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CONVERGENCE OF FINITE VOLUME APPROXIMATIONS FOR A NONLINEAR ELLIPTIC-PARABOLIC PROBLEM: A "CONTINUOUS" APPROACH*

BORIS A. ANDREIANOV†, MICHAËL GUTNIC‡, AND PETRA WITTBOLD‡

Abstract. We study the approximation by finite volume methods of the model parabolic-elliptic problem $b(v)_t = \operatorname{div}(|Dv|^{p-2}Dv)$ on $(0,T)\times\Omega \subset \mathbb{R}\times\mathbb{R}^d$ with an initial condition and the homogeneous Dirichlet boundary condition. Because of the nonlinearity in the elliptic term, a careful choice of the gradient approximation is needed. We prove the convergence of discrete solutions to the solution of the continuous problem as the discretization step h tends to 0, under the main hypotheses that the approximation of the operator $\operatorname{div}(|Dv|^{p-2}Dv)$ provided by the finite volume scheme is still monotone and coercive, and that the gradient approximation is exact on the affine functions of $x \in \Omega$. An example of such a scheme is given for a class of two-dimensional meshes dual to triangular meshes, in particular for structured rectangular and hexagonal meshes. The proof uses the rewriting of the discrete problem under a "continuous" form. This permits us to directly apply the Alt–Luckhaus variational techniques which are known for the continuous case.

Key words. doubly nonlinear elliptic-parabolic equations, finite volume methods, convergence of approximate solutions, continuous approach

AMS subject classifications. 35J60, 35K55, 35K65, 35M10, 65M12, 76M12

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1. Introduction.

Let Ω be an open bounded polygonal domain in $\mathbb{R}^d$, $d \geq 1$, and $T > 0$. We consider the initial boundary value problem for a system of nonlinear elliptic-parabolic equations:

$$(1.1)\qquad \begin{cases} b(v)_t = \operatorname{div} a_p(Dv) & \text{on } Q = (0,T)\times\Omega, \\ v = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega, \\ b(v)(0,\cdot) = u^0 & \text{on } \Omega, \end{cases}$$

where $1 < p < \infty$ and $\operatorname{div} a_p(Dv) = \operatorname{div}(|Dv|^{p-2}Dv)$ is the N-dimensional p-Laplacian, i.e.,

$$a_p : \xi = (\xi_1,\ldots,\xi_N) \in (\mathbb{R}^d)^N \mapsto |\xi|^{p-2}\xi = \left(\sum_{i,j} |\xi_i^j|^2\right)^{p/2-1} (\xi_1,\ldots,\xi_N) \in (\mathbb{R}^d)^N.$$

We assume that

$$(1.2)\ \begin{cases} b : \mathbb{R}^N \to \mathbb{R}^N \quad \text{is continuous cyclically monotone;} \quad \text{i.e.,} \\ \text{there exists a convex differentiable function } \Phi : \mathbb{R}^N \to \mathbb{R} \ \text{ s.t. } \ b = \nabla\Phi, \end{cases}$$

normalized by $b(0) = 0$ and $\Phi(0) = 0$. Moreover, we assume

$$(1.3)\qquad u^0 \in L^1(\Omega)^N \quad \text{with} \quad \Psi(u_0) \in L^1(\Omega),$$

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†LATP, CMI, Université de Provence, Technopole de Château-Gombert, 39, rue Frédéric Joliot-Curie, 13453 Marseille Cedex 13, France (borisa@cmi.univ-mrs.fr).
‡IRMA, Université Louis Pasteur, 7, rue René Descartes, 67084 Strasbourg Cedex, France (gutnic@math.u-strasbg.fr, wittbold@math.u-strasbg.fr).

where Ψ is the Legendre transform of Φ given by

$$\Psi : z \in \mathbb{R}^N \mapsto \sup_{\sigma \in \mathbb{R}^N} \int_0^1 (z - b(s\sigma))\sigma \, ds = \sup_{\sigma \in \mathbb{R}^N} (\sigma z - \Phi(\sigma)).$$

Equations of elliptic-parabolic type (1.1) arise as models of the flow of fluids through porous media (cf., e.g., [6, 12]). They have already been studied extensively in the literature in the last decade from a theoretical point of view (cf., e.g., [1, 21, 22, 12, 7, 26, 8, 10, 2]). Existence of weak solutions of general systems of elliptic-parabolic equations has been proved in [1], using Galerkin approximations and time-discretization. Similar results have been obtained later by other authors using different methods (e.g., using a semigroup approach as in [7, 8] in the case $N = 1$).

In particular, it is known that in the case of the system (1.1), for any u_0 satisfying (1.3), there exists a weak solution of (1.1), where the weak solution is defined as follows. Denote by E the Banach space $L^p(0,T;W_0^{1,p}(\Omega))^N$ and by E' its dual; $E' = L^{p'}(0,T;W^{-1,p'}(\Omega))^N$, where $p' = p/(p-1)$ is the conjugate exponent of p. Denote by $\langle \cdot, \cdot \rangle_{E',E}$ the duality pairing between E' and E.

Definition 1.1. *A function $v \in E$ is a weak solution of the problem* (1.1) *if $b(v) \in L^\infty(0,T;L^1(\Omega))^N$ and $b(v)_t \in \mathcal{D}'(Q)^N$ can be extended to a functional χ on E satisfying*

$$\langle \chi, \phi \rangle_{E',E} + \iint_Q a_p(Dv) \cdot D\phi = 0 \quad \text{for all } \phi \in E, \tag{1.4}$$

$$\langle \chi, \xi \rangle_{E',E} = -\iint_Q b(v)\, \xi_t - \int_\Omega u_0(\cdot)\, \xi(0,\cdot) \qquad \begin{array}{l} \text{for all } \xi \in E \text{ with} \\ \xi_t \in L^\infty(Q)^N,\ \xi(T,\cdot) = 0. \end{array} \tag{1.5}$$

Note that if v is a weak solution of (1.1), then, by the "chain rule" lemma of [1], one has

$$\begin{aligned} & B(v) \in L^\infty(0,T;L^1(\Omega))^N, \qquad \text{where} \\ & B : z \in \mathbb{R}^N \mapsto b(z)z - \Phi(z) \equiv \int_0^1 (b(z) - b(sz))z ds \equiv \Psi(b(z)) \in \mathbb{R}. \end{aligned} \tag{1.6}$$

From the results of [26, 10] it also follows that, in the scalar case $N = 1$, there is uniqueness of a weak solution of (1.1). To our knowledge, the question of uniqueness is open in the case $N \geq 2$.

In this paper we study the convergence of time-implicit approximations by finite volume numerical schemes for the model nonlinear elliptic-parabolic problem (1.1). Finite volume methods are well suited for numerical simulation of processes where extensive quantities are conserved, and they are popular methods among engineers in hydrology where equations of this type arise. Therefore justification of convergence of this numerical approximation process is of particular interest. In [17] the finite volume method has been studied and convergence of this approximation procedure has been proved for problem (1.1) in the particular case $p = 2$, $N = 1$. The same method has also been studied for this equation (i.e., $p = 2, N = 1$) in the presence of an additional convection term (cf. [18, 14]), and for a nonlinear diffusion problem in [16]. To our knowledge, in the case $p \neq 2$, only the convergence of finite element methods has been studied (cf. [19, 11, 5, 20] and their references).

Let us emphasize that our main object is not only to prove the convergence of some finite volume methods for (1.1), but also to develop a "continuous" approach for this proof. The main idea of this adaptation is to rewrite the discrete finite volume scheme under an equivalent continuous form and to apply known stability techniques for the continuous equation (cf. [1] and [2, Chap. V] for the version we use) in order to get convergence of discrete solutions to a solution of the continuous problem. The "continuous" approach and the convergence result have already been presented in [3].

In section 2, we describe the finite volume schemes and in particular the admissible flux approximations we use. We show the existence and uniqueness of the solution of a finite volume scheme and give some a priori estimates on discrete solutions. Then we state the convergence result. In section 3, we show in Proposition 3.3 that the solution of a finite volume scheme, originally satisfying a discrete system of algebraic equations, also verifies a "continuous" formulation similar to (1.4), (1.5). This representation makes clear in which sense finite volume schemes approximate the elliptic operator in (1.1); we prove that this approximation is consistent. In section 4 we prove the convergence theorem, passing to the limit in the "continuous" formulation of Proposition 3.3. In section 5, we analyze the two admissibility conditions imposed in section 2. For $d = 2$, we propose a scheme on meshes dual to triangular meshes that enters into our framework; in particular, we have the convergence result on structured rectangular and hexagonal meshes.

We consider the p-Laplacian as a prototype of a class of the so-called Leray–Lions-type operators; in [4], we discuss the extension of the techniques presented above to a particular case of the p-Laplacian operator with convection, studied in [12].

In order to simplify the notation, we restrict the exposition to the scalar equation ($N = 1$). The proofs of the auxiliary results used in section 4 can be found in [4].

2. The numerical method. In order to construct approximate solutions to the problem (1.1), we will use the implicit discretization in time and a finite volume scheme in space.

2.1. Finite volume meshes, discrete gradients and finite volume schemes for the problem (1.1). Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. A finite volume mesh $\mathcal{T}$ of Ω is given by a family of open polygonal convex subsets of Ω with positive measure, called "control volumes," a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, with positive $(d-1)$-measure (these are the interfaces between control volumes), and a family of points of $\overline{\Omega}$, one per control volume (these are the "centers" of the volumes). For a volume K with center $x_K \in \overline{K}$, the interfaces contained in $\partial\Omega$ are considered as additional "boundary" volumes, unless $x_K \in \partial\Omega$.

For the sake of simplicity, we shall denote by $\mathcal{T}$ the family $(K)_{K\in\mathcal{T}}$ of control volumes; $(x_K)_{K\in\mathcal{T}}$ denotes the family of their centers. The set of all volumes K such that $x_K \in \partial\Omega$ is denoted by $\mathcal{T}_{ext}$, and the set of all volumes K with $x_K \in \Omega$ is denoted by $\mathcal{T}_{int}$. The set of interfaces $K|L$ such that K or L or both belong to $\mathcal{T}_{int}$ is denoted by $\mathcal{E}$, and $K|L$ denotes the interface between two neighbors $K, L \in \mathcal{T}$. For all $K|L$, $\widehat{KL}$ denotes the "diamond" over $K|L$, i.e., the smallest convex set of $\mathbb{R}^d$ containing $K|L$, x_K and x_L. Whenever we use K, $K|L$, or n to index objects and make summations, we mean that $K \in \mathcal{T}$, $K|L \in \mathcal{E}$, and $n \in \{1, \dots, [T/k]+1\}$, where k is the time step of the scheme.

Following [15], we give the following definition.

DEFINITION 2.1. *We say that $\mathcal{T}$ is a finite volume mesh of Ω if the following hold:*

(2.1 i) *The closure of the union of all control volumes is $\overline{\Omega}$.*

(2.1 ii) *For any* $(K, L) \in (\mathcal{T})^2$ *with* $K \neq L$, *either the* $(d-1)$*-dimensional measure of* $\overline{K} \cap \overline{L}$ *is* 0 *or* $\overline{K} \cap \overline{L} = \overline{\sigma}$ *for some* $\sigma \in \mathcal{E}$ *(in which case we denote* $\sigma = K|L = L|K$*).*

(2.1 iii) *For any* $K \in \mathcal{T}$, *there exists a subset* $\mathcal{E}_K$ *of* $\mathcal{E}$ *such that* $\partial K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. *Furthermore,* $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$. *We will denote by* $\mathcal{N}_K$ *the set of volumes adjacent to* K*; i.e.,* $\mathcal{N}_K = \{L \in \mathcal{T},\ K|L \in \mathcal{E}_K\}$.

(2.1 iv) *The family of points* $(x_K)_K$ *is such that* $x_K \in \overline{K}$ *for all* $K \in \mathcal{T}$, *and it is assumed that the straight line joining* x_K *and* x_L *is orthogonal to* $K|L$ *whenever* $L \in \mathcal{N}_K$.

Denote by $\mathfrak{m}(K)$ and $\mathfrak{d}(K)$ the d-dimensional measure and the diameter of $K \in \mathcal{T}$, respectively; and denote by $m(K|L)$ the $(d-1)$-dimensional measure of $K|L \in \mathcal{E}$. A mesh $\mathcal{T}$ is characterized, in particular, by the following numbers:

$$\text{size}(\mathcal{T}) = \max_K \mathfrak{d}(K), \qquad \zeta^*(\mathcal{T}) = \min_K \min_{\sigma \in \mathcal{E}_K} \frac{\text{dist}\,(x_K, \sigma)}{\mathfrak{d}(K)},$$

$$M(\mathcal{T}) = \max_K \text{card}(\mathcal{E}_K), \qquad \zeta_*(\mathcal{T}) = \frac{\min_K \min_{\sigma \in \mathcal{E}_K} \text{dist}\,(x_K, \sigma)}{\text{size}(\mathcal{T})}.$$

A finite volume method for (1.1) requires a family $\big((\mathcal{T}^h, k^h)\big)_h$ of meshes and corresponding time steps $k^h > 0$ such that both the size of the mesh and the time step go to zero. We will assume in our notation that the family is parametrized with h in some subset of $(0,1)$ whose closure contains zero, and $\text{size}(\mathcal{T}^h) + k^h \leq h$. A couple $(\mathcal{T}^h, k^h)$ will be called a space-time grid.

