Discrete duality finite volume schemes for Leray-Lions type elliptic problems on general 2D meshes

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Received

Discrete duality finite volume schemes on general meshes, introduced by Hermeline in [24] and Domelevo & Omnès in [13] for the Laplace equation, are proposed for nonlinear diffusion problems in 2D with non homogeneous Dirichlet boundary condition.
This approach allows the discretization of non linear fluxes in such a way that the discrete operator inherits the key properties of the continuous one. Furthermore, it is well adapted to very general meshes including the case of non-conformal locally refined meshes.
We show that the approximate solution exists and is unique, which is not obvious since the scheme is nonlinear. We prove that, for general $W^{-1,p'}(\Omega)$ source term and $W^{1-\frac{1}{p},p}(\partial\Omega)$ boundary data, the approximate solution and its discrete gradient converge strongly towards the exact solution and its gradient respectively in appropriate Lebesgue spaces.
Finally, error estimates are given in the case where the solution is assumed to be in $W^{2,p}(\Omega)$. Numerical examples are given, including those on locally refined meshes. © 2006 John Wiley & Sons, Inc.

Keywords: Finite-volume methods, Error estimates, Leray-Lions operators.

Numerical Methods for Partial Differential Equations , 1–50 (2006)

I. INTRODUCTION

I.A Nonlinear elliptic equations

In this paper, we are interested in the study of a finite volume approximation of solutions to the nonlinear diffusion problem with non homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\mathrm{div}\,\big(\varphi(z,\nabla u_e(z))\big) = f(z), & \text{in } \Omega, \\ u_e = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a given bounded polygonal domain in $\mathbb{R}^2$.

We first recall the usual functional framework ensuring that the problem above is well-posed. Let $p \in]1,\infty[$ and $p' = \frac{p}{p-1}$. The flux $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ in equation (1.1) is supposed to be a Caratheodory function which is strictly monotonic with respect to $\xi \in \mathbb{R}^2$:

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) > 0, \text{ for all } \xi \neq \eta, \text{ for } a.e.\ z \in \Omega\,. \tag{$\mathcal{H}_1$}$$

We also assume that there exist $C_1, C_2 > 0$, $b_1 \in L^1(\Omega)$, $b_2 \in L^{p'}(\Omega)$ such that

$$(\varphi(z,\xi),\xi) \geq C_1|\xi|^p - b_1(z), \text{ for all } \xi \in \mathbb{R}^2,\ a.e.\ z \in \Omega, \tag{$\mathcal{H}_2$}$$

$$|\varphi(z,\xi)| \leq C_2|\xi|^{p-1} + b_2(z), \text{ for all } \xi \in \mathbb{R}^2,\ a.e.\ z \in \Omega. \tag{$\mathcal{H}_3$}$$

These assumptions ensure that $u \mapsto -\mathrm{div}\,(\varphi(\,\cdot\,,\nabla u))$ is a Leray-Lions operator, and in particular

$$\text{the map } G \in (L^p(\Omega))^2 \mapsto \varphi(\,\cdot\,,G(\,\cdot\,)) \in (L^{p'}(\Omega))^2 \text{ is continuous.} \tag{1.2}$$

Theorem 1.1. *Under assumptions* ($\mathcal{H}_1$),($\mathcal{H}_2$),($\mathcal{H}_3$), *for any source term* $f \in W^{-1,p'}(\Omega)$ *and any boundary data* $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, *the problem* (1.1) *has a unique solution* $u_e \in W^{1,p}(\Omega)$.

Remark 1.2. *Note that, in view of the numerical approximation of the source term, it is necessary to suppose that* $f \in L^{p'}(\Omega)$. *This is not a restriction of our approach. Indeed, in order to treat the case of general source term* $f \in W^{-1,p'}(\Omega)$, *it is possible to write* $f = f_0 + \mathrm{div}\, f_1$ *with* $f_0 \in L^{p'}(\Omega)$ *and* $f_1 \in \left(L^{p'}(\Omega)\right)^2$, *so that the problem* (1.1) *is equivalent to*

$$-\mathrm{div}\,(\tilde{\varphi}(z,\nabla u_e)) = f_0(z),$$

where $\tilde{\varphi} : (z,\xi) \to \varphi(z,\xi) + f_1(z)$. *It is easily seen that, if* φ *satisfies* ($\mathcal{H}_1$)-($\mathcal{H}_3$), *so does the new flux* $\tilde{\varphi}$. *It is worth noticing that the couple* (f_0, f_1) *is not unique and each choice will lead to a different approximation of the original equation (see [15]). As a consequence, from now on we always assume at least that* $f \in L^{p'}(\Omega)$.

I.B Examples

Our framework includes classical elliptic operators like the linear anisotropic Laplace equation

$$-\mathrm{div}\,(A(z)\nabla u_e) = f, \tag{1.3}$$

$A(z)$ being a uniformly coercive symmetric matrix-valued map, or the p-laplacian

$$-\mathrm{div}\,(|\nabla u_e|^{p-2}\nabla u_e) = f. \tag{1.4}$$

One can also encounter, for instance in the modelling of non-newtonian fluids flows in a porous medium, equations like

$$-\text{div}\left(k(z)|F(z) + \nabla u_e|^{p-2}(F(z) + \nabla u_e)\right) = f, \tag{1.5}$$

where F is a vector-valued map and k a positive scalar map bounded from below. Notice that F is not necessarily a gradient, so that this problem may not reduce to the p-laplacian (1.4) through a change of variables. The models presented in [12] are even more general, since the flux φ depends also on the unknown u_e. In [22], the authors also propose such nonlinear elliptic problem for the study of glacier flows. In this case, the flux φ depends only on ξ but in an implicit way.

We recall the key technical lemma which implies the monotonicity and continuity properties of the two non-linear model problems above (see [5]).

Lemma 1.3. *For any $p \in]1, +\infty[$ and $\delta \geq 0$, there exists $\underline{C}, \overline{C} > 0$ such that for any $n \in \mathbb{N}$ we have*

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \geq \underline{C}|\xi - \eta|^{2+\delta}(|\xi| + |\eta|)^{p-2-\delta}, \quad \forall \xi, \eta \in \mathbb{R}^n,$$

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq \overline{C}|\xi - \eta|^{1-\delta}(|\xi| + |\eta|)^{p-2+\delta}, \quad \forall \xi, \eta \in \mathbb{R}^n.$$

I.C Finite volume approach

Finite elements approximation of problems like (1.1) are now quite classical (see for instance [5, 8, 21, 26]). Nevertheless, it is also natural to consider finite volume methods for these problems. Indeed, finite volume methods allow more flexibility on the geometry of the meshes and ensure the local consistency of the numerical fluxes inside the domain. Furthermore, this kind of discretization is well-adapted if one adds a convective term in the problem (1.1).

The nonlinearity and the possible anisotropy of the flux φ with respect to ∇u_e makes it difficult to approximate the problem by standard cell-centered finite volume methods (as presented, e.g., in [18]), since only the normal component of ∇u_e on the interface between two adjacent volumes can be easily approximated on conformal meshes.

In the case of linear anisotropic Laplace equation, some studies are available in the literature for instance in [10, 14, 20], with various approaches. To our knowledge, three kinds of gradient reconstruction were proposed for the finite volume approximation of the fully non-linear equation. The one of [4] applies for the p-laplacian on meshes that are dual to triangular ones, moreover, the triangular meshes should be close enough to the structured mesh. The one of [1] applies on rectangular meshes. Finally, for general grids it was recently proposed in [17] to handle fluxes on the edges of a control volume as new unknowns, and reconstruct the discrete gradient, constant per control volume, using these fluxes. In all cases, the crucial feature is that the summation-by-parts procedure permits to reconstitute, starting from the one-dimensional finite differences $u_{\mathcal{K}} - u_{\mathcal{L}}$, the whole two-dimensional discrete gradient (see [4, Lemma 8, Proposition 4] and [1, Proposition 2.5, Lemma 2.7]). The coercivity and monotonicity properties of the continuous elliptic operator are then inherited by its discrete finite volume counterparts. For instance, the variational structure of the p-laplacian operator can be inherited by its discrete analogues.

We consider in this paper the class of finite volume schemes introduced by Hermeline in [24], by Domelevo, Omnès in [13] for the Laplace equation and in [11] for other linear equations like the Stokes or the Div-Curl system. More precisely, we show that the method can be successfully extended to the case of the nonlinear diffusion equation (1.1) we are interested in while preserving the main features of the continuous problem. In [11], these schemes are called “Discrete Duality Finite Volume” (DDFV for short) since the discrete gradient and discrete divergence operators are dual one from each other (see Lemma 4.1 below). The equation is approximated simultaneously on two interrelated meshes: the primal and the dual mesh. The number of variables and of equations doubles compared to usual cell-centered FV schemes, but the gradient approximation (the one already used by Coudière and al. [10]) becomes simple and quite efficient. Furthermore, the method is well-suited to almost arbitrary meshes since few geometrical constraints are imposed to the primal and dual control volumes. Indeed, non convex control volumes, non matching triangulations or locally refined meshes can naturally be handled by this method and also fulfill the assumptions needed for the convergence analysis given in this paper.

I.D Outline

This paper is organized as follows. The framework of DDFV meshes, the discrete gradient and the finite volume scheme associated with equation (1.1) are described in Section II.. In Section III., we present the main results of discrete functional analysis necessary for the theoretical study of the finite volume method. These results include the discrete Poincaré inequality (Lemma 3.3), the study of the mean-value projection of functions on the meshes (Proposition 3.6 and Corollary 3.7) and finally a discrete compactness result similar to the Rellich theorem (Lemma 3.8).

Existence and uniqueness of a discrete solution of the scheme as well as *a priori* estimates are given in section IV. (Theorem 4.4). The structure properties of Leray-Lions operators being inherited in the framework of DDFV schemes, the method we use is similar to the one for the continuous problem (1.1).

Section V. is devoted to the proof of Theorem 5.1 which states the convergence of the approximate solution in case of general data $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and $f \in W^{-1,p'}(\Omega)$ (see Remark 1.2). Notice that the *strong convergence* of the discrete gradient of the approximate solution towards the gradient ∇u_e of the exact solution is obtained in $(L^p(\Omega))^2$.

In Section VI., we study the stability properties of the approximate solution with respect to the data f and g (Proposition 6.1). Finally, in Section VII., we prove error estimates for the discrete gradient in $(L^p(\Omega))^2$ in the case where the exact solution lies in $W^{2,p}(\Omega)$, which is a usual assumption for the error analysis (Theorem 7.2). The convergence rate obtained is $\mathrm{size}(\mathcal{T})^{\frac{1}{p-1}}$ for $p \geq 2$ and $\mathrm{size}(\mathcal{T})^{p-1}$ for $p < 2$. These rates are the same than the one obtained in [1, 26] for different schemes. As an example, this result implies the first order convergence in the case of the anisotropic Laplace equation which Lipschitz coefficients.

Note that error estimates for general solutions of the p-laplacian equation with source term in $L^{p'}(\Omega)$ were obtained in [2], making use of the intrinsic Besov regularity of continuous and discrete solutions, in the case of structured rectangular meshes. It is an open question how to generalize this Besov approach to the unstructured DDFV schemes.

In Section VIII., we provide some numerical results which show in particular that, in the truly nonlinear case, the method behaves better than what is expected even for non conformal locally refined meshes. In the concluding Section IX., we discuss the extension of our study to some fully practical variants of the finite volume scheme and to even more general meshes than the ones described in Section II.A.

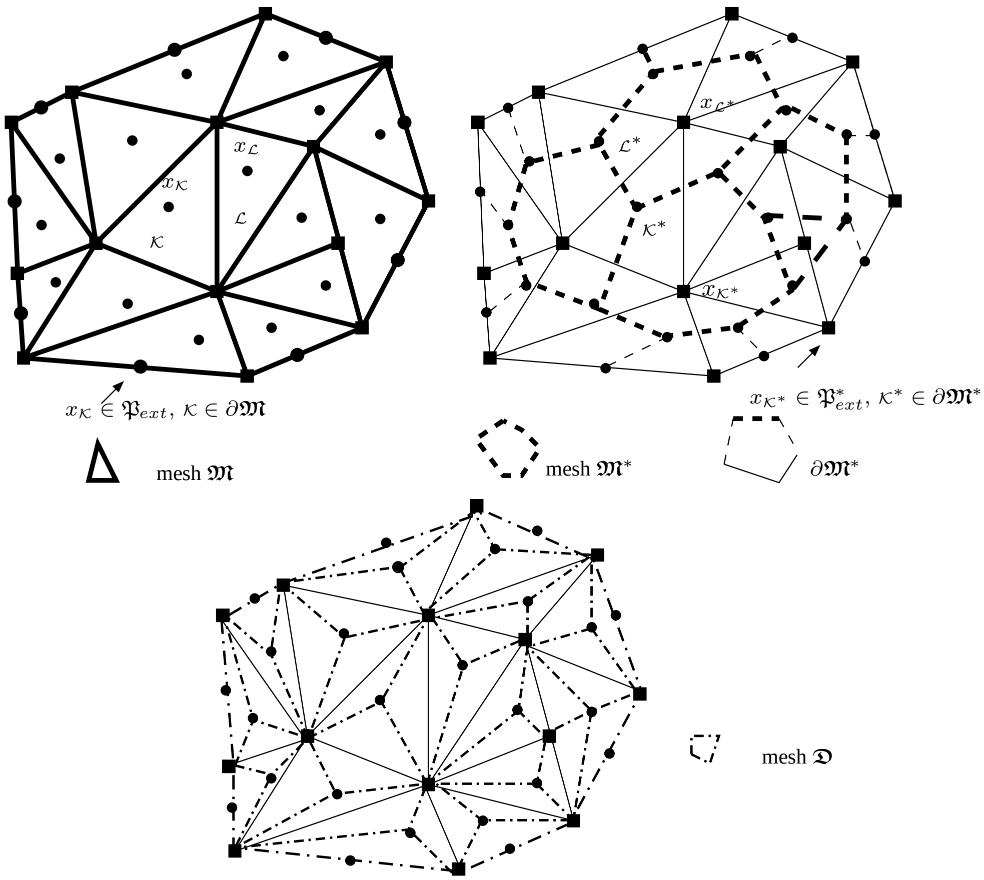

FIG. 1. Example of a DDFV mesh

II. THE FINITE VOLUME METHOD

II.A Definition of the mesh

We call $\mathcal{T}$ a triple $(\mathfrak{M}, \mathfrak{M}^*, \mathfrak{D})$ of meshes on Ω, defined as follows. The mesh $\mathfrak{M}$ is a set of disjoint polygonal control volumes $\kappa \subset \Omega$ such that $\cup\bar{\kappa} = \overline{\Omega}$. We denote by $\partial\mathfrak{M}$ the set of edges of the control volumes in $\mathfrak{M}$ included in $\partial\Omega$, which we consider as degenerate control volumes. We associate to $(\mathfrak{M}, \partial\mathfrak{M})$ a family of points $\mathfrak{P}$.

The set $\mathfrak{P} = \mathfrak{P}_{int} \cup \mathfrak{P}_{ext}$ is composed of one point per control volume $\kappa \in \mathfrak{M}$ (called $x_\kappa \in \mathfrak{P}_{int}$) and one point per degenerate control volume $\kappa \in \partial\mathfrak{M}$ (called $x_\kappa \in \mathfrak{P}_{ext}$):

$$\mathfrak{P}_{int} = \{x_\kappa,\ \kappa \in \mathfrak{M}\},\ \mathfrak{P}_{ext} = \{x_\kappa,\ \kappa \in \partial\mathfrak{M}\}.$$

Let $\mathfrak{P}^*$ denote the set of vertices of the mesh $\mathfrak{M}$. The set $\mathfrak{P}^*$ can be decomposed into $\mathfrak{P}^* = \mathfrak{P}^*_{int} \cup \mathfrak{P}^*_{ext}$ where $\mathfrak{P}^*_{int} \cap \partial\Omega = \emptyset$ and $\mathfrak{P}^*_{ext} \subset \partial\Omega$ (see Figure 1). The sets $\mathfrak{M}^*$ and $\partial\mathfrak{M}^*$ are two families of "dual" control volumes defined as follows. To any point $x_{\kappa^*} \in \mathfrak{P}^*_{int}$ (resp. $x_{\kappa^*} \in \mathfrak{P}^*_{ext}$) we associate the polygon $\kappa^* \in \mathfrak{M}^*$ (resp. $\kappa^* \in \partial\mathfrak{M}^*$) whose vertices are

$\{x_{\mathcal{K}} \in \mathfrak{P}/x_{\mathcal{K}^*} \in \bar{\mathcal{K}},\, \mathcal{K} \in \mathfrak{M}\}$ (resp. $\{x_{\mathcal{K}^*}\} \cup \{x_{\mathcal{K}} \in \mathfrak{P}/x_{\mathcal{K}^*} \in \bar{\mathcal{K}},\, \mathcal{K} \in \mathfrak{M} \cup \partial\mathfrak{M}\}$) sorted with respect to the clockwise order of the corresponding primal control volumes.

For all adjacent control volumes $\mathcal{K}$ and $\mathcal{L}$, we assume that $\partial\mathcal{K} \cap \partial\mathcal{L}$ is a segment that we call *an edge of the mesh* $\mathfrak{M}$ and that we denote by $\sigma = \mathcal{K}|\mathcal{L}$. Let $\mathcal{E}$ be the set of such edges. The corresponding notations $\sigma^* = \mathcal{K}^*|\mathcal{L}^*$ and $\mathcal{E}^*$ refer to the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$. Even though more general situations can be handled, we concentrate in this paper to meshes satisfying the following assumption.

Main assumption. *We assume either that all the primal control volumes $\mathcal{K} \in \mathfrak{M}$ are star-shaped with respect to $x_{\mathcal{K}}$ either that all the dual control volumes $\mathcal{K}^* \in \mathfrak{M}^*$ are star-shaped with respect to $x_{\mathcal{K}^*}$.*

Remark 2.1. *This hypothesis is not so restrictive and is fulfilled for instance in the case of a primal Voronoï mesh associated to the vertices $x_{\mathcal{K}^*}$, or in the dual case of a Delaunay conformal triangulation of the domain (see [19]).*

For each couple $(\sigma, \sigma^*) \in \mathcal{E} \times \mathcal{E}^*$ such that $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ and $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$, can we introduce the quadrilateral diamond cell $\mathcal{D}_{\sigma,\sigma^*}$ whose diagonals are σ and σ^*, as shown in Figure 2. Notice that the diamond cells are the union of two disjoint triangles and can be non convex. Furthermore, if $\sigma \in \mathcal{E} \cap \partial\bar{\Omega}$, then the quadrilateral $\mathcal{D}_{\sigma,\sigma^*}$ degenerate in a single triangle. The set of diamond cells is denoted by $\mathfrak{D}$ and we have

$$\overline{\Omega} = \bigcup_{\mathcal{D} \in \mathfrak{D}} \overline{\mathcal{D}}.$$

As a consequence of the main assumption above, we easily see that the interior of the diamond cells are disjoint.

For simplicity, we will also assume that the interiors of the dual control volumes are all disjoint. Nevertheless, it is possible to cope with particular meshes with overlapping dual control volumes (see [13]) but it would be necessary to introduce some more notations.

II.B Notations

For any control volume $\mathcal{K} \in \mathfrak{M}$, we define

- $m_{\mathcal{K}}$, the measure of $\mathcal{K}$.
- $\mathcal{E}_{\mathcal{K}}$, the set of edges for $\mathcal{K} \in \mathfrak{M}$ and abusively the edge $\sigma = \mathcal{K}$ for $\mathcal{K} \in \partial\mathfrak{M}$.
- $\mathfrak{D}_{\mathcal{K}} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}/\sigma \in \mathcal{E}_{\mathcal{K}}\}$.
- $\boldsymbol{\nu}_{\mathcal{K}}$, the outward unit normal vector to $\partial\mathcal{K}$.
- $\mathrm{d}_{\mathcal{K}}$, the diameter of $\mathcal{K}$.

For any degenerate control volume $\mathcal{K} \in \partial\mathfrak{M}$, $\boldsymbol{\nu}_{\mathcal{K}}$ stands for the outward unit normal vector to $\partial\Omega$. In the same way, for a "dual" control volume $\mathcal{K}^* \in \mathfrak{M}^* \cap \partial\mathfrak{M}^*$, we set

- $m_{\mathcal{K}^*}$, the measure of $\mathcal{K}^*$.
- $\mathcal{E}_{\mathcal{K}^*}$, the set of edges for $\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$.
- $\mathfrak{D}_{\mathcal{K}^*} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}/\sigma^* \in \mathcal{E}_{\mathcal{K}^*}\}$.
- $\boldsymbol{\nu}_{\mathcal{K}^*}$, the outward unit normal vector to $\partial\mathcal{K}^*$.
- $\mathrm{d}_{\mathcal{K}^*}$, the diameter of $\mathcal{K}^*$.

For a diamond cell $\mathcal{D}_{\sigma,\sigma^*}$, recall that $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ are the vertices of $\mathcal{D}_{\sigma,\sigma^*}$ and note :

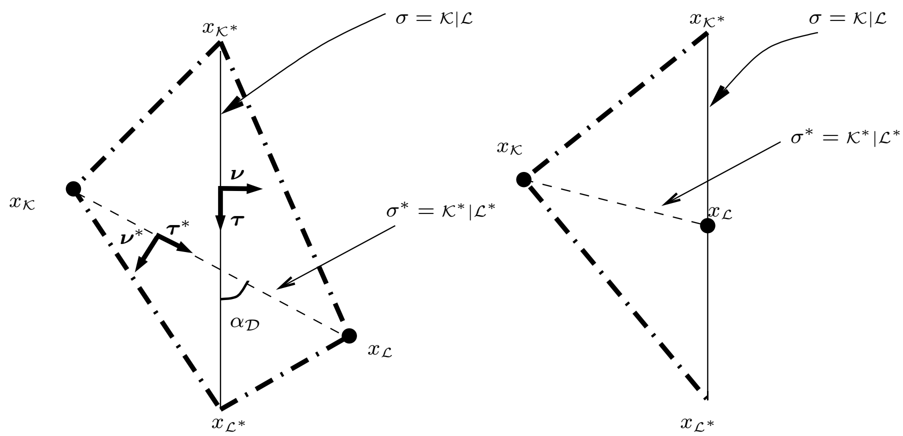

FIG. 2. Notations in a diamond cell $\mathcal{D}_{\sigma,\sigma^*}$

- m_σ, the length of σ, m_{σ^*} the length of σ^* and $m_{\mathcal{D}}$ the measure of the diamond cell.
- $\boldsymbol{\tau}$, the unit vector parallel to σ, oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$.
- $\boldsymbol{\nu}$, the unit vector normal to σ, oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$.
- $\boldsymbol{\tau}^*$, the unit vector parallel to σ^*, oriented from $x_{\mathcal{K}}$ to $x_{\mathcal{L}}$.
- $\boldsymbol{\nu}^*$, the unit vector normal to σ^*, oriented from $x_{\mathcal{K}^*}$ to $x_{\mathcal{L}^*}$.
- $\alpha_{\mathcal{D}}$, the angle between $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^*$.
- $\mathrm{d}_{\mathcal{D}}$, the diameter of $\mathcal{D}_{\sigma,\sigma^*}$.

It can happen that the point $x_{\mathcal{K}}$ does not belong to $\mathcal{K}$ (if $\mathcal{K}$ is a triangle and $x_{\mathcal{K}}$ its circumcenter for instance). As usual in that case (see [18]) it is necessary to assume that $\boldsymbol{\nu}_{\mathcal{K}}$ points from $x_{\mathcal{K}}$ towards $x_{\mathcal{L}}$, that is $\boldsymbol{\nu}_{\mathcal{K}} = \boldsymbol{\nu}$ with the notations above. Similarly we assume that $\boldsymbol{\nu}_{\mathcal{K}^*} = \boldsymbol{\nu}^*$. In the case of a triangular mesh $\mathfrak{M}$ in which $x_{\mathcal{K}}$ is the circumcenter of $\mathcal{K}$, this assumption is known as the Delaunay condition.

II.C Unknowns and boundary data

The finite volume method associates to all primal control volumes $\mathcal{K} \in \mathfrak{M}$, an unknown value $u_{\mathcal{K}}$ and to all dual control volumes $\mathcal{K}^* \in \mathfrak{M}^*$, an unknown value $u_{\mathcal{K}^*}$. We denote the approximate solution on the mesh $\mathcal{T}$ by

$$u^{\mathcal{T}} = \left((u_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*} \right). \tag{2.1}$$

The space of all discrete functions $u^{\mathcal{T}}$ in the sense of definition (2.1) is denoted by $\mathbb{R}^{\mathcal{T}}$.

In this paper we deal with non-homogeneous Dirichlet boundary condition. We describe here the way the boundary data will enter the scheme. Note first that

$$\partial\Omega = \bigcup_{\mathcal{K}\in\partial\mathfrak{M}} \overline{\mathcal{K}} = \bigcup_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \left(\overline{\mathcal{K}^*} \cap \partial\Omega \right).$$

For any boundary data $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we introduce its discrete counterpart by defining for each $\mathcal{K} \in \partial\mathfrak{M}$, a value $g_{\mathcal{K}}$ and for each $\mathcal{K}^* \in \partial\mathfrak{M}^*$, a value $g_{\mathcal{K}^*}$. The family $((g_{\mathcal{K}})_{\mathcal{K}\in\partial\mathfrak{M}}, (g_{\mathcal{K}^*})_{\mathcal{K}^*\in\partial\mathfrak{M}^*})$

is denoted by $g^{\mathcal{T}}$ and is also associated with a piecewise constant function in $L^p(\partial\Omega)$ as follows

$$g^{\mathcal{T}} \sim \frac{1}{2}\sum_{\kappa\in\partial\mathfrak{M}} \mathbf{1}_{\kappa}\, g_{\kappa} + \frac{1}{2}\sum_{\kappa^*\in\partial\mathfrak{M}^*} \mathbf{1}_{\overline{\kappa^*}\cap\partial\Omega}\, g_{\kappa^*},$$

where here and in the sequel, we denote by $\mathbf{1}_E$ the characteristic function of any set E.

