Sbornik: *Mathematics* **194**:6 793–811

Matematicheskiĭ Sbornik **194**:6 3–22
DOI 10.1070/SM2003v194n06ABEH000739

On viscous limit solutions of the Riemann problem for the equations of isentropic gas dynamics in Eulerian coordinates

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Abstract. For the problem $\rho_t+(\rho u)_x=0$, $(\rho u)_t+(\rho u^2+p(\rho))_x=0$, $(\rho,u)\big|_{t=0,\,x<0}=(\rho_-,u_-)$, $(\rho,u)\big|_{t=0,\,x>0}=(\rho_+,u_+)$ one shows the existence and uniqueness of a solution obtainable as a limit as ε tends to zero of the bounded self-similar solutions of the regularized problem with additional viscosity term $\varepsilon t u_{xx}$, $\varepsilon>0$, in the second equation. The structure of the solutions is described in detail, in particular, when they contain vacuum states.

Bibliography: 19 titles.

Introduction

Consider the problem

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p(\rho))_x=\varepsilon t u_{xx} \end{cases} \tag{1}$$

with initial condition

$$\rho(0,x)=\begin{cases} \rho_+, & x>0, \\ \rho_-, & x<0, \end{cases} \qquad u(0,x)=\begin{cases} u_+, & x>0, \\ u_-, & x<0, \end{cases} \tag{2}$$

where $\rho_+,\rho_-\in\mathbb{R}^+$ and $u_+,u_-\in\mathbb{R}$. We are interested in bounded self-similar (that is, depending only on the variable x/t) solutions of the problem (1), (2) and their convergence as $\varepsilon\downarrow 0$.

Within the framework of isentropic gas dynamics in Eulerian coordinates solutions $(\rho,u)\colon(t,x)\in\mathbb{R}^+\times\mathbb{R}\mapsto(\rho(t,x),u(t,x))\in\mathbb{R}^+\times\mathbb{R}$ correspond to the gas density and velocity, p is the pressure of the gas, and $\varepsilon>0$ is the viscosity coefficient. In what follows we assume that $\rho_\pm>0$ and $p(\,\cdot\,)$ is a continuous strictly increasing function on $\mathbb{R}^+$ normalized by the condition $p(0)=0$. Solutions of the problem (1), (2) may contain points with density ρ equal to 0; in this case we shall say that the solution 'contains a vacuum'.

Recently, the problem (1), (2) has been treated in [1] on the basis of ideas of [2] (see also [3]). Under additional assumptions prohibiting the vacuum in solutions, the solubility of the problem (1), (2) was proved. An estimate of the variation of

AMS 2000 *Mathematics Subject Classification.* Primary 76N15, 35L40.

solutions uniform in $\varepsilon > 0$ was obtained and the wave-fan structure of the limit functions as $\varepsilon \downarrow 0$ was described.

In the present paper, for a broad class of functions $p(\,\cdot\,)$ we give a description of solutions of (1), (2) in which a vacuum is allowed.

In § 3 we consider the system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0 \end{cases} \tag{3}$$

and using the hulls of $p(\,\cdot\,)$ defined below (see Definition 1) we give a formula for the unique solution of the Riemann problem (3), (2), which can be obtained as a limit of bounded self-similar solutions of (1), (2) with $\varepsilon = \varepsilon_n$ for a sequence $\varepsilon_n \downarrow 0$.

The approach through self-similar viscosity limits was used in [4]–[10] in the context of the admissibility of weak solutions of the Riemann problem for hyperbolic systems of conservation laws. Following [11] we call the corresponding admissibility criterion the *wave-fan criterion*. It has been successfully tested on various special systems and viscous regularizations; see [12], [3] for a survey of results. In particular, an analysis of the problem (3), (2) regularized with self-similar viscosity $\varepsilon t\begin{pmatrix}\rho\\ \rho u\end{pmatrix}$ is carried out in [13], covering also cases when a vacuum is allowed. Special attention to the formation of vacuum in this problem is paid in [14]. In [3] the existence of an admissible solution in the sense of the wave-fan criterion was proved for a large class of strictly hyperbolic systems with sufficiently close Riemann data with the use of the identity viscosity matrix.

The main result of this paper (Theorem 2) is as follows: in the case of the viscosity matrix $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ and under the additional assumption

$$p(\rho) \to +\infty \quad \text{as } \rho \to +\infty. \tag{4}$$

the Riemann problem for the non-strictly hyperbolic system (3) has a unique admissible weak solution in the sense of the wave-fan criterion (Definition 3). We describe the structure of this solution in § 3.

Let us introduce the notation and definitions used in what follows. For points $a, b \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ we denote the interval $(\min\{a,b\}, \max\{a,b\})$ by $I(a,b)$ and its closure by $\overline{I(a,b)}$.

A function $F(\,\cdot\,)$ on an interval $\overline{I} \subset \mathbb{R}$ is called the convex (respectively, concave) hull of a function $f(\,\cdot\,)$ on $\overline{I}$ if it is the greatest convex function $F \leqslant f$ on $\overline{I}$ (respectively, the smallest concave function $F \geqslant f$ on $\overline{I}$).

We consider the transformation

$$T\colon \bigl[F\colon V \in (0,+\infty) \mapsto F(V)\bigr] \mapsto \bigl[P\colon \rho \in (0,+\infty) \mapsto -F(1/\rho)\bigr] \tag{5}$$

and denote by T^{-1} the transformation inverse to T. In the gas dynamics model T corresponds to the transition from the Eulerian to the Lagrangian representation. The problem in Lagrangian coordinates equivalent to (1), (2) for solutions without a vacuum was studied in detail in [2] and [15] (results on the admissibility of solutions for the problem without viscosity in Lagrangian coordinates were obtained in [16], [17]). The present paper is essentially based on the analysis carried out in [15].

Definition 1. Let $p\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous strictly increasing function. Then for $a \geqslant 0$ and $b \geqslant a$ the *lower hull* of $p(\,\cdot\,)$ on $[a,b]$ is the function $P(\,\cdot\,)$ such that

$$\begin{aligned} &\text{(i)}\ \ P(\,\cdot\,) \in C[a,b] \text{ and } P \leqslant p \text{ on } [a,b];\\ &\text{(ii)}\ \ \text{the function } F = T^{-1}P \text{ is concave on } (1/b, 1/a);\\ &\text{(iii)}\ \ \text{for each function } Q(\,\cdot\,) \text{ satisfying (i) and (ii) one has } P \geqslant Q \text{ on } [a,b]. \end{aligned} \tag{6}$$

For $b > 0$ and $a \geqslant b$ the upper hull of $p(\,\cdot\,)$ on $[b,a]$ is the function $P(\,\cdot\,)$ such that

$$\begin{aligned} &\text{(i}'\text{)}\ \ P(\,\cdot\,) \in C[b,a] \text{ and } P \geqslant p \text{ on } [b,a];\\ &\text{(ii}'\text{)}\ \ \text{the function } F = T^{-1}P \text{ is convex on } (1/a, 1/b);\\ &\text{(iii}'\text{)}\ \ \text{for each function } Q(\,\cdot\,) \text{ satisfying (i}'\text{) and (ii}'\text{) one has } P \leqslant Q \text{ on } [b,a]. \end{aligned} \tag{7}$$

The *hull* of $p(\,\cdot\,)$ on $\overline{I(a,b)}$ is its lower hull $[a,b]$ if $a \leqslant b$ and its upper hull on $[b,a]$ if $a > b$.

For $a = 0$ we set formally $1/a = +\infty$ in (6). Note that for all $a, b \in \mathbb{R}^+$ there exists a unique hull $P(\,\cdot\,)$ of the function $p(\,\cdot\,)$ on $\overline{I(a,b)}$. For $a, b \neq 0$ we have $P = TF$, where $F(\,\cdot\,)$ is the concave (respectively, convex) hull of $f = T^{-1}p$ on $[1/b, 1/a]$ (respectively, on $[1/a, 1/b]$). For $a = 0$ we construct $P(\,\cdot\,)$ as the decreasing limit of the functions $P_\delta(\,\cdot\,)$ such that $P_\delta(\,\cdot\,)\big|_{[0,\delta]} \equiv p(\delta)$ and $P_\delta(\,\cdot\,)\big|_{[\delta,b]}$ satisfies (6) on $[\delta, b]$; the properties (6) follow easily by the monotonicity and the continuity of $P(\,\cdot\,)$.

§ 1. Some properties of viscous approximations

We fix $\varepsilon > 0$.

Definition 2. A pair of functions $(\rho, u)\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ is a *solution of the Riemann problem* (1), (2) if $(\rho, u) \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$, for all $k > 0$ one has $(\rho, u)(t,x) = (\rho, u)(kt, kx)$ for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, (1) holds in the distribution space $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$, and

$$\operatorname*{ess\,lim}_{t\downarrow 0}\bigl(\|\rho(t,\,\cdot\,) - \rho(0,\,\cdot\,)\|_{L^1(-R,R)} + \|u(t,\,\cdot\,) - u(0,\,\cdot\,)\|_{L^1(-R,R)}\bigr) = 0 \tag{8}$$

for all $R > 0$, where $\rho(0,\,\cdot\,)$ and $u(0,\,\cdot\,)$ are defined by (2).

We shall denote x/t by ξ and use the same notation for a self-similar function of the variables (t,x) and the corresponding function of ξ.

Lemma 1. *A pair (ρ, u) is a solution of* (1), (2) *in the sense of Definition* 2 *if and only if the following conditions are fulfilled*:

(i) *there exist continuous bounded functions $\rho, u\colon \mathbb{R} \to \mathbb{R}$ with continuous $u'(\,\cdot\,)$ and $(\,\cdot - u(\,\cdot\,))\rho'(\,\cdot\,)$ such that $(\rho, u)(t,x) = (\rho, u)(x/t)$ for almost all points $(t,x) \in (0, +\infty) \times \mathbb{R}$;*

(ii) *there exists a constant $C \in \mathbb{R}$ such that*

$$\varepsilon u'(\xi) = -\int_0^\xi (\zeta - u(\zeta))^2 \rho'(\zeta)\, d\zeta + p(\rho(\xi)) + C, \tag{9}$$

$$\rho(\xi) u'(\xi) = (\xi - u(\xi))\rho'(\xi), \tag{10}$$

$$\lim_{\xi \to \pm\infty} \rho(\xi) = \rho_\pm, \quad \lim_{\xi \to \pm\infty} u(\xi) = u_\pm. \tag{11}$$

In addition, there exists a unique ξ_0 such that $u(\xi_0) = \xi_0$.

In the case $\rho(\xi_0) > 0$ there exist $\xi_\pm \in \overline{\mathbb{R}}$ such that $\xi_- \leqslant \xi_0 \leqslant \xi_+$ and both $\rho(\,\cdot\,)$, and $u(\,\cdot\,)$ are constant on (ξ_-, ξ_+) and strictly monotone on $(-\infty, \xi_-)$ and $(\xi_+, +\infty)$.