In relation to a family $\big((\mathcal{T}^h, k^h)\big)_h$, we define the numbers

$$M = \sup_h M(\mathcal{T}^h) \in \overline{\mathbb{N}}, \quad \zeta^* = \inf_h \zeta^*(\mathcal{T}^h) \in \mathbb{R}^+, \quad \text{and} \quad \zeta_* = \inf_h \zeta_*(\mathcal{T}^h) \in \mathbb{R}^+. \tag{2.1}$$

DEFINITION 2.2. *We say that the family of meshes* $(\mathcal{T}^h)_h$ *is weakly proportional if* $M < \infty$ *and* $\zeta^* > 0$. *We say that the family of meshes* $(\mathcal{T}^h)_h$ *is strongly proportional if, in addition,* $\zeta_* > 0$.

Weak proportionality is standard (cf. [18]). Strong proportionality is a technical assumption which ensures that $(\mathcal{T}^h)_h$ has the interpolation property (cf. sections 2.5 and 5.2).

Given a grid $(\mathcal{T}^h, k^h)$, to each time-space volume $Q_K^n = I^n \times K$, $I^n = (k^h(n-1), k^h n)$ one associates an unknown value $v_K^n \in \mathbb{R}^N$. In order to obtain a finite volume scheme for (1.1), one "integrates" the equation in (1.1) over each grid volume Q_K^n. The time derivative in the left-hand side is approximated by the corresponding finite difference. On the right-hand side, one uses the Green formula and then needs to replace the flux on the lateral boundary of Q_K^n by some function of the unknowns $(v_K^n)_{K,n}$. For problem (1.1), this amounts to finding a substitution for Dv in the expression $\int_{I^n \times K|L} a_p(Dv) \cdot \nu_{K,L}$ (where $\nu_{K,L}$ is the unit normal vector to $K|L$ pointing from K into L). We will assume that this substitution is in L^p on each interface $I^n \times K|L$, typically constant in time and piecewise constant in space. We therefore consider "discrete gradient" operators $\mathcal{D}^h$ of the form

$$\begin{cases} \mathcal{D}^h : (v_K^n)_{K,n} \mapsto (D_{K|L}^n)_{K|L,n}, \\ D_{K|L}^n \in L^p(I^n \times K|L) \quad \text{for all} \quad K|L,\ n. \end{cases} \tag{2.2}$$

It seems natural, though not necessary, to require that $\mathcal{D}^h$ be a linear operator.

A finite volume scheme for (1.1) is defined by a grid $(\mathcal{T}^h, k^h)$ and a discrete gradient $\mathcal{D}^h$ associated with the grid. Finally, a finite volume method for (1.1) is given by a family $\big((\mathcal{T}^h, k^h, \mathcal{D}^h)\big)_h$ of grids and associated discrete gradient operators $\mathcal{D}^h$. In sections 2.3 and 2.5 we state the admissibility conditions for such methods.

Now we are able to write the equations for a scheme $(\mathcal{T}^h, k^h, \mathcal{D}^h)$:

$$\mathfrak{m}(K)\frac{b(v_K^n) - b(v_K^{n-1})}{k^h} = \sum_{L\in\mathcal{N}_K} \int_{K|L} a_p(D^n_{K|L}(x))dx \cdot \nu_{K,L} \quad \text{for all } K \in \mathcal{T}^h_{int}, \quad \text{for all } n \in \{1,\dots,[T/k^h]+1\}. \tag{2.3}$$

The homogeneous Dirichlet boundary condition is taken into account by assigning

$$v_K^n = 0 \quad \text{for all } K \in \mathcal{T}^h_{ext}, \quad \text{for all } n \in \{1,\dots,[T/k^h]+1\}. \tag{2.4}$$

The initial condition is given by any values $v_K^0 \in b^{-1}(u_K^0)$, where

$$u_K^0 = \frac{1}{\mathfrak{m}(K)} \int_K u^0 \quad \text{for all } K \in \mathcal{T}^h_{int}. \tag{2.5}$$

We denote by u_0^h the piecewise constant initial function $\sum_K u_K^0 1\!\!1_K$, where $1\!\!1_K$ is the characteristic function of the set K. Other choices of u_K^0 are possible, provided one has $u_0^h \to u_0$ a.e. on Ω and $\Psi(u_0^h) \to \Psi(u_0)$ in $L^1(\Omega)$ as $h \to 0$, where Ψ is defined in the introduction. These properties hold for u_K^0 given by (2.5), due to the convexity of Ψ.

We denote by $(\mathcal{S}^h)$ the system (2.3), (2.4), (2.5) corresponding to a given finite volume scheme $(\mathcal{T}^h, k^h, \mathcal{D}^h)$.

2.2. Memento on notation. In this section we collect the most used notation related to the finite volume schemes.

$\mathcal{T}$:	a finite volume mesh;
$\mathcal{T}_{ext}, \mathcal{T}_{int}$:	the set of exterior,interior control volumes;
$\mathcal{E}$:	the set of interfaces between control volumes;
K, L:	control volumes of $\mathcal{T}$;
$K|L$:	the interface between the two neighbors K and L;
$\mathcal{E}_K$:	the set of all interfaces surrounding K;
$\mathcal{N}_K$:	the set of all neighbors of K;
x_K:	the "center" of K;
$d_{K,L}$:	the distance between x_K and x_L, $d_{K,L} = \lvert x_K - x_L\rvert$;
$d_{K,K\lvert L}$:	the distance between x_K and $K\lvert L$; one has $d_{K,K\lvert L} + d_{L,K\lvert L} = d_{K,L}$;
$\nu_{K,L}$:	the unit normal vector to $K\lvert L$ pointing from K to L;
$\widehat{KL}$:	the smallest convex set of $\mathbb{R}^d$ containing $K\lvert L, x_K$, and x_L;
$\mathfrak{d}(K), \mathfrak{m}(K)$:	the diameter and the d-dimensional measure of K, respectively;
size($\mathcal{T}$):	the size of the mesh $\mathcal{T}$, $\text{size}(\mathcal{T}) = \max_K \mathfrak{d}(K)$;
$m(K\lvert L)$:	the $(d-1)$-dimensional measure of $K\lvert L$;
$\lvert R\rvert$:	the $(d+1)$-dimensional measure of a set $R \subset \mathbb{R}^+ \times \mathbb{R}^d$;
I^n:	the time interval, $I^n = ((n-1)k, nk)$;
Q_K^n:	the time-space grid element, $Q_K^n = I^n \times K$;
Σ_K^n:	the lateral boundary of Q_K^n, $\Sigma_K^n = I^n \times \partial K$;
$\Upsilon_\varsigma(K)$:	the union of all control volumes of $(\mathcal{T})$ that are separated from K by at most $(\varsigma - 1)$ other control volumes;

$1\!\!1_A$: the characteristic function of a set A;
$(\mathcal{T}^h, k^h, \mathcal{D}^h)$: a finite volume scheme (mesh, time step, discrete gradient);
$(\mathcal{S}^h)$: the corresponding system of equations (2.3),(2.4),(2.5);
h: the discretization parameter, $h \geq \text{size}(\mathcal{T}^h) + k^h$;
M, ζ^*: the weak proportionality bounds for $(\mathcal{T}^h)_h$,
$M = \sup_h \max_K \text{card}(\mathcal{N}_K)$, and $\zeta^* = \inf_h \min_K \frac{\min_{L\in\mathcal{N}_K} d_{K,K|L}}{\mathfrak{d}(K)}$;
ζ_*: the strong proportionality bound for $(\mathcal{T}^h)_h$,
$\zeta_* = \inf_h \frac{\min_{K,L\in\mathcal{N}_K} d_{K,K|L}}{\text{size}(\mathcal{T})}$;
v_K^n: the unknown of the scheme $(\mathcal{S}^h)$ corresponding to the volume Q_K^n;
$\overline{v}^h$: a discrete solution for the scheme $(\mathcal{T}^h, k^h, \mathcal{D}^h)$, $\overline{v}^h = \sum_{K,n} v_K^n 1\!\!1_{Q_K^n}$;
u_0^h: the discrete initial data, $u_0^h = \sum_K u_K^0 1\!\!1_K$;
$D_{K|L}^n$: the discrete gradient values on $I^n \times K|L$, $D_{K|L}^n \in L^p(I^n \times K|L)$;
$\mathcal{D}^h$: the discrete gradient operator, $\mathcal{D}^h : (v_K^n)_{K,n} \mapsto (D_{K|L}^n)_{K|L,n}$;
$D_{\perp,K|L}^n$: the value $D_{\perp,K|L}^n = \frac{v_L^n - v_K^n}{d_{K,L}}$, featuring in the "discrete $L^p(0,T;W^{1,p}(\Omega))$ norm" of $\overline{v}^h$;
$\mathcal{D}_\perp^h$: the corresponding operator, $\mathcal{D}_\perp^h : (v_K^n)_{K,n} \mapsto (D_{\perp,K|L}^n)_{K|L,n}$.

It is convenient to extend $\mathcal{D}^h$ (as well as $\mathcal{D}_\perp^h$) to an operator acting from E into $L^p(Q)$. Let $\mathcal{P}^h$ be the operator from Ω to $\bigcup K|L$ which projects $x \in K$ on ∂K along the ray joining x_K to x. We define the appropriate lifting operator $\mathcal{L}^h$ and averaging operator $\mathcal{M}^h$ by

$$\mathcal{L}^h\Big[(D_{K|L}^n)_{K|L,n}\Big](t,x) = \sum_{K|L,n} D_{K|L}^n(\mathcal{P}^h(x))\, 1\!\!1_{I^n \times \widehat{KL}}(t,x),$$
$$\mathcal{M}^h : \eta \in L^1(Q) \mapsto \mathcal{M}^h[\eta] = (\eta_K^n)_{K,n} \subset \mathbb{R}^N, \quad \eta_K^n = \frac{1}{|Q_K^n|}\iint_{Q_K^n} \eta.$$

We will abusively write $\mathcal{D}^h$ for the operators $\mathcal{D}^h$, $\mathcal{L}^h \circ \mathcal{D}^h$, and $\mathcal{L}^h \circ \mathcal{D}^h \circ \mathcal{M}^h$; and the same for $\mathcal{D}_\perp^h$.

The following notations, specific to the "continuous" approach, are introduced in sections 2.5 and 3.1.

u^h: the continuous in t interpolation of $b(\overline{v}^h)$, affine on each time interval I^n;
v^h: an interpolated solution in E for $\overline{v}^h$ (cf. Definition 2.8);
$\mathcal{G}^h$: the interpolated gradient operator produced by $(\mathcal{T}^h, k^h, \mathcal{D}^h)$ (cf. Definition 3.2);
$\mathcal{A}$: the elliptic operator in (1.1), $\mathcal{A} : \eta \in E \mapsto -\text{div}\, a_p(D\eta) \in E'$;
$\mathcal{A}^h$: the finite volume approximation of $\mathcal{A}$ produced by the scheme $(\mathcal{T}^h, k^h, \mathcal{D}^h)$, given by $\mathcal{A}^h : \eta \in E \mapsto -\text{div}\, a_p(\mathcal{G}^h[\eta]) \in E'$.

2.3. Admissible flux approximations. For simplicity, we consider only the gradients that yield fully implicit schemes; in this case $\mathcal{D}^h, \mathcal{D}_\perp^h$ act independently on each set $\big(v_K^n\big)_K$, and the dependence on n does not matter for their definition.

Let us introduce the operator $\mathcal{D}_\perp^h$, which appears naturally in the a priori estimates of section 2.4:

$$\mathcal{D}_\perp^h : (v_K)_K \mapsto (D_{\perp,K|L})_{K|L}, \qquad D_{\perp,K|L} = \frac{v_L - v_K}{d_{K,L}} \in \mathbb{R}. \tag{2.6}$$

For $\varsigma \in \mathbb{N}$, denote by $\Upsilon_\varsigma(K)$ the union of all control volumes of $\mathcal{T}$ that are separated from K by at most $(\varsigma - 1)$ other control volumes; for instance, $\Upsilon_1(K) = \bigcup_{L\in\mathcal{N}_K} L$. The choice of ς corresponds to the choice of control volumes that are really involved in the construction of $\mathcal{D}^h$ on ∂K.

Now we can make precise the assumptions on discrete gradient operators of the form (2.2).

