative. □

...L^1-dissipative germs and $\mathcal{G}$-entropy solutions; uniqueness...

The above def. is perfectly suited to get uniqueness of a $\mathcal{G}$-entropy solution:

Theorem (uniqueness, comparison, L^1 contraction)

Assume that $\mathcal{G}^*$ is an L^1D germ.
If u and $\hat{u}$ are two $\mathcal{G}$-entropy solutions of the model problem, then
the comparison principle and the L^1-contractivity property hold .
In particular, for each L^∞ datum u_0
there exists at most one $\mathcal{G}$-entropy solution of the Cauchy problem .

Proof.

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Examples

Examples of germs (and some extensions)...

Example 0 : "the Kruzhkov germ" (case $f^l = f^r$).

$\mathcal{G}_{Kr} := \{ (k, k) \,|\, k \in \mathbb{R} \}$ is an L^1D germ... and it "turns out" that it is definite (this amounts to the definition of entropy solutions by Vol'pert !)

Thus it defines an L^1-contractive semigroup of $\mathcal{G}_{Kr}$-entropy solutions that "contains" all constants. But this is precisely the Kruzhkov semigroup (Kruzhkov, B.Quinn-Keyfitz).

If one looks at the dual $\mathcal{G}^*_{Kr}$ (which is the unique maximal L^1D extension of $\mathcal{G}$) one finds the celebrated Oleinik chord condition caracterizing admissible stationary jumps.

Example 0 bis : "the viscosity profiles' germ" (case $f^l = f^r$).

$$\mathcal{G}_{VV} := \left\{ (c^l, c^r) \,\middle|\, \begin{array}{l} \text{the state } c^l \text{ can be connected to the state } c^r \\ \text{by a stationary viscous profile} \end{array} \right\}.$$

NB. No calculation is needed to know that $\mathcal{G}_{VV}$ is L^1D !
this is because the approximation procedure defining $\mathcal{G}_{VV}$ (here, the vanishing viscosity method) generates an L^1-contractive semigroup.
Clearly, $\mathcal{G}_{Kr} \subset \mathcal{G}_{VV}$ thus $\mathcal{G}_{VV}$ is also definite and it has the same closure.
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Here, there is a whole family of definite germs of the form (A, B) (called (A, B)-connections by Adimurthi&Mishra&V.Gowda ; see the recent paper by Bürger&Karlsen&Towers).

At least two at least have physical meaning: one comes from the vanishing viscosity; the other, from a physically relevant capillarity (Kaasschietter).

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Here also, we see (A, B) connections; each of these connections represents a non-Kruzhkov stationary shock.

The germ $\mathcal{G}_{(A,B)} := \{(A, B)\}$ is a singleton such that $f(A) = f(B) =: F$ (with $A > B$, which means the shock is non-Kruzhkov).

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The idea is the following: the $\mathcal{G}_{(A,B)}$-entropy solutions model the traffic flow with pointwise flux restriction formally given by "$f(u) \leqslant F$" , with $F = f(A) = f(B)$. (Cf. the "bus-embedded-into-traffic" model presented by C.Lattanzio)
The case $A = B = \operatorname{argmax} f$ (constraint automatically fulfilled !) corresponds to the standard Kruzhkov entropy solutions.

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$$u_t + (u^2/2)_x = -u\delta_0.$$

Here the coupling is non-conservative ; but formally, the additional term has a good sign, it should not destroy the L^1 contraction.
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It is easy to see that any L^1-contractive semigroup for the model problem (with solutions that are Kruzhkov entropy solutions away from the interface !) corresponds to some maximal germ $\mathcal{G}$.

To summarize, we have identified the structure "responsible for the L^1-dissipation on the interface":

this is a "maximal L^1D germ", that can be viewed as the set of all possible strong trace couples $(\gamma^l u, \gamma^r u)$ at the interface.
Furthermore, it is sufficient to look at "definite L^1D germs", that are somewhat smaller (sometimes, much smaller!) subsets of $\mathbb{R} \times \mathbb{R}$.

Remark:
This definition and the uniqueness argument apply (a bit more technical) to the general case : e.g., variable germs; curved interface; multi-D problems.

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EQUIVALENT DEFINITIONS, EXISTENCE, CONVERGENCE OF APPROXIMATIONS

Reformulations, Existence&Convergence of approximations...

Let us give another formulation, which does not involve explicitly boundary traces of u. For all $(c^l, c^r) \in \mathbb{R}^2$, consider $c(x) = c^l 1\!\!1_{\{x<0\}} + c^r 1\!\!1_{\{x<0\}}$.

Equivalent Definition (global adapted entropy inequalities)

(Still $\mathfrak{f}(x,.) = f^l(.)1\!\!1_{\{x<0\}} + f^r(.)1\!\!1_{\{x>0\}}$ with $\mathcal{G}$ a definite germ associated to $f^{l,r}$)

A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is called a $\mathcal{G}$-entropy solution of the problem if,

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- *secondly, it is a solution in the sense of distributions;*
- *and thirdly, $\forall\ (c^l, c^r) \in \mathcal{G}$ and $c(x)$ defined above, the adapted entropy inequalities hold globally (with general test functions $\xi \in \mathcal{D}(\mathbb{R})$):*

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(i.e. $|u - c(x)|_t + \mathfrak{q}(x; u, c(x))_x \leqslant 0$ in the sense of distributions).

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As in Example 0 bis, one defines the germ $\mathcal{G}_{VV}$ that consists of (c^l, c^r) such that c^l,c^r can be joined by a viscous profile. This germ is definite and we describe its dual $\mathcal{G}_{VV}^*$ explicitly (cf. S.Diehl who was the first to describe it).

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In Example 1 bis, given a germ $\mathcal{G}_{(A,B)}$,
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Summary of results

- L^1-dissipative coupling of scalar conservation laws across an interface can be reduced to the description of possible trace values
- Description in terms of adapted entropy inequalities is an equivalent and the preferable way to define solutions
- Well-posedness (existence + L^1 contraction)
- Convergence of specifically designed approximations (industrial) and of some more practical, less elaborated approximations (artwork)
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Summary of contributions

- A unified approach to scalar conservation laws with discontinuous flux
- Many of the previously known entropy uniqueness criteria are tested and classified
- Theory validated by new applications, and by a better understanding of the vanishing viscosity limit

Conclusions II

Summary of results

- L^1-dissipative coupling of scalar conservation laws across an interface can be reduced to the description of possible trace values
- Description in terms of adapted entropy inequalities is an equivalent and the preferable way to define solutions
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- Convergence of specifically designed approximations (industrial) and of some more practical, less elaborated approximations (artwork)
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