In the case $\rho(\xi_0) = 0$, ξ_0 is the unique vacuum point of the solution, $\xi_\pm = \xi_0$, and the same monotonicity properties hold.

Moreover, $u'(\xi) \neq 0$ for all $\xi \in (-\infty, \xi_-) \cup (\xi_+, +\infty)$, $u'(\xi_0) = 0$ in the case $\rho(\xi_0) > 0$, and $0 \leqslant u'(\xi_0) < 1$ in the case $\rho(\xi_0) = 0$.

Remark 1. We see from Lemma 1 that for $\varepsilon > 0$ solutions of (1), (2) contain at most one vacuum point. Note that the absence of intervals with density zero for solutions of the system (1) regularized with the more conventional viscosity term εu_{xx} in the second equation has been proved in [18] for solutions of the Cauchy problem with general initial data.

Proof. The proof consists of four steps.

(I) Let (ρ, u) be a solution of (1), (2). Then $(\rho, u)(t, x) = (\rho, u)(x/t)$ and we have

$$\begin{cases} -\xi\rho' + (\rho u)' = 0, \\ -\xi(\rho u)' + (\rho u^2 + p(\rho))' = \varepsilon u'' \end{cases} \tag{12}$$

in $\mathcal{D}'(\mathbb{R})$. Hence $\varepsilon u'' = -\xi^2\rho' + (\rho u^2 + p(\rho))' = -(\xi^2\rho)' + 2\xi\rho + (\rho u^2 + p(\rho))'$ in $\mathcal{D}'(\mathbb{R})$, so that $u' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $u \in C(\mathbb{R})$. Thus, $\rho u'$ and $(\rho u)u'$ are well defined in $\mathcal{D}'(\mathbb{R})$, therefore so are $\rho' u$ and $\rho' u^2$, and we obtain

$$-(\xi - u)\rho' + \rho u' = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}) \tag{13}$$

and

$$-(\xi - u)^2\rho' + p(\rho)' = \varepsilon u'' \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{14}$$

It now follows that $(\xi - u)\rho = \displaystyle\int_0^\xi \rho(\zeta)\,d\zeta + \mathrm{const} \in C(\mathbb{R})$. Consequently, we have $(\xi - u)^2\rho + p(\rho) - \varepsilon u' = \displaystyle\int_0^\xi 2(\zeta - u(\zeta))\rho(\zeta)\,d\zeta + \mathrm{const} \in C^1(\mathbb{R})$. Thus,

$$p(\rho(\,\cdot\,)) - \varepsilon u'(\,\cdot\,) \in C(\mathbb{R}). \tag{15}$$

(II) Consider now the function $\xi \mapsto \xi - u(\xi)$. It is continuous and approaches $\pm\infty$ as $\xi \to \pm\infty$. Hence there exist finite quantities $\eta_- := \min\{\xi : u(\xi) = \xi\}$ and $\eta_+ := \max\{\xi : u(\xi) = \xi\}$. One has $\xi - u(\xi) < 0$ on $(-\infty, \eta_-)$ and $\xi - u(\xi) > 0$ on $(\eta_+, +\infty)$. By (13), $\rho' \in L^\infty_{\mathrm{loc}}$ on each of these intervals. It follows that ρ is continuous on these intervals; so are also u' by (15) and ρ' by (13).

We now claim that ρ and u are monotone on $(-\infty, \eta_-)$ and $(\eta_+, +\infty)$. We set $\xi_+ := \sup\{\xi \geqslant \eta_+ : \rho|_{(\eta_+, \xi]} \equiv \mathrm{const}\}$. We show first that u is strictly monotone on $(\xi_+, +\infty)$. Indeed, assume the contrary. Then there exist $c \in (\xi_+, +\infty)$ and δ, $0 < \delta < c - \xi_+$, such that $u'(c) = 0$ and $u' \neq 0$ on $(c - \delta, c)$. For definiteness, assume that $u' > 0$ on $(c - \delta, c)$. Then we see from (13) that $\rho' = \dfrac{\rho u'}{\xi - u} \geqslant 0$ on $(c - \delta, c)$, so that $p(\rho(\,\cdot\,))$ is non-decreasing on $[c - \delta, c]$. Hence by (13) and (14) we obtain

$$\varepsilon u'(\xi) = \int_\xi^c (\zeta - u)\rho u'\,d\zeta + p(\rho(\xi)) - p(\rho(c)) \leqslant \int_\xi^c (\zeta - u)\rho u'\,d\zeta$$

pointwise on $[c-\delta, c]$. Choosing a sequence $\xi_n \uparrow c$ such that $u'(\xi_n) = \max_{\xi\in[\xi_n,c]} u'(\xi)$ we obtain that $\varepsilon u'(\xi_n) \leqslant u'(\xi_n)\mathcal{O}(c-\xi_n)$, where $\mathcal{O}(\,\cdot\,)$ is a bounded function. As $n \to \infty$, we obtain $\varepsilon \leqslant 0$, which is a contradiction.

We conclude that u is strictly monotone on $(\xi_+, +\infty)$; by (13) and the definition of ξ_+, ρ is strictly monotone on $(\xi_+, +\infty)$. In a similar way there exists $\xi_- \leqslant \eta_-$ such that u and ρ are strictly monotone on $(-\infty, \xi_-)$ and $\rho \equiv$ const on $[\xi_-, \eta_-)$. We have $\rho \equiv$ const on $(\eta_+, \xi_+]$ and $[\xi_-, \eta_-)$; by (14), u' is constant on either of these intervals. Taking into account the continuity of u' at $\xi_\pm$ we conclude that u is monotone on $(-\infty, \eta_-)$ and $(\eta_+, +\infty)$.

(III) We now investigate the behaviour of ρ and u on the interval $[\eta_-, \eta_+]$. Note first that there exist finite limits $\rho(\eta_\pm \pm 0)$; by (15) there exist finite limits $u'(\eta_\pm \pm 0)$. Note also that $u'(\eta_+ + 0) = 0$ in the case of $\rho(\eta_+ + 0) > 0$ and $0 \leqslant u'(\eta_+ + 0) < 1$ if $\rho(\eta_+ + 0) = 0$. The same relation exists between $u'(\eta_- - 0)$ and $\rho(\eta_- - 0)$.

Indeed, assume that $\rho(\eta_+ + 0) > 0$. By (13) we obtain

$$\int_{\eta_+}^{\eta_+ + 1} \big(\ln \rho(\zeta)\big)'\, d\zeta = \int_{\eta_+}^{\eta_+ + 1} \frac{u'(\zeta)}{\zeta - u(\zeta)}\, d\zeta.$$

The integral on the left-hand side converges, therefore the limit $u'(\eta + 0)$ must be zero. Further, let $\rho(\eta_+ + 0) = 0$. Then ρ is non-decreasing on $(\eta_+, +\infty)$, therefore $u'(\eta_+ + 0) \geqslant 0$ by (13). On the other hand, the definition of η_+ immediately shows that $u'(\eta_+ + 0) \leqslant 1$. Assume that $u'(\eta_+ + 0) = 1$. From (14) and (13) we obtain

$$\varepsilon(1 - u'(\xi)) = \int_{\eta_+}^{\xi} (\zeta - u(\zeta))\rho(\zeta)u'(\zeta)\, d\zeta - p(\rho(\xi))$$

for all $\xi \in (\eta_+, +\infty)$. Since $\rho u'$ is continuous on this interval and approaches zero as $\xi \downarrow \eta_+$, there exists $\delta > 0$ such that

$$\varepsilon(\xi - u(\xi))' \leqslant \int_{\eta_+}^{\xi} (\zeta - u(\zeta))\, d\zeta \tag{16}$$

for all $\xi \in (\eta_+, \eta_+ + \delta)$. Setting $g(\xi) := \xi - u(\xi)$ and $h(\xi) := \sqrt{\varepsilon} g(\xi) + \displaystyle\int_{\eta_+}^{\xi} g(\zeta)\, d\zeta$, by (16) we obtain $h \in C^1[\eta_-, +\infty)$ and $h'(\xi) \leqslant 1/\sqrt{\varepsilon}\, h(\xi)$. Since $h(\eta_+) = 0$, it follows by Gronwall's inequality that $h \equiv 0$ on $[\eta_+, \eta_+ + \delta]$. This contradicts the definition of η_+. Thus, finally, $0 \leqslant u'(\eta_+ + 0) < 1$. The proof for $\eta_- - 0$ in place of $\eta_+ + 0$ is similar.

We now analyse separately three cases:

(a) $\eta_- = \eta_+ =: \xi_0$ and one of the limits $\rho(\xi_0 \pm 0)$ is non-zero. Then these two limits are equal. For assume that $\rho(\xi_0 + 0) > 0$. We assume first that $\rho(\xi_0 - 0) > 0$; in this case $u'(\xi_0 \pm 0) = 0$, so that $\rho(\xi_0 \pm 0)$ are equal by (15). Further, let $\rho(\xi_0 - 0) = 0$. In this case $p(\rho(\xi_0 + 0)) - \varepsilon u'(\xi_0 + 0) = p(\rho(\xi_0 + 0)) > 0$ and $p(\rho(\xi_0 - 0)) - \varepsilon u'(\xi_0 - 0) = -\varepsilon u'(\xi_0 - 0) \leqslant 0$, which contradicts (15). Hence we see that both ρ and u' are continuous on $\mathbb{R}$ and (13), (14) can be written in the form (9), (10).

We see that in this case there exists a unique ξ_0 such that $u(\xi_0) = \xi_0$, and we have $\rho(\xi_0) > 0$ and $u'(\xi_0) = 0$. Moreover, with $\xi_\pm$ defined as above we see that ρ and u are constant on $[\xi_-, \xi_+]$ and that u' — and therefore also ρ' – is non-zero on $(-\infty, \xi_-) \cup (\xi_+, +\infty)$.

(b) $\eta_- = \eta_+ =: \xi_0$ and $\rho(\xi_0 \pm 0) = 0$. Then ρ and u' are continuous on $\mathbb{R}$, which yields (9), (10). Further, $\xi = \xi_0$ is the unique vacuum point. Indeed, let $\xi_+ = \sup\{\xi : \rho(\xi) = 0\}$. We have

$$\int_{\xi_+}^{\xi_+ +1} \big(\ln \rho(\zeta)\big)' \, d\zeta = \int_{\xi_+}^{\xi_+ +1} \frac{u'(\zeta)}{\zeta - u(\zeta)} \, d\zeta;$$

but the integral on the left-hand side diverges. Hence $\zeta - u(\zeta) \to 0$ as $\xi \downarrow \xi_+$, therefore $\xi_+ = \xi_0$.