DEFINITION 2.3. *Let $(\mathcal{T}^h, \mathcal{D}^h)_h$ be a family of finite volume meshes and corresponding discrete gradient operators. The gradient approximation provided by $\mathcal{D}^h$ is admissible if the following hold.*

(2.3 i) *$\mathcal{D}^h$ is linear and injective;*

(2.3 ii) *$\mathcal{D}^h$ provides a strictly monotone scheme; i.e., for all $(v_K)_K, (\widetilde{v}_K)_K \subset (\mathbb{R}^d)^N$ that do not coincide,*

$$\frac{1}{d}\sum_{K|L}\Big((v_L - v_K) - (\widetilde{v}_L - \widetilde{v}_K)\Big)\int_{K|L}\Big(a_p(D_{K|L}(x)) - a_p(\widetilde{D}_{K|L}(x))\Big)dx\cdot\nu_{K,L} > 0,$$

where $(D_{K|L})_{K|L} = \mathcal{D}^h\big[(v_K)_K\big]$, $(\widetilde{D}_{K|L})_{K|L} = \mathcal{D}^h\big[(\widetilde{v}_K)_K\big]$;

(2.3 iii) *$\mathcal{D}^h$ provides a scheme coercive at zero; i.e., there exists a constant $C_* > 0$, independent of h, such that for all $(v_K)_K \subset (\mathbb{R}^d)^N$ and $(D_{K|L})_{K|L} = \mathcal{D}^h\big[(v_K)_K\big]$, one has*

$$\frac{1}{d}\sum_{K|L}\Big(v_L - v_K\Big)\int_{K|L} a_p(D_{K|L}(x))dx\cdot\nu_{K,L} \geq C_*\Big\|\mathcal{D}^h_\perp[(v_K)_K]\Big\|^p_{L^p(\Omega)};$$

and there exists $\varsigma \in \mathbb{N}$, independent of h, such that the following hold.

(2.3 iv) *For each h, $\mathcal{D}^h$ is consistent with affine functions. More exactly, assume that, for $K \in \mathcal{T}^h$ given, there exists an affine function w on Ω such that $v_L = \frac{1}{\mathfrak{m}(L)}\int_L w$ whenever $L \subset \Upsilon_\varsigma(K)$. Then $D_{K|L}(x) = Dw = \text{const}$ for all $x \in K|L$ for all $L \in \mathcal{N}_K$.*

(2.3 v) *There exists a constant C^*, independent of h, such that, for all $\widetilde{K} \in \mathcal{T}^h$ and all sets of values $(v_K)_K$ of $\mathbb{R}^N$,*

$$\int_{\widetilde{K}}\Big|\mathcal{D}^h[(v_K)_K]\Big|^p \leq C^*\int_{\Upsilon_\varsigma(\widetilde{K})}\Big|\mathcal{D}^h_\perp[(v_K)_K]\Big|^p.$$

Conditions (2.3 ii) and (2.3 iv) imply strong restrictions on the gradient approximation. We provide some examples of methods with admissible gradient approximation in section 5.1.

2.4. Discrete solutions. Recall that we consider as unknowns the values v^n_K on $K \in \mathcal{T}_{int}$, assigning v^n_K to be zero in $K \in \mathcal{T}_{ext}$. We will repeatedly use the following "summation by parts" formula (cf., e.g., [15]).

Remark 2.4. Let $\mathcal{T}$ be a finite volume mesh of Ω in the sense of Definition 2.1. Let $(v_K)_{K\in\mathcal{T}} \subset \mathbb{R}^N$, $(F_{K,L})_{(K,L)\in\mathcal{T}^2} \subset \mathbb{R}^N$. Assume $v_K = 0$ for all $K \in \mathcal{T}_{ext}$ and $F_{K,L} = -F_{L,K}$ for all $K|L \in \mathcal{E}$. Then

$$\sum_K v_K \sum_{L\in\mathcal{N}_K} F_{K,L} = \sum_{K|L}(v_K - v_L)\,F_{K,L}.$$

If $(v_K^n)_{K,n}$ verifies $(\mathcal{S}^h)$, we say that the function $\overline{v}^h = \sum_{K,n} v_K^n \mathbb{1}_{Q_K^n}$ is the corresponding discrete solution. We prove the discrete version of the $L^p(0,T;W_0^{1,p}(\Omega))$- a priori estimate on $\overline{v}^h$ (which is exactly the estimate on $\mathcal{D}_\perp^h[\overline{v}^h]$ in L^p), and the discrete version of (1.6).

PROPOSITION 2.5. *Let $\big((\mathcal{T}^h,k^h)\big)_h$ be a family of finite volume grids and let $\big(\mathcal{D}^h\big)_h$ be a family of corresponding discrete gradient operators satisfying property* (2.3 iii) *of Definition* 2.3. *Then, for any solution $\overline{v}^h$ of the discrete problem $(\mathcal{S}^h)$, there exists a constant C which depends only on p,d,Ω,T, on C_* in* (2.3 iii), *and on $\|\Psi(u_0)\|_{L^1(\Omega)}$ such that*

$$\text{(i)} \qquad \left\|\mathcal{D}_\perp^h[\overline{v}^h]\right\|_{L^p(Q)}^p = \frac{1}{d}\sum_{K|L,n} m(K|L) d_{K,L}\left|\frac{v_L^n - v_K^n}{d_{K,L}}\right|^p \le C;$$

$$\text{(ii)} \qquad \left\|B(\overline{v}^h)\right\|_{L^\infty(0,T;L^1(\Omega))} = \sup_{n\in\{1,\dots,[T/k^h]+1\}} \sum_K \mathfrak{m}(K)\, B(v_K^n) \le C.$$

Proof. Take $i \in \{1,\dots,[T/k^h]+1\}$ and multiply each term in (2.3) by v_K^i. By (2.3 iii), using Remark 2.4 and (2.4), one gets

$$\sum_K \mathfrak{m}(K)(b(v_K^i) - b(v_K^{i-1}))\, v_K^i + C_* k^h d\int_\Omega \left|\mathcal{D}_\perp^h[\overline{v}^h]\right|^p \le 0.$$

By the convexity of Φ, one has $(b(v_K^i) - b(v_K^{i-1}))\, v_K^i \ge B(v_K^i) - B(v_K^{i-1})$. Summing over i from 1 to $n \in \{1,\dots,[T/k^h]+1\}$ and taking into account the convexity of Ψ, we infer

$$\sum_K \mathfrak{m}(K)B(v_K^n) + C_* d\int_0^{nk^h}\!\!\int_\Omega \left|\mathcal{D}_\perp^h[\overline{v}^h]\right|^p \le \sum_K \mathfrak{m}(K)\Psi(u_K^0) = \sum_K \mathfrak{m}(K)\Psi\left(\frac{1}{\mathfrak{m}(K)}\int_K u^0\right) \le \int_\Omega \Psi(u^0). \qquad \square$$

Next, let us prove the discrete version of the Poincaré inequality and of the compact embedding of $W^{1,p}(\Omega)$ in $L^1(\Omega)$. Note that we do not need any proportionality assumptions on the mesh.

LEMMA 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain of diameter $\mathfrak{d}(\Omega)$, and let $\mathcal{T}$ be a finite volume mesh of Ω. Let $\overline{v}^h = \sum_K v_K \mathbb{1}_K$ such that $(v_K)_{K\in\mathcal{T}} \subset \mathbb{R}$ and $v_K = 0$ for all $K \in \mathcal{T}_{ext}$. Then there exists a constant C which depends only on p and d such that*

$$\text{(i)} \qquad \|\overline{v}^h\|_{L^p(\Omega)} \le C\,\mathfrak{d}(\Omega)\left\|\mathcal{D}_\perp^h[\overline{v}^h]\right\|_{L^p(\Omega)};$$

$$\text{(ii)} \qquad \textit{for all } \Delta > 0, \quad \sup_{|\Delta x|\le\Delta}\int_{\mathbb{R}^d} |\overline{v}^h(x+\Delta x) - \overline{v}^h(x)|\,dx \le \Delta \times \left\|\mathcal{D}_\perp^h[\overline{v}^h]\right\|_{L^1(\Omega)}.$$

Proof. (i) For $x \in \Omega$, set $\psi_{K|L}(x) = 1$ in the case that the orthogonal projection of $K|L$ on the hyperplane $\{x_1 = 0\}$ contains $(0,x_2,\dots,x_d)$, and set $\psi_{K|L}(x) = 0$ otherwise. One has

$$|\overline{v}^h(x)|^p \le \frac{1}{2}\sum_{K|L} \psi_{K|L}(x)\left||v_L|^p - |v_K|^p\right| \le C\sum_{K|L}\psi_{K|L}(x)\, d_{K,L}\frac{|v_L - v_K|}{d_{K,L}}\Big(|v_K|^{p-1} + |v_L|^{p-1}\Big).$$

Since $\int_\Omega \psi_{K|L}(x)\,dx \le m(K|L)\,\mathfrak{d}(\Omega)$, one has by the Hölder inequality

$$\int_\Omega |\overline{v}^h(x)|^p\,dx \le C\mathfrak{d}(\Omega)\left(\sum_{K|L}\frac{1}{d}\,m(K|L)\,d_{K,L}\left|\frac{v_L-v_K}{d_{K,L}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{K|L}\frac{1}{d}\,m(K|L)\,d_{K,L}\big(|v_K|^p+|v_L|^p\big)\right)^{\frac{p-1}{p}}.$$

Denote $h=\mathrm{size}(\mathcal{T})$. Assertion (i) will follow by the Young inequality if we show that

$$\sum_{K|L}\frac{1}{d}\,m(K|L)\,d_{K,L}\big(|v_K|^p+|v_L|^p\big) \le (1+2^p)\sum_K \mathfrak{m}(K)\,|v_K|^p + 2(2h)^p\sum_{K|L}\frac{1}{d}\,m(K|L)\,d_{K,L}\left|\frac{v_L-v_K}{d_{K,L}}\right|^p, \tag{2.7}$$

since $h\le \mathfrak{d}(\Omega)$. Denote by R the left-hand side of (2.7). We have $d_{K,L}= d_{K,K|L}+d_{L,K|L}$; thus

$$R=\sum_K \mathfrak{m}(K)|v_K|^p+\sum_{K|L}\frac{1}{d}m(K|L)(|v_K|^p d_{L,K|L}+|v_L|^p d_{K,K|L}).$$

Note that

$$|v_K|^p d_{L,K|L} \le \begin{cases} 2^p|v_L|^p d_{L,K|L} & \text{if } |v_K|\le 2|v_L|,\\ (2h)^p\left|\frac{v_L-v_K}{d_{K,L}}\right|^p d_{K,L} & \text{otherwise.}\end{cases}$$

Indeed, if $|v_K|>2|v_L|$, one has $|v_L-v_K|>\frac{1}{2}|v_K|$ so that

$$|v_K|^p d_{L,K|L}\le |v_K|^p d_{K,L}\le 2^p|v_L-v_K|^p d_{K,L}\le 2^p h^p\left|\frac{v_L-v_K}{d_{K,L}}\right|^p d_{K,L}.$$

Using the same argument for $|v_L|^p d_{K,K|L}$, we obtain the desired estimate (2.7).

(ii) Now for $x\in\mathbb{R}^d$, set $\overline{\psi}_{K|L}(x)=1$ in the case where the segment $[x,x+\Delta x]$ crosses $K|L$, and set $\overline{\psi}_{K|L}(x)=0$ otherwise. Note that $\int_{\mathbb{R}^d}\overline{\psi}_{K|L}(x)\,dx\le m(K|L)\,\Delta$; hence (ii) follows, since

$$\int_{\mathbb{R}^d}|\overline{v}^h(x)-\overline{v}^h(x+\Delta x)|\,dx \le \int_{\mathbb{R}^d}\sum_{K|L}\overline{\psi}_{K|L}(x)|v_L-v_K|\,dx \le \Delta\sum_{K|L}m(K|L)d_{K,L}\left|\frac{v_L-v_K}{d_{K,L}}\right|. \qquad \square$$

Now we can state the result for existence and uniqueness of a discrete solution.

THEOREM 2.7. *Let $\mathcal{T}^h$ be a finite volume mesh of Ω, $k^h>0$, and let $\mathcal{D}^h$ be a discrete gradient associated to $\mathcal{T}^h$. If $\mathcal{D}^h$ satisfies* (2.3 iii), *there exists a solution $(v^n_K)_{K,n}$ to the discrete problem $(\mathcal{S}^h)$. If $\mathcal{D}^h$ satisfies* (2.3 ii), *the solution is unique.*

Proof of Theorem 2.7. Using Remark 2.4 and the coercivity of the scheme, we apply the Brouwer fixed point theorem and get existence. Uniqueness follows from the monotonicity of $b(\cdot)$ and the strict monotonicity of the scheme. See [3] for more detailed proofs. $\square$

2.5. Interpolation property and main result. Consider a family of finite volume schemes $\big((\mathcal{T}^h, k^h, \mathcal{D}^h)\big)_h$ such that h tends to 0. Let $(v_K^n)_{K,n}$ be a solution to the scheme $(\mathcal{S}^h)$ and $\overline{v}^h$ the corresponding discrete solution.

We require the existence of what will be called "interpolated solutions" for $\overline{v}^h$, denoted by v^h, such that $v^h \in E$; these should be close to $\overline{v}^h$ (asymptotically as $h \to 0$) and satisfy the a priori estimate in E analogous to the estimate of Proposition 2.5(i) on $\overline{v}^h$. Moreover, the values $(v_K^n)_{K,n}$ should be recoverable from v^h. To this end, we require $\mathcal{M}^h[v^h] = (v_K^n)_{K,n}$.