We consider the discrete mean-value boundary data denoted by $\mathbb{P}_m^{\mathcal{T}} g = (\mathbb{P}_m^{\partial\mathfrak{M}} g, \mathbb{P}_m^{\partial\mathfrak{M}^*} g)$ defined by

$$\mathbb{P}_m^{\partial\mathfrak{M}} g = \left(\frac{1}{m_{\sigma_\kappa}}\int_{\sigma_\kappa} g(s)\,ds\right)_{\kappa\in\partial\mathfrak{M}}, \quad \mathbb{P}_m^{\partial\mathfrak{M}^*} g = \left(\frac{1}{m_{\sigma_{\kappa^*}}}\int_{\sigma_{\kappa^*}} g(s)\,ds\right)_{\kappa^*\in\partial\mathfrak{M}^*}. \tag{2.2}$$

Here

$$\sigma_\kappa \stackrel{\text{def}}{=} B(x_\kappa,\rho_\kappa)\cap\partial\Omega, \text{ and } \sigma_{\kappa^*} \stackrel{\text{def}}{=} B(x_{\kappa^*},\rho_{\kappa^*})\cap\partial\Omega \tag{2.3}$$

and ρ_κ and ρ_{κ^*} are positive numbers associated to the mesh $\mathcal{T}$ and such that

$$\sigma_\kappa \subset \overline{\kappa}, \quad \sigma_{\kappa^*} \subset \partial\kappa^*.$$

Finally, introduce numbers ρ_κ and ρ_{κ^*} for any $\kappa\in\mathfrak{M}$ (resp. $\kappa^*\in\mathfrak{M}^*$) such that

$$B_\kappa \stackrel{\text{def}}{=} B(x_\kappa,\rho_\kappa)\subset\Omega, \quad B_{\kappa^*} \stackrel{\text{def}}{=} B(x_{\kappa^*},\rho_{\kappa^*})\subset\Omega.$$

These balls are only introduced in order to prove the convergence of the scheme (but not to prove the error estimates). In particular, they do not enter the definition of the scheme. Of course, some assumptions are needed on the radii ρ_κ and ρ_{κ^*} as stated in the next paragraph.

II.D Regularity of meshes

We note $\mathrm{size}(\mathcal{T})$ the maximum of the diameters of the diamond cells in $\mathfrak{D}$. The following bounds follow:

$$m_\sigma \le \mathrm{size}(\mathcal{T}),\ \forall\sigma\in\mathcal{E}; \quad m_{\sigma^*} \le \mathrm{size}(\mathcal{T}),\ \forall\sigma^*\in\mathcal{E}^*;$$

$$m_\kappa \le \pi\,\mathrm{size}(\mathcal{T})^2,\ \forall\kappa\in\mathfrak{M}; \quad m_{\kappa^*} \le \pi\,\mathrm{size}(\mathcal{T})^2,\ \forall\kappa^*\in\mathfrak{M}^*; \quad m_{\mathcal{D}} \le \frac{1}{2}\mathrm{size}(\mathcal{T})^2,\ \forall\mathcal{D}\in\mathfrak{D}.$$

We introduce now a positive number that quantifies the regularity of a given mesh and is useful to perform the convergence analysis of the finite volume schemes. We first define $\alpha_{\mathcal{T}}$ to be the unique real number in $]0,\frac{\pi}{2}]$ such that

$$\sin\alpha_{\mathcal{T}} \stackrel{\text{def}}{=} \min_{\mathcal{D}\in\mathfrak{D}} |\sin\alpha_{\mathcal{D}}|, \tag{2.4}$$

that is the minimal angle between the diagonals of the diamond cells in the mesh.

Let us introduce the number

$$\begin{aligned}\mathcal{N}_{\mathcal{T}} \stackrel{\text{def}}{=} &\sup_{x\in\Omega}\#\big\{\kappa^* \text{ s.t. } x\in\widehat{\kappa^*\cup B_{\kappa^*}}\big\} + \sup_{x\in\Omega}\#\big\{\kappa \text{ s.t. } x\in\widehat{\kappa\cup B_\kappa}\big\}\\ &+ \sup_{x\in\Omega}\#\big\{\mathcal{D} \text{ s.t. } x\in\widehat{\mathcal{D}\cup B_\kappa},\ \mathcal{D}\in\mathfrak{D}_\kappa,\ \kappa\in\mathfrak{M}\big\}\\ &+ \sup_{x\in\Omega}\#\big\{\mathcal{D} \text{ s.t. } x\in\widehat{\mathcal{D}\cup B_{\kappa^*}},\ \mathcal{D}\in\mathfrak{D}_{\kappa^*},\ \kappa^*\in\mathfrak{M}^*\big\},\end{aligned} \tag{2.5}$$

where $\widehat{E}$ denotes the convex hull of any set $E \subset \mathbb{R}^2$. In the usual case where the primal control volumes, the dual control volumes and the diamond cells are convex and where $B_{\mathcal{K}} \subset \mathcal{K}$, $B_{\mathcal{K}^*} \subset \mathcal{K}^*$ then $\mathcal{N}_{\mathcal{T}}$ can be bounded by a function of the maximal number of edges per primal and dual control volumes. Since we do not impose any convexity assumption on the meshes, we need to control this number $\mathcal{N}_{\mathcal{T}}$ in the convergence analysis. We can now introduce

$$\begin{aligned}\operatorname{reg}(\mathcal{T}) \stackrel{\text{def}}{=} \max\Bigg(\frac{1}{\alpha_{\mathcal{T}}}, \mathcal{N}_{\mathcal{T}}, \max_{\mathcal{D}\in\mathfrak{D}} \frac{\mathrm{d}_{\mathcal{D}}}{\sqrt{m_{\mathcal{D}}}}, \max_{\mathcal{K}\in\mathfrak{M}} \frac{\mathrm{d}_{\mathcal{K}}}{\sqrt{m_{\mathcal{K}}}}, \max_{\mathcal{K}^*\in\mathfrak{M}^*} \frac{\mathrm{d}_{\mathcal{K}^*}}{\sqrt{m_{\mathcal{K}^*}}},\\ \max_{\mathcal{K}\in\mathfrak{M}\cup\partial\mathfrak{M}} \left(\frac{\mathrm{d}_{\mathcal{K}}}{\rho_{\mathcal{K}}} + \frac{\rho_{\mathcal{K}}}{\mathrm{d}_{\mathcal{K}}}\right), \max_{\substack{\mathcal{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*\\ x_{\mathcal{K}^*}\neq\text{corner}}} \left(\frac{\mathrm{d}_{\mathcal{K}^*}}{\rho_{\mathcal{K}^*}} + \frac{\rho_{\mathcal{K}^*}}{\mathrm{d}_{\mathcal{K}^*}}\right),\\ \max_{\substack{\mathcal{K}^*\in\partial\mathfrak{M}^*\\ x_{\mathcal{K}^*}=\text{corner}}} \left(\frac{\mathrm{d}_{\mathcal{K}^*}}{\zeta_p(\rho_{\mathcal{K}^*})} + \frac{\zeta_p(\rho_{\mathcal{K}^*})}{\mathrm{d}_{\mathcal{K}^*}}\right), \max_{\substack{\mathcal{K}\in\mathfrak{M}\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}}}} \frac{\mathrm{d}_{\mathcal{K}}}{\mathrm{d}_{\mathcal{D}}}, \max_{\substack{\mathcal{K}^*\in\mathfrak{M}^*\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}}} \frac{\mathrm{d}_{\mathcal{K}^*}}{\mathrm{d}_{\mathcal{D}}}\Bigg),\end{aligned} \tag{2.6}$$

where, for any $s \geq 0$, we have

$$\begin{cases} \zeta_p(s) = s, \text{if } p < 2, \\ \zeta_p(s) = s^{\frac{p'}{2}}, \text{if } p > 2, \\ \zeta_2(s) = s|\log s|. \end{cases} \tag{2.7}$$

This special treatment of the corner vertices of the mesh is a purely technical assumption which is used in the convergence rate analysis (see Lemma 7.4).

Let us point out that $\operatorname{reg}(\mathcal{T})$ essentially measures:

- how flat the diamond cells are.
- how large is the difference between the size of a primal control volume (resp. a dual control volume) and the size of a diamond cell as soon as they intersect.

Convention. *In any estimate given in this paper, the dependence of the constants in* $\operatorname{reg}(\mathcal{T})$ *is implicitly assumed to be non-decreasing. Furthermore, the dependence of the constants on the domain is often omitted.*

Remark 2.2. *For conformal finite volume meshes (see [18]), it is assumed that* $\alpha_{\mathcal{D}} = \frac{\pi}{2}$ *for any diamond* $\mathcal{D} \in \mathfrak{D}$*, so that* $\alpha_{\mathcal{T}} = \frac{\pi}{2}$*. In our case, not only this orthogonality condition is relaxed but also* $\mathfrak{M}$ *can present atypical edges, non convex dual control volumes and non convex diamond cells (see Figure 3).*

Remark 2.3. *The boundedness of* $\operatorname{reg}(\mathcal{T})$ *imposes only local restriction on the mesh. It is easy to construct a family of locally refined mesh such that* $\operatorname{reg}(\mathcal{T})$ *is bounded independently on the level of the refinement. Figure 4 provides a very simple example of such a construction.*

II.E Discrete gradient

We consider the discrete gradient introduced by Coudière and *al.* in [10] and applied by Hermeline [24] and Domelevo and Omnès [13] in the framework of DDFV schemes described above. For a given discrete Dirichlet data $g^{\mathcal{T}}$ as defined above, the discrete gradient operator $\nabla^{\mathcal{T}}_{g^{\mathcal{T}}}$ can be defined as follows : for any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, $\nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}$ is the function, constant on each diamond cell $\mathcal{D}_{\sigma,\sigma^*}$, given by

$$\nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}} = \sum_{\mathcal{D}\in\mathfrak{D}} \nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}} \mathbf{1}_{\mathcal{D}},$$

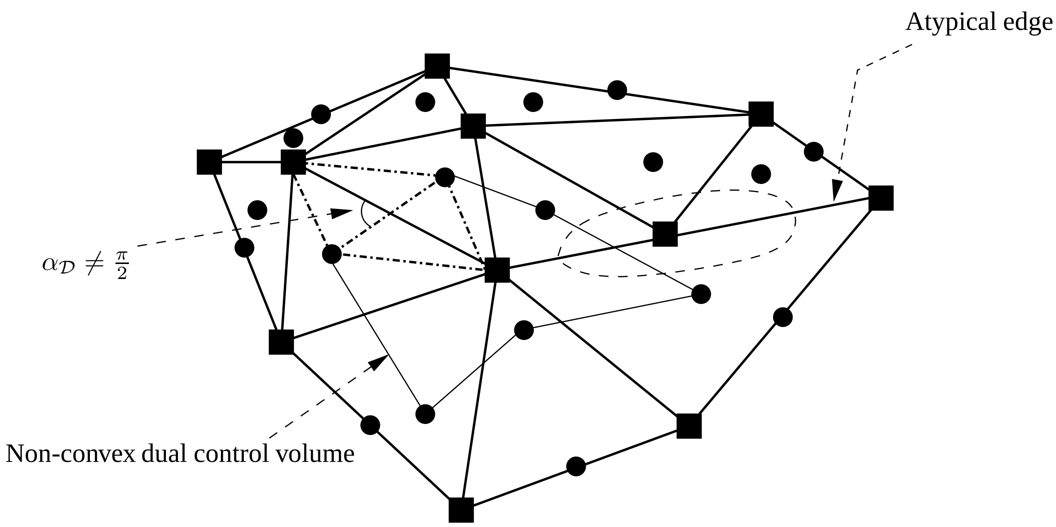

FIG. 3. Non conformal meshes

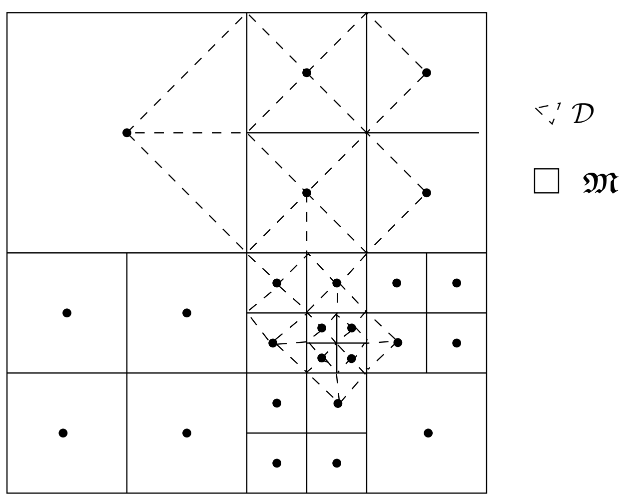

FIG. 4. Local refinement allowed by the boundedness of $\mathrm{reg}(\mathcal{T})$

with

$$\begin{cases} \left(\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}, \boldsymbol{\tau}\right) & = \dfrac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_\sigma} \\ \left(\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}, \boldsymbol{\tau}^*\right) & = \dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}, \end{cases} \tag{2.8}$$

where $\mathcal{D}_{\sigma,\sigma^*}$ is noted $\mathcal{D}$ when no confusion can arise. The dependence on $g^{\mathcal{T}}$ only appears when $\overline{\mathcal{D}_{\sigma,\sigma^*}}$ intersects $\partial\Omega$, in which case we replace the values of $u_{\mathcal{K}}$ or $u_{\mathcal{K}^*}$ by $g_{\mathcal{K}}$ or $g_{\mathcal{K}^*}$, for the points $x_{\mathcal{K}}$ or $x_{\mathcal{K}^*}$ located on the boundary. Remark that $\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}}$ can be expressed in the $(\boldsymbol{\nu}, \boldsymbol{\nu}^*)$ basis in the following way :

$$\nabla^{\mathcal{D}}_{g^{\mathcal{T}}} u^{\mathcal{T}} = \frac{1}{\sin\alpha_{\mathcal{D}}}\left(\frac{u_{\mathcal{L}}-u_{\mathcal{K}}}{m_{\sigma^*}}\,\boldsymbol{\nu} + \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{m_{\sigma}}\,\boldsymbol{\nu}^*\right) \tag{2.9}$$

thanks to the following lemma.

Lemma 2.4. *Using the notations of Section II.B and Figure 2 (in particular the orientation conventions), for any vector $\xi \in \mathbb{R}^2$ we have*

$$(\sin\alpha_{\mathcal{D}})\,\xi = (\xi, \boldsymbol{\tau})\,\boldsymbol{\nu}^* + (\xi, \boldsymbol{\tau}^*)\,\boldsymbol{\nu}.$$

In the framework of DDFV schemes on general meshes, this lemma is the crucial argument which ensures the coercivity and monotonicity properties of the finite volume approximates of Leray-Lions type operators; compare to [4, Lemma 8] and [1, Proposition 2.5], which only work due to the particular geometry of meshes.

Remark 2.5. *Our notation for the discrete gradient can be easily handled thanks to the following property: for any discrete boundary data $g_1^{\mathcal{T}}$ and $g_2^{\mathcal{T}}$ and for any discrete functions $u_1^{\mathcal{T}}$ and $u_2^{\mathcal{T}}$, we have*

$$\nabla^{\mathcal{D}}_{g_1^{\mathcal{T}}} u_1^{\mathcal{T}} + \nabla^{\mathcal{D}}_{g_2^{\mathcal{T}}} u_2^{\mathcal{T}} = \nabla^{\mathcal{D}}_{g_1^{\mathcal{T}}+g_2^{\mathcal{T}}}\left(u_1^{\mathcal{T}} + u_2^{\mathcal{T}}\right). \tag{2.10}$$

In particular, if $0^{\mathcal{T}}$ denotes the zero vector of $\mathbb{R}^{\mathcal{T}}$ then

$$\nabla^{\mathcal{D}}_{g_1^{\mathcal{T}}} u^{\mathcal{T}} - \nabla^{\mathcal{D}}_{g_2^{\mathcal{T}}} u^{\mathcal{T}} = \nabla^{\mathcal{D}}_{g_1^{\mathcal{T}}-g_2^{\mathcal{T}}} 0^{\mathcal{T}}, \forall u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

II.F The scheme

"Discrete Duality Finite Volume" schemes are obtained, as in Hermeline [24] or in Domelevo and Omnes [13], by integrating equation (1.1) on both control volumes $\mathcal{K} \in \mathfrak{M}$ and dual control volumes $\mathcal{K}^* \in \mathfrak{M}^*$:

$$\begin{aligned}
\int_{\mathcal{K}} f(z)\,dz &= \int_{\mathcal{K}} -\mathrm{div}\,\left(\varphi(z, \nabla u_{e}(z))\right)dz = -\int_{\partial\mathcal{K}} \left(\varphi(s, \nabla u_{e}(s)), \boldsymbol{\nu}_{\mathcal{K}}\right)ds \\
&= \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}}} -\int_{\sigma} \left(\varphi(s, \nabla u_{e}(s)), \boldsymbol{\nu}_{\mathcal{K}}\right)ds, \\
\int_{\mathcal{K}^*} f(z)\,dz &= \int_{\mathcal{K}^*} -\mathrm{div}\,\left(\varphi(z, \nabla u_{e}(z))\right)dz = -\int_{\partial\mathcal{K}^*} \left(\varphi(s^*, \nabla u_{e}(s^*)), \boldsymbol{\nu}_{\mathcal{K}^*}\right)ds^* \\
&= \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}^*}} -\int_{\sigma^*} \left(\varphi(s^*, \nabla u_{e}(s^*)), \boldsymbol{\nu}_{\mathcal{K}^*}\right)ds^*.
\end{aligned} \tag{2.11}$$

Let us introduce for any diamond $\mathcal{D}$ the spatial approximation $\varphi_{\mathcal{D}} : \mathbb{R}^2 \to \mathbb{R}^2$ of the flux φ defined by

$$\varphi_{\mathcal{D}}(\xi) = \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} \varphi(z, \xi)\,dz. \tag{2.12}$$

Other approximations of φ on each diamond are possible, we will discuss one of them in Section IX..

On each diamond $\mathcal{D}$, we approximate $\varphi(\cdot, \nabla u_e(\cdot))$, using the discrete gradient operator $\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}$ introduced in section II.E, by $\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}})$. Note that the choice of a constant value for the discrete flux $\varphi_{\mathcal{D}}(\xi)$ on each diamond is necessary in the calculations using Lemma 2.4. The DDFV finite volume scheme then reads

$$a_{\mathcal{K}}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}}} m_\sigma \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}), \boldsymbol{\nu}\right) = m_{\mathcal{K}} f_{\mathcal{K}},\ \forall \mathcal{K}\in\mathfrak{M}, \tag{2.13}$$

$$a_{\mathcal{K}^*}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}), \boldsymbol{\nu}^*\right) = m_{\mathcal{K}^*} f_{\mathcal{K}^*},\ \forall \mathcal{K}^*\in\mathfrak{M}^*, \tag{2.14}$$

where $f_{\mathcal{K}}$ (resp. $f_{\mathcal{K}^*}$) denotes the mean value of the function f on $\mathcal{K}$ (resp. $\mathcal{K}^*$). It is convenient for the analysis given below to introduce a notation for this kind of projections on the set of discrete functions.

Definition 2.6. *For any integrable function v on Ω, we set $\mathbb{P}^{\mathcal{T}}_m v \stackrel{\text{def}}{=} (\mathbb{P}^{\mathfrak{M}}_m v, \mathbb{P}^{\mathfrak{M}^*}_m v)$, where*

$$\mathbb{P}^{\mathfrak{M}}_m v \stackrel{\text{def}}{=} \left(\frac{1}{m_{B_{\mathcal{K}}}}\int_{B_{\mathcal{K}}} v(z)\,dz\right)_{\mathcal{K}\in\mathfrak{M}} \quad \textit{and} \quad \mathbb{P}^{\mathfrak{M}^*}_m v \stackrel{\text{def}}{=} \left(\frac{1}{m_{B_{\mathcal{K}^*}}}\int_{B_{\mathcal{K}^*}} v(z)\,dz\right)_{\mathcal{K}^*\in\mathfrak{M}^*}.$$

We call $\mathbb{P}^{\mathcal{T}}_m v$, the **mean-value projection** *of v on the space $\mathbb{R}^{\mathcal{T}}$. We also introduce the mean-value projection on the control volumes $\widetilde{\mathbb{P}}^{\mathcal{T}}_m v \stackrel{\text{def}}{=} \left(\widetilde{\mathbb{P}}^{\mathfrak{M}}_m v, \widetilde{\mathbb{P}}^{\mathfrak{M}^*}_m v\right)$, where*

$$\widetilde{\mathbb{P}}^{\mathfrak{M}}_m v \stackrel{\text{def}}{=} \left(\frac{1}{m_{\mathcal{K}}}\int_{\mathcal{K}} v(z)\,dz\right)_{\mathcal{K}\in\mathfrak{M}}, \quad \textit{and} \quad \widetilde{\mathbb{P}}^{\mathfrak{M}^*}_m v \stackrel{\text{def}}{=} \left(\frac{1}{m_{\mathcal{K}^*}}\int_{\mathcal{K}^*} v(z)\,dz\right)_{\mathcal{K}^*\in\mathfrak{M}^*}.$$

The finite volume scheme above can now be written under a compact form

$$\mathfrak{a}_g(u^{\mathcal{T}}) = \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, \tag{2.15}$$

where

$$\mathfrak{a}_g(u^{\mathcal{T}}) \stackrel{defi}{=} \left(\left(\frac{1}{m_{\mathcal{K}}} a_{\mathcal{K}}(u^{\mathcal{T}})\right)_{\mathcal{K}}, \left(\frac{1}{m_{\mathcal{K}^*}} a_{\mathcal{K}^*}(u^{\mathcal{T}})\right)_{\mathcal{K}^*}\right). \tag{2.16}$$

We postpone to section IX. a discussion concerning some variants of the proposed scheme, in particular, with respect to the choice of the discretization of the data f and g.

III. DISCRETE FUNCTIONS AND THEIR PROPERTIES

III.A Sobolev spaces on the boundary of polygonal domains

We need to recall briefly the definitions and main properties of the Sobolev spaces defined on $\partial\Omega$ and related trace theorems. A complete study of these topics can be found, for instance, in [23].

Definition 3.1. *Let $\alpha \in]0,1[$, and $p \in [1,+\infty[$. We define $W^{\alpha,p}(\partial\Omega)$, to be the space of functions $g \in L^p(\partial\Omega)$ such that*

$$\|g\|^p_{W^{\alpha,p}(\partial\Omega)} \stackrel{\text{def}}{=} \|g\|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \left|\frac{g(x)-g(y)}{|x-y|^\alpha}\right|^p \frac{d\lambda(x)\,d\lambda(y)}{|x-y|} < \infty,$$

where $d\lambda$ is the natural length measure which can be defined on the boundary $\partial\Omega$ (see [27]).

We recall that the trace operator γ is continuous from $W^{1,p}(\Omega)$ onto $W^{1-\frac{1}{p},p}(\partial\Omega)$ and that there exists a linear continuous lift operator $\mathcal{R} : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ such that

$$\gamma \circ \mathcal{R} = \mathrm{Id}_{\partial\Omega}, \quad \|\mathcal{R}(g)\|_{W^{1,p}} \leq C \, \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}, \tag{3.1}$$

where C depends only on Ω and p. For any $g \in W^{1-\frac{1}{p}}(\partial\Omega)$ we denote by $W^{1,p}_g(\Omega)$ the closed subset of $W^{1,p}(\Omega)$ of all functions whose trace on $\partial\Omega$ is equal to g.

Let us denote by $\Gamma_1, \ldots, \Gamma_k$ the sides of Ω. Since each Γ_i is a segment we can define naturally the spaces $W^{1+\alpha,p}(\Gamma_i)$ by

$$h \in W^{1+\alpha,p}(\Gamma_i) \Leftrightarrow h \in W^{1,p}(\Gamma_i), \text{ and } \int_{\Gamma_i}\int_{\Gamma_i} \left| \frac{\nabla_{\mathcal{T}} h(x) - \nabla_{\mathcal{T}} h(y)}{|x-y|^\alpha} \right|^p \frac{d\lambda(x)\, d\lambda(y)}{|x-y|} < \infty,$$

where $\nabla_{\mathcal{T}} h$ stands for the derivative of h in the direction of $\partial\Omega$ that we call "tangential gradient". Since $\partial\Omega$ is not smooth enough, it is not possible to define the space $W^{1+\alpha,p}(\partial\Omega)$ but we can introduce the following space

$$\widetilde{W}^{1+\alpha,p}(\partial\Omega) = \left\{ g \in W^{1,p}(\partial\Omega),\ g_{|\Gamma_i} \in W^{1+\alpha,p}(\Gamma_i) \right\},$$

endowed with its natural norm. We recall that the trace operator γ is continuous from $W^{2,p}(\Omega)$ onto a finite codimensional subset of $\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)$ which can be described precisely (we do not need this description here and we refer to [23] for further developments on this topic).

We also recall that for any $p > 2$ the embedding of $W^{1-\frac{1}{p},p}(\partial\Omega)$ in the Hölder space $\mathcal{C}^{0,1-\frac{2}{p}}(\partial\Omega)$ holds true and that we have the following sharp estimate.

Lemma 3.2. *Let $p > 2$, there exists a constant C depending only on p such that, for any connected subset σ of $\partial\Omega$ and any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we have*

$$|g(z) - g(z')| \leq C|z - z'|^{1-\frac{2}{p}} \left(\int_\sigma \int_\sigma \left| \frac{g(x) - g(y)}{|x-y|^{1-\frac{1}{p}}} \right|^p \frac{d\lambda(x)\, d\lambda(y)}{|x-y|} \right)^{\frac{1}{p}}, \quad \forall z, z' \in \sigma. \tag{3.2}$$

Proof. The embedding of $W^{1-\frac{1}{p},p}(\partial\Omega)$ in the Hölder space $\mathcal{C}^{0,1-\frac{2}{p}}(\partial\Omega)$ is given by the Morrey theorem. In particular, there exists $C > 0$ such that (3.2) holds for the unit segment $\sigma =]0,1[\subset \mathbb{R}$ and any $g \in W^{1-\frac{1}{p},p}(]0,1[)$. It is now easy, using a linear change of the variables, to see that (3.2) holds with the same constant C for any σ. ∎

III.B Basic notations and results

Whenever it is convenient, we associate to the discrete function $u^{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*})$ the piecewise constant function

$$u^{\mathcal{T}} \sim \frac{1}{2}\left(u^{\mathfrak{M}} + u^{\mathfrak{M}^*}\right), \tag{3.3}$$

where $u^{\mathfrak{M}} = \sum_{\mathcal{K}\in\mathfrak{M}} \mathbf{1}_{\mathcal{K}} u_{\mathcal{K}}$, $u^{\mathfrak{M}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} \mathbf{1}_{\mathcal{K}^*} u_{\mathcal{K}^*}$. As a consequence, one can define for any $r \in [1, +\infty]$ the L^r norm of $u^{\mathfrak{M}}, u^{\mathfrak{M}^*}, u^{\mathcal{T}}$. We denote by $[\![\cdot,\cdot]\!]$ the inner product on $\mathbb{R}^{\mathcal{T}}$ given by

$$[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] = \frac{1}{2} \sum_{\mathcal{K}\in\mathfrak{M}} m(\mathcal{K}) u_{\mathcal{K}} v_{\mathcal{K}} + \frac{1}{2} \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m(\mathcal{K}^*) u_{\mathcal{K}^*} v_{\mathcal{K}^*},$$

which stands for a discrete $L^2(\Omega)$ inner product, whereas the usual Euclidean inner product on $\mathbb{R}^{\mathcal{T}}$ is denoted by $(\cdot,\cdot)$:

$$(u^{\mathcal{T}}, v^{\mathcal{T}}) = \sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} u_{\mathcal{K}^*} v_{\mathcal{K}^*}.$$

Let us finally state the discrete version of the Poincaré inequality. This result is classical in the case $p = 2$ (see for example [13, 18]). When $p \neq 2$, it is proved in a slightly different context in [4], without any geometrical assumptions on the mesh. In the DDFV framework, we need to assume a lower bound on $\alpha_{\mathcal{T}}$ defined in (2.4).

To begin with, let us point out the fact that there exists $C > 0$ depending only on p and $\mathrm{reg}(\mathcal{T})$ such that for any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we have

$$\|\mathbb{P}_m^{\mathcal{T}} g\|_{L^p(\partial\Omega)} \leq \|\mathbb{P}_m^{\partial\mathfrak{M}} g\|_{L^p(\partial\Omega)} + \|\mathbb{P}_m^{\partial\mathfrak{M}^*} g\|_{L^p(\partial\Omega)} \leq C\|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}, \tag{3.4}$$

where $\mathbb{P}_m^{\mathcal{T}} g, \mathbb{P}_m^{\partial\mathfrak{M}} g$ and $\mathbb{P}_m^{\partial\mathfrak{M}^*}$ are defined in (2.2). As a consequence, for any sequence of meshes $\mathcal{T}_n$ such that $\mathrm{reg}(\mathcal{T}_n)$ is bounded and $\mathrm{size}(\mathcal{T}_n) \to 0$ we have

$$\mathbb{P}_m^{\partial\mathfrak{M}_n} g \xrightarrow[n\to\infty]{} g, \ \text{ in } L^p(\partial\Omega), \ \text{ and } \ \mathbb{P}_m^{\partial\mathfrak{M}_n^*} g \xrightarrow[n\to\infty]{} g, \ \text{ in } L^p(\partial\Omega), \tag{3.5}$$

for any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$. The proof of this result is not given since it is a straightforward adaptation of the arguments used in Section III.C for the study of the mean-value projection operators for functions defined in the whole domain Ω. Remark that we need to take $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ in (3.4) and (3.5) because of the particular treatment of the corner points (for $p \geq 2$) in (2.2) and (2.6).

We can now state and prove the main result of this section.

Lemma 3.3 (Discrete Poincaré inequality). *Let $\mathcal{T}$ be a mesh of Ω. There exists a constant C, only depending on p, on the diameter of Ω and on $\mathrm{reg}(\mathcal{T})$, such that for any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, we have*

$$\|u^{\mathcal{T}}\|_{L^p} \leq \|u^{\mathfrak{M}}\|_{L^p} + \|u^{\mathfrak{M}^*}\|_{L^p} \leq C\left(\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} u^{\mathcal{T}}\|_{L^p} + \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right), \tag{3.6}$$

where $\mathbb{P}_m^{\mathcal{T}} g$ is defined in (2.2).

Proof. We start as in the proof of the discrete Poincaré inequality given in [4] (see also [19]), taking into account the boundary conditions. It follows, using (3.4),

$$\begin{aligned}\|u^{\mathfrak{M}}\|_{L^p}^p &= \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |u_{\mathcal{K}}|^p \leq C \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma m_{\sigma^*} \left|\frac{|u_{\mathcal{K}}|^p - |u_{\mathcal{L}}|^p}{m_{\sigma^*}}\right| + C \sum_{\mathcal{K}\in\partial\mathfrak{M}} m_{\mathcal{K}} |g_{\mathcal{K}}|^p \\ &\leq C \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma m_{\sigma^*} \frac{|u_{\mathcal{K}} - u_{\mathcal{L}}|}{m_{\sigma^*}} (|u_{\mathcal{K}}|^{p-1} + |u_{\mathcal{L}}|^{p-1}) + C\|g\|^p_{W^{1-\frac{1}{p},p}(\partial\Omega)}.\end{aligned}$$

Using the definition (2.6), we see that there exists C depending on $\mathrm{reg}(\mathcal{T})$ such that

$$m_\sigma m_{\sigma^*} \leq C m_{\mathcal{D}}, \ \ m_\sigma m_{\sigma^*} \leq C m_{\mathcal{K}}, \ \text{ and } m_\sigma m_{\sigma^*} \leq C m_{\mathcal{L}}.$$

As a consequence, we can use the Hölder inequality to get

$$\|u^{\mathfrak{M}}\|_{L^p}^p \leq C \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathcal{D}} \left|\frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}}\right|^p + C\|g\|^p_{W^{1-\frac{1}{p},p}(\partial\Omega)}.$$

By definition of the discrete gradient, we have

$$\left|\frac{u_{\mathcal{K}}-u_{\mathcal{L}}}{m_{\sigma^*}}\right| = \left|(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}, \boldsymbol{\tau}^*)\right| \leq |\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}|,$$

so that, with (2.6) and (3.4), the estimate of the norm of $u^{\mathfrak{M}}$ follows. The estimate of the contribution of $u^{\mathfrak{M}^*}$ is estimated in the same way. ∎

Notice that, using the same argument than in [4], we can prove that the constant C in (3.6) depend only on $\mathrm{diam}(\Omega)$ and $\alpha_{\mathcal{T}}$ in the case where all the diamond cells are convex and $g = 0$.

III.C Properties of the mean-value projection operators

In the convergence analysis of our scheme we will have to use some discrete approximation of test functions lying in $W^{1,p}(\Omega)$. The natural projection (since these test functions may not be continuous when $p < 2$) is the mean-value projection (see Definition 2.6). We give below the main properties of such a projection onto the set of discrete functions in our framework.

To begin with, we give the following crucial result, which is similar to [16, Lemma 7.2], [18, Lemma 3.4], generalized to the case of non convex control volumes and $p \neq 2$ (see also [17, Lemma 6.1]). We do not give the proof which is a straightforward extension of the proofs that one can found in the references above.

Lemma 3.4. *For any $q \geq 1$, there exists a constant C depending only on q such that for any bounded set $\mathcal{P} \subset \mathbb{R}^2$ with positive measure, any segment $\sigma \subset \mathbb{R}^2$ and any $v \in W^{1,q}(\mathbb{R}^2)$ we have*

$$|v_{\mathcal{P}} - v_\sigma|^q \leq \frac{1}{m_\sigma m_{\mathcal{P}}}\int_\sigma\int_{\mathcal{P}} |v(x)-v(y)|^q \, dx\, dy \leq C\,\frac{\mathrm{diam}(\widehat{\mathcal{P}_\sigma})^{q+1}}{m_\sigma m_{\mathcal{P}}}\int_{\widehat{\mathcal{P}_\sigma}} |\nabla v\,(z)|^q \; dz,$$

where $v_{\mathcal{P}}$ denotes the mean value of v on $\mathcal{P}$, v_σ the mean value of v on the segment σ, and $\widehat{\mathcal{P}_\sigma}$ is the convex hull of $\mathcal{P} \cup \sigma$.

Lemma 3.5 (Mean-value projection bounds). *Let $\mathcal{T}$ be a mesh on Ω and $q \in [1, +\infty]$. There exists C depending on q and $\mathrm{reg}(\mathcal{T})$ such that*

1. *for any $v \in L^q(\Omega)$, we have*

$$\begin{cases} \|\mathbb{P}^{\mathcal{T}}_m v\|_{L^q} \leq \|\mathbb{P}^{\mathfrak{M}}_m v\|_{L^q} + \|\mathbb{P}^{\mathfrak{M}^*}_m v\|_{L^q} \leq C\|v\|_{L^q}, \\ \|\widetilde{\mathbb{P}}^{\mathcal{T}}_m v\|_{L^q} \leq \|\widetilde{\mathbb{P}}^{\mathfrak{M}}_m v\|_{L^q} + \|\widetilde{\mathbb{P}}^{\mathfrak{M}^*}_m v\|_{L^q} \leq C\|v\|_{L^q}. \end{cases} \tag{3.7}$$

2. *for any $v \in W^{1,q}(\Omega)$ we have*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_m v\|_{L^q} \leq C\|\nabla v\|_{L^q}, \tag{3.8}$$

where $g = \gamma(v)$ is the trace of v on $\partial\Omega$.

Proof.

1. This point is straightforward consequence of the Jensen inequality and of (2.5) and (2.6).

2. Recall that

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_m v\|^q_{L^q} \;\leq\; C\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\left|\frac{\mathbb{P}^{\mathcal{T}}_m v_{\mathcal{K}} - \mathbb{P}^{\mathcal{T}}_m v_{\mathcal{L}}}{m_{\sigma^*}}\right|^q + \left|\frac{\mathbb{P}^{\mathcal{T}}_m v_{\mathcal{K}^*} - \mathbb{P}^{\mathcal{T}}_m v_{\mathcal{L}^*}}{m_\sigma}\right|^q\right).$$

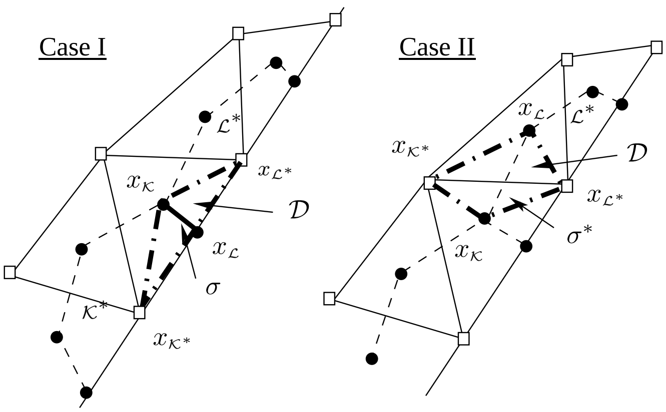

FIG. 5. Boundary dual control volumes and diamonds

For interior control volumes, we have

$$\begin{aligned}\left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}} - \mathbb{P}_m^{\mathcal{T}} v_{\mathcal{L}}}{m_{\sigma^*}}\right| \le \left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}} - v_\sigma}{m_{\sigma^*}}\right| + \left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{L}} - v_\sigma}{m_{\sigma^*}}\right|, \\ \left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}^*} - \mathbb{P}_m^{\mathcal{T}} v_{\mathcal{L}^*}}{m_{\sigma}}\right| \le \left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}^*} - v_{\sigma^*}}{m_{\sigma}}\right| + \left|\frac{\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{L}^*} - v_{\sigma^*}}{m_{\sigma}}\right|,\end{aligned} \tag{3.9}$$

where $v_\sigma = \frac{1}{m_\sigma}\int_\sigma v(s)\ ds$ and $v_{\sigma^*} = \frac{1}{m_{\sigma^*}}\int_{\sigma^*} v(s^*)\ ds^*$. Lemma 3.4 can be applied to each of the terms in the right-hand side of (3.9). The case of boundary control volumes can also be reduced, as shown in Figure 5, to estimates of differences between the mean values on the balls $B_{\mathcal{K}}$ and the mean values on edges. But as for $\mathcal{K} \in \partial\mathfrak{M}$ and $\mathcal{K}^* \in \partial\mathfrak{M}^*$, $\mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}}, \mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}^*}$ are mean value of the function v on some edges, we need to insert mean values on appropriate balls.

Thanks to (2.6) we have

$$\operatorname{diam}(\mathcal{K} \cup B_{\mathcal{K}}) \le C(\operatorname{reg}(\mathcal{T}))\,\mathrm{d}_{\mathcal{K}} \le \widetilde{C}(\operatorname{reg}(\mathcal{T}))\ \min(m_\sigma, m_{\sigma^*}),$$

$$\operatorname{diam}(\mathcal{K}^* \cup B_{\mathcal{K}^*}) \le C(\operatorname{reg}(\mathcal{T}))\,\mathrm{d}_{\mathcal{K}^*} \le \widetilde{C}(\operatorname{reg}(\mathcal{T}))\ \min(m_\sigma, m_{\sigma^*}),$$

and

$$m_{\mathcal{D}} \le C(\operatorname{reg}(\mathcal{T}))\, m_{B_{\mathcal{K}}}, \quad m_{\mathcal{D}} \le C(\operatorname{reg}(\mathcal{T}))\, m_{B_{\mathcal{K}^*}},$$

so that

$$\begin{aligned}\|\nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} \mathbb{P}_m^{\mathcal{T}} v\|_{L^q}^q &\le C \sum_{\mathcal{D}\in\mathfrak{D}} \left(\int_{\widehat{\mathcal{K}\cup B_{\mathcal{K}}}} |\nabla v(z)|^q\, dz + \int_{\widehat{\mathcal{L}\cup B_{\mathcal{L}}}} |\nabla v(z)|^q\, dz \right. \\ &\qquad \left. + \int_{\widehat{\mathcal{K}^*\cup B_{\mathcal{K}^*}}} |\nabla v(z)|^q\, dz + \int_{\widehat{\mathcal{L}^*\cup B_{\mathcal{L}^*}}} |\nabla v(z)|^q\, dz \right) \\ &\le 2C \left(\sum_{\mathcal{K}\in\mathfrak{M}} \int_{\widehat{\mathcal{K}\cup B_{\mathcal{K}}}} |\nabla v(z)|^q\, dz + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} \int_{\widehat{\mathcal{K}^*\cup B_{\mathcal{K}^*}}} |\nabla v(z)|^q\, dz \right) \\ &\le \mathcal{N}_{\mathcal{T}} C \int_\Omega |\nabla v(z)|^q\, dz,\end{aligned}$$

and the claim is proved.

■

Proposition 3.6 (Convergence of the mean-value projections). *Let $\mathcal{T}$ be a mesh on Ω and $q \in [1, +\infty]$. There exists C depending on q and $\mathrm{reg}(\mathcal{T})$ such that*

$$\|\mathbb{P}_m^{\mathcal{T}} v - v\|_{L^q} \leq \|\mathbb{P}_m^{\mathfrak{M}} v - v\|_{L^q} + \|\mathbb{P}_m^{\mathfrak{M}^*} v - v\|_{L^q} \leq C\,\mathrm{size}(\mathcal{T})\|\nabla v\|_{L^q},\ \ \forall v \in W^{1,q}(\Omega), \quad (3.10)$$

$$\|\widetilde{\mathbb{P}}_m^{\mathcal{T}} v - v\|_{L^q} \leq \|\widetilde{\mathbb{P}}_m^{\mathfrak{M}} v - v\|_{L^q} + \|\widetilde{\mathbb{P}}_m^{\mathfrak{M}^*} v - v\|_{L^q} \leq C\,\mathrm{size}(\mathcal{T})\|\nabla v\|_{L^q},\ \ \forall v \in W^{1,q}(\Omega), \quad (3.11)$$

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} \mathbb{P}_m^{\mathcal{T}} v - \nabla v\|_{L^q} \leq C\,\mathrm{size}(\mathcal{T})\|\nabla v\|_{W^{1,q}},\ \ \forall v \in W^{2,q}(\Omega),\ \ g = \gamma(v). \quad (3.12)$$

We postpone the proof of this result to Section VII.A3.

Corollary 3.7. *Let $q \in [1, +\infty[$ and $(\mathcal{T}_n)_n$ a sequence of meshes such that $\mathrm{size}(\mathcal{T}_n) \to 0$ and $\mathrm{reg}(\mathcal{T}_n)$ is bounded. Then, we have*

- *For any $v \in L^q(\Omega)$, all the sequences $(\mathbb{P}_m^{\mathfrak{M}_n} v)_n$, $(\mathbb{P}_m^{\mathfrak{M}^*_n} v)_n$, $(\mathbb{P}_m^{\mathcal{T}_n} v)_n$, $(\widetilde{\mathbb{P}}_m^{\mathfrak{M}_n} v)_n$, $(\widetilde{\mathbb{P}}_m^{\mathfrak{M}^*_n} v)_n$ and $(\widetilde{\mathbb{P}}_m^{\mathcal{T}_n} v)_n$ converge towards v in $L^q(\Omega)$.*
- *For any $v \in W^{1,q}(\Omega)$, the sequence $(\nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}_m^{\mathcal{T}_n} v)_n$ converge towards ∇v in $(L^q(\Omega))^2$.*

Proof. The two claims of the corollary can be shown in the same way. Let us give, for instance, the proof of the second point. For any $v \in W^{1,q}(\Omega)$, by density of $W^{2,q}(\Omega)$ in $W^{1,q}(\Omega)$, for any $\varepsilon > 0$ there exists $v^\varepsilon \in W^{2,q}(\Omega)$ such that $\|v - v^\varepsilon\|_{W^{1,q}} \leq \varepsilon$. We denote its trace by $g^\varepsilon = \gamma(v^\varepsilon)$ and its mean-value projection by $g_n^\varepsilon = \mathbb{P}_m^{\mathcal{T}} g^\varepsilon$. Thanks to Lemma 3.5 and Proposition 3.6 we have

$$\begin{aligned}
\|\nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}_m^{\mathcal{T}_n} v - \nabla v\|_{L^q} \leq& \|\nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}_m^{\mathcal{T}_n} v - \nabla^{\mathcal{T}_n}_{g_n^\varepsilon} \mathbb{P}_m^{\mathcal{T}_n} v^\varepsilon\|_{L^q} + \|\nabla^{\mathcal{T}_n}_{g_n^\varepsilon} \mathbb{P}_m^{\mathcal{T}_n} v^\varepsilon - \nabla v^\varepsilon\|_{L^q} \\
&+ \|\nabla v^\varepsilon - \nabla v\|_{L^q} \\
\leq& C\|\nabla v - \nabla v^\varepsilon\|_{L^q} + C\,\mathrm{size}(\mathcal{T}_n)\|\nabla v^\varepsilon\|_{W^{1,q}} \\
\leq& C\varepsilon + C\,\mathrm{size}(\mathcal{T}_n)\|\nabla v^\varepsilon\|_{W^{1,q}}.
\end{aligned}$$

The real number ε being fixed, for n large enough we have $\mathrm{size}(\mathcal{T}_n)\|\nabla v^\varepsilon\|_{W^{1,q}} \leq \varepsilon$ so that we obtain

$$\|\nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}_m^{\mathcal{T}_n} v - \nabla v\|_{L^q} \leq 2C\varepsilon,$$

and the result follows. ■

III.D A compactness result

As usual, in the convergence analysis of finite volume schemes (see [18] for instance) one needs to prove a discrete compactness result, which is a discrete counterpart of the Rellich compactness theorem.

Lemma 3.8 (Discrete compactness). *Consider a sequence of meshes $(\mathcal{T}_n)_n$ such that $\mathrm{size}(\mathcal{T}_n)$ tends to zero and $\mathrm{reg}(\mathcal{T}_n)$ is bounded. Let $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ be the boundary data and $g_n = \mathbb{P}_m^{\mathcal{T}_n} g$ its mean-value discretization on the mesh $\mathcal{T}_n$. Let $u^{\mathcal{T}_n} \in \mathbb{R}^{\mathcal{T}_n}$ be a sequence satisfying the discrete $W^{1,p}_g$ bound*

$$\|\nabla^{\mathcal{T}_n}_{g_n} u^{\mathcal{T}_n}\|_{L^p} \leq C,\ \ \forall n \in \mathbb{N}. \quad (3.13)$$

Then, there exists $u \in W^{1,p}_g(\Omega)$ such that, up to a subsequence,

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} u \text{ in } L^p(\Omega),$$
$$\nabla^{\mathcal{T}_n}_{g_n} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} \nabla u \text{ weakly in } \left(L^p(\Omega)\right)^2.$$

Notice that we will prove in fact that, up to a subsequence, $(u^{\mathfrak{M}_n})_n$ and $(u^{\mathfrak{M}^*_n})_n$ both converge in $L^p(\Omega)$ but in general their two limits can be different.

Proof.

1. For any $n \in \mathbb{N}$, consider $v^{\mathcal{T}_n} = u^{\mathcal{T}_n} - \mathbb{P}^{\mathcal{T}_n}_m \mathcal{R}(g)$, where $\mathcal{R}$ is the lift operator satisfying (3.1). By Lemma 3.5, we know that $\nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}^{\mathcal{T}_n}_m \mathcal{R}(g)$ is bounded in $(L^p(\Omega))^2$ so that, using the bound (3.13) we deduce that

$$\|\nabla^{\mathcal{T}_n}_0 v^{\mathcal{T}_n}\|_{L^p} \leq C, \ \ \forall n \in \mathbb{N}.$$

Hence, by Lemma 3.3, the sequence $v^{\mathcal{T}_n}$ is bounded in $L^p(\Omega)$. Let us now consider the sequence of discrete functions $w^{\mathcal{T}_n}$ defined by

$$w^{\mathcal{T}_n}_{\mathcal{K}} = |v^{\mathcal{T}_n}_{\mathcal{K}}|^{p-1} v^{\mathcal{T}_n}_{\mathcal{K}}, \ \ \forall \mathcal{K} \in \mathfrak{M}_n,$$

$$w^{\mathcal{T}_n}_{\mathcal{K}^*} = |v^{\mathcal{T}_n}_{\mathcal{K}^*}|^{p-1} v^{\mathcal{T}_n}_{\mathcal{K}^*}, \ \ \forall \mathcal{K}^* \in \mathfrak{M}^*_n,$$

and extended by 0 outside Ω. This sequence of functions is of course bounded in $L^1(\mathbb{R}^d)$ and vanishes outside a bounded subset of $\mathbb{R}^2$. For any $x, \eta \in \mathbb{R}^2$, and any edge $\sigma = \mathcal{K}|\mathcal{L}$ we define

$$\psi_\sigma(x,\eta) = \begin{cases} 1, & \text{if } \sigma \cap [x, x+\eta] \neq \emptyset, \\ 0, & \text{elsewhere.} \end{cases}$$

Hence, with the notations of Section III.B and by Lemma 1.3, we have for any $x \in \mathbb{R}^2$,

$$\begin{aligned} |w^{\mathfrak{M}_n}(x+\eta) - w^{\mathfrak{M}_n}(x)| &\leq \sum_{\sigma=\mathcal{K}|\mathcal{L}} \psi_{\mathcal{K}|\mathcal{L}}(x,\eta) |w_{\mathcal{L}} - w_{\mathcal{K}}| \\ &\leq C \sum_{\sigma=\mathcal{K}|\mathcal{L}} m_{\sigma^*} \psi_{\mathcal{K}|\mathcal{L}}(x,\eta) \left| \frac{v_{\mathcal{L}} - v_{\mathcal{K}}}{m_{\sigma^*}} \right| (|v_{\mathcal{L}}|^{p-1} + |v_{\mathcal{K}}|^{p-1}). \end{aligned}$$

Now we remark that $\displaystyle\int_{\mathbb{R}^2} \psi_\sigma(x,\eta)\,dx \leq m_\sigma |\eta|$ so that we have

$$\begin{aligned} \|w^{\mathfrak{M}_n}(\cdot+\eta) - w^{\mathfrak{M}_n}(\cdot)\|_{L^1(\mathbb{R}^2)} &\leq C \mathrm{reg}(\mathcal{T}_n) |\eta| \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}}_0 v^{\mathcal{T}_n}| (|v_{\mathcal{L}}|^{p-1} + |v_{\mathcal{K}}|^{p-1}) \\ &\leq C \mathrm{reg}(\mathcal{T}_n) \|\nabla^{\mathcal{T}_n}_0 v^{\mathcal{T}_n}\|_{L^p} \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} (|v_{\mathcal{L}}|^p + |v_{\mathcal{K}}|^p) \right)^{\frac{p-1}{p}}. \end{aligned}$$

The last factor in this inequality can be treated, as in [4], as follows:

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} (|v_{\mathcal{L}}|^p + |v_{\mathcal{K}}|^p) \leq C \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |v_{\mathcal{K}}|^p + C \mathrm{reg}(\mathcal{T}_n) \mathrm{size}(\mathcal{T}_n)^p \|\nabla^{\mathcal{T}_n}_0 v^{\mathcal{T}_n}\|^p_{L^p}.$$

Hence, there exists $C > 0$ such that for all $\eta \in \mathbb{R}^2$ and $n \geq 0$ we have

$$\|w^{\mathfrak{M}_n}(\cdot+\eta) - w^{\mathfrak{M}_n}\|_{L^1(\mathbb{R}^2)} \leq C|\eta|.$$

Thanks to the Kolmogorov theorem, we deduce that there exists a subsequence of $(w^{\mathfrak{M}_{n_k}})_k$ which converges towards a function $w \in L^1(\mathbb{R}^2)$ which vanishes outside Ω. The definition of $w^{\mathcal{T}_n}$ reads as

$$v^{\mathfrak{M}_n}(x) = T(w^{\mathfrak{M}_n}(x)),\ \forall x \in \Omega,$$

where T is the nonlinear map defined by $T(\xi) = |\xi|^{\frac{1-p}{p}}\xi$. By Lemma 1.3, we know that T is $\frac{1}{p}$-Hölder continuous so that we have

$$\|v^{\mathfrak{M}_n} - T(w)\|_{L^p(\Omega)} \leq C\|w^{\mathfrak{M}_n} - w\|_{L^1(\Omega)}^{\frac{1}{p}},$$

which proves that $(v^{\mathfrak{M}_{n_k}})_k$ converges strongly in $L^p(\Omega)$. We can now apply the same technique to the subsequence $(v^{\mathfrak{M}^*_{n_k}})_k$ defined on the dual meshes. We deduce, using (3.3), that there exists a function $v \in L^p(\Omega)$ such that

$$v^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} v \text{ in } L^p(\Omega).$$

By Corollary 3.7, we know that $\mathbb{P}_m^{\mathcal{T}_n}\mathcal{R}(g)$ tends to $\mathcal{R}(g)$ in $L^p(\Omega)$, so that we finally have

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} v + \mathcal{R}(g) \stackrel{\text{def}}{=} u \text{ in } L^p(\Omega). \tag{3.14}$$

2. It remains to show that $u \in W_g^{1,p}(\Omega)$ and that the discrete gradient weakly converges. Thanks to the bound (3.13), there exists $\chi \in (L^p(\Omega))^2$ and a subsequence which is still indexed by n such that

$$\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} \chi \text{ weakly in } (L^p(\Omega))^2. \tag{3.15}$$

Let $\psi \in \left(\mathcal{C}^\infty(\overline{\Omega})\right)^2$. Using (3.14) and (3.15), we have

$$\begin{aligned} I_n \stackrel{\text{def}}{=} &\int_\Omega \left(\nabla_{g_n}^{\mathcal{T}_n} u_n^{\mathcal{T}_n}(z), \psi(z)\right) dz + \int_\Omega u_n^{\mathcal{T}_n}(z) \text{div}\, \psi(z)\, dz \\ &\xrightarrow[n\to\infty]{} \int_\Omega (\chi(z), \psi(z))\, dz + \int_\Omega u(z) \text{div}\, \psi(z)\, dz. \end{aligned} \tag{3.16}$$

By definition of the discrete gradient we have

$$\int_\Omega \left(\nabla_{g_n}^{\mathcal{T}_n} u_n^{\mathcal{T}_n}(z), \psi(z)\right) dz = \sum_{\mathcal{D}\in\mathfrak{D}} m_\mathcal{D} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \psi_\mathcal{D}\right), \tag{3.17}$$

where $\psi_\mathcal{D} = \dfrac{1}{m_\mathcal{D}} \displaystyle\int_\mathcal{D} \psi(z)\, dz$. For each diamond $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$ let us introduce

$$\psi_\sigma = \frac{1}{m_\sigma}\int_\sigma \psi(s)\, ds, \quad \psi_{\sigma^*} = \frac{1}{m_{\sigma^*}}\int_{\sigma^*} \psi(s)\, ds,$$

and finally $\widetilde{\psi}_\mathcal{D}$ uniquely defined by

$$(\widetilde{\psi}_\mathcal{D}, \boldsymbol{\nu}) = (\psi_\sigma, \boldsymbol{\nu}), \quad (\widetilde{\psi}_\mathcal{D}, \boldsymbol{\nu}^*) = (\psi_{\sigma^*}, \boldsymbol{\nu}^*).$$

The test function ψ being smooth enough we have, using Lemma 2.4,

$$\begin{aligned} |\psi_\mathcal{D} - \widetilde{\psi}_\mathcal{D}| &\leq \frac{1}{\sin \alpha_{\mathcal{T}_n}} \left(|\psi_\mathcal{D} - \psi_\sigma| + |\psi_\mathcal{D} - \psi_{\sigma^*}|\right) \\ &\leq 2\text{reg}(\mathcal{T}_n)\text{size}(\mathcal{T}_n)\|\nabla\psi\|_{L^\infty}. \end{aligned} \tag{3.18}$$

Coming back to (3.17) we deduce

$$\int_\Omega \left(\nabla_{g_n}^{\mathcal{T}_n} u_n^{\mathcal{T}_n}(z), \psi(z)\right) \, dz = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \widetilde{\psi}_{\mathcal{D}}\right) + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \psi_{\mathcal{D}} - \widetilde{\psi}_{\mathcal{D}}\right),$$

and using (3.18) and the bound (3.13), we see that the second term tends to zero as n goes to infinity. As far as the first term is concerned, we use (2.9) to obtain

$$\begin{aligned}
\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \widetilde{\psi}_{\mathcal{D}}\right) &= \frac{1}{2}\sum_{\mathcal{D}\in\mathfrak{D}} m_\sigma m_{\sigma^*} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}\boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma^*}}\boldsymbol{\nu}^*, \widetilde{\psi}_{\mathcal{D}}\right) \\
&= -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma \left(\widetilde{\psi}_{\mathcal{D}}, \boldsymbol{\nu}_{\mathcal{K}}\right) - \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} u_{\mathcal{K}^*} \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} \left(\widetilde{\psi}_{\mathcal{D}}, \boldsymbol{\nu}^*_{\mathcal{K}^*}\right) \\
&\quad + \frac{1}{2}\sum_{\mathcal{K}\in\partial\mathfrak{M}} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma \left(\widetilde{\psi}_{\mathcal{D}}, \boldsymbol{\nu}_{\mathcal{K}}\right) - \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}^*} \sum_{\substack{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}\\ \sigma^*\subset\Omega}} m_{\sigma^*} \left(\widetilde{\psi}_{\mathcal{D}}, \boldsymbol{\nu}^*_{\mathcal{K}^*}\right).
\end{aligned}$$

We recall here that the two boundary terms above have different forms since the elements of $\partial\mathfrak{M}$ are degenerate control volumes whereas the elements $\partial\mathfrak{M}^*$ are plain dual control volumes located near the boundary of the domain. Thanks to the definition of $\widetilde{\psi}_{\mathcal{D}}$ we have

$$\begin{aligned}
&\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \widetilde{\psi}_{\mathcal{D}}\right) = \\
&\qquad -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \int_\sigma (\psi(s), \boldsymbol{\nu}_{\mathcal{K}})\, ds - \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} u_{\mathcal{K}^*} \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \int_{\sigma^*} (\psi(s), \boldsymbol{\nu}_{\mathcal{K}^*})\, ds \\
&\qquad + \frac{1}{2}\sum_{\mathcal{K}\in\partial\mathfrak{M}} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}} \int_{\mathcal{K}} (\psi(s), \boldsymbol{\nu}_{\mathcal{K}})\, ds - \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}^*} \sum_{\substack{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}\\ \sigma^*\subset\Omega}} \int_{\sigma^*} (\psi(s), \boldsymbol{\nu}_{\mathcal{K}^*})\, ds.
\end{aligned}$$

Let us emphasize the fact that in the last term, only the edges σ^* which are not on the boundary of the domain are taken into account. Hence, using Stokes formula in the first two terms and in the last one, it follows

$$\begin{aligned}
\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\nabla_{g_n}^{\mathcal{D}} u_n^{\mathcal{T}_n}, \widetilde{\psi}_{\mathcal{D}}\right) &= -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}} \int_{\mathcal{K}} \operatorname{div}\psi(z)\, dz - \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} u_{\mathcal{K}^*} \int_{\mathcal{K}^*} \operatorname{div}\psi(z)\, dz \\
&\quad + \frac{1}{2}\sum_{\mathcal{K}\in\partial\mathfrak{M}} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}} \int_{\mathcal{K}} (\psi(s), \boldsymbol{\nu}_{\mathcal{K}})\, ds - \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}^*} \int_{\mathcal{K}^*} \operatorname{div}\psi(z)\, dz \\
&\quad + \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}^*} \sum_{\substack{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}\\ \sigma^*\subset\partial\Omega}} \int_{\sigma^*} (\psi(s), \boldsymbol{\nu}_{\mathcal{K}^*})\, ds \\
&= -\int_\Omega u_n^{\mathcal{T}_n}(z) \operatorname{div}\psi(z)\, dz + \int_{\partial\Omega} \mathbb{P}_m^{\mathcal{T}_n} g\, (\psi(z), \boldsymbol{\nu})\, ds \\
&\quad - \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} \mathbb{P}_m^{\mathcal{T}_n} g_{\mathcal{K}^*} \int_{\mathcal{K}^*} \operatorname{div}\psi(z)\, dz
\end{aligned}$$

Notice that the last term tends to zero thanks to (3.4) since $\left|\int_{\mathcal{K}} \operatorname{div}\psi(z)\, dz\right| \leq C\|\nabla\psi\|_\infty \operatorname{size}(\mathcal{T}_n)^2$. Gathering all the computations above and using the property (3.5), we find that I_n (defined in

(3.16)) converges towards

$$\int_{\partial\Omega} g(s)(\psi(s),\boldsymbol{\nu})\,ds,$$

so that we finally proved that, for any $\psi \in (\mathcal{C}^\infty(\overline{\Omega}))^2$ we have

$$\int_\Omega (\chi(z),\psi(z))\,\,dz + \int_\Omega u(z)\operatorname{div}\psi(z)\,dz = \int_{\partial\Omega} g(s)\,(\psi(s),\boldsymbol{\nu})\,ds.$$

This proves that $u \in W^{1,p}(\Omega)$ with $\nabla u = \chi$ and that $\gamma(u) = g$.

■

IV. PROPERTIES OF THE SCHEME

In this section we show that the finite volume scheme (2.15) inherits from the properties of the continuous problem (1.1). In particular, we show the existence and uniqueness of a solution to this scheme. In a second paragraph we concentrate on the very important, in view of many applications such as (1.3)-(1.5), variational case.

IV.A The general case

Let us begin with a basic lemma which express the duality, through the discrete summation-by-parts procedure, of the discrete gradient and discrete divergence operators on DDFV meshes. Let us recall that the nonlinear map $\mathfrak{a}_g$ defining the scheme is introduced in (2.16).

Lemma 4.1 (Summation by parts). *For any $(u^\mathcal{T}, v^\mathcal{T}) \in \mathbb{R}^\mathcal{T} \times \mathbb{R}^\mathcal{T}$, we have*

$$[\![\mathfrak{a}_g(u^\mathcal{T}), v^\mathcal{T}]\!] = \sum_{\mathcal{D}\in\mathfrak{D}} m_\mathcal{D}\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \nabla^\mathcal{D}_0 v^\mathcal{T}\right).$$

Proof. Performing the summation-by-parts from the definition (2.15) of $\mathfrak{a}_g$ we deduce

$$\begin{aligned}
[\![\mathfrak{a}_g(u^\mathcal{T}), v^\mathcal{T}]\!] =& \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} a_\mathcal{K}(u^\mathcal{T}) v_\mathcal{K} + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} a_{\mathcal{K}^*}(u^\mathcal{T}) v_{\mathcal{K}^*}\\
=& -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} v_\mathcal{K} \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_\mathcal{K}} m_\sigma\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \boldsymbol{\nu}_\mathcal{K}\right)\\
& -\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} v_{\mathcal{K}^*} \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*}\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \boldsymbol{\nu}_{\mathcal{K}^*}\right).
\end{aligned}$$

Reorganizing the summation over the set of diamonds, we get using the definition (2.9) of the discrete gradient

$$\begin{aligned}
[\![\mathfrak{a}_g(u^\mathcal{T}), v^\mathcal{T}]\!] =& -\frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \boldsymbol{\nu}\right)(v_\mathcal{K} - v_\mathcal{L})\\
& -\frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\sigma^*}\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \boldsymbol{\nu}^*\right)(v_{\mathcal{K}^*} - v_{\mathcal{L}^*})\\
=& \frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma m_{\sigma^*}\left(\varphi_\mathcal{D}(\nabla^\mathcal{D}_{\mathbb{P}^\mathcal{T}_m g} u^\mathcal{T}), \left(\left(\nabla^\mathcal{D}_0 v^\mathcal{T}, \boldsymbol{\tau}^*\right)\boldsymbol{\nu} + \left(\nabla^\mathcal{D}_0 v^\mathcal{T}, \boldsymbol{\tau}\right)\boldsymbol{\nu}^*\right)\right).
\end{aligned}$$

Thanks to Lemma 2.4, we conclude that

$$\begin{aligned}[\![\mathfrak{a}_g(u^{\mathcal{T}}), v^{\mathcal{T}}]\!] &= \frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_\sigma m_{\sigma^*}\sin\alpha_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}), \nabla^{\mathcal{D}}_0 v^{\mathcal{T}}\right)\\ &= \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}), \nabla^{\mathcal{D}}_0 v^{\mathcal{T}}\right).\end{aligned}$$

∎

It is now possible to prove the coercivity of the nonlinear map $\mathfrak{a}_g$ from $\mathbb{R}^{\mathcal{T}}$ into itself.

Lemma 4.2 (Coercivity). *Assume that the flux φ satisfies $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and let $\mathcal{T}$ be a mesh on Ω. There exists $C > 0$ depending on C_1, C_2 and $\mathrm{reg}(\mathcal{T})$ such that for any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $f \in L^{p'}(\Omega)$ and any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we have*

$$\begin{aligned}\left[\!\!\left[\mathfrak{a}_g(u^{\mathcal{T}}) - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}}_m \mathcal{R}(g)\right]\!\!\right] &\geq C_1\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^p_{L^p}\\ &\quad - C\left(\|g\|^p_{W^{1-\frac{1}{p},p}} + \|f\|^{p'}_{L^{p'}} + \|b_1\|_{L^1} + \|b_2\|^{p'}_{L^{p'}}\right).\end{aligned}$$

Proof. By Lemma 4.1, we have for any $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$

$$\begin{aligned}\left[\!\!\left[\mathfrak{a}_g(u^{\mathcal{T}}) - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, u^{\mathcal{T}} - v^{\mathcal{T}}\right]\!\!\right] &= \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}), \nabla^{\mathcal{D}}_0(u^{\mathcal{T}} - v^{\mathcal{T}})\right) - \left[\!\!\left[\widetilde{\mathbb{P}}^{\mathcal{T}}_m f, u^{\mathcal{T}} - v^{\mathcal{T}}\right]\!\!\right]\\ &\geq \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\right) + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}v^{\mathcal{T}}\right)\\ &\quad - \|\widetilde{\mathbb{P}}^{\mathfrak{M}}_m f\|_{L^{p'}}\|u^{\mathfrak{M}} - v^{\mathfrak{M}}\|_{L^p} - \|\widetilde{\mathbb{P}}^{\mathfrak{M}^*}_m f\|_{L^{p'}}\|u^{\mathfrak{M}^*} - v^{\mathfrak{M}^*}\|_{L^p}.\end{aligned}$$

We derive thanks to assumptions $(\mathcal{H}_2)$,$(\mathcal{H}_3)$ and to the inequality (3.7)

$$\begin{aligned}&\left[\!\!\left[\mathfrak{a}_g(u^{\mathcal{T}}) - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, u^{\mathcal{T}} - v^{\mathcal{T}}\right]\!\!\right] \geq C_1\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^p_{L^p} - \|b_1\|_{L^1}\\ &-C\left(\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^{p-1}_{L^p} + \|b_2\|_{L^{p'}}\right)\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}v^{\mathcal{T}}\|_{L^p} - C\|f\|_{L^{p'}}(\|u^{\mathfrak{M}} - v^{\mathfrak{M}}\|_{L^p} + \|u^{\mathfrak{M}^*} - v^{\mathfrak{M}^*}\|_{L^p}).\end{aligned}$$

Using the Young inequality and the discrete Poincaré inequality (Lemma 3.3 applied to $u^{\mathcal{T}} - v^{\mathcal{T}}$ and $g = 0$) we deduce

$$\left[\!\!\left[\mathfrak{a}_g(u^{\mathcal{T}}) - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, u^{\mathcal{T}} - v^{\mathcal{T}}\right]\!\!\right] \geq C_1\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^p_{L^p} - C\|b_1\|_{L^1} - C\|b_2\|^{p'}_{L^{p'}} - C\|f\|^{p'}_{L^{p'}} - C\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}v^{\mathcal{T}}\|^p_{L^p}.$$

The claim is then proved by taking $v^{\mathcal{T}} = \mathbb{P}^{\mathcal{T}}_m \mathcal{R}(g)$ and by using the continuity of the operator $\mathcal{R}$ given in (3.1) and the estimate (3.8). ∎

Lemma 4.3 (Monotonicity). *Assume that the flux φ satisfies $(\mathcal{H}_1)$. For any mesh $\mathcal{T}$ on Ω and any distinct elements $u^{\mathcal{T}}$ and $v^{\mathcal{T}}$ of $\mathbb{R}^{\mathcal{T}}$, we have*

$$[\![\mathfrak{a}_g(u^{\mathcal{T}}) - \mathfrak{a}_g(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}}]\!] > 0.$$

Proof. By Lemma 4.1, and using (2.10), it follows

$$[\![\mathfrak{a}_g(u^{\mathcal{T}}) - \mathfrak{a}_g(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}}]\!] = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}v^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g}v^{\mathcal{T}}\right).$$

Thus, the claim derives from assumption ($\mathcal{H}_1$). ∎

We can now prove the main result of this section, that is the existence and uniqueness of the approximate solution.

Theorem 4.4. *Assume that the flux φ satisfies ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$). For any $f \in L^{p'}(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and any mesh $\mathcal{T}$ on Ω, the finite volume scheme* (2.15) *admits a unique solution $u^\tau \in \mathbb{R}^{\mathcal{T}}$. Furthermore, there exists $C > 0$ depending only on C_1, C_2 and $\mathrm{reg}(\mathcal{T})$, such that the following estimate holds*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^\tau\|_{L^p} \leq C \left(\|g\|_{W^{1-\frac{1}{p},p}} + \|f\|_{L^{p'}}^{\frac{1}{p-1}} + \|b_1\|_{L^1}^{\frac{1}{p}} + \|b_2\|_{L^{p'}}^{\frac{1}{p-1}} \right). \tag{4.1}$$

Proof. The continuity of the map $u^\tau \mapsto \mathfrak{a}_g(u^\tau)$ follows from (1.2). Thanks to the coercivity property (Lemma 4.2) and the Poincaré inequality (Lemma 3.3) we can use one of the classical consequences of the Brouwer fixed point theorem (see [25]) to obtain the existence of a solution of the scheme. The uniqueness of the solution follows readily from the strict monotonicity of the map $\mathfrak{a}_g$ (Lemma 4.3).

Finally, since $\mathfrak{a}_g(u^\tau) = \widetilde{\mathbb{P}}^{\mathcal{T}}_m f$, the estimate (4.1) is a straightforward consequence of Lemma 4.2. ∎

IV.B The potential case

Let us pay special attention to the case where the flux φ derives from a convex potential Φ:

$$\begin{cases} \varphi(z,\xi) = \nabla_\xi \Phi(z,\xi), & \text{for all } \xi \in \mathbb{R}^2 \text{ and a.e. } z \in \Omega, \\ \Phi(z,0) = 0, & \text{for a.e. } z \in \Omega. \end{cases} \tag{4.2}$$

For instance, the p-laplacian (1.4) derives from $\Phi(z,\xi) = \frac{1}{p}|\xi|^p$, the anisotropic laplacian (1.3) from $\Phi(z,\xi) = \frac{1}{2}(A(z)\xi,\xi)$ for a symmetric matrix A and the general model (1.5) from $\Phi(z,\xi) = \frac{1}{p}k(z)|\xi + F(z)|^p$. Remarking that we have $\varphi_{\mathcal{D}}(\xi) = \nabla_\xi \Phi_{\mathcal{D}}(\xi)$, where $\Phi_{\mathcal{D}}$ is naturally defined by

$$\Phi_{\mathcal{D}}(\xi) = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \Phi(z,\xi)\, dz,$$

we can define on $\mathbb{R}^{\mathcal{T}}$ the discrete energy $J_{g,\mathcal{T}}$ associated to the scheme by:

$$J_{g,\mathcal{T}}(u^\tau) = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \Phi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g} u^\tau) - \left[\!\left[u^\tau, \widetilde{\mathbb{P}}^{\mathcal{T}}_m f \right]\!\right] = \int_\Omega \Phi(z, \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^\tau)\, dz - \left[\!\left[u^\tau, \widetilde{\mathbb{P}}^{\mathcal{T}}_m f \right]\!\right].$$

Proposition 4.5 (Variational structure of the scheme). *Assume that φ has the form* (4.2) *and satisfies* ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$), *then $J_{g,\mathcal{T}}$ is a strictly convex coercive functional. Furthermore, the scheme* (2.15) *is the Euler-Lagrange equation associated to the minimization problem for $J_{g,\mathcal{T}}$. More precisely, we have*

$$(\nabla J_{g,\mathcal{T}}(u^\tau), v^\tau) = \left[\!\left[\mathfrak{a}_g(u^\tau) - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f, v^\tau \right]\!\right], \quad \forall u^\tau, v^\tau \in \mathbb{R}^{\mathcal{T}}.$$

The proof is straightforward using Lemma 4.1.

Corollary 4.6. *Under assumptions* ($\mathcal{H}_1$),($\mathcal{H}_2$), ($\mathcal{H}_3$) *and* (4.2), *the solution $u^\tau \in \mathbb{R}^{\mathcal{T}}$ of the scheme* (2.15) *is the unique minimizer of the functional $J_{g,\mathcal{T}}$ on the set $\mathbb{R}^{\mathcal{T}}$.*

The practical computation of the approximate solution can take advantage of the particular structure (4.2), for instance, by using the Polak-Ribière nonlinear conjugate gradient methods. In fact, for the computations shown in Section VIII., we used a similar saddle-point penalized formulation of the discrete problem to the one proposed by Glowinski and Marrocco in [21] for the $P1$ finite element approximation of the p-laplacian. This formulation allows the computation of the minimizer of $J_{g,\mathcal{T}}$ through a lagrangian algorithm which appears to be much more efficient than nonlinear conjugate gradient methods.

V. CONVERGENCE OF THE SCHEME

The aim of this section is to prove the convergence of the solution of the finite volume scheme given by Theorem 4.4 towards the solution u_e to the continuous problem (1.1). More exactly, we prove the strong convergence of both "components" $u^{\mathfrak{M}}, u^{\mathfrak{M}^*}$ of u^{τ} to the approximate solution in $L^p(\Omega)$, the strong convergence of the discrete gradients towards ∇u_e in $(L^p(\Omega))^2$, and the strong convergence of the discrete fluxes towards $\varphi(\cdot, \nabla u_e)$ in $(L^{p'}(\Omega))^2$. This last convergence is crucial in the applications since the flux $\varphi(\cdot, \nabla u_e)$ is often an important physical quantity that one may want to compute precisely. For instance, in the context of the modelling of non-newtonian flows in a porous medium, this flux is nothing but the velocity of the fluid.

Theorem 5.1. *Assume that the flux φ satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and consider a family of meshes $\mathcal{T}_n$ such that $\mathrm{size}(\mathcal{T}_n)$ tends to zero and $\mathrm{reg}(\mathcal{T}_n)$ is bounded. For any $f \in L^{p'}(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$, the solution $u^{\mathcal{T}_n}$ to the scheme (2.15) on the mesh $\mathcal{T}_n$ converges towards the solution u_e of the problem (1.1) as n goes to infinity. More precisely, if we note to simplify $g_n = \mathbb{P}_m^{\mathcal{T}_n} g$, we have*

$$\begin{cases} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} u_e \text{ strongly in } L^p(\Omega), \\ \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} \nabla u_e \text{ strongly in } (L^p(\Omega))^2 \\ \varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) \xrightarrow[n\to\infty]{} \varphi(\cdot, \nabla u_e) \text{ strongly in } \left(L^{p'}(\Omega)\right)^2. \end{cases} \tag{5.1}$$

Moreover, the two sequences $(u^{\mathfrak{M}_n})_n$ and $(u^{\mathfrak{M}_n^*})_n$ both converge towards u_e in $L^p(\Omega)$.

Proof. As usual (see for instance [6, 7, 25]) the key-point of the proof is to take advantage of the monotonicity properties in order to pass to the limit in the nonlinear terms (this is known in the literature as the *Minty-Browder argument*, see [25]).

1. Using the estimate (4.1) and thanks to assumption $(\mathcal{H}_3)$, we see that the families of functions $u^{\mathcal{T}_n}$, $\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}}$ and $z \to \varphi(z, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}(z))$ are bounded in $L^p(\Omega)$, $(L^p(\Omega))^2$, $(L^{p'}(\Omega))^2$ respectively. Hence by the discrete compactness result of Lemma 3.8, there exists a function $u \in W_g^{1,p}(\Omega)$ such that up to a subsequence,

$$u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} u \text{ in } L^p(\Omega),$$
$$\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n} \rightharpoonup_{n\to\infty} \nabla u \text{ weakly in } (L^p(\Omega))^2,$$

and a function $\zeta \in \left(L^{p'}(\Omega)\right)^2$ such that

$$\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}) \rightharpoonup_{n\to\infty} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2. \tag{5.2}$$

Let $w \in \mathcal{C}_c^\infty(\Omega)$ and take $\mathbb{P}_m^{\mathcal{T}_n} w$ as a discrete test function in the scheme (2.15). By Lemma 4.1 it follows

$$0 = \left[\!\left[\mathfrak{a}_{g_n}(u^{\mathcal{T}_n}) - \widetilde{\mathbb{P}}_m^{\mathcal{T}_n} f, \mathbb{P}_m^{\mathcal{T}_n} w \right]\!\right] = \int_\Omega \Big(\varphi(z, \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}(z)), \nabla_0^{\mathcal{T}} \mathbb{P}_m^{\mathcal{T}_n} w(z) \Big)\, dz - \left[\!\left[\widetilde{\mathbb{P}}_m^{\mathcal{T}_n} f, \mathbb{P}_m^{\mathcal{T}_n} w \right]\!\right].$$

We can pass to the limit in this equality using (5.2) and Corollary 3.7. We get

$$\int_\Omega f(z) w(z)\, dz = \int_\Omega \left(\zeta(z), \nabla w(z) \right)\, dz. \tag{5.3}$$

By density, we deduce that for any function $v \in W_g^{1,p}(\Omega)$, we have

$$\int_\Omega f(z)(u(z) - v(z))\, dz = \int_\Omega \left(\zeta(z), \nabla u(z) - \nabla v(z) \right)\, dz. \tag{5.4}$$

2. Thanks to the monotonicity of the scheme (Lemma 4.3), we have

$$[\![\mathfrak{a}_{g_n}(u^{\mathcal{T}_n}) - \mathfrak{a}_{g_n}(\mathbb{P}_m^{\mathcal{T}_n} v), u^{\mathcal{T}_n} - \mathbb{P}_m^{\mathcal{T}_n} v]\!] \geq 0. \tag{5.5}$$

Let us pass to the limit as $n \to \infty$, in this inequality. First, using the definition of the scheme (2.15) and Corollary 3.7, we find

$$[\![\mathfrak{a}_{g_n}(u^{\mathcal{T}_n}), u^{\mathcal{T}_n} - \mathbb{P}_m^{\mathcal{T}_n} v]\!] = \left[\!\left[\widetilde{\mathbb{P}}_m^{\mathcal{T}_n} f, u^{\mathcal{T}_n} - \mathbb{P}_m^{\mathcal{T}_n} v \right]\!\right] \xrightarrow[n\to\infty]{} \int_\Omega f(z)(u(z) - v(z))\, dz.$$

Using Lemma 4.1 and (2.12), we can write

$$[\![\mathfrak{a}_{g_n}(\mathbb{P}_m^{\mathcal{T}_n} v), u^{\mathcal{T}_n} - \mathbb{P}_m^{\mathcal{T}_n} v]\!] = \int_\Omega \Big(\varphi(z, \nabla_{g_n}^{\mathcal{T}_n} \mathbb{P}_m^{\mathcal{T}_n} v(z)), \nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}(z) - \nabla_{g_n}^{\mathcal{T}_n} \mathbb{P}_m^{\mathcal{T}_n} v(z) \Big)\, dz.$$

Using Corollary 3.7 and the property (1.2), we see that the function $\varphi(\cdot, \nabla_{g_n}^{\mathcal{T}_n} \mathbb{P}_m^{\mathcal{T}_n} v)$ converges strongly in $(L^{p'}(\Omega))^2$ towards the function $\varphi(\cdot, \nabla v)$. As a consequence, from the weak convergence of $\nabla_{g_n}^{\mathcal{T}_n} u^{\mathcal{T}_n}$ towards ∇u it follows that $[\![\mathfrak{a}_{g_n}(\mathbb{P}_m^{\mathcal{T}_n} v), u^{\mathcal{T}_n} - \mathbb{P}_m^{\mathcal{T}_n} v]\!]$ converges to the integral $\int_\Omega \Big(\varphi(z, \nabla v(z)), \nabla u(z) - \nabla v(z) \Big)\, dz$. Hence, taking the limit as n goes to infinity in (5.5) gives

$$\int_\Omega f(z)(u(z) - v(z))\, dz - \int_\Omega \Big(\varphi(z, \nabla v(z)), \nabla u(z) - \nabla v(z) \Big)\, dz \geq 0,$$

for all functions $v \in W_g^{1,p}(\Omega)$. By (5.4) it follows that

$$\int_\Omega \Big(\zeta(z) - \varphi(z, \nabla v(z)), \nabla u(z) - \nabla v(z) \Big)\, dz \geq 0. \tag{5.6}$$

3. Let us take in (5.6) $v = u \pm tw$ with $w \in \mathcal{C}_c^\infty(\Omega)$ and $t > 0$, dividing by t we get

$$\pm \int_\Omega \Big(\zeta(z) - \varphi(z, \nabla u \pm t \nabla w), \nabla w(z) \Big)\, dz \geq 0.$$

When t tends to zero, using (1.2) we obtain that for all $w \in \mathcal{C}_c^\infty(\Omega)$

$$\int_\Omega \Big(\zeta(z) - \varphi(z, \nabla u), \nabla w(z) \Big)\, dz = 0.$$

We conclude using (5.3) that $\operatorname{div} \varphi(\cdot, \nabla u) = \operatorname{div} \zeta = -f$. Thus $u \in W^{1,p}_g(\Omega)$ is nothing but the unique solution u_e of the problem (1.1). Finally, the uniqueness of u_e also guarantees that the convergence of $u^{\mathcal{T}_n}$ towards u_e, in the sense of (5.1), holds without extracting a subsequence.

4. Let us show the strong convergence properties of the discrete gradients following the techniques developed in [7, 6]. Let us note $G_n(z) \overset{\text{def}}{=} \nabla^{\mathcal{T}_n}_{g_n} u^{\mathcal{T}_n}(z)$, $H_n(z) \overset{\text{def}}{=} \nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}^{\mathcal{T}_n}_m u_e(z)$ and $\Psi_n(z) \overset{\text{def}}{=} (\varphi(z, G_n(z)) - \varphi(z, H_n(z)), G_n(z) - H_n(z))$. By assumption $(\mathcal{H}_1)$, we know that $\Psi_n \geq 0$. Furthermore, the first part of the proof above shows that the left-hand side term in (5.5) tends to zero which reads, by Lemma 4.1, as

$$\int_\Omega \Psi_n(z)\, dz \xrightarrow[n\to\infty]{} 0. \tag{5.7}$$

Hence, $(\Psi_n)_n$ tends to 0 in $L^1(\Omega)$. Furthermore, by Corollary 3.7, $(H_n)_n$ converges towards ∇u_e in $(L^p(\Omega))^2$. Thus, there exists a set $E \subset \Omega$ such that $\Omega \backslash E$ has a zero Lebesgue measure and a subsequence, always indexed by n, such that $\Psi_n(z) \to 0$ and $H_n(z) \to \nabla u_e(z)$ for any $z \in E$. We can also assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold for any $z \in E$.

First of all, using $(\mathcal{H}_3)$ and $(\mathcal{H}_1)$ we have

$$\Psi_n(z) \geq \frac{C_1}{2}|G_n(z)|^p - C|H_n(z)|^p - b_1(z) - 2b_2(z)(|H_n(z)| + |G_n(z)|),\ \ \forall z \in E. \tag{5.8}$$

For $z \in E$ fixed, the sequence $(H_n(z))_n$ is bounded and $(\Psi_n(z))_n$ tends to 0. By (5.8), we deduce that $(G_n(z))_n$ is a bounded sequence in $\mathbb{R}^2$. Moreover, if $\widetilde{G}$ is the limit of any subsequence $(G_{n_k}(z))_k$, we have

$$0 = \lim_{k\to\infty} \Psi_{n_k}(z) = (\varphi(z, \widetilde{G}) - \varphi(z, \nabla u_e(z)), \widetilde{G} - \nabla u_e(z)).$$

Using the monotonicity assumption $(\mathcal{H}_1)$, we deduce that $\widetilde{G} = \nabla u_e(z)$. Thus, we deduce that for any $z \in E$, the whole sequence $(G_n(z))_n$ converges towards $\nabla u_e(z)$ in $\mathbb{R}^2$.

In addition, (5.7) and the already established convergences imply that

$$\int_\Omega (\varphi(z, G_n(z)), G_n(z))\, dz \xrightarrow[n\to\infty]{} \int_\Omega (\varphi(z, \nabla u_e(z)), \nabla u_e(z))\, dz.$$

Furthermore, by $(\mathcal{H}_2)$, for all $n \in \mathbb{N}$ we have $(\varphi(\cdot, G_n(\cdot)), G_n(\cdot)) \geq -b_1(\cdot) \in L^1(\Omega)$. As in [6, Lemma 5], together with the a.e. convergence of G_n this implies the strong $L^1(\Omega)$ convergence of the sequence $\big(\varphi(\cdot, G_n(\cdot)), G_n(\cdot)\big)_n$. The coercivity assumption $(\mathcal{H}_2)$ implies the equi-integrability of the sequence $(|G_n|^p)_n$. Since we have already proved that $(G_n)_n$ weakly converges towards ∇u_e in $(L^p(\Omega))^2$, using the Vitali theorem we deduce that the sequence $(G_n)_n$ (which is in fact a subsequence of the initial sequence) strongly converges towards ∇u_e in $(L^p(\Omega))^2$. At the present stage, we have proved that $(G_n)_n$ is relatively compact in the strong topology of $(L^p(\Omega))^2$ and that ∇u_e is its unique accumulation point. Thus, the whole sequence $(G_n)_n$ converges strongly in $(L^p(\Omega))^2$ towards ∇u_e.

Finally, the strong convergence of the fluxes $\varphi(\,\cdot\,, G_n(\,\cdot\,))$ towards $\varphi(\,\cdot\,, \nabla u_e(\,\cdot\,))$ comes from the property (1.2).

5. We now have $\nabla^{\mathcal{T}_n}_0(u^{\mathcal{T}_n} - \mathbb{P}^{\mathcal{T}_n}_m u_e) = \nabla^{\mathcal{T}_n}_{g_n} u^{\mathcal{T}_n} - \nabla^{\mathcal{T}_n}_{g_n} \mathbb{P}^{\mathcal{T}_n}_m u_e \to 0$ in $(L^p(\Omega))^2$ as $n \to \infty$. Thanks to the discrete Poincaré inequality (Lemma 3.3) we deduce that the two sequences $(u^{\mathfrak{M}_n} - \mathbb{P}^{\mathfrak{M}_n}_m u_e)_n$ and $(u^{\mathfrak{M}^*_n} - \mathbb{P}^{\mathfrak{M}^*_n}_m u_e)_n$ tend to zero in $L^p(\Omega)$. The last claim of the theorem follows by using Corollary 3.7 with $v = u_e$.

■

Note that the proof of the strong convergence of discrete gradients and fluxes can be notably simplified, if a stronger monotonicity assumption for the flux φ is assumed:

- If $1 < p \leq 2$, there exist $C_3 > 0$ and $b_3 \in L^1(\Omega)$ such that for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ and almost every $z \in \Omega$,

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq C_3 |\xi - \eta|^2 (b_3(z) + |\xi|^p + |\eta|^p)^{\frac{p-2}{p}}. \qquad (\mathcal{H}_{1'a})$$

- If $p > 2$, there exists $C_3 > 0$ such that for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ and almost every $z \in \Omega$,

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq C_3 |\xi - \eta|^p. \qquad (\mathcal{H}_{1'b})$$

These assumptions express some kind of Hölder continuity of the inverse $[\varphi(z,\cdot)]^{-1}$ of the flux φ; they will be needed for our subsequent results on stability of discrete solutions and on error estimates.

Notice that most of the usual examples (like those given in Section I.B) satisfy these stronger assumptions (see Lemma 1.3 and [5, 8, 26]).

VI. STABILITY WITH RESPECT TO THE DATA

In this section we address the problem of the continuous dependence of the approximate solution with respect to the data. More precisely, we show that, as for the continuous problem (1.1), the discrete gradient of the solution to the finite volume scheme is Hölder continuous with respect to the source term f and the boundary data g uniformly with respect to the mesh. This property is important because it ensures, for instance, that the numerical method is stable with respect to the fully practical computation of the discretization of the data through quadrature formulae. Notice that the computations below will also be useful in the proof of the error estimate theorem in section VII..

From now on, we need to assume some kind of Hölder regularity with respect to ξ for the flux $\varphi(z,\xi)$. More precisely, we consider the following assumptions:

- If $1 < p \leq 2$, there exists $C_4 > 0$ such that for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ and almost every $z \in \Omega$,

$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_4 |\xi - \eta|^{p-1}. \qquad (\mathcal{H}_{4a})$$

- If $p > 2$, there exist $C_4 > 0$ and $b_4 \in L^{\frac{p}{p-2}}(\Omega)$ such that for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$, and almost every $z \in \Omega$,

$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_4 \left(b_4(z) + |\xi|^{p-2} + |\eta|^{p-2} \right) |\xi - \eta|. \qquad (\mathcal{H}_{4b})$$

We want to point out that, once more, these assumptions are classical in this context and are satisfied for the various examples given in Section I.B (see Lemma 1.3 and [8, 12]). Furthermore, these new assumptions do not involve regularity of φ with respect to the space variable z. This allows, for instance, the presence of spatial discontinuities in the coefficients of the problem we are studying.

Proposition 6.1. *Let $\mathcal{T}$ be a mesh on Ω and $f_1, f_2 \in L^{p'}(\Omega)$, $g_1, g_2 \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Under assumptions ($\mathcal{H}_2$), ($\mathcal{H}_3$), ($\mathcal{H}_{1'a}$)-($\mathcal{H}_{1'b}$) and ($\mathcal{H}_{4a}$)-($\mathcal{H}_{4b}$), if $u_1^{\mathcal{T}}$ and $u_2^{\mathcal{T}}$ are the solutions of the* *scheme* (2.15) *corresponding respectively to the data* $\mathbb{P}_m^{\mathcal{T}}g_1, \mathbb{P}_m^{\mathcal{T}}f_1$ *and* $\mathbb{P}_m^{\mathcal{T}}g_2, \mathbb{P}_m^{\mathcal{T}}f_2$*, then we have*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}\|_{L^p} \le C\Bigg(M^{2-p}\|f_1-f_2\|_{L^{p'}} + M^{\frac{2-p}{3-p}}\|g_1-g_2\|^{\frac{1}{3-p}}_{W^{1-\frac1p,p}} + M^{\frac{2-p}{2}}\|f_1-f_2\|^{\frac12}_{L^{p'}}\|g_1-g_2\|^{\frac12}_{W^{1-\frac1p,p}}\Bigg), \ \text{if } 1<p<2,$$

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}\|_{L^p} \le C\Bigg(\|f_1-f_2\|^{\frac{1}{p-1}}_{L^{p'}} + \|g_1-g_2\|_{W^{1-\frac1p,p}} + M^{\frac{p-2}{p-1}}\|g_1-g_2\|^{\frac{1}{p-1}}_{W^{1-\frac1p,p}}\Bigg), \ \text{if } p>2,$$

where C *depends only on* $\mathrm{reg}(\mathcal{T})$, $(C_i)_{1\le i\le 4}$, $(b_i)_{1\le i\le 4}$*, and* M *is defined by*

$$M = C + \|g_1\|_{W^{1-\frac1p,p}} + \|g_2\|_{W^{1-\frac1p,p}} + \|f_1\|^{\frac{1}{p-1}}_{L^{p'}} + \|f_2\|^{\frac{1}{p-1}}_{L^{p'}}.$$

Proof. Let us introduce $v_1^{\mathcal{T}} = \mathbb{P}_m^{\mathcal{T}}\mathcal{R}(g_1)$ and $v_2^{\mathcal{T}} = \mathbb{P}_m^{\mathcal{T}}\mathcal{R}(g_2)$, $\mathcal{R}$ being the lift operator (see (3.1)). Testing the two schemes with $u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}}$, we obtain by Lemma 4.1

$$\begin{aligned} 0 &= [\![\mathfrak{a}_{g_1}(u_1^{\mathcal{T}}) - \mathfrak{a}_{g_2}(u_2^{\mathcal{T}}), u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}}]\!] - \Big[\!\!\Big[\widetilde{\mathbb{P}}_m^{\mathcal{T}}f_1 - \widetilde{\mathbb{P}}_m^{\mathcal{T}}f_2, u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}}\Big]\!\!\Big], \\ &= \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}), \nabla_0^{\mathcal{D}}(u_1^{\mathcal{T}} - v_1^{\mathcal{T}}) - \nabla_0^{\mathcal{D}}(u_2^{\mathcal{T}} - v_2^{\mathcal{T}})\right) \\ &\qquad - \Big[\!\!\Big[\widetilde{\mathbb{P}}_m^{\mathcal{T}}f_1 - \widetilde{\mathbb{P}}_m^{\mathcal{T}}f_2, u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}}\Big]\!\!\Big]. \end{aligned}$$

Using (2.10), we obtain

$$\begin{aligned} &\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}\right) \\ &\quad = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}v_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}v_2^{\mathcal{T}}\right) \\ &\qquad + \Big[\!\!\Big[\widetilde{\mathbb{P}}_m^{\mathcal{T}}f_1 - \widetilde{\mathbb{P}}_m^{\mathcal{T}}f_2, u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}}\Big]\!\!\Big]. \end{aligned} \tag{6.1}$$

Case $1<p\le 2$**.** Assumption $(\mathcal{H}_{1'a})$ gives

$$\begin{aligned} &\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}\right) \\ &\quad = \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} \left(\varphi(z,\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}) - \varphi(z,\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}\right) dz \\ &\quad = \int_{\Omega} \left(\varphi(z,\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}(z)) - \varphi(z,\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}(z)), \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}(z) - \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}(z)\right) dz \\ &\ge \frac{1}{C}\int_{\Omega}\left(b_3(z) + |\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}(z)|^p + |\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}(z)|^p\right)^{\frac{p-2}{p}} |\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_1}u_1^{\mathcal{T}}(z) - \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}}g_2}u_2^{\mathcal{T}}(z)|^2\,dz. \end{aligned}$$

By the Hölder inequality, we get

$$
\begin{aligned}
&\int_\Omega |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p \, dz \\
&\quad = \int_\Omega \frac{|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p}{\left(b_3 + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}|^p + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p\right)^{\frac{2-p}{2}}} \left(b_3(z) + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}|^p + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p\right)^{\frac{2-p}{2}} \, dz \\
&\quad \le \left(\int_\Omega |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^2 \left(b_3 + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}|^p + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p\right)^{\frac{p-2}{p}} \, dz \right)^{\frac{p}{2}} \\
&\qquad \times \left(\int_\Omega b_3(z) + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}|^p + |\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^p \, dz \right)^{\frac{2-p}{2}} .
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^2_{L^p} &\le C \left(\|b_3\|_{L^1}^{\frac{2-p}{p}} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}\|^{2-p}_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^{2-p}_{L^p} \right) \\
&\times \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}} \right) \right). \quad (6.2)
\end{aligned}
$$

Thanks to assumption ($\mathcal{H}_{4a}$), (6.1) gives

$$
\begin{aligned}
&\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \big(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}} \big) \\
&\qquad \le C \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^{p-1}_{L^p} \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} v_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} v_2^{\mathcal{T}}\|_{L^p} \\
&+ C \left(\|v_1^{\mathfrak{M}} - v_2^{\mathfrak{M}}\|_{L^p} + \|v_1^{\mathfrak{M}^*} - v_2^{\mathfrak{M}^*}\|_{L^p} + \|u_1^{\mathfrak{M}} - u_2^{\mathfrak{M}}\|_{L^p} + \|u_1^{\mathfrak{M}^*} - u_2^{\mathfrak{M}^*}\|_{L^p} \right) \|f_1 - f_2\|_{L^{p'}} .
\end{aligned}
$$

Combining the last two inequalities and using the Poincaré inequality, we get the result.

Case $p > 2$**.** Using assumption ($\mathcal{H}_{1'b}$), we have

$$
\begin{aligned}
&\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}) - \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}} \right) \\
&\qquad = \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} \left(\varphi(z, \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}) - \varphi(z, \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}), \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}} \right) dz \\
&\qquad \ge \frac{1}{C} \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left| \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}} \right|^p dz = \frac{1}{C} \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^p_{L^p}. \quad (6.3)
\end{aligned}
$$

Denote by $b_4^{\mathcal{D}}$ the mean value of b_4 on $\mathcal{D}$. By ($\mathcal{H}_{4b}$) and the Young inequality, (6.1) implies

$$\begin{aligned}
\frac{1}{C}\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^p_{L^p} \le \sum_{\mathcal{D}\in\mathfrak{D}} & m_{\mathcal{D}} |\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}||\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} v_1^{\mathcal{T}} - \nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} v_2^{\mathcal{T}}| \\
& \times \left(b_4^{\mathcal{D}} + |\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}|^{p-2} + |\nabla^{\mathcal{D}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}|^{p-2}\right) \\
& - \left| \left[\!\left[\widetilde{\mathbb{P}}^{\mathcal{T}}_m f_1 - \widetilde{\mathbb{P}}^{\mathcal{T}}_m f_2, u_1^{\mathcal{T}} - v_1^{\mathcal{T}} - u_2^{\mathcal{T}} + v_2^{\mathcal{T}} \right]\!\right] \right| \\
\le & \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|_{L^p} \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} v_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} v_2^{\mathcal{T}}\|_{L^p} \\
& \times \left(\|b_4\|_{L^{\frac{p}{p-2}}} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}\|^{p-2}_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^{p-2}_{L^p}\right) \\
& + C\left(\|u_1^{\mathfrak{M}} - u_2^{\mathfrak{M}}\|_{L^p} + \|u_1^{\mathfrak{M}^*} - u_2^{\mathfrak{M}^*}\|_{L^p}\right) \|f_1 - f_2\|_{L^{p'}} \\
& + C\left(\|v_1^{\mathfrak{M}} - v_2^{\mathfrak{M}}\|_{L^p} + \|v_1^{\mathfrak{M}^*} - v_2^{\mathfrak{M}^*}\|_{L^p}\right) \|f_1 - f_2\|_{L^{p'}}.
\end{aligned}$$

The discrete Poincaré inequality and (3.8) then lead to

$$\begin{aligned}
&\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}} - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|_{L^p} \le \\
&\quad C\Bigg(\|g_1 - g_2\|^{\frac{1}{p-1}}_{W^{1-\frac{1}{p},p}} \left(\|b_4\|_{L^{\frac{p}{p-2}}} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_1} u_1^{\mathcal{T}}\|^{p-2}_{L^p} + \|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g_2} u_2^{\mathcal{T}}\|^{p-2}_{L^p}\right)^{\frac{1}{p-1}} \\
&\qquad\qquad + \|g_1 - g_2\|_{W^{1-\frac{1}{p},p}} + \|f_1 - f_2\|^{\frac{1}{p-1}}_{L^{p'}} \Bigg).
\end{aligned}$$

The claim follows thanks to the estimate (4.1) applied to $u_1^{\mathcal{T}}$ and $u_2^{\mathcal{T}}$.

∎

VII. ERROR ESTIMATES FOR $W^{2,P}$ SOLUTIONS

We conclude the study of the convergence of the solution to the finite volume scheme (2.15) by providing an error estimate in the case where the exact solution of the problem (1.1) lies in the space $W^{2,p}(\Omega)$ and the flux φ is smooth enough with respect to the spatial variable z. More precisely, we consider in this section the following additional assumptions on φ

- If $1 < p \le 2$, there exist $C_5 > 0$ and $b_5 \in \left(W^{1,p}(\Omega)\right)^2$ such that for all $\xi \in \mathbb{R}^2$ and almost every $(z, z') \in \Omega^2$,

$$|\varphi(z,\xi) - \varphi(z',\xi)| \le C_5(1 + |\xi|^{p-1})|z - z'|^{p-1} + |b_5(z) - b_5(z')|^{p-1}. \qquad (\mathcal{H}_{5a})$$

- If $p > 2$, there exist $C_5 > 0$ and $b_6 \in L^{p'}(\Omega)$ such that for all $\xi \in \mathbb{R}^2$ and almost every $z \in \Omega$,

$$\left|\frac{\partial\varphi}{\partial z}(z,\xi)\right| \le C_5\left(b_6(z) + |\xi|^{p-1}\right). \qquad (\mathcal{H}_{5b})$$

Remark 7.1. *In the following result we will assume that u_e lies in $W^{2,p}(\Omega)$ so that $\nabla u_e \in (W^{1,p}(\Omega))^2$. Hence, using the previous assumptions, we see that the map*

$$z \mapsto \varphi(z, \nabla u_e(z)),$$

lies in $(W^{1,p'}(\Omega))^2$ *if* $p \geq 2$, *and in* $(W^{p-1,p'}(\Omega))^2$ *if* $p \in]1,2[$. *These regularity properties will justify all the computations in the following proof.*

In particular, assumptions $(\mathcal{H}_{5a})$-$(\mathcal{H}_{5b})$ *do not allow to consider non regular data* $f \in W^{-1,p'}(\Omega)$ *through the manipulation of Remark 1.2.*

Let us comment on these assumptions in the case of the examples given in section I.B. For the anisotropic Laplace equation (1.3), assumption $(\mathcal{H}_{5a})$ is fulfilled as soon as the map A is Lipschitz. In the nonlinear example (1.5), the assumptions above are satisfied if the map k is Lipschitz and if the vector-field F lies in $(W^{1,p}(\Omega))^2$.

Our main result is the following.

Theorem 7.2. *Assume that the flux* φ *satisfies not only* $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, *but also* $(\mathcal{H}_{1'a})$-$(\mathcal{H}_{1'b})$, $(\mathcal{H}_{4a})$-$(\mathcal{H}_{4b})$, $(\mathcal{H}_{5a})$-$(\mathcal{H}_{5b})$. *Let* $f \in L^{p'}(\Omega)$ *and assume that the solution* u_e *to* (1.1) *belongs to* $W^{2,p}(\Omega)$, *which implies that* $g \in \widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)$.

Let $\mathcal{T}$ *be a mesh on* Ω. *There exists* $C > 0$ *depending on* $\|u_e\|_{W^{2,p}}$, *on* $\operatorname{reg}(\mathcal{T})$, *on the norms of the functions* $f,\ g,\ (b_i)_{1\leq i\leq 6}, i = 1,\dots,6$ *in their natural spaces and on* $(C_i)_{1\leq i\leq 5}$, *such that*

$$\begin{cases} \|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq C \operatorname{size}(\mathcal{T})^{p-1}, & \text{if } 1 \leq p \leq 2, \\ \|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq C \operatorname{size}(\mathcal{T})^{\frac{1}{p-1}}, & \text{if } p > 2. \end{cases} \tag{7.1}$$

Recall that J.W. Barrett and W.B. Liu proved in [5], in the particular case of the p-laplacian on a convex domain Ω, that if $f \in L^{p'}(\Omega)$ and if $1 < p \leq 2$, then u_e belongs to $H^2(\Omega)$ and then to $W^{2,p}(\Omega)$, so that the assumption in the previous theorem is fulfilled. On the other hand, when $p > 2$, there exist solutions of (1.1) with $f \in L^{p'}(\Omega)$ which are not in $W^{2,p}(\Omega)$ but in Besov space $B_\infty^{1+\frac{1}{p-1},p}(\Omega)$. In this last case, optimal error estimates were obtained in [2], in the framework of cartesian meshes.

VII.A Center-value projection of continuous functions

In the proof of the convergence result (Theorems 5.1) we have shown that the difference of the discrete gradient of the approximate solution and the discrete gradient of the mean-value projection of the exact solution $\mathbb{P}^{\mathcal{T}}_m u_e$ tends to 0. Using the properties of the mean-value projection of any function in $W^{1,p}(\Omega)$ (Corollary 3.7), we were able to conclude to the convergence of the discrete gradient towards the exact gradient ∇u_e.

We are now in the case where u_e is assumed to be in $W^{2,p}(\Omega)$, in particular, u_e is Hölder continuous. Hence, it is possible to define a more natural projection of this function on $\mathbb{R}^{\mathcal{T}}$ by simply taking the values of the function at the control points $(x_\mathcal{K})$ and $(x_{\mathcal{K}^*})$. This choice appears to be well adapted to the computations below. Let us state some of the properties of this new projection operator.

Definition 7.3 (Center-value projection on the mesh $\mathcal{T}$). *For any continuous function* v *on* $\overline{\Omega}$, *set*

$$\mathbb{P}^{\mathcal{T}}_c v = \left((v(x_\mathcal{K}))_{\mathcal{K}\in\mathfrak{M}}, (v(x_{\mathcal{K}^*}))_{\mathcal{K}^*\in\mathfrak{M}^*} \right).$$

We call $\mathbb{P}^{\mathcal{T}}_c$ *the* **center-value projection** *of* v *on the space* $\mathbb{R}^{\mathcal{T}}$ *of discrete functions.*

In the same way, any $g \in \widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)$ is Hölder continuous and we can consider its central-value discretization on the boundary $\mathbb{P}^{\mathcal{T}}_c g = ((g_\mathcal{K}), (g_{\mathcal{K}^*}))$ to be defined by

$$g_\mathcal{K} = g(x_\mathcal{K}),\ \ \forall \mathcal{K} \in \partial\mathfrak{M},\ \ g_{\mathcal{K}^*} = g(x_{\mathcal{K}^*}),\ \ \forall \mathcal{K}^* \in \partial\mathfrak{M}^*. \tag{7.2}$$

VII.A1 Discrete boundary data As stated before, we use $\mathbb{P}_c^{\mathcal{T}} u_e$ to compute the consistency error of our scheme. Hence, since the boundary data g enters the scheme through its mean-value projection $\mathbb{P}_m^{\mathcal{T}} g$, it is needed to evaluate the contribution in the error of the difference $\mathbb{P}_c^{\mathcal{T}} g - \mathbb{P}_m^{\mathcal{T}} g$ between the two possible discretizations of the boundary data.

Lemma 7.4. *Let $\mathcal{T}$ be a mesh on Ω.*

1. *For any $p > 2$, there exists C depending on p and $\operatorname{reg}(\mathcal{T})$ such that for any $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we have*
$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g - \mathbb{P}_c^{\mathcal{T}} g} 0^{\mathcal{T}}\|_{L^p} \leq C\|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}. \tag{7.3}$$
2. *For any $p > 1$, there exists C depending on p and $\operatorname{reg}(\mathcal{T})$ such that for any $g \in \widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)$ we have*
$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g - \mathbb{P}_c^{\mathcal{T}} g} 0^{\mathcal{T}}\|_{L^p} \leq C\operatorname{size}(\mathcal{T})\|g\|_{\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)}. \tag{7.4}$$

Proof. We want to estimate $G^{\mathcal{D}} \stackrel{\text{def}}{=} \nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}} g - \mathbb{P}_c^{\mathcal{T}} g} 0^{\mathcal{T}}$ for any diamond cell $\mathcal{D}$ near the boundary of Ω since in the other cases this term is zero. Using the definition of $\mathbb{P}_m^{\mathcal{T}} g$ given in (2.2), (2.3) and the one of $\mathbb{P}_c^{\mathcal{T}} g$ given in (7.2), we see that we have two kinds of terms to estimate in each diamond cell:

Term along the direction of τ^*. If $\kappa \subset \partial\mathfrak{M} \cap \mathcal{D}$ (case I in Figure 5), we have
$$\left(G^{\mathcal{D}}, \tau^*\right) = -\frac{1}{m_{\sigma^*}}\left(g(x_{\mathcal{K}}) - \frac{1}{m_{\sigma_{\mathcal{K}}}}\int_{\sigma_{\mathcal{K}}} g(s)\,ds\right), \tag{7.5}$$
whereas in the case II in Figure 5, this term is zero.

Term along the direction of τ. Two situations may occur a shown in Figure 5. In the case II, we have
$$\left(G^{\mathcal{D}}, \tau\right) = -\frac{1}{m_{\sigma}}\left(g(x_{\mathcal{L}^*}) - \frac{1}{m_{\sigma_{\mathcal{L}^*}}}\int_{\sigma_{\mathcal{L}^*}} g(s)\,ds\right), \tag{7.6}$$
and in the case I,
$$\left(G^{\mathcal{D}}, \tau\right) = \frac{1}{m_{\sigma}}\left(\left[g(x_{\mathcal{K}^*}) - \frac{1}{m_{\sigma_{\mathcal{K}^*}}}\int_{\sigma_{\mathcal{K}^*}} g(s)\,ds\right] - \left[g(x_{\mathcal{L}^*}) - \frac{1}{m_{\sigma_{\mathcal{L}^*}}}\int_{\sigma_{\mathcal{L}^*}} g(s)\,ds\right]\right). \tag{7.7}$$

1. The first point is a consequence of the embedding of $W^{1-\frac{1}{p},p}(\partial\Omega)$ in the Hölder class $\mathcal{C}^{0,1-\frac{2}{p}}(\partial\Omega)$. Indeed, each of the terms in (7.5)-(7.7) can be treated in the same way. For instance, by Lemma 3.2, the term (7.5) is bounded as follows
$$\begin{aligned} m_{\mathcal{D}}\left|\left(G^{\mathcal{D}}, \tau^*\right)\right|^p &\leq \frac{m_{\mathcal{D}}}{m_{\sigma^*}^p}\sup_{z,z'\in\sigma_{\mathcal{K}}}|g(z)-g(z')|^p \\ &\leq \frac{m_{\mathcal{D}} m_{\sigma_{\mathcal{K}}}^{p-2}}{m_{\sigma^*}^p}\int_{\sigma_{\mathcal{K}}}\int_{\sigma_{\mathcal{K}}}\left|\frac{|g(x)-g(y)|}{|x-y|^{1-\frac{1}{p}}}\right|^p d\lambda(x)\,d\lambda(y) \\ &\leq C(\operatorname{reg}(\mathcal{T}))\int_{\sigma_{\mathcal{K}}}\int_{\sigma_{\mathcal{K}}}\left|\frac{|g(x)-g(y)|}{|x-y|^{1-\frac{1}{p}}}\right|^p d\lambda(x)\,d\lambda(y). \end{aligned}$$

Summing over the boundary diamond cells, we get the estimate (7.3).

2. Let us prove the second point.

- We consider here the terms of the form (7.5). Let us suppose that $\sigma_{\mathcal{K}} =]-h_{\mathcal{K}}, h_{\mathcal{K}}[\times\{0\}$ and $x_{\mathcal{K}} = (0,0)$ (recall that $\sigma_{\mathcal{K}}$ is defined in (2.3) and is chosen in such a way that $x_{\mathcal{K}}$ is located at the middle of the edge $\sigma_{\mathcal{K}}$). We can write

$$m_{\sigma^*}\left(G^{\mathcal{D}},\tau^*\right) = \frac{1}{2h_{\mathcal{K}}}\int_{-h_{\mathcal{K}}}^{h_{\mathcal{K}}}\int_0^1 \nabla_{\mathcal{T}} g(tx)x\,dx\,dt = \int_0^1 \frac{1}{2t^2h_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\nabla_{\mathcal{T}} g(s)s\,ds\,dt.$$

Now since $\displaystyle\int_{-a}^{a}\nabla_{\mathcal{T}} g(y)s\,ds = \nabla_{\mathcal{T}} g(y)\int_{-a}^{a} s\,ds = 0$ for any $a>0$, integrating in $y\in[-th_{\mathcal{K}}, th_{\mathcal{K}}]$ we get

$$\begin{aligned}\frac{1}{2t^2h_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\nabla_{\mathcal{T}} g(s)s\,ds &= \frac{1}{4t^3h_{\mathcal{K}}^2}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}(\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y))s\,ds\,dy\\ &= \frac{1}{4t^3h_{\mathcal{K}}^2}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|}|s-y|s\,ds\,dy.\end{aligned}$$

It follows that

$$\begin{aligned}&\left|\left(G^{\mathcal{D}},\tau^*\right)\right|^p\\ &\le \int_0^1\frac{1}{(tm_{\sigma^*})^p}\left(\frac{1}{(2th_{\mathcal{K}})^2}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\frac{|\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)|}{|s-y|}|s-y||s|\,ds\,dy\right)^p dt\\ &\le \int_0^1\frac{1}{(tm_{\sigma^*})^p}\frac{1}{(2th_{\mathcal{K}})^2}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|}\right|^p|s-y|^p|s|^p\,ds\,dy\,dt\\ &\le C\int_0^1|th_{\mathcal{K}}|^{p-2}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\int_{-th_{\mathcal{K}}}^{th_{\mathcal{K}}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|}\right|^p ds\,dy\,dt\\ &\le Ch_{\mathcal{K}}^{p-2}\left(\int_0^1 t^{p-2}\,dt\right)\left(\int_{-h_{\mathcal{K}}}^{h_{\mathcal{K}}}\int_{-h_{\mathcal{K}}}^{h_{\mathcal{K}}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|^{1-\frac1p}}\right|^p\frac{ds\,dy}{|s-y|}\right).\end{aligned}$$

Finally we have

$$\begin{aligned}\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\left|\left(G^{\mathcal{D}},\tau^*\right)\right|^p &\le C\mathrm{size}(\mathcal{T})^p\sum_{\mathcal{K}\in\partial\mathfrak{M}}\left(\int_{\sigma_{\mathcal{K}}}\int_{\sigma_{\mathcal{K}}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|^{1-\frac1p}}\right|^p\frac{ds\,dy}{|s-y|}\right)\\ &\le C\mathrm{size}(\mathcal{T})^p\sum_{i=1}^{k}\int_{\Gamma_i}\int_{\Gamma_i}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|^{1-\frac1p}}\right|^p\frac{ds\,dy}{|s-y|}\\ &\le C\mathrm{size}(\mathcal{T})^p\|g\|^p_{\widetilde{W}^{2-\frac1p,p}(\partial\Omega)}.\end{aligned}$$

- Provided that neither $x_{\mathcal{L}^*}$ nor $x_{\mathcal{K}^*}$ are corners of Ω, we have as previously, in the case (7.6),

$$m_{\mathcal{D}}\left|\left(G^{\mathcal{D}},\tau\right)\right|^p \le C\mathrm{size}(\mathcal{T})^p\int_{\sigma_{\mathcal{L}^*}}\int_{\sigma_{\mathcal{L}^*}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|^{1-\frac1p}}\right|^p\frac{ds\,dy}{|s-y|}$$

and in the case (7.7),

$$m_{\mathcal{D}}\left|\left(G^{\mathcal{D}},\tau\right)\right|^p \le C\mathrm{size}(\mathcal{T})^p\int_{\sigma_{\mathcal{K}^*}\cup\sigma_{\mathcal{L}^*}}\int_{\sigma_{\mathcal{K}^*}\cup\sigma_{\mathcal{L}^*}}\left|\frac{\nabla_{\mathcal{T}} g(s)-\nabla_{\mathcal{T}} g(y)}{|s-y|^{1-\frac1p}}\right|^p\frac{ds\,dy}{|s-y|}.$$

If $x_{\mathcal{L}^*}$, for instance, is a corner of the domain Ω, say the corner between the edges Γ_j and Γ_k, we estimate separately the contributions of $\sigma_{\mathcal{L}^*} \cap \Gamma_j$ and $\sigma_{\mathcal{L}^*} \cap \Gamma_k$. More precisely, for $p < 2$ we use the embedding of $W^{2-\frac{1}{p},p}(\Gamma_i)$ in $\mathcal{C}^{0,2(1-\frac{1}{p})}(\Gamma_i)$ for $i \in \{j,k\}$, to find that

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \left|\left(G^{\mathcal{D}},\tau\right)\right|^p \;\leq\; C\mathrm{size}(\mathcal{T})^p \|g\|^p_{\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)}.$$

In the case $p > 2$, we recall that, thanks to (2.6) and (2.7), $\sigma_{\mathcal{L}^*} \cap \Gamma_j$ and $\sigma_{\mathcal{L}^*} \cap \Gamma_k$ are of size $C\mathrm{size}(\mathcal{T})^{2-\frac{2}{p}}$. We use the embedding of $W^{2-\frac{1}{p},p}(\Gamma_i)$ in $\mathcal{C}^{0,1}(\Gamma_i)$ to conclude.

Finally, in the case $p = 2$ we use the embedding of $H^{\frac{3}{2}}(\Gamma_i)$ in the set of Log-Lipschitz functions and the definition (2.6) of $\mathrm{reg}(\mathcal{T})$ with (2.7).

■

VII.A2 Properties of the center-value projection We sum up in this section the properties of the center-value projection operator which are used in the estimate of the consistency error of our finite volume scheme.

Lemma 7.5 (Center-value projection estimates). *Let $\mathcal{T}$ be a mesh on Ω. There exists a constant $C > 0$, depending only on p and $\mathrm{reg}(\mathcal{T})$, such that for any function v in $W^{1,p}(\Omega)$, denoting by $g = \gamma(v)$ its trace, we have*

1. *For $p \geq 2$,*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g}\mathbb{P}^{\mathcal{T}}_c v\|_{L^p} \leq C\|\nabla v\|_{L^p};$$

2. *If, in addition, $v \in W^{2,p}(\Omega)$,*

$$\|\nabla v - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g}\mathbb{P}^{\mathcal{T}}_c v\|_{L^p} \leq C\mathrm{size}(\mathcal{T})\|\nabla v\|_{W^{1,p}}, \quad \text{for } p > 1, \tag{7.8}$$

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g}\mathbb{P}^{\mathcal{T}}_c v\|_{L^p} \leq C\left(\|\nabla v\|_{L^p} + \mathrm{size}(\mathcal{T})\|\nabla v\|_{W^{1,p}}\right), \quad \text{for } 1 < p < 2. \tag{7.9}$$

Proof.

• Let $\mathcal{D}$ be a diamond cell, we use the notations of Figure 2. For a smooth function v and for all $z \in \mathcal{D}$ we get by first-order Taylor expansion of v,

$$\begin{aligned}&(v(x_{\mathcal{L}}) - v(x_{\mathcal{K}}))\\ &\quad= \left(\int_0^1 \nabla v((1-t)z + tx_{\mathcal{L}})(x_{\mathcal{L}} - z)\,dt - \int_0^1 \nabla v((1-t)z + tx_{\mathcal{K}})(x_{\mathcal{K}} - z)\,dt\right)\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}&(v(x_{\mathcal{L}^*}) - v(x_{\mathcal{K}^*}))\\ &\quad= \left(\int_0^1 \nabla v((1-t)z + tx_{\mathcal{L}^*})(x_{\mathcal{L}^*} - z)\,dt - \int_0^1 \nabla v((1-t)z + tx_{\mathcal{K}^*})(x_{\mathcal{K}^*} - z)\,dt\right).\end{aligned} \tag{7.11}$$

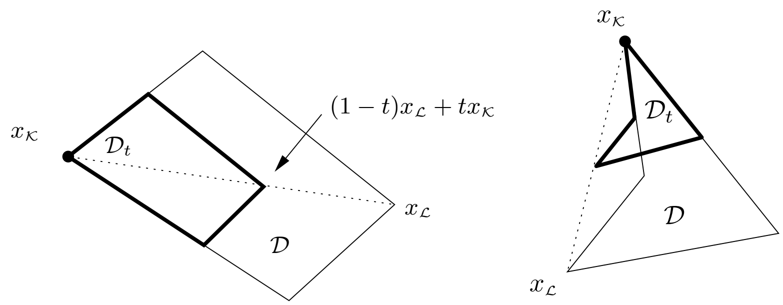

FIG. 6. The rescaled diamond D_t

Using (2.9) and Lemma 2.4, we get

$$\begin{aligned}
&\sin\alpha_{\mathcal{D}}\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}}g}\mathbb{P}_c^{\mathcal{T}}v \\
&\quad=\frac{\boldsymbol{\nu}}{m_{\sigma^*}}\left(\int_0^1\nabla v((1-t)z+tx_{\mathcal{L}})(x_{\mathcal{L}}-z)\,dt-\int_0^1\nabla v((1-t)z+tx_{\mathcal{K}})(x_{\mathcal{K}}-z)\,dt\right)\\
&\quad+\frac{\boldsymbol{\nu}^*}{m_{\sigma}}\left(\int_0^1\nabla v((1-t)z+tx_{\mathcal{L}^*})(x_{\mathcal{L}^*}-z)\,dt-\int_0^1\nabla v((1-t)z+tx_{\mathcal{K}^*})(x_{\mathcal{K}^*}-z)\,dt\right).
\end{aligned}\tag{7.12}$$

Let us define the quantity $I_{\mathcal{K},\sigma^*}$ (and, by appropriate permutations of the subscripts, $I_{\mathcal{L},\sigma^*}$, $I_{\mathcal{K}^*,\sigma}$, $I_{\mathcal{L}^*,\sigma}$) as

$$I_{\mathcal{K},\sigma^*}\stackrel{\text{def}}{=}\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}\left|\frac{1}{m_{\sigma^*}}\int_0^1\nabla v((1-t)z+tx_{\mathcal{K}})(x_{\mathcal{K}}-z)\,dt\right|\,dz.$$

Averaging (7.12) over $\mathcal{D}$ and using (2.6), we obtain

$$|\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}}g}\mathbb{P}_c^{\mathcal{T}}v|\le C(\operatorname{reg}(\mathcal{T}))\Big(I_{\mathcal{K},\sigma^*}+I_{\mathcal{L},\sigma^*}+I_{\mathcal{K}^*,\sigma}+I_{\mathcal{L}^*,\sigma}\Big).$$

Using the change of variables $z\mapsto z'=(1-t)z+tx_{\mathcal{K}}$, we can now bound each of the four terms in this inequality. For instance, we have

$$\begin{aligned}
I_{\mathcal{K},\sigma^*}&\le\frac{1}{m_{\mathcal{D}}m_{\sigma^*}}\int_{\mathcal{D}}\int_0^1|\nabla v((1-t)z+tx_{\mathcal{K}})(x_{\mathcal{K}}-z)|\,dt\,dz,\\
&\le\frac{C}{m_{\mathcal{D}}}\int_0^1\int_{\mathcal{D}}|\nabla v((1-t)z+tx_{\mathcal{K}})|\,dz\,dt\\
&\le\frac{C}{m_{\mathcal{D}}}\int_0^1\frac{1}{(1-t)^2}\int_{\mathcal{D}_t}|\nabla v(z')|\,dz'\,dt,
\end{aligned}$$

where $\mathcal{D}_t$ is the rescaled diamond as shown in Figure 6. Notice that $\mathcal{D}_t$ is included in $\widehat{\mathcal{D}}$. We have

$m_{\mathcal{D}_t} = (1-t)^2 m_{\mathcal{D}}$; since $p > 2$, by Hölder inequalities we obtain

$$\begin{aligned} I_{\mathcal{K},\sigma^*} &\leq \frac{C}{m_{\mathcal{D}}} \int_0^1 \frac{1}{(1-t)^2} \left(\int_{\mathcal{D}_t} |\nabla v(z')|^p \; dz' \right)^{\frac{1}{p}} m_{\mathcal{D}_t}^{\frac{p-1}{p}} \; dt \\ &\leq \frac{C m_{\mathcal{D}}^{\frac{p-1}{p}}}{m_{\mathcal{D}}} \left(\int_{\widehat{\mathcal{D}}} |\nabla v(z)|^p \; dz \right)^{\frac{1}{p}} \int_0^1 (1-t)^{-\frac{2}{p}} \; dt \\ &\leq C m_{\mathcal{D}}^{-\frac{1}{p}} \left(\int_{\widehat{\mathcal{D}}} |\nabla v(z)|^p \; dz \right)^{\frac{1}{p}}. \end{aligned}$$

With similar calculations for $I_{\mathcal{L},\sigma^*}$, $I_{\mathcal{K}^*,\sigma}$, $I_{\mathcal{L}^*,\sigma}$, we deduce that

$$m_{\mathcal{D}} |\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v|^p \leq C \int_{\widehat{\mathcal{D}}} |\nabla v(z)|^p \; dz.$$

We conclude by summing this estimate over the diamonds set that

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p}^p = \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v|^p \leq C \sum_{\mathcal{D} \in \mathfrak{D}} \int_{\widehat{\mathcal{D}}} |\nabla v|^p \; dz \leq C' \int_{\Omega} |\nabla v|^p \, dz,$$

using the fact that the number $\mathcal{N}_{\mathcal{T}}$ defined in (2.5) is bounded by $\mathrm{reg}(\mathcal{T})$.

• Assume that $v \in \mathcal{C}^2(\overline{\Omega})$, the claim will follow by density. Let $\mathcal{D}$ be a diamond cell; the Taylor expansions (7.10),(7.11) can be replaced by the to second-order ones, so that

$$\begin{aligned} \frac{1}{m_{\sigma^*}} (v(x_{\mathcal{L}}) - v(x_{\mathcal{K}})) = (\nabla v(z), \boldsymbol{\tau}^*) & \\ + \frac{1}{m_{\sigma^*}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{L}})(x_{\mathcal{L}} - z), (x_{\mathcal{L}} - z) \big) \; dt & \\ - \frac{1}{m_{\sigma^*}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{K}})(x_{\mathcal{K}} - z), (x_{\mathcal{K}} - z) \big) \; dt, & \\ \frac{1}{m_{\sigma}} (v(x_{\mathcal{L}^*}) - v(x_{\mathcal{K}^*})) = (\nabla v(z), \boldsymbol{\tau}) & \\ + \frac{1}{m_{\sigma}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{L}^*})(x_{\mathcal{L}^*} - z), (x_{\mathcal{L}^*} - z) \big) \; dt & \\ - \frac{1}{m_{\sigma}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{K}^*})(x_{\mathcal{K}^*} - z), (x_{\mathcal{K}^*} - z) \big) \; dt, & \end{aligned}$$

for any point $z \in \mathcal{D}$. Using (2.9) and Lemma 2.4, we get

$$\begin{aligned} \sin \alpha_{\mathcal{D}} (\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v - \nabla v(z)) & \\ = \frac{\boldsymbol{\nu}}{m_{\sigma^*}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{K}})(x_{\mathcal{K}} - z), (x_{\mathcal{K}} - z) \big) \; dt & \\ - \frac{\boldsymbol{\nu}}{m_{\sigma^*}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{L}})(x_{\mathcal{L}} - z), (x_{\mathcal{L}} - z) \big) \; dt & \\ + \frac{\boldsymbol{\nu}^*}{m_{\sigma}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{K}^*})(x_{\mathcal{K}^*} - z), (x_{\mathcal{K}^*} - z) \big) \; dt & \\ - \frac{\boldsymbol{\nu}^*}{m_{\sigma}} \int_0^1 (1-t) \big(D^2 v((1-t)z + t x_{\mathcal{L}^*})(x_{\mathcal{L}^*} - z), (x_{\mathcal{L}^*} - z) \big) \; dt. & \end{aligned} \tag{7.13}$$

As in the proof of the first point, we take the average of (7.13) over $\mathcal{D}$. It follows that it is sufficient to control by means of $\int_{\widehat{\mathcal{D}}} |D^2 v|^p$ the four similar quantities $II_{\mathcal{K},\sigma^*}, II_{\mathcal{L},\sigma^*}, II_{\mathcal{K}^*,\sigma}, II_{\mathcal{L}^*,\sigma}$ where, for instance,

$$\begin{aligned} II_{\mathcal{K},\sigma^*} &\stackrel{\text{def}}{=} \int_{\mathcal{D}} \left| \frac{1}{m_{\sigma^*}} \int_0^1 (1-t) \left(D^2 v((1-t)z + t x_{\mathcal{K}})(x_{\mathcal{K}} - z), (x_{\mathcal{K}} - z) \right) dt \right|^p dz, \\ &\leq \frac{1}{m_{\sigma^*}^p} \int_{\mathcal{D}} \int_0^1 (1-t)^p \left| D^2 v((1-t)z + t x_{\mathcal{K}})(x_{\mathcal{K}} - z), (x_{\mathcal{K}} - z) \right|^p dt\, dz. \end{aligned}$$

The Jacobian determinant of the change of variables $z \mapsto z' = (1-t)z + t x_{\mathcal{K}}$ equals $(1-t)^2$. Hence,

$$II_{\mathcal{K},\sigma^*} \leq \frac{\mathrm{d}_{\mathcal{D}}^{2p}}{m_{\sigma^*}^p} \left(\int_0^1 (1-t)^{p-2}\, dt \right) \int_{\widehat{\mathcal{D}}} \left| D^2 v(z') \right|^p dz'.$$

Since $p - 2 > -1$, using (2.6), we find

$$II_{\mathcal{K},\sigma^*} \leq \frac{1}{p-1} \frac{\mathrm{d}_{\mathcal{D}}^{2p}}{m_{\sigma^*}^p} \int_{\widehat{\mathcal{D}}} |D^2 v(z)|^p\, dz, \leq C(\mathrm{reg}(\mathcal{T})) \mathrm{d}_{\mathcal{D}}^p \int_{\widehat{\mathcal{D}}} |D^2 v(z)|^p\, dz.$$

With similar calculations for $II_{\mathcal{L},\sigma^*}, II_{\mathcal{K}^*,\sigma}$, and $II_{\mathcal{L}^*,\sigma}$, we have

$$\int_{\mathcal{D}} |\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v(z) - \nabla v(z)|^p\, dz \leq C(\mathrm{reg}(\mathcal{T})) \mathrm{d}_{\mathcal{D}}^p \int_{\widehat{\mathcal{D}}} |D^2 v(z)|^p\, dz, \tag{7.14}$$

and the claim is proved by summing (7.14) over the diamonds set.

• The estimate (7.9) is a straightforward consequence of (7.8). ∎

Corollary 7.6. *Let $\mathcal{T}$ be a mesh on Ω. There exists a constant $C > 0$, depending only on p and $\mathrm{reg}(\mathcal{T})$, such that*

1. *For any $p > 2$ and any $v \in W^{1,p}(\Omega)$, denoting by $g = \gamma(v)$ its trace, we have*

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p} \leq C \|v\|_{W^{1,p}} \text{ and } \|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_m^{\mathcal{T}} v\|_{L^p} \leq C \|v\|_{W^{1,p}}.$$

2. *For any $p > 1$ and any $v \in W^{2,p}(\Omega)$, denoting by $g = \gamma(v)$ its trace, we have*

$$\left\{ \begin{array}{lcl} \|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p} & \leq & C \|\nabla v\|_{L^p} + C \mathrm{size}(\mathcal{T}) \|\nabla v\|_{W^{1,p}}, \\ \|\nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_m^{\mathcal{T}} v\|_{L^p} & \leq & C \|\nabla v\|_{L^p} + C \mathrm{size}(\mathcal{T}) \|\nabla v\|_{W^{1,p}}, \\ \|\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} \mathbb{P}_m^{\mathcal{T}} v - \nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v\|_{L^p} & \leq & C \mathrm{size}(\mathcal{T}) \|\nabla v\|_{W^{1,p}}. \end{array} \right. \tag{7.15}$$

Proof. By (2.10), we have

$$\nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v = \nabla^{\mathcal{T}}_{\mathbb{P}_c^{\mathcal{T}} g} \mathbb{P}_c^{\mathcal{T}} v + \nabla^{\mathcal{T}}_{\mathbb{P}_m^{\mathcal{T}} g - \mathbb{P}_c^{\mathcal{T}} g} 0^{\mathcal{T}}.$$

Using the results of Lemmas 7.4 and 7.5, we deduce the first estimate of the first point. The other estimates are proved in the same way, using Proposition 3.6. ∎

VII.A3 Convergence of the mean-value projection In this section, we return to the proof of Proposition 3.6.

• Let us prove (3.10). The case $q = +\infty$ is straightforward so that we only treat the case of finite values of q. We first remark that

$$\int_\Omega |v(x) - \mathbb{P}_m^{\mathfrak{M}} v(x)|^q \, dx \ \leq \ C \left(\sum_{\mathcal{K}\in\mathfrak{M}} \int_{\mathcal{K}} |v(x) - \mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}}|^q \, dx \right).$$

Furthermore, using Jensen's inequality we get

$$\begin{aligned}\int_{\mathcal{K}} |v(x) - \mathbb{P}_m^{\mathcal{T}} v_{\mathcal{K}}|^q \, dx &\leq \frac{1}{m_{B_{\mathcal{K}}}} \int_{\mathcal{K}} \int_{B_{\mathcal{K}}} |v(x) - v(y)|^q \, dx \, dy \\ &\leq \frac{C}{m_{B_{\mathcal{K}}}} \int_{\mathcal{K}} \int_{B_{\mathcal{K}}} \left(|v(x) - v_\sigma|^q + |v_\sigma - v(y)|^q \right) \, dx \, dy, \\ &\leq \frac{C}{m_\sigma} \int_{\mathcal{K}} \int_\sigma |v(x) - v(s)|^q \, ds \, dx + \frac{C m_{\mathcal{K}}}{m_\sigma m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} \int_\sigma |v(x) - v(s)|^q \, ds \, dx,\end{aligned}$$

where $\sigma \in \mathcal{E}_{\mathcal{K}}$ and $v_\sigma = \dfrac{1}{m_\sigma} \displaystyle\int_\sigma v(s) \, ds$. Thanks to Lemma 3.4 and to (2.6), we get

$$\int_{\mathcal{K}} |v(x) - \mathbb{P}_m^{\mathcal{T}} v(x)|^q \, dx \ \leq \ C \mathrm{d}_{\mathcal{K}}^q \int_{\widehat{\mathcal{K}\cup B_{\mathcal{K}}}} |\nabla v(z)|^q \, dz.$$

Hence, (3.10) follows since the the number $\mathcal{N}_{\mathcal{T}}$ defined in (2.5) is bounded by $\mathrm{reg}(\mathcal{T})$. The same argument let us show (3.11)

• Let us now sketch the proof of (3.12). We follow the same lines as that of (7.8) in Lemma 7.5. Note that it is crucial that the mean-value projection operator $\mathbb{P}_m^{\mathcal{T}}$ averages v over balls centered at $x_{\mathcal{K}}$ and $x_{\mathcal{K}^*}$.

Assume first that $v \in \mathcal{C}^2(\overline{\Omega})$. Let $y_1 \in B_{\mathcal{K}}$, $y_2 \in B_{\mathcal{L}}$ and $z \in \mathcal{D}$ and use the second-order Taylor expansion, we have

$$\begin{aligned}(v(y_2) - v(y_1)) = (\nabla v(z), y_2 - y_1) + \int_0^1 (1-t) \left(D^2 v((1-t)z + t y_2)(y_2 - z), (y_2 - z) \right) \, dt \\ - \int_0^1 (1-t) \left(D^2 v((1-t)z + t y_1)(y_1 - z), (y_1 - z) \right) \, dt.\end{aligned}$$

Take the average of this relation with respect to $y_1 \in B_{\mathcal{K}}$ and $y_2 \in B_{\mathcal{L}}$ and integrate in $z \in \mathcal{D}$. In particular, we point out that

$$\frac{1}{m_{B_{\mathcal{K}}}} \frac{1}{m_{B_{\mathcal{L}}}} \int_{B_{\mathcal{K}}} \int_{B_{\mathcal{L}}} \left(\nabla v(z), y_2 - y_1 \right) \, dy_1 \, dy_2 = \left(\nabla v(z), x_{\mathcal{L}} - x_{\mathcal{K}} \right) = m_{\sigma^*} \left(\nabla v(z), \boldsymbol{\nu}^* \right).$$

Proceeding as for the estimate of $II_{\mathcal{K},\sigma^*}$ in the proof of Lemma 7.5, we find out that

$$\begin{aligned}&\int_{\mathcal{D}} \left| \frac{1}{m_{\sigma^*}} \left(\frac{1}{m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} v(y_2) \, dy_2 - \frac{1}{m_{B_{\mathcal{L}}}} \int_{B_{\mathcal{L}}} v(y_1) \, dy_1 \right) - (\nabla v(z), \boldsymbol{\nu}^*) \right|^q \, dz \\ &\leq \frac{1}{m_{B_{\mathcal{L}}}} \int_{B_{\mathcal{L}}} \int_{\mathcal{D}} \left| \frac{1}{m_{\sigma^*}} \int_0^1 (1-t) \left(D^2 v((1-t)z + t y_2)(y_2 - z), (x_{\mathcal{K}} - z) \right) \, dt \right|^q \, dz \, dy_2 \\ &+ \frac{1}{m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} \int_{\mathcal{D}} \left| \frac{1}{m_{\sigma^*}} \int_0^1 (1-t) \left(D^2 v((1-t)z + t y_1)(y_1 - z), (x_{\mathcal{K}} - z) \right) \, dt \right|^q \, dz \, dy_1 \\ &\leq \left(C(\mathrm{reg}(\mathcal{T})) \, \mathrm{size}(\mathcal{T}) \right)^q \int_{\widehat{B_{\mathcal{K}}\cup\mathcal{D}}} |D^2 v|^q \, dz + \left(C(\mathrm{reg}(\mathcal{T})) \, \mathrm{size}(\mathcal{T}) \right)^q \int_{\widehat{B_{\mathcal{L}}\cup\mathcal{D}}} |D^2 v|^q \, dz.\end{aligned}$$

We conclude using (2.5) and (2.6). The general case of $v \in W^{2,q}(\Omega)$ follows by density. Finally, the case $v \in W^{2,\infty}(\Omega)$ follows from the limit $q \to +\infty$. ∎

VII.B Consistency error of the scheme

As usual, for the error analysis of finite volume methods (see e.g. [18]), the consistency error which has to be studied is the error on the numerical fluxes across each of the edges and dual edges in the mesh. We first give the precise definition of these terms, then we state the various estimates needed to prove Theorem 7.2 in section VII.C.

Definition 7.7 (Pointwise consistency error). *For any diamond cell $\mathcal{D} \in \mathfrak{D}$, we define the pointwise consistency error in $\mathcal{D}$ by*

$$R_{\mathcal{D}}(z) = \varphi_{\mathcal{D}}(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}} g}\mathbb{P}_c^{\mathcal{T}} u_e) - \varphi(z, \nabla u_e(z)), \quad \forall z \in \Omega.$$

The pointwise consistency error $R_{\mathcal{D}}$ can be split into three different contributions $R_{\mathcal{D}}^{grad}$, $R_{\mathcal{D}}^{bound}$, and $R_{\mathcal{D}}^{\varphi}$. They originate, respectively, from the errors due to the approximation of the gradient, to the discretization of the boundary data, and to the approximation with respect to the spatial variable of the flux $\varphi(\cdot, \nabla u_e(\cdot))$:

$$R_{\mathcal{D}}(z) = R_{\mathcal{D}}^{bound} + R_{\mathcal{D}}^{grad} + R_{\mathcal{D}}^{\varphi}(z), \tag{7.16}$$

where

$$\begin{aligned}
R_{\mathcal{D}}^{bound} &= \varphi_{\mathcal{D}}\left(\nabla^{\mathcal{D}}_{\mathbb{P}_m^{\mathcal{T}} g}\mathbb{P}_c^{\mathcal{T}} u_e\right) - \varphi_{\mathcal{D}}\left(\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g}\mathbb{P}_c^{\mathcal{T}} u_e\right),\\
R_{\mathcal{D}}^{grad} &= \varphi_{\mathcal{D}}\left(\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}} g}\mathbb{P}_c^{\mathcal{T}} u_e\right) - \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} \varphi\left(z', \nabla u_e(z')\right) dz',\\
R_{\mathcal{D}}^{\varphi}(z) &= \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} \varphi\left(z', \nabla u_e(z')\right) dz' - \varphi\left(z, \nabla u_e(z)\right).
\end{aligned}$$

Recall that σ and σ^* are the diagonals of $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$. Let us introduce the following consistency errors on the numerical fluxes:

$$R_{\sigma,\mathcal{K}} \stackrel{\text{def}}{=} -R_{\sigma,\mathcal{L}} \stackrel{\text{def}}{=} \frac{1}{m_\sigma}\int_\sigma (R_{\mathcal{D}}(s), \boldsymbol{\nu})\, ds, \quad R_\sigma \stackrel{\text{def}}{=} |R_{\sigma,\mathcal{K}}| = |R_{\sigma,\mathcal{L}}| \tag{7.17}$$

$$R_{\sigma^*,\mathcal{K}^*} \stackrel{\text{def}}{=} -R_{\sigma^*,\mathcal{L}^*} \stackrel{\text{def}}{=} \frac{1}{m_{\sigma^*}}\int_{\sigma^*} (R_{\mathcal{D}}(s^*), \boldsymbol{\nu}^*)\, ds^*, \quad R_{\sigma^*} \stackrel{\text{def}}{=} |R_{\sigma^*,\mathcal{K}^*}| = |R_{\sigma^*,\mathcal{L}^*}|. \tag{7.18}$$

Notice that these integrals make sense since the map $R_{\mathcal{D}}$ is smooth enough to give a sense to its traces on edges (see Remark 7.1).

In order to control R_σ and R_{σ^*}, let us estimate separately the different terms in the right-hand side of (7.16).

Proposition 7.8 (Error due to the discrete gradient). *Assume that φ satisfies $(\mathcal{H}_{4a})$-$(\mathcal{H}_{4b})$ and that $u_e \in W^{2,p}(\Omega)$. For any mesh $\mathcal{T}$ on Ω, there exists a constant $C > 0$, depending only on p, C_4 and $\mathrm{reg}(\mathcal{T})$, such that*

- *in the case $1 < p \leq 2$,*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} |R_{\mathcal{D}}^{grad}|^{\frac{p}{p-1}} \leq C \mathrm{size}(\mathcal{T})^p \|D^2 u_e\|_{L^p}^p;$$

- *in the case $p > 2$,*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\mathcal{D}}^{grad}|^{\frac{p}{p-1}} \leq C\Bigg(\text{size}(\mathcal{T})^p\|D^2u_e\|_{L^p}^p + \text{size}(\mathcal{T})^{\frac{p}{p-1}}\left(\|b_4\|_{L^{\frac{p}{p-2}}}^{\frac{p}{p-1}} + \|\nabla u_e\|_{L^\infty}^{\frac{(p-2)p}{p-1}}\right)\|D^2u_e\|_{L^p}^{\frac{p}{p-1}}\Bigg).$$

Proof. Let $\varepsilon(z) \stackrel{\text{def}}{=} \nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}}g}\mathbb{P}_c^{\mathcal{T}}u_e - \nabla u_e(z)$ be the error of approximation of the gradient.

Case $p \leq 2$**.** Using the definition (2.12) of $\varphi_{\mathcal{D}}$, by assumption $(\mathcal{H}_{4a})$ and the Jensen inequality we have

$$|R_{\mathcal{D}}^{grad}| \leq C\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}|\varepsilon(z)|^{p-1}\,dz \leq C\left(\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}|\varepsilon(z)|^p\,dz\right)^{\frac{p-1}{p}}.$$

Using the estimate of ε given in (7.14), we get

$$m_{\mathcal{D}}|R_{\mathcal{D}}^{grad}|^{\frac{p}{p-1}} \leq C\mathrm{d}_{\mathcal{D}}^p\int_{\widehat{\mathcal{D}}}|D^2u_e(z)|^p\,dz,$$

and the claim is proved by summing this inequality over the diamonds set and using (2.5) and (2.6).

Case $p > 2$**.** We use (2.12) and the assumption $(\mathcal{H}_{4b})$ to obtain

$$\begin{aligned}|R_{\mathcal{D}}^{grad}| &\leq \frac{C}{m_{\mathcal{D}}}\int_{\mathcal{D}}\left(b_4(z) + |\nabla u_e(z)|^{p-2} + |\nabla^{\mathcal{D}}_{\mathbb{P}_c^{\mathcal{T}}g}\mathbb{P}_c^{\mathcal{T}}u_e|^{p-2}\right)|\varepsilon(z)|\,dz\\ &\leq \frac{C}{m_{\mathcal{D}}}\int_{\mathcal{D}}\left(b_4(z) + |\nabla u_e(z)|^{p-2} + |\varepsilon(z)|^{p-2}\right)|\varepsilon(z)|\,dz.\end{aligned}$$

Using the embedding of $W^{1,p}(\Omega)$ into $L^\infty(\Omega)$ and the Hölder inequality, we deduce

$$|R_{\mathcal{D}}^{grad}| \leq C\left(\left(\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}|b_4(z)|^{\frac{p}{p-2}}\,dz\right)^{\frac{p-2}{p}} + \|\nabla u_e\|_{L^\infty}^{p-2}\right)\left(\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}|\varepsilon(z)|^p\,dz\right)^{\frac{1}{p}} + C\left(\frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}}|\varepsilon(z)|^p\,dz\right)^{\frac{p-1}{p}}.$$

Using once more (7.14) to estimate ε, and summing $m_{\mathcal{D}}|R_{\mathcal{D}}^{grad}|^{\frac{p}{p-1}}$ over the diamonds, we conclude the proof.

■

Proposition 7.9 (Error due to the boundary data). *Assume that φ satisfies $(\mathcal{H}_{4a})$-$(\mathcal{H}_{4b})$ and that $g \in \widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)$. For any mesh $\mathcal{T}$ on Ω, there exists a constant $C > 0$, depending only on p, C_4 and $\mathrm{reg}(\mathcal{T})$, such that*

- *in the case $1 < p \leq 2$,*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\mathcal{D}}^{bound}|^{\frac{p}{p-1}} \leq C\text{size}(\mathcal{T})^p\|g\|^p_{\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)};$$

- *in the case $p > 2$,*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\mathcal{D}}^{bound}|^{\frac{p}{p-1}} \leq C\text{size}(\mathcal{T})^{\frac{p}{p-1}}\|g\|_{\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)}^{\frac{p}{p-1}} \times \left(\|b_4\|_{L^{\frac{p}{p-2}}} + \|\nabla u_e\|_{L^p}^{p-2} + \text{size}(\mathcal{T})^{p-2}\|D^2 u_e\|_{L^p}^{p-2}\right)^{\frac{p}{p-1}}.$$

Proof. We just have to use assumptions ($\mathcal{H}_{4a}$) and ($\mathcal{H}_{4b}$) and the estimates (7.4) and (7.15). ∎

We define R_σ^φ and $R_{\sigma^*}^\varphi$ to be the respective contributions of $R_{\mathcal{D}}^\varphi(z)$ to R_σ and R_{σ^*}, that is

$$R_\sigma^\varphi = \left|\frac{1}{m_\sigma}\int_\sigma (R_{\mathcal{D}}^\varphi(s), \boldsymbol{\nu})\, ds\right|, \text{ and } R_{\sigma^*}^\varphi = \left|\frac{1}{m_{\sigma^*}}\int_{\sigma^*} (R_{\mathcal{D}}^\varphi(s^*), \boldsymbol{\nu}^*)\, ds^*\right|. \tag{7.19}$$

Proposition 7.10 (Error due to the approximate flux). *Assume that φ satisfies ($\mathcal{H}_{4a}$)-($\mathcal{H}_{4b}$) and ($\mathcal{H}_{5a}$)-($\mathcal{H}_{5b}$). For any mesh $\mathcal{T}$ on Ω, there exists a constant $C > 0$, depending only on p, C_4, C_5 and $\text{reg}(\mathcal{T})$, such that*

- *in the case $1 < p \leq 2$*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_\sigma^\varphi|^{\frac{p}{p-1}} + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\sigma^*}^\varphi|^{\frac{p}{p-1}} \leq C\text{size}(\mathcal{T})^p\left(1 + \|\nabla u_e\|_{W^{1,p}}^p + \|\nabla b_5\|_{L^p}^p\right);$$

- *in the case $p > 2$*

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_\sigma^\varphi|^{\frac{p}{p-1}} + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\sigma^*}^\varphi|^{\frac{p}{p-1}} \leq C\text{size}(\mathcal{T})^{\frac{p}{p-1}}\left(\|\nabla u_e\|_{W^{1,p}}^{p-1} + \|b_4\|_{L^{\frac{p}{p-2}}}^{\frac{p-1}{p-2}} + \|b_6\|_{L^{p'}}\right)^{\frac{p}{p-1}}.$$

Proof. Let us give the proof for the terms involving the edges σ; the terms involving the dual edges σ^* are estimated in the same way. First, by definition of $R_{\mathcal{D}}^\varphi(z)$, for each $z \in \mathcal{D}$ we have

$$|R_{\mathcal{D}}^\varphi(z)| \leq \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} |\varphi(z', \nabla u_e(z')) - \varphi(z, \nabla u_e(z))|\, dz'$$
$$\leq \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} |\varphi(z', \nabla u_e(z')) - \varphi(z, \nabla u_e(z'))|\, dz' + \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} |\varphi(z, \nabla u_e(z')) - \varphi(z, \nabla u_e(z))|\, dz'.$$

- If $1 < p \leq 2$, assumptions ($\mathcal{H}_{4a}$) and ($\mathcal{H}_{5a}$) yield

$$|R_{\mathcal{D}}^\varphi(z)| \leq \frac{1}{m_{\mathcal{D}}}\int_{\mathcal{D}} |b_5(z') - b_5(z)|^{p-1}\, dz' + \frac{C}{m_{\mathcal{D}}}\int_{\mathcal{D}} (1 + |\nabla u_e(z')|^{p-1})|z - z'|^{p-1}\, dz' + \frac{C}{m_{\mathcal{D}}}\int_{\mathcal{D}} |\nabla u_e(z') - \nabla u_e(z)|^{p-1}\, dz'.$$

Averaging this inequality over the edge σ and summing over the diamonds set give

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R_\sigma^\varphi|^{\frac{p}{p-1}} \leq C\Bigg(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\frac{1}{m_{\mathcal{D}} m_\sigma}\int_{\mathcal{D}}\int_\sigma |b_5(z) - b_5(s)|^p\, ds\, dz + \text{d}_{\mathcal{D}}^p \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} (1 + |\nabla u_e(z)|^p)\, dz + \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\frac{1}{m_{\mathcal{D}} m_\sigma}\int_{\mathcal{D}}\int_\sigma |\nabla u_e(z) - \nabla u_e(s)|^p\, ds\, dz\Bigg).$$

Applying Lemma 3.4 to b_5 and ∇u_e, we obtain

$$\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|R^{\varphi}_{\sigma}|^{\frac{p}{p-1}} \ \le\ C\mathrm{size}(\mathcal{T})^p\left(1+\|\nabla b_5\|^p_{L^p}+\|\nabla u_e\|^p_{L^p}+\|D^2u_e\|^p_{L^p}\right).$$

• If $p>2$, thanks to assumptions $(\mathcal{H}_{4b})$ and $(\mathcal{H}_{5b})$, we see by the chain rule that the map $\psi: z\mapsto\varphi(z,\nabla u_e(z))$ belongs to $(W^{1,p'}(\Omega))^2$ and that

$$\|\nabla\psi\|_{L^{p'}}\le C\left(\|D^2u_e\|^{p-1}_{L^p}+\|b_4\|^{\frac{p-1}{p-2}}_{L^{\frac{p}{p-2}}}+\|b_6\|_{L^{p'}}+\|\nabla u_e\|^{p-1}_{L^p}\right).$$

Applying the notations and the result of Lemma 3.4 to the function ψ, we deduce that

$$|R^{\varphi}_{\sigma}|=|\psi_\sigma-\psi_{\mathcal{D}}|\le C\mathrm{d}_{\mathcal{D}}\left(\frac{1}{m_{\mathcal{D}}}\int_{\widehat{\mathcal{D}}}|\nabla\psi(z)|^{p'}\,dz\right)^{\frac{1}{p'}},$$

and the claim follows by summing $m_{\mathcal{D}}|R^{\varphi}_{\sigma}|^{p'}$ over the diamond cells. ■

VII.C Proof of Theorem 7.2

We are now in the position to prove the error estimate (7.1) stated in Theorem 7.2. First of all, we have

$$\|\nabla u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|_{L^p}\le\|\nabla u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g}\mathbb{P}^{\mathcal{T}}_c u_e\|_{L^p}+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g-\mathbb{P}^{\mathcal{T}}_c g}0^{\mathcal{T}}\|_{L^p}+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|_{L^p}. \tag{7.20}$$

Thanks to Lemma 7.5, the first term is controlled by $C\mathrm{size}(\mathcal{T})\|\nabla u_e\|_{W^{1,p}}$ and thanks to Lemma 7.4, the second term is controlled by $C\mathrm{size}(\mathcal{T})\|g\|_{\widetilde{W}^{2-\frac{1}{p},p}(\partial\Omega)}$. Therefore, in order to prove Theorem 7.2 it is sufficient to estimate the last term in (7.20). To this end, let us prove the following inequalities:

Case $1<p\le 2$.

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|_{L^p}\le C\mathrm{size}(\mathcal{T})^{p-1}\left(1+\|u_e\|^{p-1}_{W^{2,p}}+\|\nabla b_5\|^{p-1}_{L^p}\right)$$
$$\times\left(\|u_e\|^{2-p}_{W^{2,p}}+\|f\|^{\frac{2-p}{p-1}}_{L^{p'}}+\|b_1\|^{\frac{2-p}{p}}_{L^1}+\|b_2\|^{\frac{2-p}{p-1}}_{L^{p'}}+\|b_3\|^{\frac{2-p}{p}}_{L^1}\right); \tag{7.21}$$

Case $p>2$.

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|_{L^p}\le C\mathrm{size}(\mathcal{T})^{\frac{1}{p-1}}\left(\|u_e\|_{W^{2,p}}+\|b_4\|^{\frac{1}{p-2}}_{L^{\frac{p}{p-2}}}+\|b_6\|^{\frac{1}{p-1}}_{L^{p'}}\right). \tag{7.22}$$

For $p>2$, taking $u^{\mathcal{T}}_1=\mathbb{P}^{\mathcal{T}}_c u_e$, $u^{\mathcal{T}}_2=u^{\mathcal{T}}$ and $g_1=g_2=g$ in formula (6.3), we obtain

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^p_{L^p}\le C\,[\![\mathfrak{a}_g(\mathbb{P}^{\mathcal{T}}_c u_e)-\mathfrak{a}_g(u^{\mathcal{T}}),\mathbb{P}^{\mathcal{T}}_c u_e-u^{\mathcal{T}}]\!].$$

Similarly, for $1<p\le 2$ we use (6.2) to obtain

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e-\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^2_{L^p}\le C\left(\|b_3\|^{\frac{2-p}{p}}_{L^1}+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}u^{\mathcal{T}}\|^{2-p}_{L^p}+\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e\|^{2-p}_{L^p}\right)$$
$$\times[\![\mathfrak{a}_g(\mathbb{P}^{\mathcal{T}}_c u_e)-\mathfrak{a}_g(u^{\mathcal{T}}),\mathbb{P}^{\mathcal{T}}_c u_e-u^{\mathcal{T}}]\!].$$

Set $I\stackrel{\mathrm{def}}{=}[\![\mathfrak{a}_g(\mathbb{P}^{\mathcal{T}}_c u_e)-\mathfrak{a}_g(u^{\mathcal{T}}),\mathbb{P}^{\mathcal{T}}_c u_e-u^{\mathcal{T}}]\!]$. Let us express I through the consistency errors $R_{\sigma,\kappa}$ and R_{σ^*,κ^*}. Integrating equation (1.1) over the control volumes κ and the dual control volumes

$\mathcal{K}^*$ leads to (2.11), which is the exact counterpart of the finite volume scheme (2.13)-(2.14). This computation is valid since $z \mapsto \varphi(z, \nabla ue(z))$ is smooth enough as seen in Remark 7.1.

Subtraction of these equations, together with the definitions (7.16), (7.17) and (7.18), yield

$$a_{\mathcal{K}}(u^{\mathcal{T}}) - a_{\mathcal{K}}(\mathbb{P}_c^{\mathcal{T}} ue) = \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathcal{K}}} m_\sigma R_{\sigma,\mathcal{K}}, \ \ \forall \mathcal{K} \in \mathfrak{M},$$

$$a_{\mathcal{K}^*}(u^{\mathcal{T}}) - a_{\mathcal{K}^*}(\mathbb{P}_c^{\mathcal{T}} ue) = \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} R_{\sigma^*,\mathcal{K}^*}, \ \ \forall \mathcal{K}^* \in \mathfrak{M}^*.$$

Therefore,

$$I = \sum_{\mathcal{K} \in \mathfrak{M}} \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathcal{K}}} m_\sigma R_{\sigma,\mathcal{K}} \left(ue(x_{\mathcal{K}}) - u_{\mathcal{K}} \right) + \sum_{\mathcal{K}^* \in \mathfrak{M}^*} \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} R_{\sigma^*,\mathcal{K}^*} \left(ue(x_{\mathcal{K}^*}) - u_{\mathcal{K}^*} \right).$$

Let us rewrite I using the conservativity property of the fluxes in (7.17) and (7.18), the definitions (2.8) and (2.9) of the discrete gradient, and the summation-by-parts Lemma 4.1. We get

$$\begin{aligned}
I = & \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma = \mathcal{K}|\mathcal{L}}} m_\sigma R_{\sigma,\mathcal{K}} (ue(x_{\mathcal{K}}) - u_{\mathcal{K}} - ue(x_{\mathcal{L}}) + u_{\mathcal{L}}) \\
& \qquad + \sum_{\substack{\sigma^* \in \mathcal{E}^* \\ \sigma^* = \mathcal{K}^*|\mathcal{L}^*}} m_{\sigma^*} R_{\sigma^*,\mathcal{K}^*} (ue(x_{\mathcal{K}^*}) - u_{\mathcal{K}^*} - ue(x_{\mathcal{L}^*}) + u_{\mathcal{L}^*}) \\
= & \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_\sigma m_{\sigma^*} \left(R_{\sigma,\mathcal{K}} \left(\nabla_0^{\mathcal{D}} (\mathbb{P}_c^{\mathcal{T}} ue - u^{\mathcal{T}}), \boldsymbol{\tau}^* \right) + R_{\sigma^*,\mathcal{K}^*} \left(\nabla_0^{\mathcal{D}} (\mathbb{P}_c^{\mathcal{T}} ue - u^{\mathcal{T}}), \boldsymbol{\tau} \right) \right) \\
\leq & \frac{1}{\sin \alpha_{\mathcal{T}}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |R_\sigma|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla_0^{\mathcal{D}} (\mathbb{P}_c^{\mathcal{T}} ue - u^{\mathcal{T}})|^p \right)^{\frac{1}{p}} \\
& \qquad + \frac{1}{\sin \alpha_{\mathcal{T}}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |R_{\sigma^*}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla_0^{\mathcal{D}} (\mathbb{P}_c^{\mathcal{T}} ue - u^{\mathcal{T}})|^p \right)^{\frac{1}{p}} \\
\leq & \frac{1}{\sin \alpha_{\mathcal{T}}} \left(\left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |R_\sigma|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left(\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} |R_{\sigma^*}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right) \\
& \qquad \times \| \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} \mathbb{P}_c^{\mathcal{T}} ue - \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} u^{\mathcal{T}} \|_{L^p}.
\end{aligned}$$

Hence, we have for $1 < p \leq 2$,

$$\begin{aligned}
\| \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} \mathbb{P}_c^{\mathcal{T}} ue - \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} u^{\mathcal{T}} \|_{L^p} \leq C & \left(\|b_3\|_{L^1}^{\frac{2-p}{p}} + \| \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} \mathbb{P}_c^{\mathcal{T}} ue \|_{L^p}^{2-p} + \| \nabla_{\mathbb{P}_m^{\mathcal{T}} g}^{\mathcal{T}} u^{\mathcal{T}} \|_{L^p}^{2-p} \right) \\
& \times \left(\left(\sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |R_\sigma|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left(\sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |R_{\sigma^*}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right), \quad (7.23)
\end{aligned}$$

and for $p \geq 2$,

$$\|\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}\mathbb{P}^{\mathcal{T}}_c u_e - \nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g} u^{\mathcal{T}}\|_{L^p} \leq C\left(\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathcal{D}}|R_\sigma|^{\frac{p}{p-1}}\right)^{\frac{1}{p}} + C\left(\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathcal{D}}|R_{\sigma^*}|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}, \tag{7.24}$$

where C depends only on $\mathrm{reg}(\mathcal{T})$ and the other quantities allowed in the statement of Theorem 7.2. Combining (7.16), (7.17), (7.18) and (7.19) with the estimates shown in Propositions 7.8, 7.9 and 7.10, we deduce (7.21) and (7.22) from (7.23) and (7.24) respectively. This ends the proof of Theorem 7.2.

VIII. NUMERICAL RESULTS

In this section, we illustrate our theoretical study by showing the results of some numerical experiments. We suppose that Ω is the square $]-1,1[^2$. We will consider two kinds of analytic radial solutions for which the corresponding boundary data and source term are computed explicitly in order to test the accuracy of the method. For any real parameter $\alpha \in \mathbb{R}$, we define

$$u^1_\alpha(z) = |z|^\alpha, \quad u^2_\alpha(z) = \exp\left(-\frac{|z|^2}{\alpha^2}\right).$$

We compare the results obtained on two kinds of meshes. The mesh 1 is a family of rectangular locally refined meshes obtained by successive global refinements of the original mesh shown in Figure 7. The mesh 2 is a family of standard unstructured triangulations of Ω.

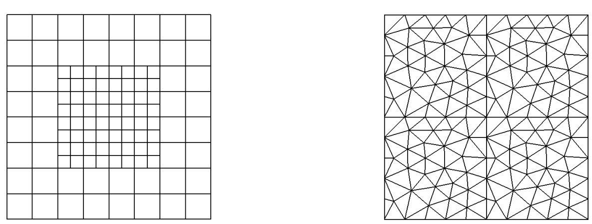

FIG. 7. Picture of the mesh 1 and the mesh 2

In all the figures below, we plot in a logarithmic scale the L^p relative error defined by $\dfrac{\|u_e - u^{\mathcal{T}}\|_{L^p}}{\|u_e\|_{L^p}}$ and the $W^{1,p}$ error defined by $\dfrac{\|\nabla u_e - \nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}\|_{L^p}}{\|\nabla u_e\|_{L^p}}$ as functions of the size of the mesh $\mathrm{size}(\mathcal{T})$ (top plot) but also as a function of the number of unknowns for the discrete problem N (bottom plot).

Let us point out that for the locally refined mesh 1, the size of the control volumes in the refined zone equals $0.5\,\mathrm{size}(\mathcal{T})$.

VIII.A Anisotropic Laplace operator

We consider here the anisotropic Laplace operator (1.3) with a diffusion tensor

$$A(z) = \frac{1}{|z|^2} \begin{pmatrix} z_1^2 + 2z^2 & -z_1 z_2 \\ -z_1 z_2 & 2z_1^2 + z_2^2 \end{pmatrix},$$

which is diagonalisable in a rotating frame around the origin with eigenvalues 1 and 2.

For smooth solutions, we observed the first order convergence in the H^1 norm (given by Theorem 7.2 for $p = 2$) and the second order convergence in the L^2 norm (this superconvergence is proved for the Laplace equation in [13]). We only present here the results obtained for the radial function $u^1_{0.5}$ which is not in $H^2(\Omega)$ but only in $H^{\frac{3}{2}+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. We find here a convergence rate of 0.4 in the H^1 norm and 0.5 in the L^2 norm. This convergence rate is the one expected, at least for the usual Laplace operator and cell-centered finite volume schemes on admissible meshes (see [16]).

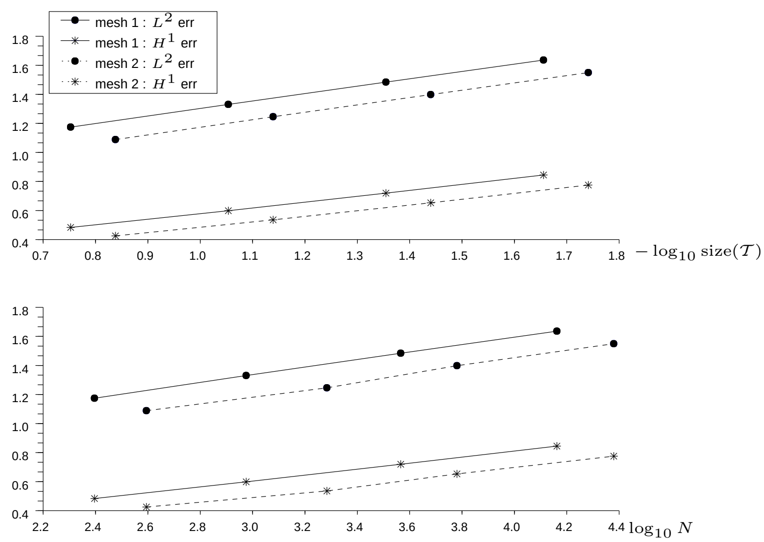

FIG. 8. Test case 1 : Anisotropic laplace equation (1.3) for the exact solution $u^1_{0.5}$

VIII.B Fully non-linear operators

First, we consider the model (1.5) for $p = 3.0$, $k(z) = 1$ and $F(z) = (z_2, -z_1)$ for any $z = (z_1, z_2) \in \Omega$. Notice that F is not a gradient field. The exact solution we used is $u^1_{1.35}$. The coefficient $\alpha = 1.35$ is chosen just greater than $\frac{4}{3}$ to ensure that $u^1_{1.35}$ is not much more smooth than $W^{2,p}(\Omega)$.

First of all, we observe an order of 1.73 in L^3 norm and an order of 0.98 in $W^{1,3}$ for both kinds of meshes. Thus, the theoretical convergence order given by Theorem 7.2 is pessimistic just like for many other studies in this field (see [1, 5, 26, 8]). To our knowledge, very few optimal error estimates for nonlinear diffusion problems are available in the literature. Some of them can be found, with in [26] for the $P1$ finite element approximation of the p-laplacian and in [3] for the FV approach on cartesian meshes.

Nevertheless, an important feature is that the convergence rate is not sensitive to the presence of non conforming control volumes in the mesh 1. Furthermore, in the second plot, we observe that, a number of unknowns being fixed, the mesh 1 (that is the one refined near the singularity) gives better results than the non refined triangular mesh 2. This means that the finite volume scheme presented in this paper for nonlinear equations, can be successfully used in conjunction with local refinement methods in order to save some CPU time without loss of precision. Of course, the analysis of possible *a posteriori* error indicators and adaptative refinement techniques would be of great interest in this framework.

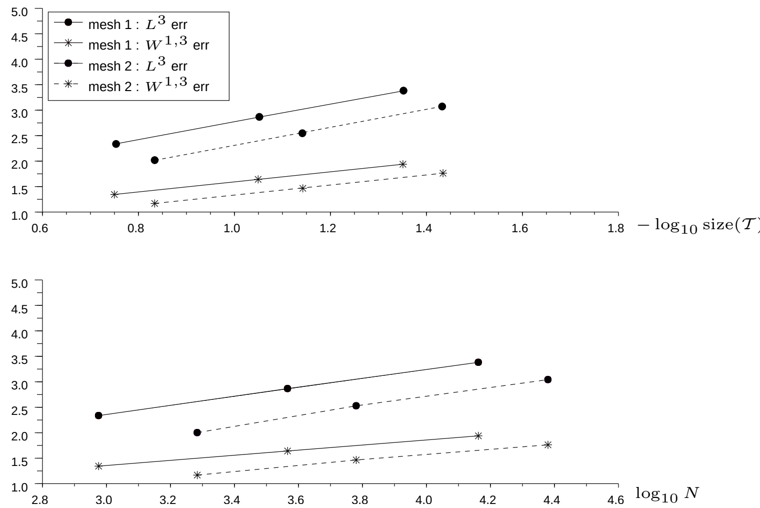

FIG. 9. Comparison between the meshes 1 and 2 for the model (1.5) and the exact solution $u^1_{1.35}$

We finally collected in table I the numerical convergence orders obtained for various values of p on the exact solution $u^1_{1.35}$ on the triangular mesh 2. We observe that the convergence order decreases as p increases. Notice that, of course, for high values of p this exact solution does not lie in the space $W^{2,p}$.

TABLE I. Convergence orders for the model (1.5) and the exact solution $u^1_{1.35}$ on mesh 2

Value of p	3.0	4.0	5.0	6.0
Numerical order in L^p	1.73	1.83	1.73	1.65
Numerical order in $W^{1,p}$	0.98	0.84	0.74	0.65

IX. EXTENSIONS AND CONCLUSIONS

IX.A Remarks on the discrete data

In the scheme studied above, one takes into account the boundary data through its mean-value projection. This choice permits to cope with the general situation with possibly discontinuous boundary data. In the case where g is a more regular (say, Hölder continuous) function, it is natural to consider a slightly different scheme by replacing the operator $\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_m g}$ by $\nabla^{\mathcal{T}}_{\mathbb{P}^{\mathcal{T}}_c g}$ in the definition of the scheme, that is in the choice of the discrete gradient appearing in (2.13) and (2.14). Similarly, for a smooth enough data f, one can replace $\mathbb{P}^{\mathcal{T}}_m f$ by $\mathbb{P}^{\mathcal{T}}_c f$ in the source terms of the scheme (that is in (2.15)).

For such smooth enough data, it is easily seen that our analysis also holds. In particular Lemmas 4.2 and 4.3 can be easily adapted to any discrete source data $f^{\mathcal{T}}$ and any discrete boundary condition $g^{\mathcal{T}}$, and then Theorem 4.4 also holds for any discrete data.

In the same spirit, if φ is smooth enough with respect to the spatial variable z, one can replace the integral definition (2.12) of $\varphi_{\mathcal{D}}$ by a pointwise definition $\varphi_{\mathcal{D}}(\xi) = \varphi(z_{\mathcal{D}}, \xi)$, where $z_{\mathcal{D}}$ is a particular point in the diamond cell $\mathcal{D}$.

IX.B Remarks on the meshes

The mesh $\mathcal{T}$ can be constructed in a more general way, starting from the diamond cells $\mathfrak{D}$. Indeed, let $\mathfrak{D} = (\mathcal{D})$ be a family of disjoint quadrilaterals such that

- $\cup\overline{\mathcal{D}} = \Omega$,
- If $(\mathcal{D}_1, \mathcal{D}_2) \in \mathfrak{D}^2$ and if d_1 and d_2 denote respectively an edge of $\mathcal{D}_1$ and an edge of $\mathcal{D}_2$, then either the two edges coincide or they have at most one common point.

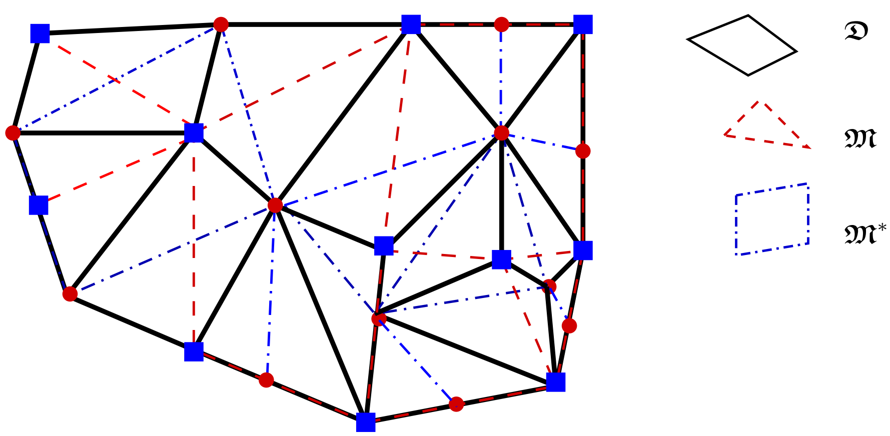

FIG. 10. Construction of $\mathcal{T}$ from the diamond set $\mathfrak{D}$

We note $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ the vertices of a diamond $\mathcal{D}$ as shown in Figure 2. A diamond can degenerate in a triangle, that is three of the points $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ can be aligned. One can prove by induction in $card(\mathfrak{D})$ that the set of the diagonals of this set of quadrilaterals is made of two connected subsets $\mathcal{S}$ and $\mathcal{S}^*$. The meshes $\mathfrak{M}$ or $\mathfrak{M}^*$ can be associated to $\mathcal{S}$ and $\mathcal{S}^*$, respectively. Indeed, the set $\mathcal{S}$ of edges partition $\mathbb{R}^2 = \mathcal{P}_0 \cup \mathcal{P}_1$, where $\mathcal{P}_0$ is unbounded and $\mathcal{P}_1$ is a finite union of disjoints polygons. These polygons are the control volumes of $\mathfrak{M}$: $\mathcal{P}_1 = \cup_{\mathcal{K}\in\mathfrak{M}}\bar{\mathcal{K}}$. The set of vertices of $\mathcal{S}$ (resp. $\mathcal{S}^*$) is noted $\mathfrak{P}^*$ (resp. $\mathfrak{P}$), $\mathfrak{P}$ (resp. $\mathfrak{P}^*$) can be split into parts $\mathfrak{P}_{int}$ (resp. $\mathfrak{P}^*_{int}$) and $\mathfrak{P}_{ext}$ (resp. $\mathfrak{P}^*_{ext}$) corresponding to the interior points and to the points on the boundary. The mesh $\mathfrak{M}^*$ is obtained in the same way from $\mathcal{S}^*$. Note that

- in each control volume $\kappa \in \mathfrak{M}$, there is a unique point of $\mathfrak{P}^*$, and conversely;
- on the boundary, the points of $\mathfrak{P}_{ext}$ and $\mathfrak{P}^*_{ext}$ interleave, as shown on Figure 10.

It is then easy to see that $\mathfrak{D}$ is the diamond set associated to the meshes $\mathfrak{M}$ and $\mathfrak{M}^*$.

IX.C Possible extensions

Let us mention here some of the possible extensions of the present work to more general situations.

- It is possible, in a quite straightforward way, to extend the finite volume method to Neumann boundary conditions on a part Γ_N of $\partial\Omega$. To this end, it is necessary to assume that $g_{\mathcal{K}}$ for $\mathcal{K} \in \partial\mathfrak{M}$ such that $\mathcal{K} \cap \Gamma_N \neq \emptyset$ and $g_{\mathcal{K}^*}$ for $\mathcal{K}^* \in \partial\mathfrak{M}^*$ such that $\partial\mathcal{K}^* \cap \Gamma_N \neq \emptyset$ are new unknowns for the problem. New equations for these supplementary unknowns are obtained by integrating the equation (1.1) over the corresponding non-degenerate boundary dual control volumes $\mathcal{K}^* \in \partial\mathfrak{M}^*$ and by imposing the value of $\varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{g^{\mathcal{T}}} u^{\mathcal{T}}) \cdot \boldsymbol{\nu}$ on each of the corresponding degenerate boundary control volumes $\mathcal{K} \in \partial\mathfrak{M}$. This is possible thanks to the coercivity and monotonicity of the map $\varphi_{\mathcal{D}}$ for each diamond $\mathcal{D}$.
- In this paper, we did not allow the flux φ to depend on u_e. Such a study can be carried out, within the framework of pseudomonotone Leray-Lions type operators $-\mathrm{div}\,(\varphi(z,\,\cdot\,,\nabla\,\cdot\,))$, see e.g. [17] for a study of another kind of finite volume approximation for such nonlinear problems.
- Finally, the extension to the DDFV approach to the 3D case is possible under some additional geometrical conditions on the meshes. At least for linear equations, this extension was proposed and studied in [9, 28].

IX.D Conclusions

We proposed in this paper to use the framework of DDFV schemes for the numerical approximation of fully nonlinear elliptic problems of Leray-Lions kind. This method is well-adapted to general 2D meshes, even locally refined ones, and ensures that the discrete problem has the same properties than the continuous one. In particular, if the problem (1.1) derives from a potential so does the discrete equations. This feature is very useful to provide a fully practical algorithm to compute the approximate solution.

We proved that, under very general assumptions including possibly source terms in $W^{-1,p'}(\Omega)$, the scheme converges. More precisely, the approximate solution, its discrete gradient and the corresponding discrete fluxes converge towards u_e, ∇u_e and $\varphi(\cdot, \nabla u_e)$, respectively, strongly in the appropriate Lebesgue spaces. We proved that the discrete solution is stable with respect to the data and provided an error analysis as soon as the exact solution lies in $W^{2,p}(\Omega)$.

Finally, we have shown numerical evidences that the method behaves better, even for locally refined meshes, than the theoretical convergence order.

The authors would like to thank the referees for their careful reading of the paper and their valuable remarks.

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