We see that in this case there exists a unique ξ_0 such that $u(\xi_0) = \xi_0$, and we have $\rho(\xi_0) = 0$ and $0 \leqslant u'(\xi_0) < 1$; moreover, u' and ρ' are non-zero on $\mathbb{R} \setminus \{\xi_0\}$.

(c) $\eta_- < \eta_+$. We shall show that this case is actually impossible. Indeed, from equality (13) we obtain $\rho = ((\xi - u)\rho)'$ in $\mathcal{D}'(\mathbb{R})$. Hence

$$\int_{\eta_-}^{\eta_+} \rho(\zeta) \, d\zeta = \big[(\xi - u(\xi))\rho(\xi)\big]\Big|_{\eta_-}^{\eta_+} = 0,$$

so that $\rho\big|_{(\eta_-, \eta_+)} \equiv 0$. Hence $u'\big|_{(\eta_-, \eta_+)} \equiv \text{const}$ by (14); taking into account the equality $u(\eta_\pm) = \eta_\pm$ we see that $u'(\eta_\pm \mp 0) = 1$. Furthermore, $0 \leqslant u'(\eta_\pm \pm 0) < 1$ in all cases; arguing as in case (a) we arrive at a contradiction with (15).

We conclude that $\rho(\,\cdot\,)$ and $u(\,\cdot\,)$ satisfy (9), (10) and have all the continuity and monotonicity properties indicated in Lemma 1. It immediately follows from the monotonicity of ρ and u that (11) holds if and only if the self-similar functions $\rho(\,\cdot\,,\cdot\,)$ and $u(\,\cdot\,,\cdot\,)$ satisfy (8).

(IV) Conversely, (9)–(11) in combination with the continuity of $\rho(\,\cdot\,)$, $u(\,\cdot\,)$, $u'(\,\cdot\,)$, and $(\,\cdot - u(\,\cdot\,))\rho'(\,\cdot\,)$ show that $(\rho, u)(t, x) := (\rho, u)(x/t)$ is a solution of (1), (2). Indeed, (12) is straightforward. Furthermore, (9), (10) yield the monotonicity of $\rho(\,\cdot\,)$ and $u(\,\cdot\,)$ at infinity. Thus, (8) also holds.

Using the results of Lemma 1 we set $\rho_0 := \rho(\xi_0)$ and $k := u'(\xi_0)$; let $\sigma := \rho_0 - k$. Note that $\sigma \in (-1, +\infty)$ and

$$\rho_0 = (\sigma)^+ = \max\{\sigma, 0\}, \qquad k = (\sigma)^- = \max\{-\sigma, 0\}. \tag{17}$$

Next, we partition $\mathbb{R}$ into the three (possibly empty) intervals $(-\infty, \xi_-)$, (ξ_-, ξ_+), and (ξ_+, ∞). We invert $\rho(\,\cdot\,)$ on $(-\infty, \xi_-)$ and $(\xi_+, +\infty)$ and consider the functions

$$\rho_-^{-1} \colon I(\rho_0, \rho_-) \to (-\infty, \xi_-), \qquad \rho_+^{-1} \colon I(\rho_0, \rho_+) \to (\xi_+, +\infty);$$

in what follows we shall use the same notation for the function $\rho(\,\cdot\,)$ and the independent variable $\rho \in \mathbb{R}^+$. We set

$$\Pi_\pm^\varepsilon(\rho; \sigma) := \int_{(\sigma)^+}^{\rho} \big[\rho_\pm^{-1}(r) - u(\rho_\pm^{-1}(r))\big]^2 \, dr - C \tag{18}$$

for $\rho \in I(\rho_0, \rho_\pm)$, where $C = \varepsilon u'(\xi_0) - p(\rho_0)$. We shall use the simplified notation $\Pi_\pm(\,\cdot\,)$ for $\Pi^\varepsilon_\pm(\,\cdot\,;\sigma)$ in the case when ε and σ are fixed; we also denote by a dot differentiation with respect to ρ. We can now write (9) in the following form:

$$\varepsilon u'(\xi) = p(\rho(\xi)) - \Pi_\pm(\rho(\xi)). \tag{19}$$

Since $0 \notin I(\rho_0, \rho_\pm)$ and $u'(\xi)$ is shown to be non-zero on $(-\infty, \xi_-)$ and $(\xi_+, +\infty)$ we conclude that $\Pi_\pm \in C^2(I(\rho_0, \rho_\pm))$ and

$$\rho\ddot{\Pi}_\pm + 2\dot{\Pi}_\pm = \frac{2\varepsilon\dot{\Pi}_\pm}{p - \Pi_\pm}\,, \quad \dot{\Pi}_\pm > 0 \quad \text{and} \quad \operatorname{sign}(p - \Pi_\pm) = \operatorname{sign}(\rho_\pm - \rho_0) \tag{20}$$

on $I(\rho_0, \rho_\pm)$. Further, $\Pi_\pm$ can be extended to $\overline{I(\rho_0, \rho_\pm)}$ by continuity. By (19) and Lemma 1, $\Pi_\pm(\,\cdot\,)$ have finite limits at ρ_0 and we can set

$$\Pi_\pm(\rho_0) = p(\rho_0) - \varepsilon k, \tag{21}$$

where ρ_0 and k are defined by relation (17). Furthermore,

$$\Pi_\pm(\rho_\pm) = p(\rho_\pm). \tag{22}$$

Indeed, the right-hand side of (19) has finite limits as $\rho(\xi) \to \rho_\pm$ because in the case when $\rho_0 < \rho_\pm$ the functions $\Pi_\pm(\,\cdot\,)$ are increasing and bounded above, while in the case when $\rho_0 > \rho_\pm$ they are concave and bounded below. We see that $u'(\xi) \to 0$ as $\xi \to \pm\infty$ since u has finite limits at $\pm\infty$.

In addition, to cover the case $\xi_- = -\infty$ (that is, of $\rho_0 = \rho_-$) and/or $\xi_+ = +\infty$ (that is, $\rho_0 = \rho_+$), we merely define $\Pi_-(\,\cdot\,)$ and/or $\Pi_+(\,\cdot\,)$ by means of (22).

Finally, (18) and (10) yield $u'(\xi) = \pm\sqrt{\dot{\Pi}_\pm(\rho)}\,\dfrac{\rho'(\xi)}{\rho}$ for all $\xi \in (-\infty, \xi_-)$ and all $\xi \in (\xi_+, +\infty)$, respectively. Since $u(\xi_-) = u(\xi_+)$, it follows by (11) that

$$u_+ - u_- = \int_{\rho_0}^{\rho_+} \sqrt{\dot{\Pi}_+(r)}\,\frac{dr}{r} + \int_{\rho_0}^{\rho_-} \sqrt{\dot{\Pi}_-(r)}\,\frac{dr}{r} \tag{23}$$

and the integrals on the right-hand side of (23) are finite.

We have established the following result.

Proposition 1. *Let (ρ, u) be a solution of* (1), (2) *in the sense of Definition* 2. *Then there exist $\sigma \in (-1, +\infty)$ and functions $\Pi_\pm \in C(\overline{I(\rho_0, \rho_\pm)}) \cap C^2(I(\rho_0, \rho_\pm))$ such that relations* (20)–(23) *hold for $\rho_0 \in [0, +\infty]$ and $k \in [0, 1)$ defined by* (17).

The converse result is also true.

Proposition 2. *Let $\sigma \in (-1, +\infty)$ and let ρ_0 and k be as defined by* (17). *Let $\Pi_\pm \in C(\overline{I(\rho_0, \rho_\pm)}) \cap C^2(I(\rho_0, \rho_\pm))$ be functions satisfying* (20)–(23). *Then there* *exists a solution* (ρ, u) *of the problem* (1), (2) *in the sense of Definition* 2; *it is described by the formulae*

$$\rho(t,x) = \rho(x/t) = \begin{cases} \left[\Xi^\varepsilon_-\right]^{-1}(x/t), & x/t < \xi_-, \\ \left[\Xi^\varepsilon_+\right]^{-1}(x/t), & \xi_+ < x/t, \end{cases}$$
$$\equiv \begin{cases} \left[\Xi^\varepsilon_-\right]^{-1}(x/t), & x/t < \xi_-, \\ \rho_0, & \xi_- < x/t < \xi_+, \\ \left[\Xi^\varepsilon_+\right]^{-1}(x/t), & \xi_+ < x/t, \end{cases} \tag{24}$$

$$u(t,x) = u(x/t) = \begin{cases} U^\varepsilon_- \circ \left[\Xi^\varepsilon_-\right]^{-1}(x/t), & x/t < \xi_-, \\ U^\varepsilon_-(\rho_0) = U^\varepsilon_+(\rho_0), & \xi_- < x/t < \xi_+, \\ U^\varepsilon_+ \circ \left[\Xi^\varepsilon_+\right]^{-1}(x/t), & \xi_+ < x/t, \end{cases} \tag{25}$$

where

$$U^\varepsilon_\pm(\rho) := u_\pm \mp \int_\rho^{\rho_\pm} \sqrt{\dot\Pi_\pm(r)}\,\frac{dr}{r} \quad \textit{for } \rho \in \overline{I(\rho_0, \rho_\pm)}, \tag{26}$$

$$\Xi^\varepsilon_\pm(\rho) := U^\varepsilon_\pm(\rho) \pm \sqrt{\dot\Pi_\pm(\rho)} \qquad \textit{for } \rho \in I(\rho_0, \rho_\pm), \tag{27}$$

and the $\xi_\pm$ *are defined by the formulae*

$$\begin{aligned} &\xi_\pm := \lim_{\rho \in I(\rho_0,\rho_\pm),\, \rho\to\rho_0} \Xi^\varepsilon_\pm(\rho) \qquad \textit{for } \rho_0 \neq \rho_\pm, \\ &\xi_- := -\infty \ \textit{ and/or } \ \xi_+ := +\infty \quad \textit{for } \rho_0 = \rho_- \ \textit{ and/or } \ \rho_0 = \rho_+. \end{aligned} \tag{28}$$