Definition 2.8. *A family of grids $\big(\mathcal{T}^h, k^h\big)_h$ has the interpolation property in E if, for any family $(\overline{v}^h)_h$ of functions such that $\overline{v}^h|_{Q_K^n} = v_K^n \equiv \text{const}$ for each $K \in \mathcal{T}^h$, $n \in \{1, \dots, [T/k^h]+1\}$, with $v_K^n = 0$ for $K \in \mathcal{T}^h_{ext}$ and with $\big\|\mathcal{D}_\perp^h[\overline{v}^h]\big\|_{L^p(Q)} \le C$ for all h, there exists a family $(v^h)_h \subset E$ such that*

$$\|v^h - \overline{v}^h\|_{L^p(Q)} \to 0 \quad \text{as} \quad h \to 0, \tag{2.8}$$

$$\mathcal{M}^h[v^h] = (v_K^n)_{K,n}, \tag{2.9}$$

$$\|v^h\|_E \le I(C) \quad \text{with some function } I : \mathbb{R}^+ \mapsto \mathbb{R}^+ \text{ independent of } h. \tag{2.10}$$

If $\overline{v}^h$ is a solution to a finite volume scheme, we say that v^h is an interpolated solution for $\overline{v}^h$.

The interpolation property is the main technical assumption required by the "continuous" approach. In section 5.2 we give two conditions ensuring this property. Now let us state the main result of this paper.

Theorem 2.9. *Let $\big((\mathcal{T}^{h_m}, k^{h_m}, \mathcal{D}^{h_m})\big)_{m\in\mathbb{N}}$ be a sequence of finite volume schemes, where $k^{h_m} + \text{size}(\mathcal{T}^{h_m}) \le h_m \to 0$ as $m \to \infty$. Assume that the family of meshes is weakly proportional, the gradient approximation is admissible, and the interpolation property holds (cf. Definitions* 2.2, 2.3, and 2.8*).*

For $m \in \mathbb{N}$, let $\overline{v}^{h_m}$ be a discrete solution of (S^{h_m}). Then there exists a subsequence $(h_{m_l})_{l\in\mathbb{N}}$ such that $\overline{v}^{h_{m_l}} \rightharpoonup v$ in $L^p(Q)$ as $l \to \infty$, where v is a weak solution of the problem (1.1)*.*

Note that it suffices to strengthen slightly assumption (2.3 ii) of Definition 2.3 in order to get the strong convergence of $\overline{v}^{h_{m_l}}$ to v in $L^p(Q)$ (cf. [4, Corollary 1]). Moreover, in the case when $N = 1$ the whole sequence converges to the unique solution of (1.1). In this case error estimates can be proved (cf., e.g., [15] for the linear case), but this is not the purpose of the present paper.

In what follows, we write k instead of k^h and omit subscripts in sequences (h_m) and(h_{m_l}), simply writing that h tends to zero.

3. The "continuous" approach. Take the discrete solution $\overline{v}^h = \sum_{K,n} v_K^n 1\!\!1_{Q_K^n}$ produced by the finite volume scheme $(\mathcal{S}^h)$. Let $v^h \in E$ be a corresponding interpolated solution. We will show that there exist functions $u^h \in L^1(Q)$ and $G^h \in L^p(Q)$ such that $u^h(0,\cdot) = u_0^h(\cdot)$ and ${u^h}_t = \operatorname{div} a_p(G^h)$ in the weak sense of Definition 1.1, and the functions u^h, G^h can be recovered from the interpolated solution. More exactly, we prove in Proposition 3.3 below that ${u^h}_t \in \mathcal{D}'$ can be extended to $\chi^h \in E'$ and

$$\langle \chi^h, \phi \rangle_{E',E} + \iint_Q a_p(\mathcal{G}^h[v^h]) \cdot D\phi = 0 \quad \text{for all } \phi \in E, \tag{3.1}$$

$$(3.2)\quad \langle \chi^h, \xi\rangle_{E',E} = -\iint_Q u^h\,\xi_t - \int_\Omega u_0^h(\cdot)\,\xi(0,\cdot) \qquad \begin{array}{l}\text{for all } \xi\in E \text{ with}\\ \xi_t\in L^\infty(Q)^N,\ \xi(T,\cdot)=0,\end{array}$$

with an operator $\mathcal{G}^h : E \mapsto L^p(Q)$ to be defined.

The analogy of (3.1), (3.2) with (1.4), (1.5) in Definition 1.1 plays the key role in the proof of the convergence result of Theorem 2.9.

3.1. Interpolated gradient and the "continuous" form of the scheme. First define u^h as the piecewise affine in t interpolation of $b(\overline{v}^h)$:

$$u^h = \sum_{K,n}\left(b(v_K^n) + \frac{t-kn}{k}(b(v_K^n) - b(v_K^{n-1}))\right)1\!\!1_{Q_K^n}. \tag{3.3}$$

Then (3.2) holds, since $u^h(0,\cdot) = u_0^h(\cdot)$ and the piecewise constant function ${u^h}_t$ extends to $\chi^h \in E'$ by

$$\langle \chi^h, \phi\rangle_{E',E} = \iint_Q {u^h}_t\,\phi \quad \text{for all } \phi\in E. \tag{3.4}$$

Next, note that in $(\mathcal{S}^h)$ the numerical flux is prescribed on the boundary of each control volume; we will extend it to Q as follows. For given $K \in \mathcal{T}^h_{int}, n$ and a function $F_K^n : \partial K \mapsto \mathbb{R}$, consider the following Neumann problem in the factor space $\mathcal{W}(K) = W^{1,p}(K)/\mathbb{R}$:

$$\begin{cases} \operatorname{div} a_p(Dw) = \dfrac{1}{\mathfrak{m}(K)} \displaystyle\sum_{K\in\mathcal{N}_K} \int_{K|L} F_K^n \quad \text{on } K,\\ a_p(Dw)\cdot \nu_K|_{\partial K} = F_K^n, \end{cases} \tag{3.5}$$

where ν_K is the exterior unit normal vector to ∂K. For $K \in \mathcal{T}^h_{ext}$ with $\mathfrak{m}(K) > 0$, we drop in (3.5) the Neumann boundary condition on $\partial K \cap \partial\Omega$ and seek $w \in W^{1,p}(K)$ with $w|_{\partial K\cap\partial\Omega} = 0$.

LEMMA 3.1. *Let $F_K^n \in L^{p'}(\partial K)$ ($F_K^n \in L^{p'}(\partial K\setminus\partial\Omega)$, if $K \in \mathcal{T}^h_{ext}$). Then* (3.5) *admits a unique solution.*

The proof is standard, using the coercivity and monotonicity argument [25, Chap. 2, Th. 2.1]. Now we can introduce the interpolated gradient operator.

DEFINITION 3.2. *The interpolated gradient operator $\mathcal{G}^h : E \mapsto L^p(Q)$ maps $\eta\in E$ into $\mathcal{G}^h[\eta]$ given by*

$$\begin{cases} \mathcal{G}^h[\eta] = \sum_{K,n} D\eta_K^n 1\!\!1_{Q_K^n}, \quad \textit{where} \quad \eta_K^n \in \mathcal{W}(K) \ \textit{ solves}\\ -\displaystyle\int_K a_p(D\eta_K^n)\cdot D\varphi + \sum_{L\in\mathcal{N}_K}\int_{K|L} \varphi\, a_p(D^n_{K|L})\cdot\nu_K = \frac{1}{\mathfrak{m}(K)}\int_K \varphi \sum_{L\in\mathcal{N}_K}\int_{K|L} a_p(D^n_{K|L})\cdot\nu_K\\ \textit{for all } \varphi\in W^{1,p}(K) \ (\textit{for all } \varphi\in W^{1,p}(K) \textit{ with } \varphi|_{\partial K\cap\partial\Omega} = 0, \ \textit{in case } K\in\mathcal{T}^h_{ext})\\ \textit{and the values } D^n_{K|L}(x) \textit{ are given by } (D^n_{K|L})_{K|L,n} = \mathcal{D}^h[\eta]. \end{cases}$$

If $\overline{v}^h$ solves $(\mathcal{S}^h)$, we set $G^h = \mathcal{G}^h[\overline{v}^h]$. We remark that $G^h = \mathcal{G}^h[v^h]$ by property (2.9) of interpolated solutions v^h. We show that (3.1) follows from (3.4) and the conservation of fluxes.

PROPOSITION 3.3. *Assume that $(v_K^n)_{K,n}$ is a solution of $(\mathcal{S}^h)$. Let v^h be a corresponding interpolated solution, let u^h and χ^h be defined by* (3.3) *and* (3.4), *respectively,* *and let* $\mathcal{G}^h$ *be the interpolated gradient operator of Definition* 3.2*. Then* (3.1), (3.2) *hold.*

Proof. It remains to check (3.1). By (3.3), for all $K \in \mathcal{T}^h$ and n, we have

$$u^h{}_t - \operatorname{div} a_p(G^h) = \frac{b(v_K^n) - b(v_K^{n-1})}{k} - \frac{1}{\mathfrak{m}(K)} \sum_{L \in \mathcal{N}_K} \int_{K|L} a_p(D^n_{K|L}(x))dx \cdot \nu_{K,L} = 0$$

everywhere on Q^n_K because of (2.3). Therefore, using (3.4) and integrating by parts in each Q^n_K, we have

$$\begin{aligned} \langle \chi^h, \phi \rangle_{E',E} + \int_Q a_p(\mathcal{G}^h[v^h]) \cdot D\phi \iint_Q u^h{}_t \phi + a_p(G^h) \cdot D\phi \\ = \sum_{K,n} \iint_{Q^n_K} (u^h{}_t - \operatorname{div} a_p(G^h))\phi + \sum_{K,n} \sum_{L \in \mathcal{N}_K} \iint_{I^n \times K|L} \phi \, a_p(D^n_{K|L}) \cdot \nu_{K,L} \\ = 0 + \sum_{K|L,n} \iint_{I^n \times K|L} \phi \, a_p(D^n_{K|L}) \cdot (\nu_{K,L} + \nu_{L,K}) = 0. \end{aligned}$$ ☐

3.2. Properties of the interpolated gradient and consistency. In view of (3.1) and (1.4), it is natural to compare the elliptic operator in (1.1),

$$\mathcal{A} : \eta \in E \mapsto -\operatorname{div} a_p(D\eta) \in E', \tag{3.6}$$

with the operators

$$\mathcal{A}^h : \eta \in E \mapsto -\operatorname{div} a_p(\mathcal{G}^h[\eta]) \in E'. \tag{3.7}$$

Indeed, $\mathcal{A}^h$ can be considered as the finite volume approximation of $\mathcal{A}$, whence the following definition.

DEFINITION 3.4. *Let* $\big((\mathcal{T}^h, k^h, \mathcal{D}^h)\big)_h$ *be a family of finite volume schemes for the problem* (1.1)*, with* $\operatorname{size}(\mathcal{T}^h) + k^h \le h \to 0$*. We say that the approximation of* (1.1) *by these schemes is consistent if, for all* $\eta \in E$*, one has* $\mathcal{A}^h[\eta] \to A[\eta]$ *in* E' *as* $h \to 0$.

In this section we prove the following result.

THEOREM 3.5. *Let* $\big((\mathcal{T}^h, k^h, \mathcal{D}^h)\big)_h$ *be a family of finite volume schemes with a weakly proportional family of meshes and an admissible gradient approximation (cf. Definitions* 2.2 *and* 2.3*). Then it provides a consistent approximation of* (1.1)*, in the sense of Definition* 3.4.

The proof of Theorem 3.5 is based upon the following properties of the interpolated gradient operator $\mathcal{G}^h$.