Proof. The cases $\rho_0 = \rho_+$ and $\rho_0 = \rho_-$ are trivial; assume that $\rho_0 \neq \rho_\pm$. Let $U^\varepsilon_\pm$ and $\Xi^\varepsilon_\pm$ be defined by relations (26), (27). Note that both integrals in (26) must converge as $\rho \to \rho_0$, $\rho \in I(\rho_0, \rho_\pm)$. Indeed, if $\rho_0 = 0$, then both integrals are positive, and therefore finite, by (23). If, on the other hand, $\rho_0 > 0$ then the functions $\dot\Pi_\pm$ are bounded as $\rho \to \rho_0$ by the monotonicity of $\rho^2\dot\Pi_\pm$, which is obvious from (20). Hence $U^\varepsilon_\pm(\rho_0)$ are well defined; by (23) they are equal. Note also that the functions

$$\dot\Xi^\varepsilon_\pm(\rho) = \pm\frac{\rho\ddot\Pi_\pm(\rho) + 2\dot\Pi_\pm(\rho)}{2\rho\sqrt{\dot\Pi_\pm(\rho)}} = \pm\frac{\varepsilon\sqrt{\dot\Pi_\pm(\rho)}}{\rho(p(\rho) - \Pi_\pm(\rho))} \tag{29}$$

are continuous and non-zero on $I(\rho_0, \rho_\pm)$, so that there exist well-defined quantities $\xi_\pm$, and the functions $\left[\Xi^\varepsilon_\pm\right]^{-1} \colon I(\xi_\pm, \pm\infty) \to \mathbb{R}^+$ are also well defined in the sense of graphs. We see from (23), (26), and (27) that

$$\xi_+ - \xi_- = \lim_{\rho\in I(\rho_0,\rho_-),\, \rho\to\rho_0} \sqrt{\dot\Pi_-(\rho)} + \lim_{\rho\in I(\rho_0,\rho_+),\, \rho\to\rho_0} \sqrt{\dot\Pi_+(\rho)} \geqslant 0. \tag{30}$$

Clearly, $\rho, u \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{\xi_-, \xi_+\})$; furthermore, (24)–(27), (29), and (22) yield (11) and (19) for all $\xi \neq \xi_\pm$. In fact, we can show that (22) and (20) yield the relation $\sqrt{\dot\Pi_\pm(\rho)} \to +\infty$ as $\rho \to \rho_\pm$, $\rho \in I(\rho_0,\rho_\pm)$, so that the $\Xi^\varepsilon_\pm$ map $I(\rho_0,\rho_\pm)$ onto $(\xi_+,+\infty)$ and $(-\infty,\xi_-)$, respectively. However, even if we admit that $\Xi^\varepsilon_\pm$ can be bounded, equality (22) yields (19) for ξ outside the range of $\Xi^\varepsilon_\pm$. Furthermore, by (29), (19), (27), and (24), (25) for all $\xi \neq \xi_\pm$ we obtain

$$(\xi - u(\xi))\rho'(\xi) = \pm\frac{\rho(p(\rho(\xi)) - \Pi_\pm(\rho(\xi)))}{\pm\varepsilon} = \rho(\xi)u'(\xi); \tag{31}$$

hence (10) follows for $\xi \neq \xi_\pm$.

Consider now two possible cases.

(a) $\rho_0 > 0$. Then $u'(\xi_\pm \pm 0) = 0$ by (19) and (21), while if $\xi_- < \xi_+$, then we obtain $u'(\xi_\pm \mp 0) = 0$ from (25). Thus, $u \in C^1(\mathbb{R})$ and by (31), $(\,\cdot - u(\,\cdot\,))\rho'(\,\cdot\,)$ is continuous on $\mathbb{R}$. It now follows that (9) and (10) hold everywhere; the property (11) is obvious.

(b) $\rho_0 = 0$. Then $\dot\Pi_\pm(\rho) \to 0$ as $\rho \to \rho_0$, $\rho \in I(\rho_0,\rho_\pm)$. Indeed, by (20) for $\rho \in I(\rho_0,\rho_\pm)$ we obtain

$$\int_\rho^{(\rho_0+\rho_+)/2} \bigl(\ln\dot\Pi_\pm(r)\bigr)^{\cdot}\,dr = \int_\rho^{(\rho_0+\rho_+)/2} \frac{2}{r}\biggl(\frac{\varepsilon}{p(r)-\Pi_\pm(r)} - 1\biggr)\,dr.$$

By (21) and (17) it follows that $\varepsilon/(p(r) - \Pi_\pm(r)) - 1 \to 1/k - 1 > 0$ as $\rho \to \rho_0$. Hence the last integral diverges and $\ln\dot\Pi_\pm(\rho) \to -\infty$ as $\rho \to \rho_0$, $\rho \in I(\rho_0,\rho_\pm)$. We conclude from (30) that $\xi_- = \xi_+$; from (19) and (21) we see that $u'(\xi_\pm \pm 0) = k$. Thus, $u'(\,\cdot\,)$ and $(\,\cdot - u(\,\cdot\,))\rho'(\,\cdot\,)$ are continuous on $\mathbb{R}$ again and (9)–(11) hold.

By Lemma 1, (ρ,u) is a solution of (1), (2).

§ 2. Existence and uniqueness of viscous approximations

In § 1 we reduced the problem (1), (2) to finding $\sigma \in (-1,+\infty)$, ρ_0, k and a pair of functions $\Pi_\pm \in C(\overline{I(\rho_0,\rho_\pm)}) \cap C^2(I(\rho_0,\rho_\pm))$ satisfying (17) and (20)–(23). In this section we prove that such σ, ρ_0, k, $\Pi_\pm(\,\cdot\,)$ (uniquely) exist if $p(\,\cdot\,)$ satisfies (4) (see Proposition 3 and Lemmas 4–6). We also prove two preliminary convergence results (see Lemma 3 and Proposition 4).

We start by fixing $\sigma \in [-1,+\infty)$ and $b \in (0,+\infty)$. We set $a := (\sigma)^+$, $k := (\sigma)^-$ and consider the problem of finding a function $\Pi \in C(\overline{I(a,b)}\,) \cap C^2(I(a,b))$ such that

$$\begin{gathered} \rho\ddot\Pi(\rho) + 2\dot\Pi(\rho) = \frac{2\varepsilon\dot\Pi(\rho)}{p(\rho)-\Pi(\rho)}\,, \\ \dot\Pi(\rho) > 0, \quad (b-a)(p(\rho)-\Pi(\rho)) > 0 \quad \text{for all } \rho \in I(a,b), \\ \Pi(a) = p(a) - \varepsilon k, \qquad \Pi(b) = p(b). \end{gathered} \tag{32}$$

Proposition 3. *The problem* (32) *is uniquely soluble.*

Let $\Pi^\varepsilon(\,\cdot\,)$ be the solution of (32) corresponding to fixed positive ε.

Proposition 4. *As* $\varepsilon \downarrow 0$, Π^ε *converges to the hull of* $p(\,\cdot\,)$ *uniformly on* $\overline{I(a,b)}$ (*see Definition* 1).

For the proof of these results note first that a maximum principle holds for the equation in (32). We state it in the case $a < b$.

Lemma 2 (maximum principle). *Let* $\Pi, \Upsilon \in C[a,b] \cap C^2(a,b)$ *be functions satisfying the equations* $\ddot{\Pi}(\rho) = G(\rho, \Pi(\rho), \dot{\Pi}(\rho))$ *and* $\ddot{\Upsilon}(\rho) = H(\rho, \Upsilon(\rho), \dot{\Upsilon}(\rho))$, *respectively, for all* $\rho \in (a,b)$, *with some* $G, H : (a,b) \times \mathbb{R} \times (0,+\infty) \to (0,+\infty]$.

(a) *Assume that* $G(\rho, z, w) < H(\rho, \zeta, w)$ *for all* $\rho \in (a,b)$ *such that* $\Pi(\rho) < \Upsilon(\rho)$ *and all* z, ζ, w *such that* $z < \zeta$. *Then* $\Pi \geqslant \Upsilon$ *on* $[a,b]$ *whenever* $\Pi(a) \geqslant \Upsilon(a)$ *and* $\Pi(b) \geqslant \Upsilon(b)$.

(b) *Assume that* $G(\rho, z, w)$ *coincides with* $H(\rho, z, w)$ *and strictly increases in* z; *let* $\Pi(a) = \Upsilon(a)$ *or* $\Pi(b) = \Upsilon(b)$. *Then* $\Pi - \Upsilon$ *is monotone on* $[a,b]$.

The proof is standard.

Secondly, we point out the following result which will be particularly important in § 3. Let T be the transformation defined by (5); similarly to Definition 1 we set $1/a = +\infty$ for $a = 0$.

Lemma 3. *Let* $[a,b] \subset \mathbb{R}^+$ *and let* $\{P^\varepsilon\}_{\varepsilon \geqslant 0} \subset C[a,b]$ *be a set of functions such that the functions* $F^\varepsilon = T^{-1}P^\varepsilon$ *are concave on* $(1/b, 1/a)$. *Assume further that for all* $\rho \in [a,b]$ *the* $P^\varepsilon(\rho)$ *converge to* $P^0(\rho)$ *as* $\varepsilon \downarrow 0$. *Then the following assertions hold.*

(a) *This convergence is uniform on each closed subinterval* $[c,d]$ *of* (a,b), *and for each* ε, P^ε *is absolutely continuous on each closed subinterval* $[c,d]$ *of* (a,b). *Moreover, if the* P^ε *are increasing functions, then their derivatives* $\dot{P}^\varepsilon$ *are uniformly bounded in* ε *on each subinterval* $[c,d]$ *of* $[a,b)$ *such that* $c > 0$.

(b) *For an arbitrary sequence* $\varepsilon_n \downarrow 0$, $\dot{P}^{\varepsilon_n} \to \dot{P}^0$ *a.e. on* (a,b); *moreover, if* $P^{\varepsilon_n}, P^0 \in C^1(a,b)$, *then the convergence holds everywhere on* (a,b).

(c) *Let the* P^ε *be increasing functions and let*

$$\Xi^\varepsilon_\pm(\rho) := \text{const} \mp \int_\rho^b \sqrt{\dot{P}^\varepsilon(r)}\, \frac{dr}{r} \pm \sqrt{\dot{P}^\varepsilon(\rho)}. \tag{33}$$

Then the $\Xi^\varepsilon_\pm(\,\cdot\,)$ *are monotone functions defined a.e. on* (a,b), *so that* $\left[\Xi^\varepsilon_\pm\right]^{-1}$ *are well-defined monotone set-valued functions.*

(d) *If the* P^ε *are increasing functions, then on each closed subinterval* $[c,d]$ *of* $[a,b]$ *such that* $c > 0$, *for an arbitrary sequence* $\varepsilon_n \downarrow 0$ *the convergence*

$$\int_\rho^b \sqrt{\dot{P}^{\varepsilon_n}(r)}\, \frac{dr}{r} \to \int_\rho^b \sqrt{\dot{P}^0(r)}\, \frac{dr}{r}$$

holds uniformly in $\rho \in [c,d]$.

(e) *With the notation of* (c), $\left[\Xi^{\varepsilon_n}_+\right]^{-1}(\xi) \to \left[\Xi^0_+\right]^{-1}(\xi)$ *for an arbitrary sequence* $\varepsilon_n \downarrow 0$ *and for all* ξ *such that* $\left[\Xi^0_+\right]^{-1}$ *is continuous at the point* ξ. *The same holds for* $\left[\Xi^{\varepsilon_n}_-\right]^{-1}$ *and* $\left[\Xi^0_-\right]^{-1}$ *in place of* $\left[\Xi^{\varepsilon_n}_+\right]^{-1}$ *and* $\left[\Xi^0_+\right]^{-1}$, *respectively.*

Similar properties hold if the P^ε *are defined on* $[b,a] \subset \mathbb{R}^+ \backslash \{0\}$ *and the* $F^\varepsilon = T^{-1}P^\varepsilon$ *are convex on* $(1/a, 1/b)$.

Proof. Since the functions $F^\varepsilon = T^{-1}P^\varepsilon$ are concave, they are differentiable almost everywhere. Moreover, the convergence of the $P^\varepsilon(\rho)$ to $P^0(\rho)$ shows that the

$F^\varepsilon(1/\rho)$ converge to $F^0(1/\rho)$ as $\varepsilon \downarrow 0$. Hence $F^\varepsilon \to F^0$ uniformly on each closed subinterval $[1/d, 1/c]$ of $(1/b, 1/a)$ and $\dfrac{d}{dV}F^{\varepsilon_n} \to \dfrac{d}{dV}F^0$ a.e. on $(1/b, 1/a)$ as $\varepsilon \downarrow 0$. Since $\dot{P}^\varepsilon(\rho) = \dfrac{1}{\rho^2}\dfrac{d}{dV}F^\varepsilon(V)$ whenever $\dfrac{d}{dV}F^\varepsilon(V)$ exists, (a) and (b) are obvious.

Further, $\dfrac{d}{dV}F^\varepsilon \geqslant 0$ in case (c). Setting $v = 1/r$ in the integral in (33) we obtain

$$\Xi^\varepsilon_\pm(\rho) = \text{const} \pm \int_{1/b}^{1/\rho} \left(\sqrt{\frac{d}{dV}F^\varepsilon\left(\frac{1}{\rho}\right)} - \sqrt{\frac{d}{dV}F^\varepsilon(v)} \right) dv \pm \frac{1}{b}\sqrt{\frac{d}{dV}F^\varepsilon\left(\frac{1}{\rho}\right)}.$$

This is a monotone function because $\dfrac{d}{dV}F^\varepsilon$ is monotone; this yields (c). Property (d) follows from the continuity and the convergence of the $F^\varepsilon(\,\cdot\,)$ at $V = 1/b$. Indeed, for $\rho > c > 0$ one has

$$\left| \int_\rho^b \sqrt{\dot{P}^{\varepsilon_n}(r)}\,\frac{dr}{r} - \int_\rho^b \sqrt{\dot{P}^0(r)}\,\frac{dr}{r} \right| \leqslant \int_{1/b}^{1/\rho} \left| \sqrt{\frac{d}{dV}F^{\varepsilon_n}(v)} - \sqrt{\frac{d}{dV}F^0(v)} \right| dv. \quad (34)$$

We fix $\delta > 0$ and integrate $\left| \sqrt{\dfrac{d}{dV}F^{\varepsilon_n}(V)} - \sqrt{\dfrac{d}{dV}F^0(V)} \right|$ separately over the intervals $(1/b, 1/b+\delta)$ and $(1/b+\delta, 1/c)$. For each $\delta > 0$ the second integral vanishes as $\varepsilon_n \downarrow 0$ by property (a) and Lebesgue's bounded convergence theorem. In addition, the first integral can be made arbitrarily small by one's choice of sufficiently small δ because

$$\begin{aligned} \int_{1/b+\delta}^{1/b} \sqrt{\frac{d}{dV}F^{\varepsilon_n}(v)}\,dv &\leqslant \int_{1/b+\delta}^{1/b} \left(1 + \frac{d}{dV}F^{\varepsilon_n}(v)\right) dv \\ &= \delta + \left(F^{\varepsilon_n}\left(\frac{1}{b}+\delta\right) - F^{\varepsilon_n}\left(\frac{1}{b}\right) \right) \\ &\leqslant 2\delta + \left(F^0\left(\frac{1}{b}+\delta\right) - F^0\left(\frac{1}{b}\right) \right) \end{aligned}$$

for sufficiently small ε_n. Hence the left-hand side of (34) can be made arbitrarily small uniformly in $\rho \in [c, d]$. Moreover, if $a > 0$, then we can set $c = a$ in the above argument and prove (d).

Finally, it follows by (b) and (d) that $\Xi^{\varepsilon_n}_\pm \to \Xi^0_\pm$ a.e. on (a, b). It is well known in basic probability theory that the convergence a.e. of a sequence of monotone functions (interpreted as random variables) implies the pointwise convergence of their inverse functions (interpreted as their distribution functions) at the continuity points of the limit. This proves (e).

The case of convex functions F^ε is similar.

Proof of Proposition 3. The case $a = b$ is trivial; for definiteness assume that $a < b$, which is the most complex case. The uniqueness is immediate by Lemma 2. For each $a > 0$ we could prove the solubility directly by penalization of the right-hand side of (32), applying Lemmas 2, 3 and following the corresponding proof in [15]. Instead, we perform the transformation

$$T^{-1}\colon \big[\rho \in (a,b) \mapsto \Pi(\rho)\big] \mapsto \big[V \in (1/b, 1/a) \mapsto -\Pi(1/V)\big].$$

It reduces the equation in (32) to the following one:

$$\frac{d^2}{dV^2}\Phi(V) = \frac{1}{V}\,\frac{2\varepsilon \frac{d}{dV}\Phi(V)}{f(V) - \Phi(V)} \tag{35}$$

with $\dfrac{d}{dV}\Phi > 0$ and $\Phi > f$ on $(1/b, 1/a)$, where $\Phi = T^{-1}\Pi$ and $f = T^{-1}p$. This equation differs from the one considered in [15] only by the coefficient $1/V$ on the right-hand side. This coefficient is continuous and bounded on $(1/b, 1/a)$ since $a > 0$, so that the proof in [15] applies without modification. Hence the problem (35) has a strictly increasing concave solution $\Phi \in C[1/b, 1/a] \cap C^2(1/b, 1/a)$ such that $\Phi(1/a) = f(1/a)$, $\Phi(1/b) = f(1/b)$. Then $\Pi = T\Phi \in C[a,b] \cap C^2(a,b)$ and Π satisfies (32).

For $a = 0$, first, we find for each $\delta \in (0,b)$ a function $\Pi_\delta \in C[0,b] \cap C^2(\delta, b)$ such that

$$\rho\ddot{\Pi}_\delta + 2\dot{\Pi}_\delta = \frac{2\varepsilon\dot{\Pi}_\delta}{p - \Pi_\delta}, \qquad \dot{\Pi}_\delta > 0, \quad p - \Pi_\delta > 0 \quad \text{on } [\delta, b),$$
$$\Pi_\delta(b) = p(b), \qquad \Pi_\delta\big|_{[0,\delta]} = -k\varepsilon.$$

The existence proof proceeds as above. By Lemma 2(b) there exists a function Π on $[0,b]$ such that $\Pi_\delta \uparrow \Pi$ as $\delta \downarrow 0$. Furthermore, applying again the proof from [15] to the functions $T^{-1}\Pi_\delta$ we conclude that $T^{-1}\Pi \in C^2(1/d, 1/c)$ and $T^{-1}\Pi$ satisfies (35) on $(1/d, 1/c)$ for each closed subinterval $[1/d, 1/c]$ of $(1/b, +\infty)$. Hence $\Pi \in C^2(0,b)$ and the equation in (32) holds. In addition, similarly to [15], the continuity of Π at 0 and b follows from Lemma 2(a) by comparing with special solutions of (32). Thus, $\Pi(0) = -k\varepsilon$ and $\Pi(b) = p(b)$ by the construction of Π_δ.

Proof of Proposition 4. Assume that $a < b$ and let $P(\,\cdot\,)$ be the hull of $p(\,\cdot\,)$ on $[a,b]$. We fix $\alpha > 0$ and construct an auxiliary barrier function $\Upsilon_\alpha \in C^2(a,b) \cap C[a,b]$ such that $\rho\ddot{\Upsilon}_\alpha + 2\dot{\Upsilon}_\alpha \geqslant m(\alpha) > 0$ and $\alpha/2 \leqslant P - \Upsilon_\alpha < \alpha$ on (a,b). After that, we apply Lemma 2(a) to Π^ε and Υ_α. By Lemma 3(a), $\frac{1}{\rho}\dot{\Pi}^\varepsilon$ is uniformly bounded on each closed interval $[c,d]$ of (a,b). It follows that $P \geqslant \Pi^\varepsilon \geqslant \Upsilon_\alpha$ on $[a,b]$ for all sufficiently small ε.

Finally, we note the following property of solutions of (32).

Lemma 4. *Assume that* $\Pi(\,\cdot\,;\sigma)$ *satisfies* (32) *with* $a = 0$ *and* $k = -\sigma \in [0,1)$. *Then* $\dot{\Pi}(\rho;\sigma) \to 0$ *as* $\rho \to 0$ *and the integral* $S(\sigma) = \displaystyle\int_0^b \sqrt{\dot{\Pi}(r;\sigma)}\,\frac{dr}{r}$ *is finite. In addition,* $S(\sigma) \uparrow +\infty$ *as* $\sigma \downarrow -1$.

Proof. We shall prove that the integral $S(\sigma)$ converges (at the lower limit of integration). We set $\varkappa = 1 - (1-k)/2 \in [1/2, 1)$. Since Π and p are continuous on

$[0, b]$ and $p(0) - \Pi(0; \sigma) = k\varepsilon$, there exists $\delta > 0$ such that $p - \Pi \leqslant \varkappa\varepsilon$ on $(0, \delta)$. By (32), on $(0, \delta)$ we obtain

$$\dot{\Pi}(\delta; \sigma) \geqslant \dot{\Pi}(\rho; \sigma) + \int_{\rho}^{\delta} \left(\frac{\varepsilon}{\varkappa\varepsilon} - 1 \right) \frac{2}{r} \dot{\Pi}(r; \sigma) \, dr.$$

Hence $\dot{\Pi}(\rho; \sigma) \leqslant \text{const} \, \rho^{2(1/\varkappa - 1)}$ by Gronwall's inequality, so that the first two assertions of the lemma are obvious.

Consider now the function $\dot{\Pi}(\,\cdot\,; -1)$. If $S(-1)$ diverges, then $\lim_{\sigma \downarrow -1} S(\sigma) = +\infty$ by Lemma 2(b) and Levi's theorem. We assume that $S(-1) < +\infty$ and shall arrive at a contradiction.