PROPOSITION 3.6. *Let* $\big((\mathcal{T}^h, k^h, \mathcal{D}^h)\big)_h$ *be a family of finite volume schemes with admissible gradient approximation and weakly proportional family of meshes.*

(i) *There exists a constant* C *such that for all* $\eta \in E$ *and* $H \subset Q$ *such that* $H = \bigcup_{i=1}^m Q^{n_i}_{K_i}$,

$$\iint_H \Big| \mathcal{G}^h[\eta] \Big|^p \le C \iint_{\Upsilon_{\varsigma+1}(H)} |D\eta|^p,$$

where $\Upsilon_{\varsigma+1}(H) = \bigcup_{i=1}^m I^{n_i} \times \Upsilon_{\varsigma+1}(K_i)$*. In particular,* $\big(\mathcal{G}^h\big)_h$ *are uniformly bounded on* E *and*

$$\Big\| \mathcal{G}^h[\eta] \Big\|_{L^p(Q)} \le C \, \|\eta\|_E. \tag{3.8}$$

(ii) *The operators $(\mathcal{G}^h)_h$ are locally equicontinuous on E. More exactly, there exists a constant $C(R)$ such that, whenever $\|\eta\|_E \le R$ and $\|\mu\|_E \le R$,*

$$\left\|\mathcal{G}^h[\eta] - \mathcal{G}^h[\mu]\right\|_{L^p(Q)} \le C(R)\big(\|\eta-\mu\|_E\big)^{\min\{p-1,\frac{1}{p-1}\}}, \tag{3.9}$$

$$\left\|a_p(\mathcal{G}^h[\eta]) - a_p(\mathcal{G}^h[\mu])\right\|_{L^{p'}(Q)} \le C(R)\big(\|\eta-\mu\|_E\big)^{\min\{(p-1)^2,\frac{1}{p-1}\}}. \tag{3.10}$$

In the statement above and in the rest of this section, C denotes a generic constant that depends only on p, d, Ω, on M, ζ^* of (2.1), and on C_*, C^*, ς of Definition 2.3, unless the additional dependence on R is specified. The proof uses the standard properties of the function $a_p(\cdot)$ (cf. [19, 12]): for all $y_1, y_2 \in \mathbb{R}^d$,

$$\begin{cases} |a_p(y_1) - a_p(y_2)|^{p'} \le C\,|y_1-y_2|^p, & 1<p\le 2; \\ |a_p(y_1) - a_p(y_2)|^{p'} \le C\,|y_1-y_2|^{p'}\Big(|y_1|^p + |y_2|^p\Big)^{\frac{p-2}{p-1}}, & p\ge 2; \end{cases} \tag{3.11}$$

$$\begin{cases} |y_1-y_2|^p \le C\Big[(a_p(y_1)-a_p(y_2))\cdot(y_1-y_2)\Big]^{\frac{p}{2}}\Big[|y_1|^p+|y_2|^p\Big]^{\frac{2-p}{2}}, & 1<p\le 2; \\ |y_1-y_2|^p \le C\,(a_p(y_1)-a_p(y_2))\cdot(y_1-y_2), & p\ge 2. \end{cases} \tag{3.12}$$

Before turning to the proofs of Proposition 3.6 and Theorem 3.5, note the following three lemmas.

LEMMA 3.7. *Let $K \subset \mathbb{R}^d$ be a bounded convex domain of $\mathbb{R}^d$ of diameter $\mathfrak{d}(K)$ and d-dimensional measure $\mathfrak{m}(K)$. Assume that K contains a ball of radius $\zeta^*\mathfrak{d}(K) > 0$. Then there exists a constant C such that, assigning $\overline{w} = \frac{1}{\mathfrak{m}(K)}\int_K w$, one has*

$$\int_{\partial K} |w-\overline{w}|^p \le C\,(\mathfrak{d}(K))^{p-1}\int_K |Dw|^p$$

for all $w \in W^{1,p}(K)$, where $w|_{\partial K}$ is understood in the sense of traces.

Proof. Applying, e.g., the proofs of [13, Theorems 59, 60, and 76] with $p=2$ replaced by a general $p\in(1,+\infty)$, we obtain the claim of the lemma with C depending on p, d, and the Lipschitz continuity of ∂K. Due to the convexity of K, C actually depends only on p, d, and ζ^*. ∎

LEMMA 3.8. *Let $(\mathcal{T}^h)_h$ be a weakly proportional family of meshes, and let $(\mathcal{D}^h_\perp)_h$ be the operators defined by (2.6). Then there exists a constant C such that for all K, n for all $\eta\in E$,*

$$\iint_{Q_K^n} \left|\mathcal{D}^h_\perp[\eta]\right|^p \le C\iint_{I^n\times\Upsilon_1(K)} |D\eta|^p.$$

Proof. Let $\left(\eta_K^n\right)_{K,n} = \mathcal{M}^h[\eta]$ and $\eta^n_{K|L} = \frac{1}{k m(K|L)}\int_{I^n\times K|L}\eta$ in the sense of traces. By definition,

$$\begin{aligned} \iint_{Q_K^n}\left|\mathcal{D}^h_\perp[\eta]\right|^p &= \sum_{K|L}\frac{1}{d}\,k\,m(K|L)\,d_{K,K|L}\left|\frac{\eta^n_L-\eta^n_K}{d_{K,L}}\right|^p \\ &\le C\sum_{K|L}\frac{1}{d}\,k\,m(K|L)\,d_{K,K|L}\left(\frac{\left|\eta^n_{K|L}-\eta^n_K\right|^p}{(d_{K,K|L})^p} + \frac{\left|\eta^n_{K|L}-\eta^n_L\right|^p}{d_{K,K|L}\,(d_{L,K|L})^{p-1}}\right) \\ &\le C\sum_{K|L}\frac{1}{d}\,k\,m(K|L)\,d_{K,K|L}\left|\frac{\eta^n_{K|L}-\eta^n_K}{d_{K,K|L}}\right|^p + C\sum_{K|L}\frac{1}{d}\,k\,m(K|L)\,d_{L,K|L}\left|\frac{\eta^n_{K|L}-\eta^n_L}{d_{L,K|L}}\right|^p. \end{aligned}$$

By the convexity of the function $z \mapsto |z|^p$ and Lemma 3.7,

$$k\, m(K|L)\, |\eta^n_{K|L} - \eta^n_K|^p \le \iint_{I^n \times K|L} |\eta - \eta^n_K|^p \le C\, \mathfrak{d}(K)^{p-1} \iint_{Q^n_K} |D\eta|^p,$$

and the same holds if K and L are exchanged. Hence by (2.1) we have

$$\iint_{Q^n_K} \left|\mathcal{D}^h_\perp[\eta]\right|^p \le C \sum_{L \in \mathcal{N}_K} \left(\iint_{Q^n_K} |D\eta|^p + \iint_{Q^n_L} |D\eta|^p \right) \le C \iint_{I^n \times \Upsilon_1(K)} |D\eta|^p. \qquad \square$$

LEMMA 3.9. *Let $\left((\mathcal{T}^h, k^h, \mathcal{D}^h)\right)_h$ be a family of finite volume schemes with a weakly proportional family of meshes and an admissible gradient approximation. Then the following hold:*

(i) *For all $R > 0$ there exists a constant $C(R)$ such that, whenever $\|\eta\|_E \le R$ and $\|\mu\|_E \le R$,*

$$\sum_{K,n} \mathfrak{d}(K) \iint_{\Sigma^n_K} \left| a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu]) \right|^{p'} \le C(R) \Big(\|\eta - \mu\|_E \Big)^{\min\{p,p'\}}.$$

(ii) *There exists a constant C such that for all $\eta \in E$ and $H, \Upsilon_{\varsigma+1}(H)$ as in Proposition 3.6 one has*

$$\sum_{i=1}^m \mathfrak{d}(K_i) \iint_{\Sigma^{n_i}_{K_i}} \left|\mathcal{D}^h[\eta]\right|^p \le C \iint_{\Upsilon_{\varsigma+1}(H)} |D\eta|^p.$$

Proof. First take K and consider $\varphi^K = \left|a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu])\right|^{p'} \in L^1(\partial K)$. Recall that the values of $\mathcal{D}^h$ have been extended from ∂K inside K by means of the projection operator $\mathcal{P}^h$ (cf. section 2.2). Hence by (2.1) we have

$$\begin{aligned} \mathfrak{d}(K) \int_{\partial K} |\varphi^K| &= \mathfrak{d}(K) \sum_{L \in \mathcal{N}_K} \int_{K|L} |\varphi^K| \\ &= d \sum_{L \in \mathcal{N}_K} \frac{\mathfrak{d}(K)}{d_{K,K|L}} \int_{\widehat{KL} \cap K} |\varphi^K \circ \mathcal{P}^h| \le \frac{d}{\zeta^*} \int_K |\varphi^K \circ \mathcal{P}^h|. \end{aligned} \tag{3.13}$$

If $1 < p \le 2$, (3.13) and (3.11) yield

$$\sum_{K,n} \mathfrak{d}(K) \iint_{\Sigma^n_K} \left| a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu]) \right|^{p'} \le C \sum_{K,n} \iint_{Q^n_K} \left| \mathcal{D}^h[\eta] - \mathcal{D}^h[\mu] \right|^p.$$

In turn, (2.3 i), (2.3 iv), Lemma 3.8, and (2.1) imply that

$$\begin{aligned} \sum_{K,n} \iint_{Q^n_K} \left| \mathcal{D}^h[\eta] - \mathcal{D}^h[\mu] \right|^p &\le C \sum_{K,n} \iint_{I^n \times \Upsilon_\varsigma(K)} \left| \mathcal{D}^h_\perp[\eta - \mu] \right|^p \\ \le C \sum_{K,n} \iint_{I^n \times \Upsilon_{\varsigma+1}(K)} \left| D(\eta - \mu) \right|^p &\le C \iint_Q |D\eta - D\mu|^p = C \left(\|\eta - \mu\|_E \right)^p, \end{aligned} \tag{3.14}$$

which was the claim of (i) for $1 < p \le 2$. Furthermore, we remark that (3.14) also holds for $p \ge 2$, in particular with $\eta = 0$ or $\mu = 0$. Therefore for $p \ge 2$, using (3.13), and then (3.11) and the Hölder inequality, we get

$$\sum_{K,n} \mathfrak{d}(K) \iint_{\Sigma^n_K} \left| a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu]) \right|^{p'} \le C \left(\sum_{K,n} \iint_{Q^n_K} \left| \mathcal{D}^h[\eta] - \mathcal{D}^h[\mu] \right|^p \right)^{\frac{p'}{p}} \times \left(R^p\right)^{\frac{p-2}{p-1}}.$$

Thus, in the case $p \geq 2$, (i) also follows from (3.14).

The proof of (ii) is similar, using the identity $|a_p(y)|^{p'} = |y|^p$ instead of inequalities (3.11). □

Proof of Proposition 3.6. Recalling Definition 3.2, for all $K \in \mathcal{T}_{int}$ and n, denote by η_K^n (respectively, by μ_K^n) a function in $W^{1,p}(K)$ that solves (3.5) with $F_K^n(x) = a_p(D_{K|L}^n(x)) \cdot \nu_K$ for $x \in K|L \in \mathcal{E}_K$, where $(D_{K|L}^n)_{K|L,n} = \mathcal{D}^h[\eta]$ (respectively, $(D_{K|L}^n)_{K|L,n} = \mathcal{D}^h[\mu]$). In other words, each of η_K^n, μ_K^n verifies the integral identity corresponding to (3.5) with all test functions in $W^{1,p}(K)$. Taking for the test function $(\eta_K^n - \mu_K^n)$, subtracting the two identities, and integrating in $t \in I^n$, we obtain

$$\begin{aligned} &\iint_{Q_K^n} \Big(a_p(D\eta_K^n) - a_p(D\mu_K^n)\Big) \cdot \Big(D\eta_K^n - D\mu_K^n\Big) \\ &\qquad = \int_{I^n}\int_{\partial K} \Big(\eta_K^n - \mu_K^n - \overline{\eta_K^n - \mu_K^n}\Big) \Big(a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu])\Big) \cdot \nu_K, \end{aligned} \tag{3.15}$$

where $\overline{\eta_K^n - \mu_K^n} = \frac{1}{\mathfrak{m}(K)} \int_K \eta_K^n - \mu_K^n$ for a.a. $t \in I^n$. Summing over K, n, using the Hölder inequality and Lemmas 3.7 and 3.9(i), we have from (3.15)

$$\begin{aligned} &\iint_Q \Big(a_p(\mathcal{G}^h[\eta]) - a_p(\mathcal{G}^h[\mu])\Big) \cdot \Big(\mathcal{G}^h[\eta] - \mathcal{G}^h[\mu]\Big) \\ &\quad \leq \sum_{K,n} \int_{I^n} \int_{\partial K} \mathfrak{d}(K)^{\frac{-1}{p'}} \,|\, \eta_K^n - \mu_K^n - \overline{\eta_K^n - \mu_K^n} \,| \\ &\qquad\qquad \times \mathfrak{d}(K)^{\frac{1}{p'}} \Big| a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu]) \Big| \\ &\quad \leq \left(\sum_{K,n} \int_{I^n} \mathfrak{d}(K)^{1-p} \int_{\partial K} |\, \eta_K^n - \mu_K^n - \overline{\eta_K^n - \mu_K^n} \,|^p \right)^{\frac{1}{p}} \\ &\qquad\qquad \times \left(\sum_{K,n} \mathfrak{d}(K) \iint_{\Sigma_K^n} \Big| a_p(\mathcal{D}^h[\eta]) - a_p(\mathcal{D}^h[\mu]) \Big|^{p'} \right)^{\frac{1}{p'}} \\ &\quad \leq \left(\sum_{K,n} \iint_{Q_K^n} |D\eta_K^n - D\mu_K^n|^p \right)^{\frac{1}{p}} \|\eta - \mu\|_E^{\min\{p/p',1\}} \\ &\quad = \Big\| \mathcal{G}^h[\eta] - \mathcal{G}^h[\mu] \Big\|_{L^p(Q)} \|\eta - \mu\|_E^{\min\{p/p',1\}}. \end{aligned} \tag{3.16}$$

In the same manner, taking $\mu = 0$ and using Lemma 3.9(ii), we get

$$\iint_H \Big|\mathcal{G}^h[\eta]\Big|^p C \left(\iint_H \Big|\mathcal{G}^h[\eta]\Big|^p \right)^{\frac{1}{p}} \left(\iint_{\Upsilon_{\varsigma+1}(H)} |D\eta|^p \right)^{\frac{1}{p'}},$$

which proves (i).