Consider the function $\Xi_+(\rho) = \sqrt{\dot{\Pi}_+(\rho; -1)} - \int_{\rho}^{b} \sqrt{\dot{\Pi}_+(r; -1)} \, \frac{dr}{r}$. By (32) we obtain $\dot{\Xi}_+ > 0$ on $(0, b]$. Since $\Xi_+(\rho) \geqslant -S(-1) > -\infty$, Ξ_+ has a finite limit as $\rho \to 0$. Hence $\sqrt{\dot{\Pi}_+(\rho; -1)}$ approaches zero as $\rho \to 0$. We now set $\xi_+ = \lim_{\rho \to 0} \Xi_+(\rho)$ and consider the C^1-functions $\rho, u \colon \xi \in [\xi_+, +\infty) \mapsto \rho(\xi), u(\xi)$ defined by formulae (24)–(26) with $u_+ = 0$. Writing (32) in terms of ξ, ρ, and u we obtain $\rho(\xi_+) = 0$, $\xi_+ - u(\xi_+) = 0$, $u'(\xi_+ + 0) = 1$, and

$$\varepsilon\big(u'(\xi) - u'(\xi_+ + 0)\big) = -\int_{\xi_+}^{\xi} (\zeta - u(\zeta))\rho(\zeta)u'(\zeta) \, d\zeta + p(\rho(\xi))$$

for all $\xi > \xi_+$. As shown in part (III) of the proof of Lemma 1, these properties are incompatible.

For $\sigma \in (-1, +\infty)$ we denote by $S^{\varepsilon}_{\pm}(\sigma)$ the integrals $\int_{(\sigma)^+}^{\rho_\pm} \sqrt{\dot{\Pi}^{\varepsilon}_{\pm}(r; \sigma)} \, \frac{dr}{r}$, where $\Pi^{\varepsilon}_{\pm}(\,\cdot\,; \sigma)$ are the unique solutions of (20)–(22), which exist by Proposition 3. By Lemma 4, the $S^{\varepsilon}_{\pm}(\sigma)$ are finite for $\sigma \in (-1, 0]$; the same obviously holds for σ lying in $(0, +\infty)$.

The next step consists in finding σ such that (23) holds, that is,

$$u_+ - u_- = S^{\varepsilon}_+(\sigma) + S^{\varepsilon}_-(\sigma),$$

which we have shown above to be a meaningful condition. We claim that for fixed $\rho_\pm$ and ε equality (23) establishes a bijection between $\sigma \in (-1, +\infty)$ and $u_+ - u_- \in \mathbb{R}$, provided that $p(\,\cdot\,)$ satisfies (4). For $\rho_0 = (\sigma)^+ \in [0, +\infty)$ let $P_\pm(\,\cdot\,; \rho_0)$ be the hulls of $p(\,\cdot\,)$ on $\overline{I(\rho_0, \rho_\pm)}$, respectively. It will be convenient to extend $\Pi^{\varepsilon}_{\pm}(\,\cdot\,; \sigma)$ and $P_\pm(\,\cdot\,; \rho_0)$ to continuous functions on $\mathbb{R}^+$ by setting them constant on each component of $\mathbb{R}^+ \setminus I(\rho_0, \rho_\pm)$.

Lemma 5. *With the above notation the following assertions hold for* $\sigma \in (-1, +\infty)$, $\rho_0 = (\sigma)^+$:

- (a) *for all* $\rho \in \mathbb{R}^+$ *and* $\varepsilon > 0$ *the functions* $\sigma \mapsto \Pi^{\varepsilon}_{\pm}(\rho; \sigma)$ *do not decrease*; *nor do the functions* $\rho_0 \mapsto P_\pm(\rho; \rho_0)$;
- (b) *for all* $\rho \in \mathbb{R}^+$ *and* $\varepsilon > 0$ *the functions* $\sigma \mapsto \operatorname{sign}(\rho_\pm - \rho_0)\dot{\Pi}^{\varepsilon}_{\pm}(\rho; \sigma)$ *do not increase*; *nor do the functions* $\rho_0 \mapsto \operatorname{sign}(\rho_\pm - \rho_0)\dot{P}_\pm(\rho; \rho_0)$;

(c) *for all* $\varepsilon > 0$ *the maps* $\sigma \mapsto \Pi^\varepsilon_\pm(\,\cdot\,;\sigma)$ *are continuous in the* $L^\infty(\mathbb{R}^+)$ *topology; so also are the maps* $\rho_0 \mapsto P_\pm(\,\cdot\,;\rho_0)$;

(d) *for all* $\varepsilon > 0$ *the functions* $\sigma \mapsto S^\varepsilon_\pm(\sigma)$ *are continuous and strictly decreasing; so also are the functions* $\rho_0 \mapsto \displaystyle\int_{\rho_0}^{\rho_\pm} \sqrt{\dot P_\pm(r;\rho_0)}\,\frac{dr}{r}$.

Proof. Properties (a)–(c) for $\varepsilon > 0$ follow from Lemma 2. Hence (a)–(c) for $P_\pm$ follow by Proposition 4 and Lemma 3(b). Now, (b) and Levi's theorem yield (d).

Remark 2. The following extremality property holds: the upper hull $P(\,\cdot\,)$ of the function $p(\,\cdot\,)$ on $[a,b] \subset (0,+\infty)$ maximizes the integral $\displaystyle\int_a^b \sqrt{\dot Q(r)}\,\frac{dr}{r}$ in the class of all $Q(\,\cdot\,)$ satisfying assumptions (7) (i′), (ii″) such that $Q(a) = p(a)$ and $Q(b) = p(b)$.

It is sufficient for the proof to pass to $F(\,\cdot\,) = T^{-1}P(\,\cdot\,)$ and $G(\,\cdot\,) = T^{-1}Q(\,\cdot\,)$. We have $F(1/a) = G(1/a)$, $F(1/b) = G(1/b)$, and the functions F and G are convex on $[1/b, 1/a]$. Hence

$$\begin{aligned}\int_a^b \Big(\sqrt{\dot P(r)} - \sqrt{\dot Q(r)}\Big)\frac{dr}{r} &= \int_{1/b}^{1/a}\bigg(\sqrt{\frac{d}{dV}F(v)} - \sqrt{\frac{d}{dV}G(v)}\bigg)\,dv \\ &= \int_{1/b}^{1/a} \frac{\frac{d}{dV}(F(v) - G(v))}{\sqrt{\frac{d}{dV}F(v)} + \sqrt{\frac{d}{dV}G(v)}}\,dv \\ &= -\int_{1/b}^{1/a}(F(v) - G(v))\,d\Bigg(\frac{1}{\sqrt{\frac{d}{dV}F(v)} + \sqrt{\frac{d}{dV}G(v)}}\Bigg)\end{aligned}$$

in the sense of Stieltjes integrals. Since $\dfrac{d}{dV}F(V)$ and $\dfrac{d}{dV}G(V)$ increase on $[1/b, 1/a]$, we see in view of the definition of $P(\,\cdot\,)$ that $\displaystyle\int_a^b\Big(\sqrt{\dot P(r)} - \sqrt{\dot Q(r)}\Big)\frac{dr}{r} \geqslant 0$.

Lemma 6. *For fixed* $\rho_\pm$ *and* $u_\pm$ *assume that* (4) *holds. Then*

(a) $S^\varepsilon_\pm(\sigma) \to -\infty$ *as* $\sigma \to +\infty$ *for each* $\varepsilon > 0$;

(b) $\displaystyle\int_{\rho_0}^{\rho_\pm} \sqrt{\dot P_\pm(r;\rho_0)}\,\frac{dr}{r} \leqslant S^\varepsilon_\pm(\rho_0)$ *for all* $\rho_0 > 0$ *and* $\varepsilon > 0$; *in particular,* $\displaystyle\int_{\rho_0}^{\rho_\pm} \sqrt{\dot P_\pm(r;\rho_0)}\,\frac{dr}{r} \to -\infty$ *as* $\rho_0 \to +\infty$.

Proof. Assume that $S^+(\sigma)$ is bounded below by a constant $-M \in \mathbb{R}^-$. We set $V_0 := 1/(\sigma)^+$, $V_+ := 1/\rho_+ > V_0$, and perform the transformation $T^{-1}p = f$, $T^{-1}\Pi^\varepsilon_+(\,\cdot\,;\sigma) = \Phi(\,\cdot\,;V_0)$. Then the function $\Phi(\,\cdot\,;V_0)$ is convex on $[V_0, V_+]$, we have $\Phi(V_0;V_0) = f(V_0)$, $\Phi(V_+;V_0) = f(V_+)$, and $\Phi(\,\cdot\,;V_0)$ satisfies equation (35) on (V_0, V_+). Note that $f(V_0) \to -\infty$ as $V_0 \to 0$ by (4). Hence the convexity of $\Phi(\,\cdot\,;V_0)$ means that $\Phi(V_+/2;V_0) \to -\infty$ as $V_0 \to 0$; on the other hand,

$$\frac{V_+}{2}\sqrt{\frac{d}{dV}\Phi\Big(\frac{V_+}{2};V_0\Big)} \leqslant -S^\varepsilon_+(\sigma) \leqslant M$$

for all $V_0 \in (0, V_+/2)$. Thus, the family of functions $\{\Phi(\,\cdot\,; V_0)\}_{V_0\in(0,V_+/2)}$ has the following properties: $\Phi(V_+/2; V_0) \downarrow -\infty$ as $V_0 \downarrow 0$, $\dot{\Phi}(V_+/2; V_0) \leqslant (2M/V_+)^2$ uniformly in V_0, $\Phi(V_+; V_0) = f(V_+)$, and $\Phi(\,\cdot\,; V_0)$ satisfies (35) on $(V_+/2, V_+)$. We now compare $\Phi(\,\cdot\,; V_0)$ with the solution $\Psi(\,\cdot\,)$ of the Cauchy problem for (35) with initial data $\Psi(V_+/2) = \Phi(V_+/2; V_0)$, $\dot{\Psi}(V_+/2) = (2M/V_+)^2 + 1$. For sufficiently small V_0, $\Psi(\,\cdot\,)$ is defined on $[V_+/2, V_+]$ and $\Psi(V_+) < f(V_+)$. We can apply the maximum principle (Lemma 2(b)) to equation (35) and conclude that $\Phi(V_+; V_0) < \Psi(V_+) < f(V_+)$ for sufficiently small V_0. This contradiction proves (a).

Assertion (b) follows from (a) and Remark 2.

Combining Propositions 1–3 and Lemmas 4–6 we deduce easily the main result of this section.

Theorem 1. *Let $p(\,\cdot\,)$ be a continuous strictly increasing function on $\mathbb{R}^+$ and assume that* (4) *holds. Then the problem* (1), (2) *is uniquely soluble in the sense of Definition* 2 *for all $\rho_\pm > 0$, $u_\pm \in \mathbb{R}$, and $\varepsilon > 0$.*

Remark 3. It is easy to see that if $p(\,\cdot\,)$ fails to satisfy condition (4), then in the general case the problem (1), (2) has no bounded self-similar solution for arbitrary data $\rho_\pm$, $u_\pm$. By Lemma 6(b) the precise condition for the difference $u_+ - u_-$ ensuring the solubility for positive ε in some non-empty neighbourhood of 0 is as follows:

$$-(u_+ - u_-) < \lim_{\rho_0\to+\infty} \int_{\rho_+}^{\rho_0} \sqrt{\dot{P}_+(r;\rho_0)}\,\frac{dr}{r} + \lim_{\rho_0\to+\infty} \int_{\rho_-}^{\rho_0} \sqrt{\dot{P}_-(r;\rho_0)}\,\frac{dr}{r}\,, \qquad (36)$$

where $P_\pm(\,\cdot\,;\rho_0)$ are the hulls of $p(\,\cdot\,)$ on $[\rho_+, \rho_0]$ and $[\rho_-, \rho_0]$, respectively.

§ 3. Admissible solution of the problem (3), (2)

In this section we study the global (with respect to the data $\rho_\pm$, $u_\pm$) solubility of the problem (3), (2) in the class of admissible solutions in the sense of the wave-fan criterion (see [9], [11], [3]).

Definition 3. Let (ρ, u) be a pair of functions such that $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$ and $u \in L^\infty(\{\rho > 0\}; \mathbb{R})$, where $\{\rho > 0\} := \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} : \overline{\rho}(t,x) > 0\}$, $\overline{\rho}$ being an almost everywhere defined representative of ρ. Then (ρ, u) is a *wave-fan admissible solution* of (3), (2) if

(i) the equations $\rho_t + q_x = 0$ and $q_t + (e + p(\rho))_x = 0$ hold in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$, where $q = \rho u$, $e = \rho u^2$ on the set $\{\rho > 0\}$ and $q = 0$, $e = 0$ on its complement;

(ii) $\operatorname{ess\,lim}_{t\downarrow 0}\bigl(\|\rho(t,\,\cdot\,) - \rho(0,\,\cdot\,)\|_{L^1(-R,R)} + \|u(t,\,\cdot\,) - u(0,\,\cdot\,)\|_{L^1(-R,R)}\bigr) = 0$ for all $R > 0$, where $\rho(0,\,\cdot\,)$ and $u(0,\,\cdot\,)$ are defined in (2);

(iii) in addition, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$, $\varepsilon_n \to 0$ as $n \to \infty$, such that $\rho^{\varepsilon_n} \to \rho$, $\rho^{\varepsilon_n} u^{\varepsilon_n} \to q$, and $\rho^{\varepsilon_n}(u^{\varepsilon_n})^2 \to e$ almost everywhere on $\mathbb{R}^+ \times \mathbb{R}$ as $n \to \infty$, where $(\rho^{\varepsilon_n}, u^{\varepsilon_n})$ is a solution of (1), (2) in the sense of Definition 2.

Note that in accordance with (i) of Definition 3, u remains undefined within the vacuum state. While (i) and (ii) define a weak solution, (iii) is an additional selection criterion, which is more restrictive than (i) unless $p(\,\cdot\,)$ degenerates on a subinterval of $\mathbb{R}^+$ to $p(\rho) = \mathrm{const}/\rho$. If there is no degeneration, one can construct infinitely many solutions that are not admissible (cf. [17]).

In view of (iii), a wave-fan admissible solution is actually self-similar. By formulae (38)–(43) and Theorem 2 below it is unique and has the following wave-fan structure. The solution contains at most three 'main' constant states: (ρ_-, u_-) at $-\infty$; the vacuum state $\rho \equiv 0$ or a constant state (ρ_0, u_0), $\rho_0 > 0$, in a neighbourhood of the unique point ξ_0 such that there exists a limit $\lim_{\xi\to\xi_0} u(\xi) = \xi_0$ (in the case when there is no vacuum state); and (ρ_+, u_+) at $+\infty$. These states are separated by the two wave fans of the first and second families, respectively (see [19]). Each wave fan is a sequence of shocks, rarefactions, contact discontinuities, and (if $p(\,\cdot\,)$ is not smooth) constant states that are in fact degenerate rarefaction waves. There is no vacuum state in the solution unless the intermediate 'main' state is one. In that case $u(\xi) - \xi \to 0$ as ξ enters the vacuum state from any side (see (39), (41) and (46) below). A necessary and sufficient condition for the existence of a vacuum state is as follows:

$$u_+ - u_- \geqslant \int_0^{\rho_+} \sqrt{\dot{P}_+(r;0)}\,\frac{dr}{r} + \int_0^{\rho_-} \sqrt{\dot{P}_-(r;0)}\,\frac{dr}{r}\,, \tag{37}$$

where the $P_\pm(\,\cdot\,;0)$ are defined below. All these properties can be deduced from the following formulae for the wave-fan admissible solution:

$$\begin{aligned} \rho(t,x) = \rho(x/t) &= \begin{cases} \big[\Xi_-\big]^{-1}(x/t), & x/t < \xi_-, \\ \big[\Xi_+\big]^{-1}(x/t), & \xi_+ < x/t, \end{cases} \\ &\equiv \begin{cases} \big[\Xi_-\big]^{-1}(x/t), & x/t < \xi_-, \\ \rho_0, & \xi_- < x/t < \xi_+, \\ \big[\Xi_+\big]^{-1}(x/t), & \xi_+ < x/t, \end{cases} \end{aligned} \tag{38}$$

$$u(t,x) = u(x/t) = \begin{cases} U_- \circ \big[\Xi_-\big]^{-1}(x/t), & x/t < \xi_-, \\ U_-(\rho_0) = U_+(\rho_0), & \xi_- < x/t < \xi_+(\text{ if } \rho_0 > 0), \\ U_+ \circ \big[\Xi_+\big]^{-1}(x/t), & \xi_+ < x/t, \end{cases} \tag{39}$$

where

$$U_\pm(\rho) := u_\pm \mp \int_\rho^{\rho_\pm} \sqrt{\dot{P}_\pm(r;\rho_0)}\,\frac{dr}{r} \quad \text{for } \rho \in \overline{I(\rho_0,\rho_\pm)}, \tag{40}$$

$$\Xi_\pm(\rho) := U_\pm(\rho) \pm \sqrt{\dot{P}_\pm(\rho;\rho_0)} \qquad \text{for } \rho \in I(\rho_0,\rho_\pm), \tag{41}$$

$\xi_\pm$ are defined by the formulae

$$\begin{aligned} &\xi_\pm := \lim_{\rho\in I(\rho_0,\rho_\pm),\,\rho\to\rho_0} \Xi_\pm(\rho) && \text{if } \rho_0 \neq \rho_\pm, \\ &\xi_- := -\infty \ \text{ and/or } \ \xi_+ := +\infty && \text{if } \rho_0 = \rho_- \ \text{ and/or } \ \rho_0 = \rho_+, \end{aligned} \tag{42}$$

$\rho_0 = 0$ if (37) holds, and ρ_0 is the unique point in the interval $(0, +\infty)$ satisfying the relation

$$u_+ - u_- = \int_{\rho_0}^{\rho_+} \sqrt{\dot{P}_+(r;\rho_0)}\,\frac{dr}{r} + \int_{\rho_0}^{\rho_-} \sqrt{\dot{P}_-(r;\rho_0)}\,\frac{dr}{r} \tag{43}$$

if (37) fails; finally, $P_\pm(\,\cdot\,;\rho_0)$ are the hulls of the graph of $p(\,\cdot\,)$ on $\overline{I(\rho_0, \rho_\pm)}$, respectively (see Definition 1).

We now prove the main result of this paper.

Theorem 2. *Let $p(\,\cdot\,)$ be a continuous strictly increasing function satisfying* (4). *Assume that $\rho_\pm > 0$ and let $u_\pm \in \mathbb{R}$. Then as $\varepsilon \downarrow 0$ the solution $(\rho^\varepsilon, u^\varepsilon)$ of the problem* (1), (2) *approaches* (*in the sense of Definition* 3(iii)) *the pair (ρ, u) defined by formulae* (38)–(43). *The pair (ρ, u) is the unique solution of the Riemann problem* (3), (2) *in the sense of Definition* 3.

Proof. We fix $\varepsilon > 0$. By Theorem 1 the problem (1), (2) is uniquely soluble. Let $(\rho^\varepsilon, u^\varepsilon)$ be its solution. By Propositions 1 and 2, $(\rho^\varepsilon, u^\varepsilon)$ corresponds to some $\sigma^\varepsilon \in (-1, +\infty)$ and $\Pi^\varepsilon_\pm(\,\cdot\,;\sigma^\varepsilon)$ such that (20)–(23) and (24)–(28) hold. We consider now the set $\{(\sigma^\varepsilon)^+\} \subset \mathbb{R}^+$; it has an accumulation point ρ_0 in $\overline{\mathbb{R}^+}$. Consider a sequence $\varepsilon_n \downarrow 0$ such that $\sigma^{\varepsilon_n} \to \sigma^0$ as $n \to \infty$; we shall drop the subscript n because we shall show below that σ^0 does not depend on the choice of the particular subsequence. We consider separately the three possible cases $\rho_0 \in (0, +\infty)$, $\rho_0 = +\infty$, and $\rho_0 = 0$.

(a) $\rho_0 \in (0, +\infty)$. As in [15], it follows from Proposition 4 and Lemma 5(a), (c) that

$$\Pi^\varepsilon_\pm(\,\cdot\,;\sigma^\varepsilon) \to P_\pm(\,\cdot\,;\rho_0) \quad \text{in } L^\infty(\mathbb{R}^+) \quad \text{as } \varepsilon \to 0, \tag{44}$$

where the functions $\Pi^\varepsilon_\pm$ and $P_\pm$ are extended in $\mathbb{R}^+$ as in Lemma 5. In a similar way we extend $U^\varepsilon_\pm$ and U_ε in (26) and (40) to continuous functions on $\mathbb{R}^+$ of the same variation. By Lemma 3(d), the $U^\varepsilon_\pm(\,\cdot\,)$ converge to $U_\pm(\,\cdot\,)$ uniformly on $\overline{I(\rho_0, \rho_\pm)}$. Since one can write (23) and (43) as $U^\varepsilon_-((\sigma^\varepsilon)^+) = U^\varepsilon_+((\sigma^\varepsilon)^+)$ and $U_-(\rho_0) = U_+(\rho_0)$, respectively, one arrives at (43). It follows by Lemma 5 that in case (a) ρ_0 is uniquely determined by $p(\,\cdot\,)$, $\rho_\pm$, and $u_\pm$.

Further, $\Xi_\pm(\rho_1) \leqslant \xi_- \leqslant U_-(\rho_0) = U_+(\rho_0) \leqslant \xi_+ \leqslant \Xi_+(\rho_2)$ for all $\rho_1 \in \overline{I(\rho_0, \rho_-)}$ and $\rho_2 \in \overline{I(\rho_0, \rho_+)}$ by Lemma 3(c) and (41), (42). By a separate consideration of the cases $\rho_0 > \rho_+$, $\rho_0 = \rho_+$, and $\rho_0 < \rho_+$, using Lemma 3(e) one deduces from (44), (24), and (38) that $\rho^\varepsilon(\,\cdot\,) \to \rho(\,\cdot\,)$ a.e. on $(\xi_-, +\infty)$. In a similar way $\rho^\varepsilon(\,\cdot\,) \to \rho(\,\cdot\,)$ a.e. on $(-\infty, \xi_+)$, so that the convergence holds actually for almost all $\xi \in \mathbb{R}$. Consequently, $u^\varepsilon(\,\cdot\,) \to u(\,\cdot\,)$ a.e. on $\mathbb{R}$.