Now if $1 < p \leq 2$, (3.12), (3.16), and the Hölder inequality yield

$$\begin{aligned} &\Big\| \mathcal{G}^h[\eta] - \mathcal{G}^h[\mu] \Big\|_{L^p(Q)}^p \\ &\quad \leq C \left(\Big\| \mathcal{G}^h[\eta] - \mathcal{G}^h[\mu] \Big\|_{L^p(Q)} \|\eta - \mu\|_E^{\frac{p}{p'}} \right)^{\frac{p}{2}} \left(\Big\| \mathcal{G}^h[\eta] \Big\|_{L^p(Q)}^p + \Big\| \mathcal{G}^h[\mu] \Big\|_{L^p(Q)}^p \right)^{\frac{2-p}{2}}. \end{aligned}$$

Using (3.8), we obtain (3.9). Now (3.10) follows by (3.11).

If $p \geq 2$, (3.12) and (3.16) readily yield (3.9); hence (3.10) follows by (3.11). □

Proof of Theorem 3.5. We have to prove that $\left\|a_p(D\eta) - a_p(\mathcal{G}^h[\eta])\right\|_{L^{p'}(Q)} \to 0$ as $h \to 0$.

Let us first prove the theorem for the case of $\eta \in E$ that is piecewise constant in t and piecewise affine in x. Let $J \subset Q$ be the set of discontinuities of $D\eta$. Clearly, J is of finite d-dimensional Hausdorff measure $\mathcal{H}^d(J)$.

For ς given in Definition 2.3, let us introduce $H^h = \bigcup_{\{K,n \mid I^n \times \Upsilon_\varsigma(K) \cap J \neq \emptyset\}} Q^n_K$. Note that $|H^h| \le (\varsigma+1)h\,\mathcal{H}^d(J) \to 0$ as $h \to 0$; likewise, $|\Upsilon_{\varsigma+1}(H^h)| \to 0$ as $h \to 0$. Therefore, by Proposition 3.6(i), we have

$$\iint_{H^h} \left|a_p(D\eta) - a_p(\mathcal{G}^h[\eta])\right|^{p'} \le C \left(\iint_{H^h} |D\eta|^p + \iint_{\Upsilon_{\varsigma+1}(H^h)} |D\eta|^p \right) \to 0$$

as $h \to 0$. Moreover, for all Q^n_K such that $Q^n_K \cap H^h = \emptyset$, we have $\mathcal{G}^h[\eta] \equiv D\eta$ on Q^n_K. Indeed, we have $D[\eta] \equiv \mathrm{const}$ on $\Upsilon_{\varsigma+1}(Q^n_K)$. Therefore $\mathcal{D}^h[\eta]|_{Q^n_K} \equiv D\eta = \mathrm{const}$ by property (2.3 iv) of admissible gradient approximations. Hence $Dw = D\eta$ satisfies the boundary condition in (3.5); the equation is also satisfied, since $\operatorname{div} a_p(D\eta) \equiv 0$ on K and $\frac{1}{\mathfrak{m}(K)} \int_{\partial K} a_p(\mathcal{D}^h[\eta]) \cdot \nu_K = a_p(D\eta) \cdot \int_{\partial K} \nu_K = 0$.

It follows that $\left\|a_p(D\eta) - a_p(\mathcal{G}^h[\eta])\right\|_{L^{p'}(Q)} \to 0$ as $h \to 0$, which was our claim.

Now let us approximate an arbitrary function η in E by functions η_l that are piecewise constant in t and piecewise affine in x. Note that we can always choose this sequence η_l in E such that $\eta_l \to \eta$ in E and a.e. on Q as $l \to \infty$, and $|D\eta_l|^p$ are dominated by an $L^1(Q)$ function independent of l. We have

$$\begin{aligned} \left\|a_p(D\eta) - a_p(\mathcal{G}^h[\eta])\right\|_{L^{p'}(Q)} &\le \left\|a_p(D\eta) - a_p(D\eta_l)\right\|_{L^{p'}(Q)} \\ &+ \left\|a_p(D\eta_l) - a_p(\mathcal{G}^h[\eta_l])\right\|_{L^{p'}(Q)} + \left\|a_p(\mathcal{G}^h[\eta_l]) - a_p(\mathcal{G}^h[\eta])\right\|_{L^{p'}(Q)}. \end{aligned} \tag{3.17}$$

As $l \to 0$, the first term in the right-hand side of (3.17) converges to zero by the Lebesgue dominated convergence theorem, independently of h. The second one converges to zero as $h \to 0$ for all l fixed. Finally, by Proposition 3.6(ii), the third one converges to zero as $l \to \infty$ uniformly in h. Hence the result follows. □

4. Proof of Theorem 2.9. In the context of continuous dependence upon the data of weak solutions to "general" elliptic-parabolic problems (cf. [2, Chap.V]), the proof of convergence of weak solutions of approximating problems is based upon the three essential arguments (A), (B), and (C) below.

(A) A priori estimates, using (1.2) and the Alt–Luckhaus chain rule lemma (cf. [1, 26, 10]).

(B) Strong compactness in the parabolic term, using a variant of the Kruzhkov lemma (cf. [23]):

LEMMA 4.1 (cf. [4], [2, Chap. V]). *Let Ω be an open domain in $\mathbb{R}^d$, $Q = (0,T)\times\Omega$, and let the families of functions $(u^h)_h, (F^h_\alpha)_{h,\alpha}$ be bounded in $L^1(Q)$ and satisfy $\frac{\partial}{\partial t}u^h = \sum_{|\alpha|\le m} D^\alpha F^h_\alpha$ in $\mathcal{D}'(Q)$. Assume that u^h can be extended by zero outside Q, and one has*

$$\sup_{|\Delta x|\le\Delta} \iint_{\mathbb{R}^{d+1}} |u^h(t,x+\Delta x) - u^h(t,x)|\,dxdt \le \omega(\Delta), \quad \text{with } \lim_{\Delta\to 0}\omega(\Delta) = 0, \tag{4.1}$$

where $\omega(\cdot)$ does not depend on h. Then $(u^h)_h$ is relatively compact in $L^1(Q)$.

(C) Convergence in the elliptic term, using a variant of the Minty–Browder argument (cf., e.g., [25]).

LEMMA 4.2 (cf. [4], [2, Chap. V]). *Let E be a Banach space, E' its dual and $\langle\cdot,\cdot\rangle_{E',E}$ denote the duality product of elements of E' and E. Let $(v^h)_h \subset E$ and $v^h \rightharpoonup v$ as $h \to 0$. Let $\mathcal{A}^h$ be a sequence of monotone operators from E to E' such that $\mathcal{A}^h[v^h] \overset{*}{\rightharpoonup} -\chi$ for some $\chi \in E'$. Assume that $\mathcal{A}^h$ converge pointwise to some operator $\mathcal{A}$, and $\mathcal{A}$ is hemicontinuous (i.e., continuous in the weak-$*$ topology of E' along each direction). Assume that*

$$\liminf_{h\to 0}\langle \mathcal{A}^h[v^h], v^h\rangle_{E',E} \;\leq\; \langle -\chi, v\rangle_{E',E}. \tag{4.2}$$

Then $\chi + \mathcal{A}[v] = 0$, and (4.2) necessarily holds with equality.

Taking advantage of the "continuous" form (3.1), (3.2) of the discrete problem $(\mathcal{S}^h)$, we can prove the convergence of finite volume approximate solutions in the same way, using the discrete a priori estimates shown in Propositions 2.5 and 3.6(i), using next Lemma 4.1, and then using finally Lemma 4.2 together with the essential consistency result of Theorem 3.5.

Proof of Theorem 2.9. Let $\overline{v}^h$ be the solution of $(\mathcal{S}^h)$. Let v^h be a corresponding interpolated solution, and let $\mathcal{A}^h$ be the finite volume approximate of the operator $\mathcal{A}$ in (1.1) (cf. (3.6), (3.7)). Note that all the convergences we state below take place up to extraction of a subsequence.

(A) By Proposition 2.5(i), $\left\|\mathcal{D}^h_\perp[\overline{v}^h]\right\|_{L^p(Q)} \leq$ const uniformly in h so that the family $(v^h)_h$ is bounded in E, by (2.10). Hence there exists a function $v \in E$ such that $v^h \rightharpoonup v$ in E as $h \to 0$. By (2.8), one also has $\overline{v}^h \rightharpoonup v$ in $L^p(Q)$.

(B) We claim that the family $(u^h)_h$ given by (3.3) is relatively compact in $L^1(Q)$. Indeed, let us check the assumptions of Lemma 4.1. We have ${u^h}_t = \operatorname{div} a_p(\mathcal{G}^h[v^h])$ in $\mathcal{D}'(Q)$ by (3.1), (3.2), and the family $\left(a_p(\mathcal{G}^h[v^h])\right)_h$ is bounded in $L^{p'}(Q)$ by Proposition 2.5(i), equation (2.10), and Proposition 3.6(i) (note that $\left(\mathcal{A}^h[v^h]\right)_h$ is thus bounded in E'). Furthermore, (3.3) yields

$$\|u^h\|_{L^1(Q)} \leq 2\iint_Q |b(\overline{v}^h)| + k^h \sum_K \mathfrak{m}(K)|u^0_K|,$$

and one has $|b(z)| \leq \delta B(z) + \sup_{|\zeta|\leq 1/\delta}|b(\zeta)|$ for all $\delta > 0$ (cf., e.g., [1]). By Proposition 2.5(ii) and since $u^h_0 = \sum_K \mathfrak{m}(K)|u^0_K| \to u_0$ in $L^1(\Omega)$ as $h \to 0$, it follows that $(u^h)_h$ is bounded in $L^1(Q)$.

Finally, by Proposition 2.5(i) and Lemma 2.6(ii), we obtain (4.1) with u^h replaced by $\overline{v}^h$. Hence the estimate (4.1) for u^h follows by (3.3), as in the continuous case (cf. [1]); see [4] for the detailed proof.

Thus the claim of (B) follows, and there exists a function $u \in L^1(Q)$ such that $u^h \to u$ in $L^1(Q)$ and a.e. on Q. In addition, we claim that $u = b(v)$, where v is the weak limit of v^h in E. It suffices to show that $\overline{v}^h \rightharpoonup v$ in $L^1(Q)$ and $b(\overline{v}^h) \to u$ in $L^1(Q)$, and then apply the monotonicity argument of [9]; see [4] for the detailed proof.

(C) By (A), we have $v^h \rightharpoonup v$ in E. We claim that v is a weak solution of (1.1).

By Proposition 3.3, $\chi^h + \mathcal{A}^h[v^h] = 0$ in E' and the initial condition (3.2) is verified for all h. The family $\left(\mathcal{A}^h[v^h]\right)_h$ is bounded in E' (cf. (B)), thus $(\chi^h)_h$ is weak-$*$ relatively compact in E'. By (3.4), (B), and Definition 1.1, we also have $\chi^h = {u^h}_t \to b(v)_t = \chi$ in $\mathcal{D}'(Q)$. Hence $\mathcal{A}^h[v^h] = -\chi^h \overset{*}{\rightharpoonup} -\chi$ in E'.

Moreover, passing to the limit in (3.2), using (B) and the convergence of u_0^h to u_0 in $L^1(\Omega)$, we get (1.5). Consequently, by the chain rule argument [1, Lemma 1.5] we have

$$\langle -\chi, v\rangle_{E',E} = -\int_\Omega \Psi(b(v(T,\cdot))) + \int_\Omega \Psi(u^0). \tag{4.3}$$

On the other hand, by (3.4), (3.3), (2.9), and the monotonicity of $b(\cdot)$, we have

$$\begin{aligned}\langle -\chi^h, v^h\rangle_{E',E} &= -\frac{1}{k}\sum_{K,n}(b(v_K^n) - b(v_K^{n-1}))\iint_{Q_K^n} v^h \\ &= -\sum_{K,n}\mathfrak{m}(K)(b(v_K^n) - b(v_K^{n-1}))\, v_K^n \\ &\le -\sum_K \mathfrak{m}(K)\Psi(b(v_K^{[T/k^h]+1})) + \sum_K \mathfrak{m}(K)\Psi(u_K^0) \\ &= -\int_\Omega \Psi(b(\overline{v}^h(T,\cdot))) + \int_\Omega \Psi(u_0^h).\end{aligned}$$

Recall that $\Psi(u_0^h) \to \Psi(u_0)$ in $L^1(\Omega)$. Without loss of generality, we can assume that $\overline{v}^h(T,\cdot) \to v(T,\cdot)$ a.e. on Ω; hence by the Fatou lemma and (4.3) we get (4.2).

Next, the operators $\mathcal{A}^h$ are monotone. Indeed, take $\varphi \in E$ and $(\varphi_K^n)_{K,n} = \mathcal{M}^h[\varphi]$. Arguing as in the proof of Proposition 3.3, integrating by parts in Q_K^n, and cancelling the boundary terms, we get

$$\begin{aligned}\langle \mathcal{A}^h[\eta], \varphi\rangle_{E',E} &= \iint_Q a_p(\mathcal{G}^h[\eta])\cdot D\varphi = -\sum_{K,n}\iint_{Q_K^n}\varphi \operatorname{div} a_p(\mathcal{G}^h[v^h]) \\ &= -\sum_{K,n}\iint_{Q_K^n}\varphi \times \frac{1}{\mathfrak{m}(K)}\sum_{L\in\mathcal{N}_K}\int_{K|L} a_p(D_{K|L}^n(x))dx\cdot \nu_{K,L} \\ &= -k\sum_{K,n}\varphi_K^n \sum_{L\in\mathcal{N}_K}\int_{K|L} a_p(D_{K|L}^n(x))dx\cdot\nu_{K,L}.\end{aligned} \tag{4.4}$$

Substituting (4.4) and applying Remark 2.4, we infer by property (2.3 ii) of Definition 2.3 that

$$\begin{aligned}&\langle \mathcal{A}^h[\eta] - \mathcal{A}^h[\widetilde{\eta}], \eta - \widetilde{\eta}\rangle_{E',E} \\ &\quad = \frac{1}{d}k\sum_{K|L,n}\Big((\eta_L^n - \eta_K^n) - (\widetilde{\eta}_L^n - \widetilde{\eta}_K^n)\Big)\int_{K|L}\Big(a_p(D_{K|L}^n(x)) - a_p(\widetilde{D}_{K|L}^n(x))\Big)dx\cdot\nu_{K,L} \ge 0,\end{aligned}$$

where $(\eta_K^n)_{K,n} = \mathcal{M}^h[\eta]$, $(D_{K|L}^n)_{K|L,n} = \mathcal{D}^h[\eta]$, and the same for $\widetilde{\eta}$.

Finally, by Theorem 3.5, $\mathcal{A}^h$ converge pointwise to the hemicontinuous operator $\mathcal{A}\cdot = -\operatorname{div} a_p(D\,\cdot)$. By Lemma 4.2 we conclude that $\chi + \mathcal{A}[v] = 0$ in E'. Thus (1.4) holds and v is a weak solution of (1.1). ∎

5. Examples of admissible methods. By an admissible method, we mean a method which provides an admissible gradient approximation and weakly proportional meshes satisfying the interpolation property. Recall that in this case, we have the convergence of the finite volume approximation (cf. Theorem 2.9). In this section, we prove that such admissible methods exist.

5.1. On discrete gradients. In this section we construct an admissible gradient for a family $(\mathcal{T}^h)_h$ of finite volume meshes of the Voronoï kind dual to a family $(\widehat{\mathcal{T}}^h)_h$ of triangular meshes.

Let us introduce some notation. We use $\widehat{O}$ to denote a triangle of the mesh $\widehat{\mathcal{T}}^h$; for all $\widehat{O} \in \widehat{\mathcal{T}}^h$, there exist $K, L, M \in \mathcal{T}^h$ such that $\widehat{O} = \Delta x_K x_L x_M$ (the triangle with the corners x_K, x_L, x_M). The three interfaces $K|L$, $L|M$, $M|K$ intersect at point $x_{\widehat{O}}$, which is the center of the circumscribed circle of triangle $\widehat{O}$. We require it to be inside $\widehat{O}$. Let us denote by $S_{\widehat{O}}, S_{K,L}, S_{L,M}$, and $S_{M,K}$ the surfaces of $\Delta x_K x_L x_M, \Delta x_{\widehat{O}} x_K x_L, \Delta x_{\widehat{O}} x_L x_M$, and $\Delta x_{\widehat{O}} x_M x_K$, respectively. One has $S_{\widehat{O}} = S_{K,L} + S_{L,M} + S_{M,K}$.

Recall that $\nu_{K,L} = \overrightarrow{x_K x_L}/d_{K,L}, \nu_{L,M} = \overrightarrow{x_L x_M}/d_{L,M}, \nu_{M,K} = \overrightarrow{x_M x_K}/d_{M,K}$. Note the following elementary lemma.

LEMMA 5.1. *Let $\widehat{O} = \Delta x_K x_L x_M$ be a triangle in $\mathbb{R}^2$, let $x_{\widehat{O}}$ be the center of its circumscribed circle, and let $x_{\widehat{O}} \in \Delta x_K x_L x_M$. With the above notation, for all r in $\mathbb{R}^2$, we have*

$$r = \frac{2}{S_{\widehat{O}}} \Big\{ S_{K,L}(r \cdot \nu_{K,L})\nu_{K,L} + S_{L,M}(r \cdot \nu_{L,M})\nu_{L,M} + S_{M,K}(r \cdot \nu_{M,K})\nu_{M,K} \Big\}.$$

This property can be generalized to any polygon in $\mathbb{R}^2$ which admits the circumscribed circle.

Furthermore, for $\widehat{O} \in \widehat{\mathcal{T}}^h$ such that $\widehat{O} = \Delta x_K x_L x_M$, let $v^{h,0}{}_{\widehat{O}} : \mathbb{R}^2 \to \mathbb{R}^N$ be the affine function that takes the values v_K, v_L, v_M at the points x_K, x_L, x_M, respectively. The discrete gradient operator $\mathcal{D}^{h,0} = \mathcal{L}^h \circ \mathcal{D}^{h,0}$ is defined by

$$\mathcal{D}^{h,0} : (v_K)_K \mapsto \sum_{\widehat{O} \in \widehat{\mathcal{T}}^h} Dv^{h,0}{}_{\widehat{O}}(x) 1\!\!1_{\widehat{O}}(x).$$

In the case of structured hexagonal meshes, as well as that of structured rectangular ones, the family $(\mathcal{D}^{h,0})_h$ is admissible (this will be proved in Proposition 5.2, as a particular case). In general, this construction does not work. Indeed, if the points x_K are not the barycenters of $K \in \mathcal{T}^h$, property (2.3 iv) fails.

This can be overcome, for instance, in the following way. For all $K \in \mathcal{T}^h$, let y_K be the barycenter of K and set $\sigma_K = x_K - y_K$. For $\widehat{O} \in \widehat{\mathcal{T}}^h$ such that $\widehat{O} = \Delta x_K x_L x_M$, let $v^h_{\widehat{O}} : \mathbb{R}^2 \to \mathbb{R}^N$ be the affine function that takes the values v_K, v_L, v_M at the points y_K, y_L, y_M, respectively. The discrete gradient operator $\mathcal{D}^h = \mathcal{L}^h \circ \mathcal{D}^h$ is defined by

$$\mathcal{D}^h : (v_K)_K \mapsto \sum_{\widehat{O} \in \widehat{\mathcal{T}}^h} Dv^h_{\widehat{O}}(x) 1\!\!1_{\widehat{O}}(x); \tag{5.1}$$

i.e., the affine interpolation over the triangle $\Delta y_K y_L y_M$ is actually used in the triangle $\Delta x_K x_L x_M$.

We will take advantage of considering $\mathcal{D}^h$ as a perturbation of $\mathcal{D}^{h,0}$. For all $\widehat{O} \in \widehat{\mathcal{T}}^h$, let us define the correction operators

$$R_{\widehat{O}} : r \in \mathbb{R}^2 \mapsto \frac{2}{S_{\widehat{O}}} \Bigg\{ S_{K,L} \left(r \cdot \frac{\sigma_L - \sigma_K}{d_{K,L}} \right) \nu_{K,L} + S_{L,M} \left(r \cdot \frac{\sigma_M - \sigma_L}{d_{L,M}} \right) \nu_{L,M} + S_{M,K} \left(r \cdot \frac{\sigma_K - \sigma_M}{d_{M,K}} \right) \nu_{M,K} \Bigg\},$$

with the notation introduced above. We need to guarantee that the Euclidiean norm of $R_{\widehat{O}}$ is less than $\min\{p-1, 1/(p-1)\}$ for all $\widehat{O} \in \mathcal{T}^h$.

PROPOSITION 5.2. *Assume that $(\mathcal{T}^h)_h$ is a family of meshes dual to a family of meshes $(\widehat{\mathcal{T}}^h)_h$ such that all $\widehat{O} \in \widehat{\mathcal{T}}^h$ are triangles with angles less than or equal to $\pi/2$.*

Assume that for all h, for all $\widehat{O} \in \widehat{\mathcal{T}}^h$,

$$\frac{2}{S_{\widehat{O}}}\left\{S_{K,L}\frac{|\sigma_L-\sigma_K|}{d_{K,L}}+S_{L,M}\frac{|\sigma_M-\sigma_L|}{d_{L,M}}+S_{M,K}\frac{|\sigma_K-\sigma_M|}{d_{M,K}}\right\}<\min\{p-1,1/(p-1)\},$$

where σ_K is the difference between the “center” x_K of the volume K and its barycenter, etc.

Then the family of discrete gradient operators $(\mathcal{D}^h)_h$ on $(\mathcal{T}^h)_h$ defined by (5.1) *is admissible in the sense of Definition* 2.3.

Proof. Since, for any affine function w on K, one has $\frac{1}{\mathfrak{m}(K)}\int_K w(x)dx = w(y_K)$, where y_K is the barycenter of K, property (2.3 iv) holds for $\mathcal{D}^h$ (with $\varsigma = 1$, by construction). Next, (2.3 i) is clear.

Let us establish the relation between $\mathcal{D}^{h,0}$ and $\mathcal{D}^h$. Denote by $D^0_{\widehat{O}}, D_{\widehat{O}}$ the values on $\widehat{O}$ of $\mathcal{D}^{h,0}[(v_K)_K]$ and $\mathcal{D}^h[(v_K)_K]$, respectively. Let us show that for all $\widehat{O} \in \widehat{\mathcal{T}}^h$,

$$D^0_{\widehat{O}} = (I - R_{\widehat{O}})D_{\widehat{O}}. \tag{5.2}$$

Indeed, if $\widehat{O} = \Delta x_K x_L x_M$, one has

$$\begin{aligned} D_{\widehat{O}}\cdot\nu_{K,L} &= \frac{v^h_{\widehat{O}}(x_L)-v^h_{\widehat{O}}(x_K)}{d_{K,L}} = \frac{(v_L + D_{\widehat{O}}\cdot\sigma_L)-(v_K+D_{\widehat{O}}\cdot\sigma_K)}{d_{K,L}}\\ &= \frac{v_L-v_K}{d_{K,L}} + D_{\widehat{O}}\cdot\frac{\sigma_L-\sigma_K}{d_{K,L}} = D^0_{\widehat{O}}\cdot\nu_{K,L} + D_{\widehat{O}}\cdot\frac{\sigma_L-\sigma_K}{d_{K,L}}. \end{aligned}$$

Writing the same relation for L, M and M, K, from Lemma 5.1, we get $D_{\widehat{O}} - D^0_{\widehat{O}} = R_{\widehat{O}}D_{\widehat{O}}$, whence (5.2) follows.

By Lemma 5.1 and the definition of $\mathcal{D}^h_\perp = \mathcal{L}^h \circ \mathcal{D}^h_\perp$ for all $\widehat{O} \in \widehat{\mathcal{T}}^h$ such that $\widehat{O} = \Delta x_K x_L x_M$ we have

$$\begin{aligned} \int_{\widehat{O}}\left|\mathcal{D}^{h,0}[(v_K)_K]\right|^p &= S_{\widehat{O}}\left|\mathcal{D}^{h,0}[(v_K)_K]\right|^p\\ &\le C^*\left\{S_{K,L}\left|\frac{v_L-v_K}{d_{K,L}}\right|^p + S_{L,M}\left|\frac{v_M-v_L}{d_{L,M}}\right|^p + S_{M,K}\left|\frac{v_K-v_M}{d_{M,K}}\right|^p\right\}\\ &= C^*\int_{\widehat{O}}\left|\mathcal{D}^h_\perp[(v_K)_K]\right|^p \end{aligned} \tag{5.3}$$

with a constant C^* that depends only on p. Since for given $\widetilde{K} \in \mathcal{T}^h$ and $\widehat{O} \in \widehat{\mathcal{T}}^h$ we have $\widetilde{K} \cap \widehat{O} \neq \emptyset$ if and only if $\widehat{O} \in \Upsilon_1(\widetilde{K})$, it follows that property (2.3 v) holds for the discrete gradient $\mathcal{D}^{h,0}$, with $\varsigma = 1$. Now set $\theta_{\widehat{O}} = \|R_{\widehat{O}}\|$. We have $\theta_{\widehat{O}} < 1$. One has $|D_{\widehat{O}}| \le \|(I - R_{\widehat{O}})^{-1}\|\,|D^0_{\widehat{O}}| \le \frac{1}{1-\theta_{\widehat{O}}}|D^0_{\widehat{O}}|$; therefore (2.3 v) also holds for $\mathcal{D}^h$.

Next, each term in the sum in (2.3 iii) splits into two terms corresponding to the two parts of the interface $K|L$ included in different triangles $\widehat{O}_1, \widehat{O}_2 \in \widehat{\mathcal{T}}^h$. Let us write down all the terms corresponding to the same triangle $\widehat{O} \in \widehat{\mathcal{T}}^h$, $\widehat{O} = \Delta x_K x_L x_M$, combine them using Lemma 5.1, and estimate using (5.2):

$$S_{K,L}(a_p(D_{\widehat{O}})\cdot\nu_{K,L})(D^0_{\widehat{O}}\cdot\nu_{K,L})+S_{L,M}(a_p(D_{\widehat{O}})\cdot\nu_{L,M})(D^0_{\widehat{O}}\cdot\nu_{L,M})$$
$$+\,S_{M,K}(a_p(D_{\widehat{O}})\cdot\nu_{M,K})(D^0_{\widehat{O}}\cdot\nu_{M,K})=\frac{S_{\widehat{O}}}{2}a_p(D_{\widehat{O}})\cdot D^0_{\widehat{O}}$$
$$=\frac{S_{\widehat{O}}}{2}a_p(D_{\widehat{O}})\cdot(I-R_{\widehat{O}})D_{\widehat{O}}\geq\frac{1-\theta_{\widehat{O}}}{2}S_{\widehat{O}}|D_{\widehat{O}}|^p\geq\frac{1-\theta_{\widehat{O}}}{2(1+\theta_{\widehat{O}})^p}S_{\widehat{O}}|D^0_{\widehat{O}}|^p.$$

Property (2.3 iii) for $\mathcal{D}^h$ follows, because one has $|D^0|\geq|D^0_{\widehat{O}}\cdot\nu_{K,L}|=\frac{|v_L-v_K|}{d_{K,L}}=D^h_{\perp,K|L}$ so that

$$\sum_{\widehat{O}\in\widehat{\mathcal{T}}^h}S_{\widehat{O}}|D^0_{\widehat{O}}|^p=\left\|\mathcal{D}^{h,0}[(v_K)_K]\right\|^p_{L^p(\Omega)}\geq\left\|\mathcal{D}^h_\perp[(v_K)]\right\|^p_{L^p(\Omega)}.$$

The proof of (2.3 ii) is similar. Denoting the values $\widetilde{D}^0_{\widehat{O}},\widetilde{D}_{\widehat{O}}$ of $\mathcal{D}^{h,0}[(\widetilde{v}_K)_K]$ and $\mathcal{D}^h[(\widetilde{v}_K)_K]$, respectively, on $\widehat{o}$, one can rewrite the sum in (2.3 ii) as

$$\sum_{\widehat{O}\in\widehat{\mathcal{T}}^h}\Big\{S_{K,L}\Big((a_p(D_{\widehat{O}})-a_p(\widetilde{D}_{\widehat{O}}))\cdot\nu_{K,L}\Big)\Big((D^0_{\widehat{O}}-\widetilde{D}^0_{\widehat{O}})\cdot\nu_{K,L}\Big)$$
$$+\,S_{L,M}\Big((a_p(D_{\widehat{O}})-a_p(\widetilde{D}_{\widehat{O}}))\cdot\nu_{L,M}\Big)\Big((D^0_{\widehat{O}}-\widetilde{D}^0_{\widehat{O}})\cdot\nu_{L,M}\Big)$$
$$+\,S_{M,K}\Big((a_p(D_{\widehat{O}})-a_p(\widetilde{D}_{\widehat{O}}))\cdot\nu_{M,K}\Big)\Big((D^0_{\widehat{O}}-\widetilde{D}^0_{\widehat{O}})\cdot\nu_{M,K}\Big)\Big\}$$
$$=\frac{1}{2}\sum_{\widehat{O}\in\widehat{\mathcal{T}}^h}S_{\widehat{O}}(a_p(D_{\widehat{O}})-a_p(\widetilde{D}_{\widehat{O}}))\cdot(D^0_{\widehat{O}}-\widetilde{D}^0_{\widehat{O}}).$$

Using (5.2) and denoting by H the Hessian matrix of the function $x\in\mathbb{R}^2\mapsto\frac{1}{p}|x|^p$, we get

$$(a_p(D_{\widehat{O}})-a_p(\widetilde{D}_{\widehat{O}}))\cdot(D^0_{\widehat{O}}-\widetilde{D}^0_{\widehat{O}})$$
$$=(D_{\widehat{O}}-\widetilde{D}_{\widehat{O}})^t\left[\int_0^1H(\widetilde{D}_{\widehat{O}}+\tau(D_{\widehat{O}}-\widetilde{D}_{\widehat{O}}))\,d\tau\,(I-R_{\widehat{O}})\right](D_{\widehat{O}}-\widetilde{D}_{\widehat{O}}).$$

For all $x\in\mathbb{R}^2$, $x\neq 0$, $H(x)$ is a symmetric matrix with positive eigenvalues λ_1,λ_2 such that $\lambda_1/\lambda_2=p-1$. Thus the condition $\|R_{\widehat{O}}\|<\min\{p-1,1/(p-1)\}$ ensures that, for all $\tau\in[0,1]$,

$$r^t\left[H(\widetilde{D}_{\widehat{O}}+\tau(D_{\widehat{O}}-\widetilde{D}_{\widehat{O}}))\,(I-R_{\widehat{O}})\right]r\geq a\,r^t\left[H(\widetilde{D}_{\widehat{O}}+\tau(D_{\widehat{O}}-\widetilde{D}_{\widehat{O}}))\right]r>0$$

for all $r\in\mathbb{R}^2, r\neq 0$, with some constant $a>0$. Now (2.3 ii) follows. ∎

5.2. On interpolated solutions. First note that it is sufficient to prove the interpolation property in $W^{1,p}_0(\Omega)$ if we require, in addition to the time-independent analogs of (2.8), (2.9) (referred to as (2.8′), (2.9′)), that

$$\|v^h\|_{W^{1,p}_0(\Omega)}\leq c\times\left\|\mathcal{D}^h_\perp[\overline{v}^h]\right\|_{L^p(\Omega)} \tag{2.10′}$$

with a constant c independent of h. We obtain the interpolation property in E with the function $I:C\mapsto c\times C$ by taking v^h constant on each I^n and summing in $n\in\{1,\dots,[T/k^h]+1\}$.

LEMMA 5.3. *Let $(\mathcal{T}^h)_h$ be a strongly proportional family of finite volume meshes of $\Omega \subset \mathbb{R}^d$. Then it has the interpolation property in $W_0^{1,p}(\Omega)$.*

In order to prove the lemma, we first show that the strong proportionality allows us to majorate the L^p norm of the translates of the discrete solutions $\overline{v}^h$ in Lemma 2.6(ii) by const$\Delta\,(h+\Delta)^{p-1}$. Then we convolute $\overline{v}^h$ with the appropriate mollifier; finally, we restore the average over each mesh volume as in Lemma 5.4 below. The complete proof is given in [4].

Note that the interpolation property can fail on weakly proportional meshes, at least for $p > 2$.

Indeed, consider $\Omega = (0,1)^2$. For $s \geq 2$, let $\mathcal{T}^s$ be the finite volume mesh of Ω such that $\mathcal{T}^s_{int} = \{K^s, L^s\}$, where $K^s = \{(x,y) \in \Omega \mid x+y < 1/s\}$ with $x_{K^s} = (\frac{1}{4s}, \frac{1}{4s})$, and L^s is the interior of the complementary of K^s with $x_{L^s} = (\frac{1}{2}, \frac{1}{2})$. Take $\overline{v}^s$ such that $\overline{v}^s \equiv s^{1/p}$ on K^s and $\overline{v}^s \equiv 0$ on L^s. Then $\int_\Omega \left|\mathcal{D}^h_\perp[\overline{v}^s]\right|^p \leq$ const uniformly in s. If there exist $v^s \in W_0^{1,p}(\Omega)$ interpolated solutions for $\overline{v}^s$, we have $\|v^s\|_{W_0^{1,p}(\Omega)} \leq$ const. Hence by the standard embedding theorem, v^s are uniformly bounded. This contradicts the fact that $\frac{1}{\mathfrak{m}(K^s)}\int_{K^s} v^s = s^{1/p} \to +\infty$ as $s \to +\infty$.

Nevertheless, we have the following result in the situation close to that of Proposition 5.2.

LEMMA 5.4. *Assume that $(\mathcal{T}^h)_h$ is a weakly proportional family of meshes of $\Omega \subset \mathbb{R}^2$ dual to a family of meshes $(\widehat{\mathcal{T}}^h)_h$ such that all $\widehat{o} \in \widehat{\mathcal{T}}^h$ are triangles with angles less than or equal to $\pi/2$. Then $(\mathcal{T}^h)_h$ has the interpolation property in $W_0^{1,p}(\Omega)$.*

Proof. Take discrete solutions $\overline{v}^h = \sum_K v_K 1\!\!1_K$ on each of $\mathcal{T}^h$ such that, for all h $\left\|\mathcal{D}^h_\perp[\overline{v}^h]\right\|_{L^p(Q)} \leq C$. Denote by c the generic constant that depends only on p and ζ^*. Let $v^{h,0}$ be the continuous piecewise affine function on Ω that interpolates the values v_K, v_L, v_M at the points x_K, x_L, x_M over $\widehat{o}$ for all $\widehat{o} \in \widehat{\mathcal{T}}^h$ (we use the construction and notation of section 5.1). We have $v^{h,0} \in W_0^{1,p}(\Omega)$ and $Dv^{h,0} \equiv \mathcal{D}^{h,0}[\overline{v}^h]$ so that (5.3) yields $\|Dv^{h,0}\|_{L^p(\Omega)} \leq c \times C$. Note that for all $x \in K \in \mathcal{T}^h$,

$$|v^{h,0}(x) - \overline{v}^h(x)| = |v^{h,0}(x) - v^{h,0}(x_K)| \leq \mathfrak{d}(K)\,|Dv^{h,0}(x)|. \tag{5.4}$$

Hence $\|v^{h,0} - \overline{v}^h\|_{L^p(\Omega)} \leq h\,\|Dv^{h,0}\|_{L^p(\Omega)} \to 0$ as $h \to 0$. Now take a continuously differentiable function $\pi : \mathbb{R}^2 \to \mathbb{R}^+$ such that $\operatorname{supp}\pi = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ and $\int_{\mathbb{R}^2}\pi = 1$. For all $K \in \mathcal{T}^h_{int}$ set $\varphi_K = \frac{\mathfrak{m}(K)}{(\zeta^*\mathfrak{d}(K))^2}\,\pi\left(\frac{x-x_K}{\zeta^*\mathfrak{d}(K)}\right)$ (for boundary volumes K of nonzero measure; i.e., if $x_K \in \partial\Omega$, an easy modification is needed in order to keep the trace on $\partial\Omega$ equal to zero). Set

$$v^h = v^{h,0} + \sum_K \alpha_K\varphi_K, \qquad \text{with} \quad \alpha_K = v_K - \frac{1}{\mathfrak{m}(K)}\int_K v^{h,0}.$$

Since $\operatorname{supp}\varphi_K \subset K$, by the choice of α_K, the family $(v^h)_h$ verifies (2.9′). Moreover, since $\frac{\mathfrak{m}(K)}{\mathfrak{d}(K)^2} \leq c$ for all $K \in \mathcal{T}^h$, for all h, by the Hölder inequality we get

$$\begin{aligned}\int_\Omega |v^h - v^{h,0}|^p &= \sum_K |\alpha_K|^p \int_K |\varphi_K|^p \\ &\leq\; c\sum_K \frac{1}{\mathfrak{m}(K)^p}\left|\int_K \overline{v}^h - \int_K v^{h,0}\right|^p \mathfrak{m}(K)\left(\frac{\mathfrak{m}(K)}{\mathfrak{d}(K)^2}\right)^p \\ &\leq\; c\sum_K \frac{1}{\mathfrak{m}(K)^p}\mathfrak{m}(K)^{p'/p}\int_K |\overline{v}^h - v^{h,0}|^p\mathfrak{m}(K) = c\int_\Omega |\overline{v}^h - v^{h,0}|^p \to 0\end{aligned}$$

as $h \to 0$. Thus $(v^h)_h$ satisfies (2.8′). In the same manner, using (5.4) we have

$$\begin{aligned}\int_\Omega |Dv^h - Dv^{h,0}|^p &= \sum_K |\alpha_K|^p \int_K |D\varphi_K|^p \\ &\le \sum_K c \sum_K \frac{1}{\mathfrak{m}(K)^p} \mathfrak{m}(K)^{p'/p} \int_K |\overline{v}^h - v^{h,0}|^p \times \mathfrak{m}(K) \Big(\frac{\mathfrak{m}(K)}{\mathfrak{d}(K)^3}\Big)^p \\ &\le c \sum_K \frac{1}{\mathfrak{d}(K)^p} \int_K |\overline{v}^h - v^{h,0}|^p \le c \sum_K \frac{1}{\mathfrak{d}(K)^p} \mathfrak{d}(K)^p \int_K |Dv^{h,0}|^p = c \int_\Omega |Dv^{h,0}|^p.\end{aligned}$$

Hence $\|v^h\|_{W_0^{1,p}(\Omega)} \le c\, \|v^{h,0}\|_{W_0^{1,p}(\Omega)} \le c \times C$, so $(v^h)_h$ satisfies (2.10′). Thus $(v^h)_h$ can be chosen as interpolated solutions for $(\overline{v}^h)_h$. ☐

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