(b) $\rho_0 = +\infty$. This case is actually impossible. Indeed, for all $L > 0$ we have

$$\begin{aligned} u_+ - u_- &= \int_{\sigma^\varepsilon}^{\rho_+} \sqrt{\dot{\Pi}^\varepsilon_+(r;\sigma^\varepsilon)}\,\frac{dr}{r} + \int_{\sigma^\varepsilon}^{\rho_-} \sqrt{\dot{\Pi}^\varepsilon_-(r;\sigma^\varepsilon)}\,\frac{dr}{r} \\ &\leqslant \int_{L}^{\rho_+} \sqrt{\dot{\Pi}^\varepsilon_+(r;L)}\,\frac{dr}{r} + \int_{L}^{\rho_-} \sqrt{\dot{\Pi}^\varepsilon_-(r;L)}\,\frac{dr}{r} \end{aligned}$$

for sufficiently small ε by Lemma 5(d). Passing to the limit as $\varepsilon \to 0$ we obtain

$$u_+ - u_- \leqslant \int_L^{\rho_+} \sqrt{\dot{P}_+(r;L)}\,\frac{dr}{r} + \int_L^{\rho_-} \sqrt{\dot{P}_-(r;L)}\,\frac{dr}{r}$$

by Proposition 4 and Lemma 3(d). By Lemma 6(c), if (4) holds, then the right-hand side approaches $-\infty$ as $L \to +\infty$. Thus, $u_+ - u_- = -\infty$, which is impossible.

(c) $\rho_0 = 0$. Similarly to (a) one has (44). Hence $\dot{\Pi}^\varepsilon_\pm(\,\cdot\,;\sigma^\varepsilon) \to \dot{P}_\pm(\,\cdot\,;0)$ a.e. on $(0,\rho_\pm)$ by Lemma 3(b). It follows by Fatou's theorem that

$$\begin{aligned} u_+ - u_- &= \liminf_{\varepsilon\to 0}\biggl(\int_0^{\rho_+} \sqrt{\dot{\Pi}^\varepsilon_+(r;\sigma^\varepsilon)}\,\frac{dr}{r} + \int_0^{\rho_-} \sqrt{\dot{\Pi}^\varepsilon_-(r;\sigma^\varepsilon)}\,\frac{dr}{r}\biggr) \\ &\geqslant \int_0^{\rho_+} \sqrt{\dot{P}_+(r;0)}\,\frac{dr}{r} + \int_0^{\rho_-} \sqrt{\dot{P}_-(r;0)}\,\frac{dr}{r}\,. \end{aligned} \tag{45}$$

Note that since case (b) is impossible and in case (a) the right-hand side of (45) must be greater than $u_+ - u_-$, the accumulation point ρ_0 is unique and finite. We claim that $\rho^\varepsilon \to \rho$, $\rho^\varepsilon u^\varepsilon \to \rho u$, and $\rho^\varepsilon(u^\varepsilon)^2 \to \rho u^2$ on the set $\{\xi : \rho(\xi) > 0\}$ and that $\rho^\varepsilon \to 0$, $\rho^\varepsilon u^\varepsilon \to 0$, and $\rho^\varepsilon(u^\varepsilon)^2 \to 0$ on its complement.

Assume first that we have equality in (45). Then we still have $U_-(0) = U_+(0)$. Moreover,

$$\dot{P}(\rho;0) \to 0 \quad \text{as } \rho \to 0 \tag{46}$$

whenever a vacuum occurs. Indeed, one can show in the same way as in the proof of Lemma 3(c) that $\Xi_\pm(\rho) = u_\pm \mp \int_\rho^{\rho_\pm} \sqrt{\dot{P}_\pm(r;0)}\,\frac{dr}{r} \pm \sqrt{\dot{P}_\pm(\rho;0)}$ are monotone functions on $(0,\rho_\pm)$; on the other hand, the integrals $\int_\rho^{\rho_\pm} \sqrt{\dot{P}_\pm(r;0)}\,\frac{dr}{r}$ are also monotone functions and converge as $\rho \downarrow 0$. Hence the functions $\sqrt{\dot{P}_\pm(\rho;0)}$ converge to zero as $\rho \downarrow 0$. Thus, $\Xi_-(\rho_1) \leqslant \xi_- = U_-(0) = U_+(0) = \xi_+ \leqslant \Xi_+(\rho_2)$ for all $\rho_1 \in [0,\rho_-]$ and $\rho_2 \in [0,\rho_+]$; moreover, $\rho(\xi) > 0$ for all $\xi \neq \xi_- = \xi_+$. The convergence of ρ^ε and u^ε to ρ and u, respectively, a.e. on $\mathbb{R}$ follows as in case (a).

Assume next that we have strict inequality in (45). By (46) we obtain the inequalities $\Xi_-(\rho_1) \leqslant \xi_- = U_-(0) < U_+(0) = \xi_+ \leqslant \Xi_+(\rho_2)$ with ρ_1 and ρ_2 as above. In a similar way we establish the convergence of ρ^ε and u^ε to ρ and u, respectively, a.e. on $(-\infty,\xi_-) \cup (\xi_+,+\infty)$. Note that the $\Xi_\pm(\rho)$ are strictly monotone at $\rho = 0$ since the $P_\pm(\,\cdot\,;0)$ are strictly increasing and $\dot{P}_\pm(+0;0) = 0$. Hence $\rho(\xi) \to 0$ as $\xi \uparrow \xi_-$ or $\xi \downarrow \xi_+$. For ε sufficiently small ρ^ε has no points of maximum on $\mathbb{R}$, so that for all $\delta > 0$ we have $\max_{\xi\in[\xi_- -\delta,\xi_+ +\delta]} \rho^\varepsilon(\xi) = \max\{\rho^\varepsilon(\xi_- - \delta), \rho^\varepsilon(\xi_+ + \delta)\}$. It follows that $\rho^\varepsilon \to 0$ uniformly on $[\xi_-,\xi_+]$. In addition, the u^ε are non-decreasing functions on $\mathbb{R}$ for sufficiently small ε, therefore they are uniformly bounded, so that $\rho^\varepsilon u^\varepsilon \to 0$ and $\rho^\varepsilon(u^\varepsilon)^2 \to 0$ on $[\xi_-,\xi_+]$.

We have thus shown that $(\rho,u)(\cdot,\cdot)$ has properties (i), (iii) in Definition 3. Moreover, $(\rho,u)(\pm\infty) = (\rho_\pm,u_\pm)$ by (38)–(40). Since $\rho(\,\cdot\,)$ and $u(\,\cdot\,)$ are both monotone on $\pm\infty$, property (ii) in Definition 3 also holds.

Bibliography

[1] Y.-J. Kim, "A self-similar viscosity approach for the Riemann problem in isentropic gas dynamics and the structure of solutions", Preprint, Univ. of Madison, 1999.

[2] A. E. Tzavaras, "Elastic as limit of viscoelastic response, in a context of self-similar viscous limits", *J. Differential Equations* **123**:1 (1995), 305–341.

[3] A. E. Tzavaras, "Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws", *Arch. Rational Mech. Anal.* **135** (1996), 1–60.

[4] A. S. Kalashnikov, "The construction of generalized solutions of quasilinear first order equations without convexity condition as limits of solutions of parabolic equations with a small parameter", *Dokl. Akad. Nauk SSSR* **127** (1959), 27–30. (Russian)

[5] V. A. Tupchiev, "Collapse of an arbitrary discontinuity for a system of two quasilinear first-order equations", *Zh. Vychisl. Mat. i Mat. Fiz.* **4** (1964), 817–825; English transl. in *USSR Comput. Math. Math. Phys.* **4** (1964).

[6] V. A. Tupchiev, "Problem of the decay of discontinuity for a system of quasilinear equations without convexity conditions", *Zh. Vychisl. Mat. i Mat. Fiz.* **6** (1966), 527–547; English transl. in *USSR Comput. Math. Math. Phys.* **6** (1966).

[7] V. A. Tupčiev (Tupchiev), "On the method of introducing viscosity in the study of problems involving the decay of a discontinuity", *Dokl. Akad. Nauk SSSR* **211** (1973), 55–58; English transl. in *Soviet Math. Dokl.* **14** (1973).

[8] C. M. Dafermos, "Solutions of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method", *Arch. Rational Mech. Anal.* **52**:1 (1973), 1–9.

[9] C. M. Dafermos, "Structure of solutions of the Riemann problem for hyperbolic systems of conservation laws", *Arch. Rational Mech. Anal.* **53**:3 (1974), 203–217.

[10] C. M. Dafermos and R. DiPerna, "The Riemann problem for certain classes of hyperbolic systems of conservation laws", *J. Differential Equations* **20** (1976), 90–114.

[11] C. M. Dafermos, "Admissible wave fans in nonlinear hyperbolic systems", *Arch. Rational Mech. Anal.* **106** (1989), 243–260.

[12] A. E. Tzavaras, "Viscosity and relaxation approximations for hyperbolic systems of conservation laws", *An introduction to recent developments in theory and numerics of conservation laws*, Springer-Verlag, Berlin 1999, pp. 73–122.

[13] M. Slemrod and A. E. Tzavaras, "A limiting viscosity approach for the Riemann problem in isentropic gas dynamics", *Indiana Univ. Math. J.* **38** (1989), 1047–1074.

[14] H. T. Fan, "The structure of solutions of gas dynamic equations and the formation of the vacuum state", *Quart. Appl. Math.* **49** (1991), 29–48.

[15] B. Andreianov, "The Riemann problem for p-systems with continuous flux function", *Ann. Fac. Sci. Toulouse Math.* (6) **8**:3 (1999), 353–367.

[16] L. Leibovich, "Solutions of the Riemann problem for hyperbolic systems of quasilinear equations without convexity conditions", *J. Math. Anal. Appl.* **45**:3 (1974), 81–90.

[17] P. Krejčí and I. Straškraba, "A uniqueness criterion for the Riemann problem", *Hiroshima Math. J.* **27**:2 (1997), 307–346.

[18] D. Hoff and J. Smoller, "Non-formation of vacuum states for compressible Navier–Stokes equations", *Comm. Math. Phys.* **216**:2 (2001), 255–276.

[19] P. D. Lax, "Hyperbolic systems of conservation laws. II", *Comm. Pure Appl. Math.* **10** (1957), 537–566.

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Received 22/NOV/01 and 9/OCT/02
Translated by B. ANDREIANOV

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX