IMA Journal of Numerical Analysis Advance Access published December 14, 2005

IMA Journal of Numerical Analysis Page 1 of 31
doi:10.1093/imanum/dri047

On the finite-volume approximation of regular solutions of the p-Laplacian

BORIS ANDREIANOV
Laboratoire de Mathématiques, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France

AND

FRANCK BOYER AND FLORENCE HUBERT
Laboratoire d'Analyse, Topologie et Probabilités, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France

[Received on 11 October 2004; revised on 6 September 2005]

We consider finite-volume schemes on rectangular meshes for the p-Laplacian with Dirichlet boundary conditions. In Andreianov *et al.* (2004a, *Math. Model. Numer. Anal.*, 38, 931–959), we constructed a family of schemes and proved discrete $W^{1,p}$ error estimates in the case of $W^{2,p}$ solutions of the homogeneous problem. Here we improve these estimates in the case of $W^{4,1}$ solutions on uniform meshes for $p > 3$, using symmetry properties of the schemes. The proof also works for the Laplace equation, giving $O(R^2)$ convergence for a family of nine-point finite-volume schemes. With the same ideas, using the improved coercivity inequalities of Barrett and Liu, we obtain even better $W^{1,p}$, $W^{1,1}$ and L^∞ convergence rates for special classes of regular solutions to the inhomogeneous problem—in particular, for solutions without critical points in $\overline{\Omega}$, for all $p \in (1,\infty)$. Numerical examples are given. They suggest the optimality of the L^∞ estimates, of order h^2, obtained for solutions without critical points.

Keywords: finite-volume methods; p-Laplacian; error estimates; superconvergence.

1. Introduction

In this paper, we continue the study of finite-volume approximation of solutions to the p-Laplacian, $1 < p < +\infty$, with homogeneous Dirichlet boundary conditions on a rectangular domain Ω in $\mathbb{R}^2$:

$$\begin{cases} -\mathrm{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) = f, & \text{on } \Omega, \\ \overline{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The schemes we consider arise from minimization of a discrete functional approaching the energy functional

$$J\colon \overline{u} \in W_0^{1,p}(\Omega) \mapsto \frac{1}{p}\int_\Omega |\nabla\overline{u}|^p \,\mathrm{d}z - \int_\Omega f\overline{u}\,\mathrm{d}z$$

(see Section 2.4 and Andreianov *et al.*, 2004a, for details). Therefore, these non-linear schemes are easy to implement, using conjugate gradient type methods. For these schemes on rectangular non-uniform, but somewhat regular, meshes, a discrete $W^{1,p}$ error estimate of order $h^{\frac{1}{p-1}}$, for $p \geqslant 2$, was obtained assuming that the exact solution $\overline{u}$ belongs to $W^{2,p}(\Omega)$. Note that as $f \in L^{p'}(\Omega)$, $\overline{u}$ actually belongs to $W^{1,p}(\Omega)$ and even to the Besov space $B_\infty^{1+\frac{1}{p-1},p}(\Omega)$ (see Simon, 1978); but for $p > 2$, no condition on the right-hand side f is known to ensure that $\overline{u} \in W^{2,p}(\Omega)$ (see, for instance, Ebmeyer *et al.*, 2005, where the $W^{1+s,p}(\Omega)$ regularity of $\overline{u}$ is proved for any $s < \frac{2}{p}$ and smooth enough data).

In Andreianov *et al.* (2005), error estimates were derived for all $f \in L^{p'}(\Omega)$ (without any additional restriction on the exact solution $\overline{u}$), using the aforementioned Besov regularity and its finite-volume counterpart.

In this paper, we go in the opposite direction, looking for a proof of higher-order convergence, already observed for finite-element approximations of the p-Laplacian (see, e.g. Barrett & Liu, 1993). We study solutions as regular as required in order to obtain such an improved convergence rate. The appropriate regularity assumptions are that $\overline{u} \in W^{4,1}(\Omega)$ and that $p > 3$ (or $p = 2$). Moreover, the mesh is required to be uniform: in this case the schemes possess symmetries that lead to cancellations in the error terms.

We obtain in Theorem 3.1 a discrete $W^{1,p}$ error estimate of order $h^{\frac{m}{p-1}}$ with $m = 2$ if $p \geqslant 4$ and $m = p - 2$ if $3 < p \leqslant 4$, provided that f vanishes on the boundary. For general f, an additional error term of order $h^{\frac{1}{p}+\frac{1}{p-1}}$ arises from the boundary. In Section 3.4, we propose a slightly modified finite-volume scheme which exhibits an error estimate of order $h^{\frac{m}{p-1}}$ even if f does not vanish on $\partial\Omega$. Thanks to the Poincaré inequality (Lemma 2.1) and the discrete embedding in L^∞ (Lemma 2.2), L^p and L^∞ error estimates of the same order follow.

In Section 3.5, we study finite-volume approximations of regular solutions of the inhomogeneous Dirichlet problem for the p-Laplacian (3.29), focusing our attention on solutions without critical points (called ‘non-degenerate’) and solutions $\overline{u}$ such that $|\nabla\overline{u}|^{-\nu} \in L^1(\Omega)$ (called ‘ν-weakly degenerate’). This study is inspired by the quasi-norm approach developed in Barrett & Liu (1993) and adapted here to the finite-volume framework. The results are collected in Theorem 3.3 (see also Remarks 3.2, 3.5, 3.6 and 3.7 and Corollary 3.1, where regularity and degeneracy assumptions are discussed). For $p > 3$, the convergence order in the discrete $W^{1,p}$ norm varies, according to the value of ν, from $h^{\frac{m}{p-1}}$ for $\nu = 0$ (i.e. when no integrability of $|\nabla\overline{u}|^{-1}$ is assumed) to $h^{\frac{4}{p}}$ for $\nu = +\infty$ (i.e. in the case of non-degenerate solutions). Better estimates can be obtained in discrete $W^{1,q}$ norms for $1 < q \leqslant 2$; in particular, for non-degenerate solutions, we get the optimal order h^2 for all $q \leqslant 2$. The L^∞ convergence order varies from $h^{\frac{m}{p-1}}$ for $\nu = 0$ to h^2 for $\nu = \infty$. For $1 < p < 2$ we obtain, in the case of sufficiently regular non-degenerate solutions, the orders h^2 in the discrete $W^{1,p}$ norm and $h^{\frac{3p-2}{p}}$ in the L^∞ norm. We wish to stress that our results, in contrast with other results in the literature, may provide convergence rates higher than one (up to second order) for the very simple nine-point finite-volume scheme under consideration.

Note that the error analysis of finite-element schemes for the p-Laplacian has been previously carried out, e.g. in Glowinski & Marrocco (1975); Chow (1989); Barrett & Liu (1993). To our knowledge, for $p > 2$, the best error estimate in the $W^{1,p}$ norm for P^1 elements is Chow’s $h^{\frac{2}{p}}$ rate provided that $\overline{u} \in W^{2,p}(\Omega)$ and without any other assumption on the data f. First-order estimates in quasi-norms and classical $W^{1,q}$ norms with $q < p$ have been derived, under suitable assumptions on the source term f, in Barrett & Liu (1993) and Ebmeyer & Liu (2005). Higher-degree finite elements were for instance considered in Ainsworth & Kay (2000).

In Section 4, we compare the theoretical convergence orders proved in this paper to numerical experiments and to some of the results obtained in the finite-element framework in Barrett & Liu (1993).

Let us discuss the very special case of the Laplace equation ($p = 2$). Our results concern a one-parameter family of nine-point finite-volume schemes, which include the classical five-point finite-difference scheme (see Eymard *et al.*, 2000, for the finite volume point of view) for an appropriate choice of the parameter. For this classical scheme, the h^2 convergence rate is well-known (see, e.g. Samarskii & Andréev, 1978; Eymard *et al.*, 2000, Remark 3.1). In Theorems 3.1(ii) and 3.3(ii) (see also Remarks 3.2, 3.6), we generalize this h^2 convergence rate to our nine-point schemes on the rectangular domain Ω, for solutions $\overline{u} \in \bigcup_{s>1} W^{4,s}(\Omega)$. This regularity is achieved, for instance, for classical solutions $\overline{u} \in \mathcal{C}^2(\overline{\Omega})$ with source terms lying in $\bigcup_{s>1} W^{2,s}(\Omega)$ (see Remark 3.2).

Finally, we point out the fact that most of our analysis is valid for more general non-linear elliptic problems like

$$-\mathrm{div}(k(|\nabla\overline{u}|)\nabla\overline{u}) = f,$$

as long as the map $s \mapsto sk(s)$ has suitable monotonicity properties (see Chow, 1989, for instance), and is smooth enough.

2. The finite-volume schemes

2.1 *Notations*

Let Ω be a rectangular bounded domain of $\mathbb{R}^2$; without loss of generality, we assume that $\Omega = (0, L_x) \times (0, L_y)$. We consider a uniform mesh $\mathcal{T}$, i.e. a set of disjoint control volumes $\mathcal{K} \in \mathcal{T}$ isometric to a given reference rectangular volume $(-\frac{h}{2}, \frac{h}{2}) \times (-\frac{k}{2}, \frac{k}{2})$, such that $\overline{\Omega} = \bigcup_{\mathcal{K}\in\mathcal{T}} \bar{\mathcal{K}}$. We consider a family of meshes with h tending to zero which satisfy the following assumption:

$$\exists c_1 > 0, \quad \text{such that } c_1 \leqslant \frac{k}{h} \leqslant \frac{1}{c_1}. \tag{2.1}$$

Let us recall some notations introduced in Andreianov *et al.* (2004a). We denote by $x_{\mathcal{K}}$ the centre of the control volume $\mathcal{K}$. In order to take into account the boundary conditions, we introduce artificial points constructed by symmetry with respect to the boundaries of Ω (see Fig. 1).

The dual mesh $\mathcal{T}^*$ of $\mathcal{T}$ is defined to be the set of dual rectangular control volumes whose vertices are the points $x_{\mathcal{K}}$ and the artificial points.

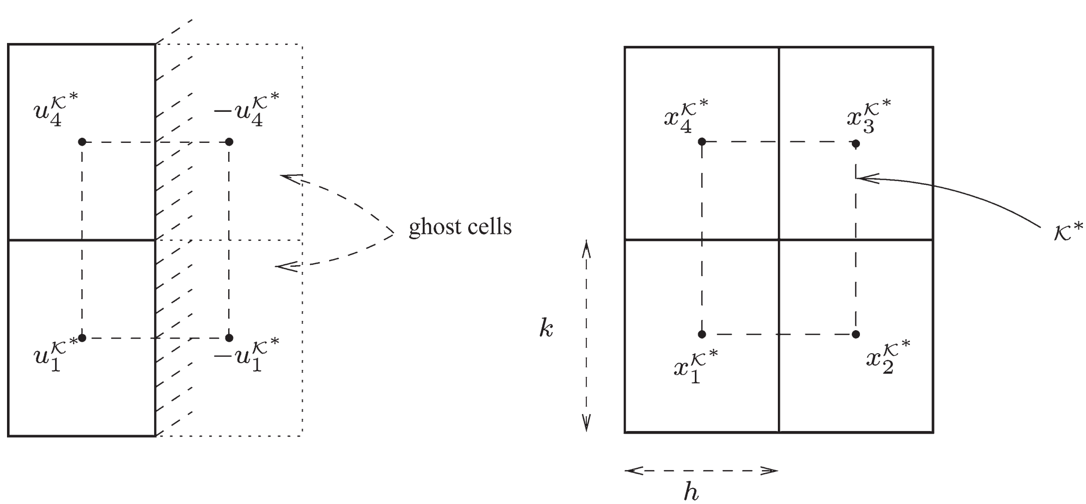

FIG. 1. Notations.

Given a dual control volume $\mathcal{K}^*$, we define (see Fig. 1):

- $(x_i^{\mathcal{K}^*})_{i=1,2,3,4}$ the vertices of the dual control volume $\mathcal{K}^*$ numbered counterclockwise starting from the lower left-hand corner;
- $(\mathcal{K}_i^{\mathcal{K}^*})_{i=1,2,3,4}$ the corresponding control volumes with centres $(x_i^{\mathcal{K}^*})_{i=1,2,3,4}$;
- $l_i^{\mathcal{K}^*}$ the distance between $x_i^{\mathcal{K}^*}$ and $x_{i+1}^{\mathcal{K}^*}$; in this paper, since the meshes are assumed to be uniform, we have $l_1^{\mathcal{K}^*} = l_3^{\mathcal{K}^*} = h$ and $l_2^{\mathcal{K}^*} = l_4^{\mathcal{K}^*} = k$;
- $\sigma_i^{\mathcal{K}^*}$ the half-edge between $\mathcal{K}_i^{\mathcal{K}^*}$ and $\mathcal{K}_{i+1}^{\mathcal{K}^*}$ located in $\mathcal{K}^*$.

In the sequel, we drop the superscripts $\mathcal{K}^*$ when the notation is not confusing. Conventionally, in a given dual control volume, the indices $i \in \mathbb{Z}$ are understood modulo 4.

The finite-volume method associates to each control volume $\mathcal{K}$ an unknown value $u_{\mathcal{K}}$. We denote the set $(u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ by $u^{\mathcal{T}}$. The discrete function $u^{\mathcal{T}}$ is called the approximate solution on the mesh $\mathcal{T}$. For any continuous function v on Ω, the discrete function $v^{\mathcal{T}} = (v_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}}$, with $v_{\mathcal{K}} = v(x_{\mathcal{K}})$, will be called the projection of v on the space $\mathbb{R}^{\mathcal{T}}$ of discrete functions. For a given discrete function $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, the boundary conditions are taken into account by using the ghost-cell method (see Fig. 1), which means that we extend the values of $u^{\mathcal{T}}$ to artificial points outside of Ω by odd symmetry with respect to the corresponding boundaries.

Given a dual control volume $\mathcal{K}^*$, we define the projection operator $T_{\mathcal{K}^*}$ which associates to each $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ its values $T_{\mathcal{K}^*}(u^{\mathcal{T}}) = (u^{\mathcal{T}}_{1,\mathcal{K}^*}, u^{\mathcal{T}}_{2,\mathcal{K}^*}, u^{\mathcal{T}}_{3,\mathcal{K}^*}, u^{\mathcal{T}}_{4,\mathcal{K}^*})$ in the four control volumes $\left(\mathcal{K}_i^{\mathcal{K}^*}\right)_i$ that intersect $\mathcal{K}^*$. Note that for boundary dual control volumes, ghost cells are used in order to give sense to the definition of $T_{\mathcal{K}^*}$. For instance, if $\mathcal{K}^*$ is located at the right-hand boundary of Ω, we have by definition

$$u^{\mathcal{T}}_{2,\mathcal{K}^*} = -u^{\mathcal{T}}_{1,\mathcal{K}^*} \quad \text{and} \quad u^{\mathcal{T}}_{3,\mathcal{K}^*} = -u^{\mathcal{T}}_{4,\mathcal{K}^*}, \quad \text{where} \quad u^{\mathcal{T}}_{1,\mathcal{K}^*} = u_{\mathcal{K}_{1,\mathcal{K}^*}},\ u^{\mathcal{T}}_{4,\mathcal{K}^*} = u_{\mathcal{K}_{4,\mathcal{K}^*}}.$$

2.2 *Discrete semi-norms and Sobolev embeddings*

Denote by $\mathbf{1}_{\mathcal{K}}$ the characteristic function of the control volume $\mathcal{K}$. Each discrete function $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ is identified with the bounded function $u^{\mathcal{T}} = \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}} \mathbf{1}_{\mathcal{K}}$, so that for $r \in [1, +\infty]$ the norms $\|u^{\mathcal{T}}\|_{L^r}$ are naturally defined. Let us define a discrete Sobolev semi-norm for the elements of $\mathbb{R}^{\mathcal{T}}$. For any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and any $\mathcal{K}^* \in \mathcal{T}^*$, we define the difference quotients

$$\delta_i^{\mathcal{K}^*}(u^{\mathcal{T}}) = \frac{u^{\mathcal{T}}_{i+1,\mathcal{K}^*} - u^{\mathcal{T}}_{i,\mathcal{K}^*}}{l_i^{\mathcal{K}^*}}, \quad i \in \{1, \ldots, 4\}. \tag{2.2}$$

DEFINITION 2.1 Consider $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$. For any $\mathcal{K}^*$, we define

$$|u^{\mathcal{T}}|_{1,\mathcal{K}^*} = \left(\frac{1}{2}\sum_{i=1}^{4} \left|\delta_i^{\mathcal{K}^*}(u^{\mathcal{T}})\right|^2\right)^{\frac{1}{2}}$$

to be an approximation of $|\nabla \bar{u}|$, so that the discrete $W_0^{1,q}$ semi-norm of $u^{\mathcal{T}}$ is defined by

$$\|u^{\mathcal{T}}\|_{1,q,\mathcal{T}} = \left(\sum_{\mathcal{K}^*\in\mathcal{T}^*} m\left(\mathcal{K}^* \cap \Omega\right) |u^{\mathcal{T}}|^q_{1,\mathcal{K}^*}\right)^{\frac{1}{q}},$$

for $1 \leqslant q < +\infty$. We denote by $W^{1,q,\mathcal{T}}$ the space $\mathbb{R}^{\mathcal{T}}$ equipped with the norm $\|\cdot\|_{1,q,\mathcal{T}}$.

Discrete versions of standard embedding and interpolation inequalities for Sobolev spaces are presented below. Note that in Lemmas 2.1–2.3, the zero boundary condition is incorporated by using the ghost-cell method. These results remain true for the ‘all-uniform’ schemes considered in Sections 3.4 and 3.5.

LEMMA 2.1 (DISCRETE POINCARÉ INEQUALITY) Let $\mathcal{T}$ be a mesh on the rectangle Ω. There exists a constant C which only depends on p such that for any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we have

$$\|u^{\mathcal{T}}\|_{L^p} \leqslant C\,\mathrm{diam}(\Omega)\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}.$$

The proof is an adaptation of the one given in Andreianov *et al.* (2004b).

LEMMA 2.2 (DISCRETE (QUASI-)EMBEDDINGS OF $W^{1,p}$ INTO L^∞) Let $\mathcal{T}$ be a uniform mesh of the rectangle Ω satisfying (2.1). There exists a constant C, depending only on Ω, p and c_1 (except for the case (i)(b) below) such that for any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$,

(i) (a) when $p > 2$, $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant C\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}$,
(b) when $p = 2$, $\|u^{\mathcal{T}}\|_{L^r} \leqslant Cr\|u^{\mathcal{T}}\|_{1,2,\mathcal{T}}$ for all $r < \infty$,
(c) when $1 < p < 2$, $\|u^{\mathcal{T}}\|_{L^r} \leqslant C\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}$ for all $r \leqslant \frac{2p}{2-p}$,

(ii) (a) when $p = 2$, $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant C|\ln h|\|u^{\mathcal{T}}\|_{1,2,\mathcal{T}}$,
(b) when $1 < p < 2$, $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^{-\frac{(2-p)}{p}}\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}$.

The proof for general admissible finite-volume meshes is given, for instance, in Coudière *et al.* (2001). We only give below a sketch of the proof in the simplest case of Cartesian meshes.

Proof.

(i) For a given Cartesian mesh $\mathcal{T}$ of Ω, consider the set of rectangles $\mathcal{K}^* \cap \Omega$, with $\mathcal{K}^* \in \mathcal{T}^*$. Split each rectangle into two triangles; we denote by $\widehat{\mathcal{T}}$ the resulting triangular mesh on Ω. For $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, denote by $\hat{u}$ the piecewise affine interpolant over the triangles of $\widehat{\mathcal{T}}$ with value $u_{\mathcal{K}}$ at the vertices $x_{\mathcal{K}}$ and equal to zero at the vertices located on $\partial\Omega$. Note that $\hat{u} \in W_0^{1,p}(\Omega)$. By the Sobolev embedding theorem, for $p > 2$, there exists a constant C such that $\|u^{\mathcal{T}}\|_{L^\infty} = \|\hat{u}\|_{L^\infty} \leqslant C\|\hat{u}\|_{W_0^{1,p}}$. It is easily seen that $\|\hat{u}\|_{W_0^{1,p}} \leqslant C\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}$, which proves (a). In the same way, (b) and (c) follow.

(ii) (a) Using (i)(b) for all $r < +\infty$, we obtain

$$\|u^{\mathcal{T}}\|_{L^\infty} = \max_{\mathcal{K}\in\mathcal{T}}|u_{\mathcal{K}}| = \frac{1}{h^{2/r}}(h^2|u_{\mathcal{K}}|^r)^{1/r} \leqslant \frac{1}{h^{2/r}}\|u^{\mathcal{T}}\|_{L^r} \leqslant \frac{r}{h^{2/r}}\|u^{\mathcal{T}}\|_{1,2,\mathcal{T}}. \tag{2.3}$$

Searching for the optimal value of r, we find $r = 2|\ln h|$ and deduce the result.

(b) As in (2.3), we get $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant \frac{1}{h^{2/r}}\|u^{\mathcal{T}}\|_{L^r}$; it suffices to take $r = \frac{2p}{2-p}$ in (i)(c).

□

REMARK 2.1 In case (ii)(a), reproducing the proof of the embedding theorem in the discrete framework, one finds a slightly sharper estimate $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant C|\ln h|^{1/2}\|u^{\mathcal{T}}\|_{1,2,\mathcal{T}}$.

LEMMA 2.3 (DISCRETE INTERPOLATION INEQUALITIES) Let $\mathcal{T}$ be a uniform mesh of the rectangle Ω satisfying (2.1). Let $1 \leqslant q \leqslant t < p$, and $\theta = \frac{q(p-t)}{t(p-q)}$. There exists a constant C, depending only on Ω, p, c_1, q and t such that $\|u^{\mathcal{T}}\|_{1,t,\mathcal{T}} \leqslant C\|u^{\mathcal{T}}\|_{1,q,\mathcal{T}}^{\theta}\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}^{1-\theta}$. As $t = 2$, $\|u^{\mathcal{T}}\|_{L^\infty} \leqslant C\|u^{\mathcal{T}}\|_{1,q,\mathcal{T}}^{\theta}\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}}^{1-\theta}$ also holds.

Proof. The results follow by the corresponding inequalities for Sobolev spaces, upon replacing $u^{\mathcal{T}}$ by its piecewise affine interpolant, as in the proof of Lemma 2.2(i). □

2.3 *Construction of the schemes*

In Andreianov *et al.* (2004a), we derived the general form of symmetric finite-volume schemes on Cartesian meshes that are consistent with piecewise affine functions and that satisfy a discrete $W^{1,p}$ estimate.

In the special case of a uniform mesh, the schemes can be written as a system of the following equations:

$$a(u^{\mathcal{T}}) \stackrel{\text{def}}{=} (a_{\mathcal{K}}(u^{\mathcal{T}}))_{\mathcal{K}\in\mathcal{T}} \stackrel{\text{def}}{=} \sum_{\mathcal{K}^*\in\mathcal{T}^*} m\left(\mathcal{K}^*\cap\Omega\right) T^t_{\mathcal{K}^*}\circ a_0 \circ T_{\mathcal{K}^*}(u^{\mathcal{T}}) = (m(\mathcal{K})\, f_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}}, \tag{2.4}$$

where $(u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}}$ are the unknowns and $f_{\mathcal{K}}$ denotes the mean value of the function f on the control volume $\mathcal{K}$. Here,

$$a_0(v) \stackrel{\text{def}}{=} (Bv, v)^{\frac{p-2}{2}} Bv \quad \forall\, v \in \mathbb{R}^4, \tag{2.5}$$

where B is a 4×4 matrix defined by the choice of a parameter ξ as follows:

$$B = \frac{1}{2hk}\begin{pmatrix} 4\xi+\frac{k}{h}+\frac{h}{k} & -4\xi-\frac{k}{h} & 4\xi & -4\xi-\frac{h}{k} \\ -4\xi-\frac{k}{h} & 4\xi+\frac{k}{h}+\frac{h}{k} & -4\xi-\frac{h}{k} & 4\xi \\ 4\xi & -4\xi-\frac{h}{k} & 4\xi+\frac{k}{h}+\frac{h}{k} & -4\xi-\frac{k}{h} \\ -4\xi-\frac{h}{k} & 4\xi & -4\xi-\frac{k}{h} & 4\xi+\frac{k}{h}+\frac{h}{k} \end{pmatrix}. \tag{2.6}$$

This choice ensures the consistency and the symmetry of the scheme.

In Andreianov *et al.* (2004a), more general schemes were studied, where the parameter ξ and consequently the matrix B and the map a_0 may depend on $\mathcal{K}^*$. The fact that ξ does not depend on $\mathcal{K}^*$ is fundamental in the present paper, since we need the symmetry properties of the scheme on each control volume.

DEFINITION 2.2 We say that a scheme defined by (2.4)–(2.6) is admissible, if

$$8\xi + \frac{k}{h} + \frac{h}{k} > 0.$$

For any control volume $\mathcal{K}$, the numerical flux $a_{\mathcal{K}}$ is naturally decomposed into the contributions of the half-edges $\sigma \subset \partial\mathcal{K}\cap\partial\mathcal{K}^*$:

$$a_{\mathcal{K}}(u^{\mathcal{T}}) = \sum_{\sigma\subset\partial\mathcal{K}\cap\partial\mathcal{K}^*} a_{\mathcal{K},\sigma}(u^{\mathcal{T}}),$$

where for $\mathcal{K} = \mathcal{K}_j^{\mathcal{K}^*}$ and $\sigma = \sigma_j^{\mathcal{K}^*}$, $a_{\mathcal{K},\sigma}$ is defined by

$$a_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = m_\sigma g_\sigma\Big(\delta_1^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_2^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_3^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_4^{\mathcal{K}^*}(u^{\mathcal{T}})\Big)$$

with

$$g_\sigma(\delta_1,\delta_2,\delta_3,\delta_4) = [q(\delta_1,\delta_2,\delta_3,\delta_4)]^{\frac{p-2}{2}} \begin{cases} -\delta_j - 2\xi\frac{h}{k}(\delta_1+\delta_3), & \text{if } j=1,3,\\ -\delta_j - 2\xi\frac{k}{h}(\delta_2+\delta_4), & \text{if } j=2,4, \end{cases} \tag{2.7}$$

and

$$q(\delta_1,\delta_2,\delta_3,\delta_4) = \frac{1}{2}\sum_{i=1}^{4}\delta_i^2 + \xi\left(\frac{h}{k}(\delta_1+\delta_3)^2 + \frac{k}{h}(\delta_2+\delta_4)^2\right).$$

Note that $q(\delta_1^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_2^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_3^{\mathcal{K}^*}(u^{\mathcal{T}}), \delta_4^{\mathcal{K}^*}(u^{\mathcal{T}})) = |B^{\frac{1}{2}}T_{\mathcal{K}^*}(u^{\mathcal{T}})|^2$.

Let us recall the following property (see Andreianov *et al.*, 2004a).

LEMMA 2.4 Let $\gamma > 0$ be such that

$$\xi \leqslant \frac{1}{\gamma} \quad \text{and} \quad 8\xi + \frac{k}{h} + \frac{h}{k} \geqslant \gamma. \tag{2.8}$$

Then, there exist $\beta_1, \beta_2 > 0$, depending only on γ and on c_1 in (2.1) such that

$$\beta_1|u^{\mathcal{T}}|_{1,\mathcal{K}^*} \leqslant \left|B^{\frac{1}{2}}T_{\mathcal{K}^*}(u^{\mathcal{T}})\right| \leqslant \beta_2|u^{\mathcal{T}}|_{1,\mathcal{K}^*} \quad \forall\,\mathcal{K}^*\in\mathcal{T}^*,\ \ \forall\,u^{\mathcal{T}}\in\mathbb{R}^{\mathcal{T}}. \tag{2.9}$$

2.4 *Discrete energy*

We call ‘discrete energy of the scheme’ the following functional $J_{\mathcal{T}}$ acting on discrete functions $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$:

$$\begin{aligned} J_{\mathcal{T}}(u^{\mathcal{T}}) &= \frac{1}{p}(a(u^{\mathcal{T}}), u^{\mathcal{T}}) - \sum_{\mathcal{K}\in\mathcal{T}} m(\mathcal{K})\, f_{\mathcal{K}} u_{\mathcal{K}} \\ &= \frac{1}{p}\sum_{\mathcal{K}^*\in\mathcal{T}^*} m\left(\mathcal{K}^*\cap\Omega\right)\left|B^{\frac{1}{2}}T_{\mathcal{K}^*}(u^{\mathcal{T}})\right|^p - \sum_{\mathcal{K}\in\mathcal{T}} m(\mathcal{K})\, f_{\mathcal{K}} u_{\mathcal{K}}. \end{aligned}$$

This functional is strictly convex and coercive, and its unique minimizing point is the unique solution of the set of discrete equations (2.4). In practice, we compute the approximate solution by minimizing $J_{\mathcal{T}}$ through standard iterative algorithms. The functional $J_{\mathcal{T}}$ inherits the well-known properties of the functional $J\colon u\in W_0^{1,p}(\Omega)\mapsto \int_\Omega \frac{1}{p}|\nabla u|^p - fu$ associated with the continuous problem (see Andreianov *et al.*, 2004a).

LEMMA 2.5 If $p \geqslant 2$, there exists a constant $C > 0$ such that for any $u^{\mathcal{T}}, v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$,

$$(\nabla J_{\mathcal{T}}(v^{\mathcal{T}}) - \nabla J_{\mathcal{T}}(u^{\mathcal{T}}), v^{\mathcal{T}} - u^{\mathcal{T}}) \geqslant C\|u^{\mathcal{T}} - v^{\mathcal{T}}\|_{1,p,\mathcal{T}}^p. \tag{2.10}$$

2.5 *Previously obtained error estimates*

We recall the error estimates obtained in our previous works.

- *For $W^{2,p}(\Omega)$ solutions*
  For $p \geqslant 2$ and non-uniform rectangular meshes, the following estimate has been obtained in Andreianov *et al.* (2004a):

  $$\|\overline{u}^{\mathcal{T}} - u^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch\|\overline{u}\|_{W^{2,p}} + Ch^{\frac{1}{p-1}}\|\overline{u}\|_{W^{2,p}}^{\frac{3p-4}{p(p-1)}}\|f\|_{L^{p'}}^{\frac{(p-2)^2}{p(p-1)^2}}, \tag{2.11}$$

  provided that the solution $\overline{u}$ of the p-Laplacian (1.1) belongs to $W^{2,p}(\Omega)$.

- *For general solutions*
  For uniform meshes and all $f \in L^{p'}(\Omega)$, it has been proved in Andreianov *et al.* (2005) that

$$\|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C h^{\frac{2}{p(p-1)}} \|f\|_{L^{p'}}^{\frac{2}{p(p-1)}} \|\overline{u}\|_{W^{1,p}}^{1-\frac{2}{p}}, \quad \text{if } p \geqslant 3,$$

$$\|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C h^{\frac{1}{p}} \|f\|_{L^{p'}}^{\frac{1}{p}} \|\overline{u}\|_{W^{1,p}}^{\frac{1}{p}}, \quad \text{if } 2 < p < 3.$$

In these estimates, the constant $C > 0$ depends only on c_1 in (2.1) and γ in (2.8).

3. Higher-order error estimates

3.1 *Principle of the proof*

In order to prove the error estimates (2.11), we used the fact that

$$\begin{aligned}(\nabla J_{\mathcal{T}}(\overline{u}^{\mathcal{T}}) - \nabla J_{\mathcal{T}}(u^{\mathcal{T}}), \overline{u}^{\mathcal{T}} - u^{\mathcal{T}}) &= \sum_{\mathcal{K}\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}) \\ &= \sum_{\sigma=\mathcal{K}|\mathcal{L}} R_{\sigma}((u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}) - (u_{\mathcal{L}} - \overline{u}_{\mathcal{L}})). \end{aligned} \tag{3.1}$$

Here $R_{\sigma,\mathcal{K}}$ is the so-called local consistency error defined by

$$R_{\sigma,\mathcal{K}} = a_{\mathcal{K},\sigma}(u^{\mathcal{T}}) - \left(-\int_{\sigma} |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nu_{\sigma}\,\mathrm{d}s\right), \tag{3.2}$$

and $R_{\sigma} = R_{\sigma,\mathcal{K}} = -R_{\sigma,\mathcal{L}}$. Now, for any σ and $\mathcal{K}$, the local consistency error $R_{\sigma,\mathcal{K}}$ is, in general, of order h^2 if $\overline{u}$ is smooth enough (say in $\mathcal{C}^2(\overline{\Omega})$). This yields the estimate (2.11). We will show that cancellations due to symmetries of the scheme imply that the term $\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}$ is of order h^4 (at least if $\mathcal{K}$ is an interior control volume; see also Section 3.4), provided that $\overline{u}$ is regular enough (roughly speaking, if it is in $\mathcal{C}^4(\overline{\Omega})$) and g_{σ} is of class $\mathcal{C}^3$. According to (2.7), we thus need to control $q^{\frac{p-4}{2}}$. Therefore, our result is restricted to the cases $p \geqslant 4$ and $p = 2$. For the case $3 < p < 4$, in the same way we, exhibit an intermediate consistency order $h^{2(p-2)}$. Another possibility is to assume that $|\nabla\overline{u}|$ does not vanish, which yields the consistency order h^4 for all $p > 1$.

The key point is to treat the right-hand side of (3.1) in a different manner, bringing together the terms which originate from all the eight half-edges which delimit a control volume $\mathcal{K} \in \mathcal{T}$. Let us rewrite (3.1) in the form

$$(\nabla J_{\mathcal{T}}(\overline{u}^{\mathcal{T}}) - \nabla J_{\mathcal{T}}(u^{\mathcal{T}}), \overline{u}^{\mathcal{T}} - u^{\mathcal{T}}) = \sum_{\mathcal{K}\in\mathcal{T}} \left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}\right)(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}). \tag{3.3}$$

DEFINITION 3.1 For a given volume $\mathcal{K} \in \mathcal{T}$, we call 'mean consistency error' $R_{\mathcal{K}}$ the sum of the local consistency errors associated with all the half-edges surrounding the control volume $\mathcal{K}$:

$$R_{\mathcal{K}} = \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}.$$

Theorem 3.1 relies upon the estimate of this mean consistency error, which is quite technical. But the principle of its proof already transpires in the case of interior volumes, and if the reader assumes that $\overline{u}$ belongs to $\mathcal{C}^4$.

3.2 *Estimate of the mean consistency error*

Denote by β the multi-index $(\beta_x, \beta_y) \in (\mathbb{Z}^+)^2$ of order $|\beta| = \beta_x + \beta_y$ and by D^β the corresponding derivative $\frac{\partial^{|\beta|}}{\partial^{\beta_x} x \partial^{\beta_y} y}$. Whenever it simplifies the notation, we may also write $w_x, w_{xy}, \ldots$ for the derivatives of the function w on Ω

In the sequel, we denote by C any constant that depends on $\|\overline{u}\|_{W^{4,1}}$, Ω, p, c_1 (appearing in (2.1)) and γ (appearing in (2.8)). Finally, we use the Landau symbol $\mathcal{O}(1)$ to denote any bounded function with respect to the mesh size h, the norm of $\overline{u}$ in $W^{4,1}(\Omega)$ and, perhaps, other variables.

The goal of this section is to state and prove Proposition 3.1 which provides an estimate of the mean consistency error $R_\mathcal{K}$ for any control volume $\mathcal{K} \in \mathcal{T}$. To this end, let us introduce some notations and give some preliminary results.

Without loss of generality, assume that $\mathcal{K} = (-\frac{h}{2}, \frac{h}{2}) \times (-\frac{k}{2}, \frac{k}{2})$ and let $\sigma_0 = \{\frac{h}{2}\} \times [0, \frac{k}{2}]$ be one of its four vertical half-edges. Denote by τ_x, τ_y the reflexions of $\mathbb{R}^2$ in the coordinate axes, namely, $\tau_x(x, y) = (-x, y)$ and $\tau_y(x, y) = (x, -y)$ for all $(x, y) \in \mathbb{R}^2$. For any smooth function w on $\overline{\mathcal{K}}$, we define

$$\delta_w^0 = \left(\delta_1^{\mathcal{K}^*}(w^\mathcal{T}), \delta_2^{\mathcal{K}^*}(w^\mathcal{T}), \delta_3^{\mathcal{K}^*}(w^\mathcal{T}), \delta_4^{\mathcal{K}^*}(w^\mathcal{T})\right),$$

$$\delta_w(s) = \left(w_x\left(\frac{h}{2}, s\right), w_y\left(\frac{h}{2}, s\right), -w_x\left(\frac{h}{2}, s\right), -w_y\left(\frac{h}{2}, s\right)\right) \quad \forall s \in \left[0, \tfrac{k}{2}\right],$$

the $\left(\delta_i^{\mathcal{K}^*}\right)_i$ being defined in (2.2) and $\mathcal{K}^*$ being the unique dual control volume containing σ_0. Let us state some properties of the function g_{σ_0} defined in (2.7) (for $j = 1$ in this particular case). For convenience, we now drop the subscript σ_0.

Notations. For any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{Z}^+)^4$ of order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, we denote by $D^\alpha g$ the corresponding derivative of the function $g(\cdot, \cdot, \cdot, \cdot)$. We will also write D_1 for $D^{(1,0,0,0)}$, D_2 for $D^{(0,1,0,0)}$, D_{11} for $D^{(2,0,0,0)}$, D_{12} for $D^{(1,1,0,0)}$, and so on.

LEMMA 3.1 Let w be a regular enough function on $\overline{\mathcal{K}}$.

(i) For any $\beta = (\beta_x, \beta_y) \in (\mathbb{Z}^+)^2$, we have

$$D^\beta(w \circ \tau_x) = (-1)^{\beta_x}[D^\beta w] \circ \tau_x, \quad D^\beta(w \circ \tau_y) = (-1)^{\beta_y}[D^\beta w] \circ \tau_y.$$

(ii) For any $\alpha \in (\mathbb{Z}^+)^4$, we have, for all $s \in \left[0, \frac{k}{2}\right]$,

$$D^\alpha g(\delta_{w \circ \tau_x}(s)) = (-1)^{\alpha_1 + \alpha_3 + 1}[D^\alpha g(\delta_w(s))] \circ \tau_x,$$
$$D^\alpha g(\delta_{w \circ \tau_y}(s)) = (-1)^{\alpha_2 + \alpha_4}[D^\alpha g(\delta_w(s))] \circ \tau_y.$$

(iii) We have, for all $s \in \left[0, \frac{k}{2}\right]$,

$$D_4 g(\delta_w(s)) = -D_2 g(\delta_w(s)), \quad D_{21} g(\delta_w(s)) = -D_{41} g(\delta_w(s)),$$
$$D_{23} g(\delta_w(s)) = -D_{43} g(\delta_w(s)), \quad \text{and also} \quad D_{22} g(\delta_w(s)) = D_{44} g(\delta_w(s)).$$

Proof. The first point is a direct consequence of the chain rule. In order to prove the second point for $w \circ T_x$, we just have to see that for any $\delta \in \mathbb{R}^4$, by (2.7), we have

$$g(-\delta_1, \delta_2, -\delta_3, \delta_4) = -g(\delta_1, \delta_2, \delta_3, \delta_4),$$

so that for any $\alpha \in (\mathbb{Z}^+)^4$,

$$D^\alpha g(-\delta_1, \delta_2, -\delta_3, \delta_4) = (-1)^{\alpha_1+\alpha_3+1} D^\alpha g(\delta_1, \delta_2, \delta_3, \delta_4).$$

The result follows if we apply this identity to $\delta_w = (w_x, w_y, -w_x, -w_y)$ and use the first point. The result concerning $w \circ \tau_y$ is shown in the same manner.

As to the third point, using (2.7) we easily compute

$$D_2 g(\delta_1, \delta_2, \delta_3, \delta_4) = -\frac{p-2}{2} q(\delta)^{\frac{p-4}{2}} \left(\delta_1 + 2\frac{h}{k}\xi(\delta_1+\delta_3)\right)\left(\delta_2 + 2\frac{k}{h}\xi(\delta_2+\delta_4)\right),$$
$$D_4 g(\delta_1, \delta_2, \delta_3, \delta_4) = -\frac{p-2}{2} q(\delta)^{\frac{p-4}{2}} \left(\delta_1 + 2\frac{h}{k}\xi(\delta_1+\delta_3)\right)\left(\delta_4 + 2\frac{k}{h}\xi(\delta_2+\delta_4)\right).$$

Hence, as $\delta = \delta_w(s)$, we obtain

$$D_2 g(\delta_w(s)) = D_2 g(w_x, w_y, -w_x, -w_y) = -\frac{p-2}{2}(w_x^2 + w_y^2)^{\frac{p-4}{2}} w_x w_y,$$
$$D_4 g(\delta_w(s)) = D_4 g(w_x, w_y, -w_x, -w_y) = -\frac{p-2}{2}(w_x^2 + w_y^2)^{\frac{p-4}{2}} w_x(-w_y) = -D_2 g(\delta_w(s)),$$

which is the first claim of (i). The other claims are shown by similar computations. □

Let us note that, calculating δ_i on $\mathcal{K}^*$ by the Taylor expansion of $\overline{u}$ at a point $x \in \mathcal{K}^* \in \mathcal{T}^*$, we readily obtain the following result.

LEMMA 3.2 Let $\overline{u} \in W^{2,\infty}(\Omega)$. Assume that the mesh $\mathcal{T}$ satisfies (2.1) and (2.8). Then, there exists a constant C, depending only on c_1, γ and $\|\overline{u}\|_{W^{2,\infty}}$ such that for all $\mathcal{K}^* \in \mathcal{T}^*$, for all $x \in \mathcal{K}^*$ one has $\|\overline{u}^{\mathcal{T}}|_{1,\mathcal{K}^*} - |\nabla \overline{u}(x)\| \leqslant Ch$.

The main result of this section is the following proposition:

PROPOSITION 3.1 Assume that $\overline{u} \in W^{4,1}(\Omega) \cap W_0^{1,p}(\Omega)$. Then, for any $\mathcal{K} \in \mathcal{T}$, we can write $R_{\mathcal{K}} = R^i_{\mathcal{K}} + R^b_{\mathcal{K}}$, so that $R^b_{\mathcal{K}} = 0$ whenever $\partial\mathcal{K} \cap \partial\Omega = \emptyset$, and the following estimates hold.

$$|R^b_{\mathcal{K}}| \leqslant Ch^2 \int_{\partial\mathcal{K}\cap\partial\Omega} \sum_{|\beta|=3} |D^\beta \overline{u}| \mathrm{d}s + Ch \int_{\mathcal{K}} \sum_{|\beta|=2,3} |D^\beta \overline{u}| \mathrm{d}z$$
$$+ Ch \int_{\partial\mathcal{K}\cap\partial\Omega} |f| \mathrm{d}s, \quad \text{for } p > 3 \text{ or } p = 2,$$

$$|R^i_{\mathcal{K}}| \leqslant Ch^2 \sum_{\mathcal{K}^* \in V_{\mathcal{K}}} \int_{\mathcal{K}^*\cap\Omega} \left[\sum_{|\beta|=4} |D^\beta \overline{u}| + \sum_{|\beta|=3} |D^\beta \overline{u}|^2 + \sum_{|\beta|=2} |D^\beta \overline{u}|^3 + 1\right] \mathrm{d}z,$$
$$\text{for } p \geqslant 4 \text{ or } p = 2; \qquad (3.4)$$

furthermore this estimate also holds for all $p > 1$ when $|\nabla \overline{u}| \geqslant \mu > 0$ on $\mathcal{K}$ and h is small enough;

$$|R^i_{\mathcal{K}}| \leqslant Ch^{p-2}\left\{\sum_{\mathcal{K}^*\in V_{\mathcal{K}}}\int_{\mathcal{K}^*\cap\Omega}\left[\sum_{|\beta|=4}|D^\beta\overline{u}| + \sum_{|\beta|=3}|D^\beta\overline{u}|^2 + \sum_{|\beta|=2}|D^\beta\overline{u}|^3 + 1\right]\mathrm{d}z \right.$$
$$\left. + h\int_{\partial\mathcal{K}}\sum_{|\beta|=2}|D^\beta\overline{u}|^{\frac{2}{4-p}}\,\mathrm{d}s\right\}, \quad \text{for } 3 < p < 4. \tag{3.5}$$

REMARK 3.1 Note that for $3 < p < 4$, a similar estimate can be obtained if we require that $\overline{u} \in \mathcal{C}^{3,p-3}(\overline{\Omega})$.

Proof. We only consider the contributions from the set $\mathcal{E}^v_{\mathcal{K}}$ of vertical half-edges of $\mathcal{K}$, as the proof for the set $\mathcal{E}^h_{\mathcal{K}}$ of horizontal half-edges of $\mathcal{K}$ can be obtained in the same way. As in the beginning of this section, we can assume without loss of generality that $\mathcal{K} = (-\frac{h}{2}, \frac{h}{2}) \times (-\frac{k}{2}, \frac{k}{2})$ and we denote by $\sigma_0 = \{\frac{h}{2}\} \times [0, \frac{k}{2}]$ one of its vertical half-edges.

Let us rewrite (3.2) for each of the four vertical half-edges of $\mathcal{K}$. We use T_x, T_y and $T_x \circ T_y$ as changes of variables to express each term as an integral along the half-edge σ_0. We obtain

$$\sum_{\sigma\in\mathcal{E}^v_{\mathcal{K}}} R_{\sigma,\mathcal{K}} = \int_0^{\frac{k}{2}}[r^{\overline{u}}(s) + r^{\overline{u}\circ T_x}(s) + r^{\overline{u}\circ T_y}(s) + r^{\overline{u}\circ T_y\circ T_x}(s)]\mathrm{d}s, \tag{3.6}$$

where, for $s \in [0, \frac{k}{2}]$ and for a given function w on $\mathcal{K}$, we define

$$r^w(s) = g_{\sigma_0}(\delta^0_w) - g_{\sigma_0}(\delta_w(s)). \tag{3.7}$$

We shall consider a Taylor expansion of each of the terms in (3.6) using (3.7). For this reason, let us denote by $\varepsilon^w(s)$ the vector $(\varepsilon^w_1(s), \varepsilon^w_2(s), \varepsilon^w_3(s), \varepsilon^w_4(s)) = \delta^0_w - \delta_w(s)$. Furthermore, let us define $\chi \in]0, 1]$ to be the index of Hölder continuity of the second derivatives of g. We easily find that $\chi = 1$ for $p \geqslant 4$, and $\chi = p - 3$ for $3 < p < 4$. In fact, when $p \geqslant 4$, the function g is at least in $\mathcal{C}^3(\mathbb{R}^4)$.

We can now use the Taylor expansion of g around $\delta_w(s)$ in order to write $r^w(s)$ as follows:

$$r^w(s) = \sum_{i=1}^4 D_i g(\delta_w(s))\varepsilon^w_i(s) + \frac{1}{2}\sum_{i,j=1}^4 D_{ij}g(\delta_w(s))\varepsilon^w_i(s)\varepsilon^w_j(s) + |\varepsilon^w(s)|^{2+\chi}\mathcal{O}(1).$$

We remark now that, thanks to Lemma 3.1, one can rewrite this expansion as

$$\begin{aligned} r^w(s) = &\sum_{i\in\{1,3\}} D_i g(\delta_w(s))\varepsilon^w_i(s) + D_2 g(\delta_w(s))(\varepsilon^w_2(s) - \varepsilon^w_4(s)) \\ &+ \frac{1}{2}\sum_{i,j\in\{1,3\}} D_{ij}g(\delta_w(s))\varepsilon^w_i(s)\varepsilon^w_j(s) + \sum_{i\in\{1,3\}} D_{2i}g(\delta_w(s))(\varepsilon^w_2(s) - \varepsilon^w_4(s))\varepsilon^w_i(s) \\ &+ \frac{1}{2}D_{22}g(\delta_w(s))(\varepsilon^w_2(s) - \varepsilon^w_4(s))^2 + [D_{22}g(\delta_w(s)) + D_{24}g(\delta_w(s))]\varepsilon^w_2(s)\varepsilon^w_4(s) \\ &+ |\varepsilon^w(s)|^{2+\chi}\mathcal{O}(1). \end{aligned} \tag{3.8}$$

Let us compute precisely each of the terms in (3.2).

Suppose that $\mathcal{K}$ is an interior control volume. By interior control volume, we mean a volume $\mathcal{K} \in \mathcal{T}$ such that $\partial\mathcal{K} \cap \partial\Omega = \emptyset$. Denote by $x_0(s)$ the point $(\frac{h}{2}, s) \in \sigma_0$. The points $x_1 = (0,0)$, $x_2 = (h,0)$, $x_3 = (h,k)$ and $x_4 = (0,k)$ are the vertices of the dual control volume $\mathcal{K}^*$ containing σ_0. Note that, as $\mathcal{K}$ is an interior control volume, the points $(x_i)_{i=1,\dots,4}$ are located in $\overline{\Omega}$.

Using the Taylor formula for w to estimate $\varepsilon^w(s)$, we have, for any $s \in [0, \frac{k}{2}]$,

$$\begin{aligned}
\varepsilon_1^w(s) &= -s w_{xy} + \frac{h^2}{24} w_{xxx} + \frac{s^2}{2} w_{xyy} \\
&\quad + \mathcal{O}(1) h^3 \sum_{j=1,2} \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t, \\
\varepsilon_3^w(s) &= -(k-s) w_{xy} - \frac{h^2}{24} w_{xxx} - \frac{(k-s)^2}{2} w_{xyy} \\
&\quad + \mathcal{O}(1) h^3 \sum_{j=3,4} \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\varepsilon_2^w(s) &= \frac{h}{2} w_{xy} + \frac{1}{2}(k-2s) w_{yy} + \frac{h^2}{8} w_{xxy} + \frac{h(k-2s)}{4} w_{xyy} + \frac{k^2 - 3ks + 3s^2}{6} w_{yyy} \\
&\quad + \mathcal{O}(1) h^3 \sum_{j=2,3} \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t, \\
\varepsilon_4^w(s) &= \frac{h}{2} w_{xy} - \frac{1}{2}(k-2s) w_{yy} - \frac{h^2}{8} w_{xxy} + \frac{h(k-2s)}{4} w_{xyy} - \frac{k^2 - 3ks + 3s^2}{6} w_{yyy} \\
&\quad + \mathcal{O}(1) h^3 \sum_{j=1,4} \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t,
\end{aligned} \tag{3.10}$$

where all the derivatives of w in the main terms of the expansion are taken at the point $x_0(s)$. In particular, we have

$$\begin{aligned}
\varepsilon_2^w(s) - \varepsilon_4^w(s) &= (k-2s) w_{yy} + \frac{h^2}{4} w_{xxy} + \frac{k^2 - 3ks + 3s^2}{3} w_{yyy} \\
&\quad + \mathcal{O}(1) h^3 \sum_{j=1,2,3,4} \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t.
\end{aligned}$$

Note that the same expansions can be truncated at orders one or two; in this case the remainder terms take the form $\mathcal{O}(1) \sum_{j=1}^4 \int_0^1 (1-t)^{n-1} \sum_{|\beta|=n} |D^\beta w(t x_j + (1-t) x_0(s))| \mathrm{d}t$ with $n = 2$ or $n = 3$, respectively. Moreover, the second derivatives of $w \in W^{4,1}(\Omega) \cap W_0^{1,p}(\Omega)$ are uniformly bounded due to the Sobolev embeddings. Hence, we also have

$$\begin{aligned}
\varepsilon_2^w(s) \varepsilon_4^w(s) &= \frac{h^2}{4} (w_{xy})^2 - \frac{1}{4} (k-2s)^2 (w_{yy})^2 \\
&\quad + \mathcal{O}(1) h^3 \left(\sum_{j=1}^4 \int_0^1 \sum_{|\beta|=2,3} (1-t)^{2(|\beta|-1)} |D^\beta w(t x_j + (1-t) x_0(s))|^2 \mathrm{d}t \right).
\end{aligned}$$

Substituting the expressions for $\varepsilon^w(s)$ into Formula (3.2), we can see that each of the terms of order one with respect to h in $r^w(s)$ can be expressed as

$$a(s)D^\alpha g(\delta_w(s))D^\beta w\left(\frac{h}{2},s\right), \quad \text{with } |a(s)| \leqslant Ch, \ |\alpha|=1, \ |\beta|=2,$$
$$\text{where } \alpha_1+\alpha_3+\beta_x \text{ is even and } \alpha_2+\alpha_4+\beta_y \text{ is odd.} \tag{3.11}$$

Similarly, each of the terms of order two in $r^w(s)$ can be written either as

$$b(s)D^\alpha g(\delta_w(s))D^\beta w\left(\frac{h}{2},s\right), \quad \text{with } |b(s)| \leqslant Ch^2, \ |\alpha|=1, \ |\beta|=3,$$
$$\text{where } \alpha_1+\alpha_3+\beta_x \text{ is even,} \tag{3.12}$$

or as

$$b(s)D^\alpha g(\delta_w(s))D^\beta w\left(\frac{h}{2},s\right)D^\gamma w\left(\frac{h}{2},s\right), \quad \text{with } |b(s)| \leqslant Ch^2, \ |\alpha|=2, \ |\beta|=|\gamma|=2,$$
$$\text{where } \alpha_1+\alpha_3+\beta_x+\gamma_x \text{ is even.} \tag{3.13}$$

- Assume first that $p \geqslant 4$. Due to the properties stated in Lemma 3.1, we see that each of the terms of order two is of the following general form:

$$b(s)\psi^w\left(\frac{h}{2},s\right), \quad \text{with } |b(s)| \leqslant Ch^2 \text{ and } \psi^w\colon \bar{\mathcal{K}} \mapsto \mathbb{R} \text{ such that } \psi^{w\circ T_x} = -\psi^w \circ T_x.$$

From (3.12) and (3.13), ψ^w has an explicit form so that we can easily compute $\frac{\partial}{\partial x}\psi^{\overline{u}}$ for each term. Using that $g \in \mathcal{C}^3(\mathbb{R})$ (since $p \geqslant 4$) and that $\overline{u}$ is supposed to be in $W^{4,1}(\Omega)$ and it is therefore also in $W^{3,2}(\Omega)$, $W^{2,3}(\Omega)$ and $W^{1,\infty}(\Omega)$, by Young's inequality we get

$$\psi^{\overline{u}} \in W^{1,1}(\mathcal{K}) \quad \text{and}$$
$$\int_{\mathcal{K}}\left|\frac{\partial}{\partial x}\psi^{\overline{u}}\right|\mathrm{d}z \leqslant C\int_{\mathcal{K}}\left[\sum_{|\beta|=4}|D^\beta\overline{u}| + \sum_{|\beta|=3}|D^\beta\overline{u}|^2 + \sum_{|\beta|=2}|D^\beta\overline{u}|^3 + 1\right]\mathrm{d}z. \tag{3.14}$$

Therefore, the total contribution to (3.6) from ψ^w and $\psi^{w\circ T_x}$ is estimated by

$$\begin{aligned}
&\left|\int_0^{\frac{k}{2}} b(s)\left(\psi^{\overline{u}}\left(\frac{h}{2},s\right) + \psi^{\overline{u}\circ T_x}\left(\frac{h}{2},s\right)\right)\mathrm{d}s\right| \\
&\quad = \left|\int_0^{\frac{k}{2}} b(s)\left(\psi^{\overline{u}}\left(\frac{h}{2},s\right) - \psi^{\overline{u}}\left(-\frac{h}{2},s\right)\right)\mathrm{d}s\right| \\
&\quad \leqslant Ch^2\int_0^{\frac{k}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left|\frac{\partial}{\partial x}\psi^{\overline{u}}(x,s)\right|\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant Ch^2\int_{\mathcal{K}}\left[\sum_{|\beta|=4}|D^\beta\overline{u}| + \sum_{|\beta|=3}|D^\beta\overline{u}|^2 + \sum_{|\beta|=2}|D^\beta\overline{u}|^3 + 1\right]\mathrm{d}z,
\end{aligned} \tag{3.15}$$

and the same estimate holds if we replace $\overline{u}$ on the left-hand side by $\overline{u} \circ \tau_y$.
The contribution to (3.6) from any first-order term in (3.2) can be written in the following form:

$$a(s)\phi^{w}\left(\frac{h}{2}, s\right), \quad \text{with } |a(s)| \leqslant Ch, \text{ and } \phi^{w} \colon \mathcal{K} \mapsto \mathbb{R}$$
$$\text{such that } \phi^{w\circ T_x} = -\phi^{w} \circ \tau_x, \text{ and } \phi^{w\circ T_y} = -\phi^{w} \circ \tau_y.$$

Using (3.11), Young's inequality and Sobolev embeddings for $\overline{u}$, we can estimate the derivatives of $\phi^{\overline{u}}$ as follows:

$$\phi^{\overline{u}} \in W^{2,1}(\mathcal{K}), \quad \text{and}$$
$$\int_{\mathcal{K}} \left| \frac{\partial^2}{\partial x \partial y} \phi^{\overline{u}} \right| \mathrm{d}z \leqslant C \int_{\mathcal{K}} \left[\sum_{|\beta|=4} |D^{\beta}\overline{u}| + \sum_{|\beta|=3} |D^{\beta}\overline{u}|^2 + \sum_{|\beta|=2} |D^{\beta}\overline{u}|^3 + 1 \right] \mathrm{d}z. \tag{3.16}$$

Collecting the contributions to (3.6) from $\phi^{\overline{u}}, \phi^{\overline{u}\circ_x}, \phi^{\overline{u}\circ T_y}, \phi^{\overline{u}\circ T_y \circ T_x}$, we get

$$\begin{aligned}
&\left| \int_0^{\frac{k}{2}} a(s) \left(\phi^{\overline{u}}\left(\frac{h}{2}, s\right) + \phi^{\overline{u}\circ T_x}\left(\frac{h}{2}, s\right) + \phi^{\overline{u}\circ T_y}\left(\frac{h}{2}, s\right) + \phi^{\overline{u}\circ T_y \circ T_x}\left(\frac{h}{2}, s\right) \right) \mathrm{d}s \right| \\
&= \left| \int_0^{\frac{k}{2}} a(s) \left(\phi^{\overline{u}}\left(\frac{h}{2}, s\right) - \phi^{\overline{u}}\left(\frac{h}{2}, s\right) - \phi^{\overline{u}}\left(\frac{h}{2}, -s\right) + \phi^{\overline{u}}\left(-\frac{h}{2}, -s\right) \right) \mathrm{d}s \right| \\
&\leqslant Ch \int_0^{\frac{k}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-s}^{s} \left| \frac{\partial^2}{\partial x \partial y} \phi^{\overline{u}}(x, y) \right| \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}s \\
&\leqslant Ch^2 \int_{\mathcal{K}} \left[\sum_{|\beta|=4} |D^{\beta}\overline{u}| + \sum_{|\beta|=3} |D^{\beta}\overline{u}|^2 + \sum_{|\beta|=2} |D^{\beta}\overline{u}|^3 + 1 \right] \mathrm{d}z.
\end{aligned} \tag{3.17}$$

By (3.6), the contributions to the first- and second-order terms in $\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}^{v}} R_{\sigma,\mathcal{K}}$ are estimated by the right-hand side of (3.4).

- If $3 < p < 4$, we cannot obtain (3.14) and (3.16) since $\frac{\partial}{\partial x} D^{\alpha} g(\nabla \overline{u})$ can be unbounded for $|\alpha| = 2$. In this case, $D^{\alpha} g$ is only Hölder continuous of index $\chi = p - 3$. Therefore, we have to estimate in a different manner the terms of the form

$$I = \int_{-\frac{k}{2}}^{\frac{k}{2}} c(s) \left[D^{\alpha} g(\nabla \overline{u}(\xi)) D^{\beta} \overline{u}(\xi) D^{\gamma} \overline{u}(\xi) |_{\xi = \left(\frac{h}{2}, s\right)} - D^{\alpha} g(\nabla \overline{u}(\xi)) D^{\beta} \overline{u}(\xi) D^{\gamma} \overline{u}(\xi) |_{\xi = \left(-\frac{h}{2}, s\right)} \right] \mathrm{d}s$$

with $|\alpha| = 2$, $|\beta| = |\gamma| = 2$ and $|c(s)| \leqslant Ch^2$. Let us denote by $\pi^{\overline{u}}$ the product $D^{\beta}\overline{u} D^{\gamma}\overline{u}$. We remark that $|\pi^{\overline{u}}| \leqslant C \sum_{|\beta|=2} |D^{\beta}\overline{u}|^2 \in L^q(\partial\mathcal{K})$ for all $q < +\infty$, and $|\frac{\partial}{\partial x}\pi^{\overline{u}}| \leqslant C \sum_{|\beta|=2,3} |D^{\beta}\overline{u}|^2$. Let us estimate I by

$$\begin{aligned}
|I| \leqslant{}& Ch^2 \int_{-\frac{k}{2}}^{\frac{k}{2}} \left| D^{\alpha} g\left(\nabla \overline{u}\left(-\frac{h}{2}, s\right)\right) \right| \left| \pi^{\overline{u}}\left(\frac{h}{2}, s\right) - \pi^{\overline{u}}\left(-\frac{h}{2}, s\right) \right| \mathrm{d}s \\
&+ Ch^2 \int_{-\frac{k}{2}}^{\frac{k}{2}} \left| D^{\alpha} g\left(\nabla \overline{u}\left(\frac{h}{2}, s\right)\right) - D^{\alpha} g\left(\nabla \overline{u}\left(-\frac{h}{2}, s\right)\right) \right| \left| \pi^{\overline{u}}\left(\frac{h}{2}, s\right) \right| \mathrm{d}s.
\end{aligned} \tag{3.18}$$

The first part in the right-hand side of (3.18) is estimated by $Ch^2\int_{\mathcal{K}}\sum_{|\beta|=2,3}|D^\beta\overline{u}|^2\mathrm{d}z$. Using Young's inequality $ab\leqslant h^{\chi-1}a^{1/\chi}+h^\chi b^{1/(1-\chi)}, \forall\, a,b\in\mathbb{R}^+$, we can bound the second part by

$$Ch^2\int_{-\frac{k}{2}}^{\frac{k}{2}}\left|\sum_{|\beta|=2}\int_{-\frac{h}{2}}^{\frac{h}{2}}|D^\beta\overline{u}(x,s)|\mathrm{d}x\right|^\chi\left|\pi^{\overline{u}}\left(\frac{h}{2},s\right)\right|\mathrm{d}s$$
$$\leqslant Ch^2\left[h^{\chi-1}\int_{\mathcal{K}}\sum_{|\beta|=2}|D^\beta\overline{u}|\mathrm{d}z+h^\chi\int_{\partial\mathcal{K}}\sum_{|\beta|=2}|D^\beta\overline{u}|^{\frac{2}{1-\chi}}\,\mathrm{d}s\right].$$

As we have $\chi=p-3$, we see that I is estimated by the right-hand side of (3.5).

- Now consider the terms in $\int_0^{\frac{k}{2}}r^w(s)\mathrm{d}s$ corresponding to the contribution of the remainders in the Taylor expansions of $\varepsilon_i^w(s)$. Suppose that $w=\overline{u}$ (estimates with $w=\overline{u}\circ T_x,\overline{u}\circ T_y,\overline{u}\circ T_y\circ T_x$ are similar). Using Young's inequality and the bounds on the second derivatives of $\overline{u}$, we can estimate all of them by

$$C(h^{2+\chi}+h^3)\int_0^{\frac{k}{2}}\sum_{j=1}^4\int_0^1(1-t)\left[\sum_{|\beta|=4}|D^\beta\overline{u}(\xi)|+\sum_{|\beta|=3}|D^\beta\overline{u}(\xi)|^2+\sum_{|\beta|=2}|D^\beta\overline{u}(\xi)|^3+1\right]\mathrm{d}t\,\mathrm{d}s, \tag{3.19}$$

  where the integrand is evaluated at $\xi=tx_j+(1-t)x_0(s)$. Changing variables (t,s) to the Cartesian variables on $\mathcal{K}$, we see that the remainder terms in $\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^v}R_{\sigma,\mathcal{K}}$ are also controlled by the right-hand side of (3.4) and (3.5).

- The case $p=2$ can be treated in the same way as the case $p\geqslant 4$, with the understanding that the linearity of the Laplace equation considerably simplifies the calculations.

- The case $p>1$, $|\nabla\overline{u}|\geqslant\mu>0$, can be treated as that of $p\geqslant 4$ if we show that (3.2) remains valid with $\chi=1$. Since $W^{4,1}(\Omega)\subset W^{2,\infty}(\Omega)$, it follows by Lemma 3.2 that $|\overline{u}^{\mathcal{T}}|_{1,\mathcal{K}^*}=\frac{1}{2}(\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2)\geqslant\mu/2$ for all h small enough. Since the function g_σ in (2.7) is of class C^3 when restricted to the set $\{(\delta_1,\delta_2,\delta_3,\delta_4)\in\mathbb{R}^4|\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2\geqslant\mu\}$, the expansion (3.2) holds true, with $\chi=1$ and $\mathcal{O}(1)$ that depends on μ.

Suppose that $\mathcal{K}$ is a boundary control volume. The previous computations are only valid for interior control volumes. Now, we have to deal with boundary control volumes. Again, it is sufficient to estimate $\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^v}R_{\mathcal{K},\sigma}$. As we are concerned with the half-edge $\sigma_0=\{\frac{h}{2}\}\times[0,\frac{k}{2}]$, there are in fact only two new situations to study: when $\mathcal{K}$ is located on the upper boundary of Ω or when $\mathcal{K}$ is located on the right-hand boundary of Ω. The other configurations can easily be treated in a similar way.

- If $\mathcal{K}$ is located on the upper boundary of Ω, the computation of $\varepsilon_1^w(s)$ is the same as that in the interior case. Thanks to the conventional treatment of the boundary (see Fig. 1), we easily see that $\varepsilon_3^w(s)=\varepsilon_1^w(s)+2w_x(\frac{h}{2},s)$. We will use the following expressions only for $w=\overline{u}$ or $w=\overline{u}\circ T_y$ both of which satisfy the Dirichlet boundary condition $w=0$ on $\partial\Omega$. As we are concerned here with the top boundary of Ω, we also have $w_x=0$ on $\partial\Omega\cap\partial\mathcal{K}$. Let us introduce the points $y_1=x_1$, $y_2=x_2$, $y_3=(h,\frac{k}{2})$ and $y_4=(0,\frac{k}{2})$.

Adding the Taylor expansions of w_x at the point $x_0(s)$ evaluated respectively at y_3 and y_4, we get

$$2w_x\left(\frac{h}{2},s\right) = -(k-2s)w_{xy} + \frac{h^2}{4}w_{xxx} + \left(\frac{k}{2}-s\right)^2 w_{xyy} \\ + \mathcal{O}(1)h^3 \sum_{j=3,4}\int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(ty_j + (1-t)x_0(s))|\mathrm{d}t.$$

Finally, we obtain

$$\varepsilon_3^w(s) = -(k-s)w_{xy} + \frac{7h^2}{24}w_{xxx} + \left(\left(\frac{k}{2}-s\right)^2 + \frac{s^2}{2}\right)w_{xyy} \\ + \mathcal{O}(1)h^3 \sum_{j=1,2}\int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(ty_j + (1-t)x_0(s))|\mathrm{d}t.$$

Furthermore, using once more that $w=0$ on $\partial\Omega$, we find that

$$\varepsilon_2^w(s) = \frac{h}{2}w_{xy} + \frac{1}{4}(k-4s)w_{yy} + \frac{h^2}{8}w_{xxy} + \frac{h}{8}(k-4s)w_{xyy} + \frac{k^2-6ks+12s^2}{24}w_{yyy} \\ + \mathcal{O}(1)h^3 \sum_{j=2,3}\int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(ty_j + (1-t)x_0(s))|\mathrm{d}t,$$

$$\varepsilon_4^w(s) = \frac{h}{2}w_{xy} - \frac{1}{4}(k-4s)w_{yy} - \frac{h^2}{8}w_{xxy} + \frac{h}{8}(k-4s)w_{xyy} - \frac{k^2-6ks+12s^2}{24}w_{yyy} \\ + \mathcal{O}(1)h^3 \sum_{j=1,4}\int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^\beta w(ty_j + (1-t)x_0(s))|\mathrm{d}t.$$

As in the case of interior volumes, we will also use the lower-order Taylor expansions of ε_i. Collecting the four first-order terms in (3.6), arising from $\phi^{\overline{u}}$, $\phi^{\overline{u}\circ T_x}$, $\phi^{\overline{u}\circ T_y}$ and $\phi^{\overline{u}\circ T_x\circ T_y}$, we have to estimate

$$I = \left|\int_0^{\frac{k}{2}} \left[a_1(s)\phi^{\overline{u}}\left(\frac{h}{2},s\right) + a_1(s)\phi^{\overline{u}\circ T_x}\left(\frac{h}{2},s\right) + a_2(s)\phi^{\overline{u}\circ T_y}\left(\frac{h}{2},s\right) \right.\right. \\ \left.\left. + a_2(s)\phi^{\overline{u}\circ T_x\circ T_y}\left(\frac{h}{2},s\right)\right]\mathrm{d}s\right|,$$

where a_1 and a_2 are two distinct functions such that

$$|a_1(s)| \leqslant Ch, \quad |a_2(s)| \leqslant Ch,$$

and ϕ^w is such that $\phi^{w\circ T_x} = -\phi^w \circ T_x$ and $\phi^{w\circ T_y} = -\phi^w \circ T_y$.

Hence (3.17) becomes, in this case,

$$
\begin{aligned}
I &= \left| \int_0^{\frac{k}{2}} a_1(s) \left[\phi^{\overline{u}}\left(\frac{h}{2}, s\right) - \phi^{\overline{u}}\left(-\frac{h}{2}, s\right) \right] + a_2(s) \left[\phi^{\overline{u}}\left(-\frac{h}{2}, -s\right) - \phi^{\overline{u}}\left(\frac{h}{2}, -s\right) \right] \mathrm{d}s \right| \\
&\leqslant \left| \int_0^{\frac{k}{2}} a_1(s) \left[\phi^{\overline{u}}\left(\frac{h}{2}, s\right) - \phi^{\overline{u}}\left(-\frac{h}{2}, s\right) + \phi^{\overline{u}}\left(-\frac{h}{2}, -s\right) - \phi^{\overline{u}}\left(\frac{h}{2}, -s\right) \right] \mathrm{d}s \right| \\
&\quad + \left| \int_0^{\frac{k}{2}} (a_2(s) - a_1(s)) \left[\phi^{\overline{u}}\left(-\frac{h}{2}, -s\right) - \phi^{\overline{u}}\left(\frac{h}{2}, -s\right) \right] \mathrm{d}s \right| \\
&= I_1 + I_2.
\end{aligned}
$$

The first term, I_1, can be treated in the same way as (3.17). Using the expression of $\phi^{\overline{u}}$, we can estimate the second term by

$$
I_2 \leqslant Ch \int_0^{\frac{k}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left| \frac{\partial \phi^{\overline{u}}}{\partial x}(x, s) \right| \mathrm{d}x\, \mathrm{d}s \leqslant Ch \int_{\mathcal{K}} \left(\sum_{|\beta|=2} |D^{\beta}\overline{u}|^2 + \sum_{|\beta|=3} |D^{\beta}\overline{u}| \right) \mathrm{d}z. \tag{3.20}
$$

Thus, we obtain the estimate for the first-order terms. Furthermore, the second-order terms in (3.2) are estimated as in the case of interior control volumes since in (3.15) we only combine the terms associated with $\overline{u}$ and $\overline{u} \circ T_x$.

- If $\mathcal{K}$ is located near the right-hand boundary of Ω, note that the half-edge σ_0 is a part of the boundary of Ω. Consequently, the computation of $\varepsilon_4^w(s)$ is again given by (3.9) and (3.10). Furthermore, we have $\varepsilon_2^w(s) = \varepsilon_4^w(s) + 2w_y\left(\frac{h}{2}, s\right) = \varepsilon_4^w(s)$ since $w = w_y = 0$ on σ_0, which is a part of the vertical boundary of Ω.

  Using that $w = w_y = w_{yy} = w_{yyy} = 0$ on σ_0, we deduce that

$$
\begin{aligned}
\varepsilon_1^w(s) = &-\frac{h}{4} w_{xx} - s w_{xy} + \frac{h^2}{24} w_{xxx} + \frac{hs}{4} w_{xxy} + \frac{s^2}{2} w_{xyy} \\
&+ \mathcal{O}(1) h^3 \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^{\beta} w(tx_1 + (1-t)x_0(s))| \mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_3^w(s) = &\frac{h}{4} w_{xx} - (k-s) w_{xy} - \frac{h^2}{24} w_{xxx} + \frac{h(k-s)}{4} w_{xxy} - \frac{(k-s)^2}{2} w_{xyy} \\
&+ \mathcal{O}(1) h^3 \int_0^1 (1-t)^3 \sum_{|\beta|=4} |D^{\beta} w(tx_4 + (1-t)x_0(s))| \mathrm{d}t.
\end{aligned}
$$

Hence, ε_1^w and ε_3^w have the same form as the corresponding terms in (3.9) and (3.10), except for the terms $\pm\frac{h}{4} w_{xx}$ and $\pm\frac{h^2}{24} w_{xxx}$ which do not possess the good symmetry properties of the other terms. We remark that the estimates of the terms which are of the same form as in (3.9) and (3.10) remain true. Let us now concentrate on the two new terms.

The contribution in (3.6) to the second-order term $\pm\frac{h^2}{24}w_{xxx}$ can be easily controlled without symmetry considerations by

$$Ch^2\int_0^{\frac{k}{2}}(|D_1g|+|D_3g|)(|\overline{u}_{xxx}|+|(\overline{u}\circ\tau_y)_{xxx}|)\mathrm{d}s\leqslant Ch^2\int_{\partial K\cap\partial\Omega}|D^3\overline{u}|\mathrm{d}s.$$

Unfortunately, the contribution to the first-order term $\pm\frac{h}{4}w_{xx}$ is not so well controlled. This contribution, denoted by I_3, can be written as

$$I_3=\frac{h}{4}\int_0^{\frac{k}{2}}(D_1g(\delta_w(s))-D_3g(\delta_w(s)))w_{xx}\left(\frac{h}{2},s\right)\mathrm{d}s,$$

and since on σ_0 we have $w_y=0$, this yields

$$\begin{aligned}I_3&=\frac{h}{4}\int_0^{\frac{k}{2}}(D_1g(w_x,0,-w_x,0)-D_3g(w_x,0,-w_x,0))w_{xx}\left(\frac{h}{2},s\right)\mathrm{d}s\\&=\frac{h}{4}\int_0^{\frac{k}{2}}\frac{\partial}{\partial x}(g(w_x,0,-w_x,0))\mathrm{d}s.\end{aligned}$$

Using (2.7), we find that $g(w_x,0,-w_x,0)=-|w_x|^{p-2}w_x$, so that

$$\frac{\partial}{\partial x}(g(w_x,0,-w_x,0))=-(p-1)|w_x|^{p-2}w_{xx}.$$

As $w_y=0$ on σ_0, we deduce that

$$\begin{aligned}\frac{\partial}{\partial x}\left(|w_x^2+w_y^2|^{\frac{p-2}{2}}w_x\right)&=|w_x^2+w_y^2|^{\frac{p-2}{2}}w_{xx}+(p-2)|w_x^2+w_y^2|^{\frac{p-4}{2}}(w_xw_{xx}+w_yw_{xy})w_x\\&=(p-1)|w_x|^{p-2}w_{xx}=-\frac{\partial}{\partial x}(g(w_x,0,-w_x,0))\end{aligned}$$

on σ_0; using the fact that $|w_x^2+w_y^2|^{\frac{p-2}{2}}w_y=0$ identically on σ_0, we finally get

$$I_3=-\frac{h}{4}\int_0^{\frac{k}{2}}\frac{\partial}{\partial x}\left(|w_x^2+w_y^2|^{\frac{p-2}{2}}w_x\right)+\frac{\partial}{\partial y}\left(|w_x^2+w_y^2|^{\frac{p-2}{2}}w_y\right)\mathrm{d}s.$$

Hence, as $w=\overline{u}$ or $w=\overline{u}\circ\tau_y$, using (1.1), we find in each case

$$|I_3|\leqslant Ch\int_{\partial K\cap\partial\Omega}|f|\mathrm{d}s. \tag{3.21}$$

Let us note that the trace of f on $\partial\Omega$ is well-defined because f is continuous over $\overline{\Omega}$. Indeed, the second derivatives of $\overline{u}$ are in $W^{2,1}(\Omega)$ which is embedded into $\mathcal{C}^0(\overline{\Omega})$. This ends the proof of the estimate for the case of volumes adjacent to the right-hand boundary of Ω.

- The estimate of the mean consistency error for the volumes adjacent to the lower and the left-hand boundaries of Ω follows by a symmetry argument. Note that the terms corresponding to the four control volumes at the corners can be easily treated using only the estimates of the local consistency error given in Andreianov *et al.* (2004a).

Collecting estimates (3.15), (3.17), (3.19), (3.20) and (3.21), we obtain the claim of the proposition. □

3.3 $W^{1,p}$ *error estimates for the homogeneous problem*

Set $m = 2$ when $p \geqslant 4$ or $p = 2$, and $m = p - 2$ when $3 < p < 4$. From Proposition 3.1, we deduce the following error estimate:

$$\|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C\|\overline{u}\|_E h^{m/(p-1)} + c\left(\sup_{\partial\Omega}|f|\right)^{\frac{1}{p-1}} h^{\frac{1}{p}+\frac{1}{p-1}}. \tag{3.22}$$

Here the *a priori* regularity $\overline{u} \in E$ is assumed (we require in most cases $E = W^{4,1}(\Omega)$; see also Remark 3.1). More precisely, we have the following result.

THEOREM 3.1 Let $\overline{u} \in W_0^{1,p}(\Omega)$ be a solution of (1.1), and let $u^{\mathcal{T}}$ be the solution of the corresponding admissible finite-volume scheme, in the sense of Definition 2.2, on a uniform mesh $\mathcal{T}$ which satisfies (2.1) and (2.8).

There exist constants C, c that depend only on Ω, p, c_1 and γ such that

(i) for $\overline{u} \in E = W^{4,1}(\Omega)$ and $p > 3$, (3.22) holds;

(ii) for $p = 2$, and $\overline{u} \in E = W^{4,q}(\Omega)$ with some $q > 1$, (3.22) holds with $m = 2$. When $\overline{u} \in E = W^{4,1}(\Omega)$, (3.22) holds with h^2 replaced by $h^2|\ln h|$.

Note that $m > 1$ in each case, so that the convergence order obtained here is improved compared to the order $h^{\frac{1}{p-1}}$ obtained in Andreianov *et al.* (2004a, Theorem 3.1) for the finite-volume approximation of less regular solutions. Under specific assumptions on the integrability of $|\nabla\overline{u}|^{-\gamma}$, for some $\gamma > 0$ the results above will be further improved in Section 3.5.

REMARK 3.2 Let $p = 2$, $f \in \bigcup_{s>1} W^{2,s}(\Omega)$, and let $\overline{u}$ be a classical solution of (1.1) on the rectangle $\overline{\Omega}$. Then, the regularity assumption $\overline{u} \in \bigcup_{s>1} W^{4,s}(\Omega)$ is automatically fulfilled, so the result of Theorem 3.1(ii) applies.

For the sake of completeness, let us give a proof relying on Grisvard (1985, Theorem 4.4.4.13).

Proof. Denote the corners of the rectangle Ω by $S_0 = S_4 = (0, 0)$, $S_1 = (L_x, 0)$, $S_2 = (L_x, L_y)$, $S_3 = (0, L_y)$ and the edges of Ω by $\Gamma_j = [S_{j-1}, S_j]$, $j = 1, \ldots, 4$. Let $f \in W^{2,s_0}(\Omega)$ with $1 < s_0 < 2$. We start with $\overline{u} \in \mathcal{C}^2(\overline{\Omega})$, the solution to the Poisson problem (1.1), $p = 2$; note that $f(S_j) = 0$, $j = 1, \ldots, 4$. The function $u_x \in H^1(\Omega)$ solves the problem

$$\begin{cases} -\Delta(u_x) = f_x, \quad \text{in } \Omega, \\ u_x = 0, \quad \text{on } \Gamma_1 \cup \Gamma_3, \\ \frac{\partial}{\partial n}(u_x) = -f - u_{yy} = -f, \quad \text{on } \Gamma_2, \quad \frac{\partial}{\partial n}(u_x) = f + u_{yy} = f, \quad \text{on } \Gamma_4. \end{cases}$$

Note that the right-hand side of the equation is in $L^{s_1}(\Omega)$, $s_1 = \frac{2s_0}{2-s_0}$; furthermore, the boundary conditions on Γ_j, $j = 1, \ldots, 4$, are regular, and the compatibility conditions at the corners S_j are fulfilled since $f(S_j) = 0$. By Theorem 4.4.4.13 of Grisvard (1985) cited above, we deduce that $u_x \in W^{2,s_1}$. The same argument applies for u_y. Now, the function $u_{xx} \in H^1(\Omega)$ solves

$$\begin{cases} -\Delta(u_{xx}) = f_{xx}, \quad \text{in } \Omega, \\ u_{xx} = 0, \quad \text{on } \Gamma_1 \cup \Gamma_3, \\ u_{xx} = -f - u_{yy} = -f, \quad \text{on } \Gamma_2 \cup \Gamma_4. \end{cases} \tag{3.23}$$

The compatibility conditions are fulfilled at the corners, and we still have a regular enough source term and boundary data to conclude that $u_{xx} \in W^{2,s_0}(\Omega)$, by Theorem 4.4.4.13 of Grisvard (1985) cited above. The same reasoning holds for u_{yy}. This is sufficient so as to conclude that all third and fourth derivatives of u are integrable at least to the power s_0. □

Proof of Theorem 3.1. Using Lemma 2.5 and (3.3), we have

$$C\|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}}^p \leqslant (\nabla J_{\mathcal{T}}(u^{\mathcal{T}}) - \nabla J_{\mathcal{T}}(\overline{u}^{\mathcal{T}}), u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}) = \sum_{\mathcal{K}\in\mathcal{T}} \left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}} \right) (u_{\mathcal{K}} - \overline{u}_{\mathcal{K}})$$
$$= \sum_{\mathcal{K}\in\mathcal{T}} R_{\mathcal{K}}^i (u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}) + \sum_{\mathcal{K}\in\mathcal{T}} R_{\mathcal{K}}^b (u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}) \equiv E_1 + E_2. \tag{3.24}$$

- **Case** $p \geqslant 4$**:** Thanks to Proposition 3.1, the first term E_1 in the right-hand side of (3.24) can be estimated by

$$|E_1| \leqslant Ch^2 \|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{L^\infty} \int_\Omega \left[\sum_{|\beta|=4} |D^\beta \overline{u}| + \sum_{|\beta|=3} |D^\beta \overline{u}|^2 + \sum_{|\beta|=2} |D^\beta \overline{u}|^3 + 1 \right] \mathrm{d}z.$$

The term E_2 is in fact a sum over the boundary control volumes only, so that

$$|E_2| \leqslant Ch^2 \|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{L^\infty} \left(\int_{\partial\Omega} \sum_{|\beta|=3} |D^\beta \overline{u}| \mathrm{d}s \right)$$
$$+ Ch^2 \sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} \left(\int_{\mathcal{K}} \sum_{|\beta|=2,3} |D^\beta \overline{u}|^2 \mathrm{d}z \right)^{\frac{1}{2}} |u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}|$$
$$+ Ch^2 \left(\sup_{\partial\Omega} |f| \right) \sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} |u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}| \equiv T_1 + T_2 + T_3. \tag{3.25}$$

The term T_1 is estimated by $|E_1|$ using the Sobolev embedding theorem. By the Hölder inequality, the term T_2 is estimated by

$$T_2 \leqslant Ch^2 \left(\int_\Omega \sum_{|\beta|=2,3} |D^\beta \overline{u}|^2 \mathrm{d}z \right)^{\frac{1}{2}} \left(\sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K}) \left| \frac{u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}}{h} \right|^2 \right)^{\frac{1}{2}}$$
$$\leqslant Ch^2 \left(\int_\Omega \sum_{|\beta|=2,3} |D^\beta \overline{u}|^2 \mathrm{d}z \right)^{\frac{1}{2}} \left(\sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K}) \left| \frac{u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}}{h} \right|^p \right)^{\frac{1}{p}} \left(\sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K}) \right)^{\frac{p-2}{2p}}.$$

However, thanks to the boundary conditions (see Fig. 1), we know that

$$\left| \frac{2(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}})}{h} \right| = \left| \frac{(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}}) - [-(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}})]}{h} \right| \leqslant C|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}|_{1,\mathcal{K}^*},$$

where $\mathcal{K}^* \in V_{\mathcal{K}}$ is such that $\mathcal{K}^* \cap \partial\Omega \neq \emptyset$. Hence,

$$T_2 \leqslant Ch^2 h^{\frac{p-2}{2p}} \|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}}.$$

In the same way, the term T_3 in (3.25) can be estimated by

$$\begin{aligned} T_3 &\leqslant C\left(\sup_{\partial\Omega}|f|\right) h \sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K})\left|\frac{u_{\mathcal{K}}-\overline{u}_{\mathcal{K}}}{h}\right| \\ &\leqslant C\left(\sup_{\partial\Omega}|f|\right) h \left(\sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K})\left|\frac{u_{\mathcal{K}}-\overline{u}_{\mathcal{K}}}{h}\right|^p\right)^{\frac{1}{p}} \left(\sum_{\mathcal{K},\partial\mathcal{K}\cap\partial\Omega\neq\emptyset} m(\mathcal{K})\right)^{\frac{1}{p'}} \\ &\leqslant C\left(\sup_{\partial\Omega}|f|\right) h^{1+\frac{1}{p'}}\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}}. \end{aligned}$$

Finally, collecting the previous results, we get

$$\begin{aligned} \|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|^p_{1,p,\mathcal{T}} \leqslant{}& C\left(\sup_{\partial\Omega}|f|\right) h^{1+\frac{1}{p'}}\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \\ &+ Ch^2(\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{L^\infty}+\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}}), \end{aligned}$$

so that using Lemma 2.2(i)(a) we deduce that

$$\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C_1\left(\sup_{\partial\Omega}|f|\right)^{\frac{1}{p-1}} h^{\frac{1}{p}+\frac{1}{p-1}} + C_2 h^{\frac{2}{p-1}},$$

with the constant C_1 depending only on p, Ω, γ and c_1, and C_2 depending also on $\|\overline{u}\|_{W^{4,1}}$.

- **Case** $3 < p < 4$**:** In this case, the new term

$$C\left(h^{2+\chi}\sum_{\mathcal{K}}\int_{\partial\mathcal{K}}\sum_{|\beta|=2}|D^\beta\overline{u}|^{\frac{2}{4-p}}\,\mathrm{d}s\right)\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{L^\infty}$$

appears in (3.24) with $\chi = p-3$. This term can be rewritten as

$$Ch^{2+\chi}\left(\sum_{\Gamma}\int_{\Gamma}\sum_{|\beta|=2}|D^\beta\overline{u}|^{\frac{2}{4-p}}\,\mathrm{d}s\right)\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{L^\infty},$$

where the sum over Γ is taken over all the horizontal and vertical lines in Ω composed of the boundaries of the control volumes. We have at most $\frac{C}{h}$ such segments Γ, and for a given Γ, the integral is estimated using the Sobolev embedding $W^{4,1}(\Omega) \subset W^{2,\frac{2}{4-p}}(\Gamma)$. Note that the constant in the embedding is the same for all Γ values. Hence, the new term is finally estimated by

$$C_2 h^{1+\chi}\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{L^\infty}.$$

Other terms are estimated as in the case $p \geqslant 4$, with h^2 replaced by $h^{1+\chi}$.

- At this point, we have obtained, for $p \geqslant 4$ and $3 < p < 4$, an estimate of the form

$$\|u^{\mathcal{T}}-\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C_1\left(\sup_{\partial\Omega}|f|\right)^{\frac{1}{p-1}} h^{\frac{1}{p}+\frac{1}{p-1}} + C_2 h^{\frac{m}{p-1}},$$

where C_2 depends on $\|\overline{u}\|_{W^{4,1}(\Omega)}$. Since both (1.1) and the discrete schemes are homogeneous in $\overline{u}$, it follows by a scaling argument that, in fact, the constant C_2 depends linearly on $\|\overline{u}\|_{W^{1,4}}$, as claimed in the theorem.

- **Case** $p = 2$**:** For $\overline{u} \in W^{4,1}(\Omega)$ the claim follows in exactly the same way as in the case $p \geqslant 4$, using Lemma 2.2(ii)(a) in order to estimate $\|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{L^\infty}$.
  When $\overline{u} \in W^{4,q}(\Omega)$, $q > 1$, note that $\overline{u}$ is estimated by $\|\overline{u}\|_{W^{4,q}}$ in $W^{3,2q}(\Omega)$ and $W^{2,3q}(\Omega)$. Thus, we can apply the Hölder inequality in order to estimate the terms E_1 and T_1 (the estimates of T_2 and T_3 remain unchanged). We find $|T_1| \leqslant C|E_1|$ and

$$|E_1| \leqslant Ch^2 \|u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}\|_{L^{\frac{q}{q-1}}} \left(\int_\Omega \left[\sum_{|\beta|=4} |D^\beta \overline{u}| + \sum_{|\beta|=3} |D^\beta \overline{u}|^2 + \sum_{|\beta|=2} |D^\beta \overline{u}|^3 + 1 \right]^q \mathrm{d}z \right)^{\frac{1}{q}}.$$

The claim follows by using Lemma 2.2(i)(b) and the scaling argument above. □

REMARK 3.3 Using the $W^{1,p,\mathcal{T}}$ estimate of Theorem 3.1, one deduces, among other bounds, error estimates in L^p, L^∞ and $W^{1,q,\mathcal{T}}$, $q < p$; also, an estimate of $|J(\overline{u}) - J_{\mathcal{T}}(u^{\mathcal{T}})|$ can be obtained.

In the following two sections, we improve the above error estimates in two ways.

3.4 *Error estimate for 'all-uniform' schemes*

First, let us slightly change the definition of the mesh. Let us define the control volumes of the mesh to be of the form $\mathcal{K} = ((i - \frac{1}{2})h, (i + \frac{1}{2})h) \times ((j - \frac{1}{2})k, (j + \frac{1}{2})k)$ with $i = 1, \ldots, \frac{L_x}{h} - 1$, $j = 1, \ldots, \frac{L_y}{k} - 1$. In this case, Ω is not covered by $\bigcup_{\mathcal{K} \in \mathcal{T}} \overline{\mathcal{K}}$, but we have $\Omega = \bigcup_{\mathcal{K}^* \in \mathcal{T}^*} \overline{\mathcal{K}^*}$. In other words, we now consider a 'vertex-centred' method and not a 'cell-centred' method. The boundary condition is taken into account by assigning the value zero to all vertices of dual control volumes located on the boundary $\partial\Omega$. Discrete gradients are reconstructed in each dual volume $\mathcal{K}^*$ in the same way as for the previously considered schemes. With the zero boundary condition imposed in this way, Lemmas 2.1 and 2.2 still hold. Then, the scheme

$$a_{\mathcal{K}}(u^{\mathcal{T}}) = m(\mathcal{K}) f_{\mathcal{K}} \quad \forall \mathcal{K} \in \mathcal{T}, \tag{3.26}$$

is equivalent to the problem of minimizing the functional

$$J_{\mathcal{T}}\colon u^{\mathcal{T}} \mapsto \frac{1}{p} \sum_{\mathcal{K}^* \in \mathcal{T}^*} m(\mathcal{K}^*) \left| B^{\frac{1}{2}} (T_{\mathcal{K}^*}(u^{\mathcal{T}})) \right|^p - \sum_{\mathcal{K} \in \mathcal{T}} m(\mathcal{K}) f_{\mathcal{K}} u_{\mathcal{K}}. \tag{3.27}$$

This functional is strictly convex and coercive on $\mathbb{R}^{\mathcal{T}}$, so there exists a unique solution to the modified scheme. In the sequel, this scheme will be called all-uniform. Indeed, even if uniform meshes are chosen in the previously considered schemes, some symmetries are broken for the boundary control volumes; thus a lower-order term depending on $\sup_{\partial\Omega} |f|$ appears in the error estimate. For the all-uniform scheme, all volumes are covered by four isometric dual control volumes lying in Ω, so the symmetry is never broken. We have the following result.

THEOREM 3.2 Let $\overline{u}$ satisfy the assumptions of Theorem 3.1, and let $u^{\mathcal{T}}$ be the solution of an all-uniform scheme satisfying (2.1) and (2.8). Then the estimate (3.22) holds with $c = 0$, in each of the cases (i), (ii).

The proof is the same as for the 'interior volume' case of Theorem 3.1.

3.5 *Error estimates for weakly degenerate solutions of the inhomogeneous problem*

Improved $W^{1,q,\mathcal{T}}$ error estimates for $1 \leqslant q < p$ can be derived if we assume that either $|f|$ is positive or $|f|^{-\gamma}$ with some $\gamma > 0$ is integrable on Ω. In turn, interpolating $\|u^{\mathcal{T}} - \bar{u}^{\mathcal{T}}\|_{L^\infty}$ between the $W^{1,p,\mathcal{T}}$ and the $W^{1,q,\mathcal{T}}$ norm with $q < 2$, we improve the $W^{1,p,\mathcal{T}}$ estimate of Theorem 3.1. The key argument is the inequality of Barrett & Liu (1993, Lemma 2.1) for $\xi, \eta \in \mathbb{R}^2$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \geqslant C(p,t)|\xi - \eta|^t(|\xi| + |\eta|)^{p-t} \quad \text{as } p > 1, \quad t \geqslant 2.$$

Proceeding as in the proof of Lemma 2.5 (cf. Andreianov *et al.*, 2004a) and using (2.9), we can therefore replace the inequality (2.10) by an error estimate in a quasi-norm:

$$\sum_{\mathcal{K}^*} m\left(\mathcal{K}^* \bigcap \Omega\right) |u^{\mathcal{T}} - \bar{u}^{\mathcal{T}}|^t_{1,\mathcal{K}^*}(|u^{\mathcal{T}}|_{1,\mathcal{K}^*} + |\bar{u}^{\mathcal{T}}|_{1,\mathcal{K}^*})^{p-t}$$
$$\leqslant C(\nabla J_{\mathcal{T}}(u^{\mathcal{T}}) - \nabla J_{\mathcal{T}}(\bar{u}^{\mathcal{T}}), u^{\mathcal{T}} - \bar{u}^{\mathcal{T}}) = \sum_{\mathcal{K}\in\mathcal{T}} \left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}\right)(u_{\mathcal{K}} - \bar{u}_{\mathcal{K}}). \tag{3.28}$$

If for some $\nu \in (0, \infty]$, one has $\sum_{\mathcal{K}^*} m(\mathcal{K}^* \cap \Omega)|\bar{u}^{\mathcal{T}}|^{-\nu}_{1,\mathcal{K}^*} \leqslant C$ uniformly in h, using the inverse Hölder inequality and Proposition 3.1, one gets $W^{1,q,\mathcal{T}}$ estimates for sufficiently small q. This motivates the following definition.

DEFINITION 3.2 For $\nu > 0$, we say that $\bar{u}$ is ν-weakly degenerate, if $|\nabla\bar{u}|^{-\nu} \in L^1(\Omega)$. We say that $\bar{u}$ is non-degenerate, if $|\nabla\bar{u}| \geqslant \mu > 0$ on Ω.

REMARK 3.4 Since we are interested in ν-weakly degenerate solutions $\bar{u}$ which belong, in particular, to $W^{2,\infty}(\Omega)$, we have either $\nu < 2$ or that $\bar{u}$ is non-degenerate.

Indeed, if $\nabla\bar{u}(x_0) = 0$, then $|\nabla\bar{u}(x)| \leqslant C|x - x_0|$. Thus, the integrability of $|\nabla\bar{u}|^{-\nu}$ implies that $|x - x_0|^{-\nu}$ is integrable in a neighbourhood of $x_0 \in \mathbb{R}^2$, which implies that $\nu < 2$.

A sufficient condition of weak degeneracy and non-degeneracy of solutions of the p-Laplacian was given in Barrett & Liu (1993, Lemma 4.2), which in our case reads as follows.

LEMMA 3.3 Let $p > 2$, $\bar{u} \in W^{2,\infty}(\Omega)$ and $-\mathrm{div}(|\nabla\bar{u}|^{p-2}\nabla\bar{u}) = f$ on Ω. Then $|\nabla\bar{u}| \geqslant \mu > 0$ if $|f|^{-1} \in L^\infty(\Omega)$, and $|\nabla\bar{u}|^{-\gamma(p-2)} \in L^1(\Omega)$ if $|f|^{-\gamma} \in L^1(\Omega)$, $\gamma > 0$.

Note that the case of non-degenerate solutions, which can be natural in a wide range of physical situations, is interesting since it yields the consistency estimate (3.4) valid not only for $3 < p < 4$ (where it improves (3.5)) but also for all $p > 2$ and even for $1 < p < 2$.

A positive bound from below for $|\nabla\bar{u}|$ being incompatible with homogeneous Dirichlet boundary conditions, let us apply (3.28), Proposition 3.1 and interpolation techniques to study finite-volume approximations of weakly degenerate solutions of the problem

$$\begin{cases} -\mathrm{div}(|\nabla\bar{u}|^{p-2}\nabla\bar{u}) = f, & \text{on } \Omega, \\ \bar{u} = g, & \text{on } \partial\Omega, \end{cases} \tag{3.29}$$

with (at least) $f \in L^{p'}(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Actually, we consider solutions $\bar{u}$ that are at least $W^{4,1}$; in particular, it follows that g can be defined pointwise on $\partial\Omega$.

For the sake of simplicity, let us consider only all-uniform schemes for (3.29) (see Section 3.4). The inhomogeneous Dirichlet boundary condition is taken into account by assigning the values taken by the function g at the vertices of dual control volumes located on the boundary $\partial\Omega$. Discrete gradients are reconstructed in each dual volume $\mathcal{K}^*$ in the same way as for the previously considered schemes. Note that Lemmas 2.1–2.3 hold for all $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ supplemented with the zero boundary condition in the above sense. Then the scheme, still written in the form (3.26), is equivalent to the problem of minimizing the functional (3.27) on $\mathbb{R}^{\mathcal{T}}$, so that it possesses a unique solution. Also, the consistency estimates (3.4), (3.5) of Proposition 3.1 remain valid (recall that the error $R^b_{\mathcal{K}}$ is zero in the case of all-uniform schemes).

The error estimates for $p < 2$ use the $W^{1,p,\mathcal{T}}$ estimates on $\overline{u}^{\mathcal{T}}$ and $u^{\mathcal{T}}$ stated below.

LEMMA 3.4 Let $\overline{u} \in W^{2,\infty}$ be a solution of (3.29) and $u^{\mathcal{T}}$ the solution of the scheme (3.26) on a uniform mesh $\mathcal{T}$ satisfying (2.1) and (2.8). There exists a constant C that only depends on Ω, p, c_1, γ and $\|\overline{u}\|_{W^{2,\infty}}$ such that $\|\overline{u}^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C$ and $\|u^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant C$.

Proof. The first estimate follows directly from Lemma 3.2. The second estimate follows from the first one, if one multiplies (3.26) by $(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}}$ and uses (2.4), (2.5) and (2.9), as in the proof of Andreianov *et al.* (2004a, Lemma 3.3). □

In the following theorem, $W^{1,p,\mathcal{T}}$ and L^∞ error estimates are collected for the all-uniform scheme for the inhomogeneous problem.

THEOREM 3.3 Let $\overline{u} \in W^{1,p}(\Omega)$ be a solution of (3.29), and $u^{\mathcal{T}}$ the solution of the corresponding admissible all-uniform finite-volume scheme, in the sense of Definition 2.2, on a mesh $\mathcal{T}$ which satisfies (2.1) and (2.8). We denote $e^{\mathcal{T}} = u^{\mathcal{T}} - \overline{u}^{\mathcal{T}}$. Under the *a priori* regularity assumptions $\overline{u} \in E$, in the cases (i)–(iv) below we have the following error estimates.

(i) For $p > 2$, assume that $\overline{u}$ is non-degenerate. Assume also that $\overline{u} \in E = W^{4,1}(\Omega)$. Then,

$$\|e^{\mathcal{T}}\|_{1,2,\mathcal{T}}^{\leqslant} Ch^2|\ln h|, \quad \|e^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch^{\frac{4}{p}}|\ln h|^{\frac{2}{p}}, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^2|\ln h|^2. \tag{3.30}$$

(ii) For $p = 2$, assume that $\overline{u} \in E = W^{4,1}(\Omega)$. Then,

$$\|e^{\mathcal{T}}\|_{1,2,\mathcal{T}} \leqslant Ch^2|\ln h|, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^2|\ln h|^2. \tag{3.31}$$

In each of the cases (i), (ii), if $\overline{u} \in E = W^{4,s}(\Omega)$ for some $s > 1$, then the factors with $|\ln h|$ in the estimates (3.30), (3.31), respectively, can be omitted.

(iii) For $1 < p < 2$, assume that $\overline{u}$ is non-degenerate and $\overline{u} \in E = W^{4,1}(\Omega)$. Then,

$$\|e^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch^{\frac{3p-2}{p}}, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^{\frac{4(p-1)}{p}}. \tag{3.32}$$

If $\overline{u}$ is non-degenerate and $\overline{u} \in E = W^{4,\frac{2p}{3p-2}}(\Omega)$, then

$$\|e^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch^2, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^{\frac{3p-2}{p}}. \tag{3.33}$$

(iv) For $p > 3$, assume that $\overline{u}$ is ν-weakly degenerate, $0 < \nu < 2$ and $\overline{u} \in E = W^{4,1}(\Omega)$.

(a) For $3 < p < 4$ and $\nu \geqslant p - 2$, set $q_0 = \frac{2\nu}{p-2+\nu}$. Set $m = p - 2$. One has

$$\|e^{\mathcal{T}}\|_{1,q_0,\mathcal{T}} \leqslant Ch^{\frac{m(p+\nu)}{2p-2+\nu}}, \quad \|e^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch^{\frac{2m(p+\nu)}{p(2p-2+\nu)}}, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^{\frac{m(2+\nu)}{2p-2+\nu}}. \tag{3.34}$$

(b) For $\nu \leqslant p-2$, set $m=2$ with $p \geqslant 4$, and $m=p-2$ for $3<p<4$. One has

$$\|e^{\mathcal{T}}\|_{1,1,\mathcal{T}} \leqslant Ch^{\frac{2m(1+\nu)}{2p-2+\nu}}, \quad \|e^{\mathcal{T}}\|_{1,p,\mathcal{T}} \leqslant Ch^{\frac{2m(p+\nu)}{p(2p-2+\nu)}}, \quad \|e^{\mathcal{T}}\|_{L^\infty} \leqslant Ch^{\frac{m(2+\nu)}{2p-2+\nu}}. \tag{3.35}$$

In each of the cases (i)–(iv), the constant C depends on $\|\overline{u}\|_E$, p, Ω, c_1 and γ, and on μ (resp., on $\int_\Omega |\nabla\overline{u}|^{-\nu}$) for the cases of non-degenerate (resp., ν-weakly degenerate) solutions.

Applying Lemma 3.3, one can ensure non-degeneracy or ν-weak degeneracy.

COROLLARY 3.1 If $p>2, \overline{u} \in E = W^{4,1}(\Omega)$ (or $\overline{u} \in W^{4,s}(\Omega)$ for some $s>1$) and $|f|^{-1} \in L^\infty(\Omega)$, then the corresponding conclusions of Theorem 3.3(i) hold.

If $p>3, \overline{u} \in E = W^{4,1}(\Omega)$ and $|f|^{-\gamma} \in L^1(\Omega)$, $\gamma>0$, then the conclusions of Theorem 3.3(iv) hold with $\nu = \gamma(p-2)$.

Further, proceeding as in the proof of Remark 3.2, in the case $p=2$ one can weaken the *a priori* regularity assumptions on $\overline{u}$ in Theorem 3.3.

REMARK 3.5 Let $p=2$, and assume that $\overline{u}$ is a classical solution of (1.1) on the rectangle $\overline{\Omega}$. Let $(\Gamma_j)_{j=1,\ldots,4}$ be the partition of $\partial\Omega$, as introduced in the proof of Remark 3.2. If $\overline{u}$ corresponds to data (f,g) such that g is continuous on $\partial\Omega$, and for some $s>1$, $f \in W^{2,s}(\Omega)$ and $g \in W^{2-\frac{1}{s},s}(\Gamma_j)$, $j=1,\ldots,4$, then $\overline{u} \in W^{4,s}(\Omega)$, and the conclusion of Theorem 3.3(ii) holds.

REMARK 3.6 We suspect that for non-degenerate solutions of the p-Laplacian, regularity results similar to Remark 3.5 hold, at least for smooth domains Ω. Indeed, $\overline{u}$ solves in this case a non-degenerate elliptic equation $-\mathrm{div}(a(x)\nabla\overline{u}) = f$ with $a = |\nabla\overline{u}|^{p-2}$, and one can bootstrap the corresponding regularity results. For instance, if $\nabla\overline{u}$ is assumed to be periodic and non-degenerate (this assumption can be relevant in the case when $\overline{u}$ is the pressure field in a non-linear porous medium), then the above argument applies.

Finally, note that if no integrability of $|\nabla\overline{u}|^{-\nu}$ is assumed, we formally set $\nu=0$ in (3.35) and the proof of Theorem 3.3(i) still applies, providing the generalization of Theorem 3.2 to the non-homogeneous case.

3.6 *Proof of Theorem 3.3*

In each of the cases (i)–(iv), we apply (3.28) and then various Hölder, or interpolation inequalities in order to estimate $\|e^{\mathcal{T}}\|_{1,q_0,\mathcal{T}}$ with suitably chosen values of t and q_0. The estimates of $\|e^{\mathcal{T}}\|_{1,p,\mathcal{T}}$ and $\|e^{\mathcal{T}}\|_{L^\infty}$ are recovered by interpolation.

Note that within our method of proof, one could perform the same calculations with all admissible choices of t and q_0 (i.e. $1<q_0 \leqslant 2 \leqslant t$). Let us point out that the convergence orders we obtain with the special choices of t and q_0 below are the best ones.

Let us denote by C a generic constant with the dependencies admitted by in Theorem 3.3.

First, note that by Lemma 3.2, one has

$$|\nabla\overline{u}(x)| \leqslant |\overline{u}^{\mathcal{T}}|_{1,\mathcal{K}^*} + Ch \tag{3.36}$$

for all $x \in \mathcal{K}^* \in \mathcal{T}^*$ and all h small enough. Take $t \in [2,p]$ in (3.28); with (3.36), we get

$$\int_\Omega |\nabla\overline{u}|^{p-t}|e^{\mathcal{T}}|_{1,\mathcal{K}^*}^t \mathrm{d}x \leqslant C \sum_{\mathcal{K}\in\mathcal{T}} \left(\sum_{\sigma\in\mathcal{E}_\mathcal{K}} R_{\sigma,\mathcal{K}} \right) (u_\mathcal{K} - \overline{u}_\mathcal{K}) + Ch^{p-t} \int_\Omega |e^{\mathcal{T}}|_{1,\mathcal{K}^*}^t \mathrm{d}x.$$

Here and in the sequel, we write $|\nabla\overline{u}|^{p-t}|e^{\mathcal{T}}|^t_{1,\mathcal{K}^*}$ for the function defined a.e. on Ω by

$$x \mapsto |\nabla\overline{u}(x)|^{p-t}|e^{\mathcal{T}}|^t_{1,\mathcal{K}^*}, \quad \text{where } \mathcal{K}^* \in \mathcal{T}^* \text{ is the dual control volume that contains } x.$$

As $\overline{u} \in W^{4,1}(\Omega)$, by Proposition 3.1 adapted to all-uniform schemes (in which case the term $\mathcal{R}^b_{\mathcal{K}}$ disappears), proceeding as in the proof of Theorem 3.1, we deduce that

$$\int_\Omega |\nabla\overline{u}|^{p-t}|e^{\mathcal{T}}|^t_{1,\mathcal{K}^*}\mathrm{d}x \leqslant Ch^m\|e^{\mathcal{T}}\|_{L^\infty} + Ch^{p-t}\|e^{\mathcal{T}}\|^t_{1,t,\mathcal{T}}. \tag{3.37}$$

Here $m = 2$, except for the case $3 < p < 4$ in (iv) where $m = p - 2$. When $\overline{u} \in W^{4,s}(\Omega)$ with $s > 1$, proceeding as in the proof of Theorem 3.1 (case $p = 2$), we can replace (3.37) by

$$\int_\Omega |\nabla\overline{u}|^{p-t}|e^{\mathcal{T}}|^t_{1,\mathcal{K}^*}\mathrm{d}x \leqslant Ch^m\|e^{\mathcal{T}}\|_{L^{\frac{s}{s-1}}} + Ch^{p-t}\|e^{\mathcal{T}}\|^t_{1,t,\mathcal{T}}. \tag{3.38}$$

Finally, we saw in the proof of Theorem 3.1 that, in the limiting case where $t = p$, the two inequalities (3.37) and (3.38) reduce, respectively, to

$$\int_\Omega |e^{\mathcal{T}}|^p_{1,\mathcal{K}^*}\mathrm{d}x \leqslant Ch^m\|e^{\mathcal{T}}\|_{L^\infty} \tag{3.39}$$

and

$$\int_\Omega |e^{\mathcal{T}}|^p_{1,\mathcal{K}^*}\mathrm{d}x \leqslant Ch^m\|e^{\mathcal{T}}\|_{L^{\frac{s}{s-1}}} \tag{3.40}$$

since the extra term due to (3.36) is no longer needed.

Case (i): Non-degenerate solutions, $p > 2$

In the case where $\overline{u} \in W^{4,1}(\Omega)$, we start with (3.37) with $t = 2$. The non-degeneracy of $\overline{u}$ and Lemma 2.2(ii)(a) yield

$$\|e^{\mathcal{T}}\|^2_{1,2,\mathcal{T}} \leqslant Ch^2|\ln h|,$$

as stated in (3.30). The L^∞ estimate follows by another application of Lemma 2.2(ii)(a). Further, the $W^{1,p,\mathcal{T}}$ estimate follows from (3.39). Finally, the L^p estimate follows by Lemma 2.2(i)(b).

In the case where $\overline{u} \in W^{4,s}(\Omega)$ for some $s > 1$, we start with (3.38) with $t = 2$. Using the non-degeneracy of $\overline{u}$ and applying Lemma 2.2(i)(b) with $r = \frac{s}{s-1}$, we get the $W^{1,2,\mathcal{T}}$ estimate without the $|\ln h|$ factor. The $W^{1,p,\mathcal{T}}$ and L^∞ estimates follow from (3.40) and Lemma 2.2.

Case (ii): $p = 2$

The proof is a simplification of the previous one, with $t = p = 2$.

Case (iii): Non-degenerate solutions, $1 < p < 2$

Here we use (3.28) with $t = 2 > p$, and apply the Hölder inequality to obtain

$$\|e^{\mathcal{T}}\|^p_{1,p,\mathcal{T}} \leqslant C\left(\sum_{\mathcal{K}\in\mathcal{T}}\left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} R_{\sigma,\mathcal{K}}\right)(u_{\mathcal{K}} - \overline{u}_{\mathcal{K}})\right)^{\frac{p}{2}}\left(\sum_{\mathcal{K}^*} m(\mathcal{K}^*)(|u^{\mathcal{T}}|_{1,\mathcal{K}^*} + |\overline{u}^{\mathcal{T}}|_{1,\mathcal{K}^*})^p\right)^{\frac{2-p}{2}}.$$

We deduce, using Lemma 3.4, Proposition 3.1 and Lemma 2.2(ii)(b),

$$\|e^{\mathcal{T}}\|^p_{1,p,\mathcal{T}} \leqslant C(h^2\|e^{\mathcal{T}}\|_{L^\infty})^{\frac{p}{2}} \leqslant C\left(h^2 h^{-\frac{2-p}{p}}\|e^{\mathcal{T}}\|_{1,p,\mathcal{T}}\right)^{\frac{p}{2}},$$

which yields the $W^{1,p,\mathcal{T}}$ estimate in (3.32). The L^∞ estimate follows by Lemma 2.2(ii)(b). The L^p estimate is obtained by Lemma 2.1.

If $\overline{u} \in W^{4,\frac{2p}{3p-2}}(\Omega)$, by the Hölder inequality, we estimate $\sum_{\mathcal{K}\in\mathcal{T}}(\sum_{\sigma\in\mathcal{E}_\mathcal{K}} R_{\sigma,\mathcal{K}})(u_\mathcal{K} - \overline{u}_\mathcal{K})$ with $Ch^2\|\overline{u}\|_{W^{4,\frac{2p}{3p-2}}}\|e^\mathcal{T}\|_{L^{\frac{2p}{2-p}}}$, as in the proof of Theorem 3.1 (case $p = 2$). Using Lemma 2.2(i)(c), we deduce the $W^{1,p,\mathcal{T}}$ estimate in (3.33). By Lemma 2.2(ii)(b), the L^∞ estimate follows.

Case (iv): ν-weakly degenerate solutions, $p > 3$

We start with (3.39). For all $q_0 \in (1, 2)$, by Lemma 2.3 we have

$$\|e^\mathcal{T}\|^p_{1,p,\mathcal{T}} \leqslant Ch^m\|e^\mathcal{T}\|_{L^\infty} \leqslant Ch^m\|e^\mathcal{T}\|^\theta_{1,q_0,\mathcal{T}}\|e^\mathcal{T}\|^{1-\theta}_{1,p,\mathcal{T}},$$

where $\theta = \frac{q_0(p-2)}{2(p-q_0)}$. Using once more the interpolation inequalities for $\|e^\mathcal{T}\|_{L^\infty}$, we get the estimates

$$\|e^\mathcal{T}\|_{1,p,\mathcal{T}} \leqslant Ch^{\frac{m}{p-1+\theta}}\|e^\mathcal{T}\|^{\frac{\theta}{p-1+\theta}}_{1,q_0,\mathcal{T}}, \quad \|e^\mathcal{T}\|_{L^\infty} \leqslant Ch^{\frac{m(1-\theta)}{p-1+\theta}}\|e^\mathcal{T}\|^{\frac{p\theta}{p-1+\theta}}_{1,q_0,\mathcal{T}}. \tag{3.41}$$

- **Case (iv)(a): $\nu > p - 2$**

  In this case, we necessarily have $3 < p < 4$ by Remark 3.4, hence $m = p - 2$. Now let us take $t = 2$ in (3.37), and choose $q_0 = \frac{2\nu}{p-2+\nu}$. By the Hölder inequality, since $\overline{u}$ is ν-weakly degenerate and $-q_0\frac{p-2}{2-q_0} = -\nu$, we deduce

$$\begin{aligned}\|e^\mathcal{T}\|^{q_0}_{1,q_0,\mathcal{T}} &= \int_\Omega |e^\mathcal{T}|^{q_0}_{1,\mathcal{K}^*}\mathrm{d}x \leqslant \left(\int_\Omega |\nabla\overline{u}|^{(p-2)}|e^\mathcal{T}|^2_{1,\mathcal{K}^*}\mathrm{d}x\right)^{\frac{q_0}{2}}\left(\int_\Omega |\nabla\overline{u}|^{-q_0\frac{p-2}{2-q_0}}\mathrm{d}x\right)^{\frac{2}{2-q_0}}\\ &\leqslant Ch^{\frac{mq_0}{2}}\|e^\mathcal{T}\|^{\frac{q_0}{2}}_{L^\infty} + Ch^{\frac{q_0(p-2)}{2}}\|e^\mathcal{T}\|^{q_0}_{1,2,\mathcal{T}}.\end{aligned} \tag{3.42}$$

  Interpolating $\|e^\mathcal{T}\|_{1,2,\mathcal{T}}$ by Lemma 2.3 and using (3.41), we get

$$\|e^\mathcal{T}\|_{1,2,\mathcal{T}} \leqslant \|e^\mathcal{T}\|^\alpha_{1,q_0,\mathcal{T}}\|e^\mathcal{T}\|^{1-\alpha}_{1,p,\mathcal{T}} \leqslant Ch^{\frac{(1-\alpha)m}{p-1+\theta}}\|e^\mathcal{T}\|^{\frac{\alpha(p-1)+\theta}{p-1+\theta}}_{1,q_0,\mathcal{T}}, \tag{3.43}$$

  with $\alpha = \theta = \frac{q_0(p-2)}{2(p-q_0)}$. Substituting (3.41), (3.43) into (3.42), we obtain

$$\|e^\mathcal{T}\|_{1,q_0,\mathcal{T}} \leqslant Ch^{\frac{mp}{2(p-1+\theta)}}\|e^\mathcal{T}\|^{\frac{p\theta}{2(p-1+\theta)}}_{1,q_0,\mathcal{T}} + Ch^{\frac{p-2}{2}+\frac{m(1-\theta)}{p-1+\theta}}\|e^\mathcal{T}\|^{\frac{\theta(p-1)+\theta}{p-1+\theta}}_{1,q_0,\mathcal{T}}.$$

  By Young's inequality, we deduce

$$\|e^\mathcal{T}\|_{1,q_0,\mathcal{T}} \leqslant Ch^{\frac{mp}{2(p-1+\theta)-p\theta}} + Ch^{\frac{(p-2)(p-1+\theta)}{2(p-1)(1-\theta)}+\frac{m}{p-1}}. \tag{3.44}$$

  Note that for h small enough, the second term in (3.44) is controlled by the first one. Indeed, since $m = p - 2$, it is sufficient to set $z = 1 - \theta \in (0, 1)$ and show the inequality

$$\frac{p}{2(p-z) - p(1-z)} \leqslant \frac{p+z}{2z(p-1)}, \tag{3.45}$$

  which amounts to $p^2(1-z) + z(1+z)(p - \frac{2z}{1+z}) \geqslant 0$. Since $\frac{2z}{1+z} \leqslant 1 < p$, (3.45) holds true.

From (3.44), using the values of $m = p-2$, $q_0 = \frac{2\nu}{p-2+\nu}$ and $\theta = \frac{q_0(p-2)}{2(p-q_0)} = \frac{\nu}{p+\nu}$, we get

$$\|e^{\mathcal{T}}\|_{1,q_0,\mathcal{T}} \leqslant Ch^{\frac{mp}{2(p-1+\theta)-p\theta}} = Ch^{\frac{(p-2)(p+\nu)}{2p-2+\nu}},$$

as stated in (3.34). The $W^{1,p,\mathcal{T}}$ and L^∞ estimates in (3.34) follow by (3.41).

- **Case (iv)(b):** $\nu \leqslant p-2$
  Now we take $t = t_0 = \frac{p+\nu}{1+\nu}$ in (3.37), and choose $q_0 = 1$. As in (3.42), we deduce

$$\|e^{\mathcal{T}}\|_{1,1,\mathcal{T}} \leqslant Ch^{\frac{m}{t_0}} \|e^{\mathcal{T}}\|_{L^\infty}^{\frac{1}{t_0}} + Ch^{\frac{(p-t_0)}{t_0}} \|e^{\mathcal{T}}\|_{1,t_0,\mathcal{T}}.$$

Interpolating $\|e^{\mathcal{T}}\|_{1,t_0,\mathcal{T}}$ between $\|e^{\mathcal{T}}\|_{1,1,\mathcal{T}}$ and $\|e^{\mathcal{T}}\|_{1,p,\mathcal{T}}$ and using (3.41), we get, in the same way as in the case (iv)(a),

$$\|e^{\mathcal{T}}\|_{1,1,\mathcal{T}} \leqslant Ch^{\frac{mp}{t_0(p-1+\theta)-p\theta}} + Ch^{\frac{(p-t_0)(p-1+\theta)}{t_0(p-1)(1-\alpha)}+\frac{m}{p-1}},$$

with $\theta = \frac{p-2}{2(p-1)}$, $\alpha = \frac{p-t_0}{t_0(p-1)} = \frac{\nu}{p+\nu}$. Substituting θ and α, after cancellations, we find

$$\|e^{\mathcal{T}}\|_{1,1,\mathcal{T}} \leqslant Ch^{\frac{2m(1+\nu)}{2p-2+\nu}} + Ch^{\frac{\nu(2p-3)}{2(p-1)}+\frac{m}{p-1}} \leqslant Ch^{\frac{m(1+\nu)}{p-1+\frac{\nu}{2}}} + Ch^{\frac{\nu m(p-1-\frac{1}{2})}{2(p-1)}+\frac{m}{p-1}} \tag{3.46}$$

since $m \leqslant 2$. Setting $z = p-1$, we have

$$\frac{1+\nu}{z+\frac{\nu}{2}} - \frac{\nu\left(z-\frac{1}{2}\right)}{2z} - \frac{1}{z} = \frac{\nu\left(z-\frac{1}{2}\right)}{z\left(z+\frac{\nu}{2}\right)} - \frac{\nu\left(z-\frac{1}{2}\right)}{2z} = \frac{\nu\left(z-\frac{1}{2}\right)\left(2-z-\frac{\nu}{2}\right)}{2z\left(z+\frac{\nu}{2}\right)} < 0$$

since $z = p-1 > 2$ and $\nu > 0$. Thus, the second term on the right-hand side of (3.46) is controlled by the first one, for h small enough. This proves the $W^{1,1,\mathcal{T}}$ estimate in (3.35); the $W^{1,p,\mathcal{T}}$ and L^∞ estimates follow by (3.41).

REMARK 3.7 In the statement of Theorem 3.3, we assume that we control *a priori* the degeneracy of the exact solution $\overline{u}$. We saw in Lemma 3.3, due to Barrett & Liu (1993), that for $p > 2$ it is possible to provide *a priori* control of the degeneracy of $\overline{u}$ if the source term f does not vanish too quickly.

We note that, in fact, we can also replace the degeneracy restrictions on $\overline{u}$ by degeneracy restrictions on the approximate solution $u^{\mathcal{T}}$ which can be easily computed for any $1 < p < +\infty$. Indeed, one can replace the non-degeneracy assumption on $\overline{u}$ by the condition

$$\exists \underline{C} > 0, \quad |u^{\mathcal{T}}|_{1,\mathcal{K}^*} \geqslant \underline{C} \quad \forall \mathcal{K}^*, \tag{3.47}$$

the constant $\underline{C}$ being independent of the mesh size. In the same way, one can replace the ν-weak degeneracy assumption by the condition

$$\exists \overline{C} > 0, \quad \sum_{\mathcal{K}^*} m(\mathcal{K}^*)|u^{\mathcal{T}}|_{1,\mathcal{K}^*}^{-\gamma} \leqslant \overline{C}. \tag{3.48}$$

The principle of the proof is exactly the same: we start with (3.28) and use Proposition 3.1, replacing the inequality (3.37) by the inequality

$$\int_\Omega |u^{\mathcal{T}}|_{1,\mathcal{K}^*}^{p-t} |e^{\mathcal{T}}|_{1,\mathcal{K}^*}^{t} \mathrm{d}x \leqslant Ch^m \|e^{\mathcal{T}}\|_{L^\infty}.$$

Using either assumption (3.47) or assumption (3.48), we can now conclude the proof in the same way using Hölder and interpolation inequalities.

4. Numerical results

In this final section, we present some numerical results obtained using the finite-volume scheme described above. The aim of this section is to investigate the sharpness of our theoretical results.

We begin with an example of a non-degenerate solution whose analytical expression is given by $\overline{u}(x, y) = \exp(x + \pi y)$. For $p = 4$, the results are collected in Table 1. In this case, estimates (3.30) of Theorem 3.3 hold without the logarithmic factor since $\overline{u}$ is smooth enough. It appears that second-order convergence is achieved in all norms considered. Hence, the second-order error estimates in the $W^{1,2}$ and L^∞ norms stated in Theorem 3.3 are sharp whereas the $W^{1,p}$ estimate does not seem to be sharp.

In the case $p = 1.5$, the estimates (3.33) in Theorem 3.3 hold. In this case, numerical results reported in Table 2 show that the second-order $W^{1,p}$ estimate is sharp whereas the $\frac{5}{3}$ convergence order in the L^∞ norm is not sharp.

Our next example is a degenerate solution defined by $\overline{u}(x, y) = \sin(3\pi x)\sin(3\pi y)$. This solution is ν-degenerate with $\nu = 2 - \varepsilon$ for any $\varepsilon > 0$. We present in Table 3 the results obtained for $p = 4$ so that estimate (3.35) in Theorem 3.3 applies. We see that the numerical convergence rates are better than the theoretical predictions in any norm. Nevertheless, the error estimate in the $W^{1,1}$ norm is not too far from being optimal. Furthermore, to our knowledge, no other result in the literature is able to predict convergence orders greater than one in any Sobolev norm for finite-volume or P^1 finite-element schemes for the p-Laplacian.

Our last example is one of the test cases studied in Barrett & Liu (1993). The analytical radial solution we consider is $\overline{u}(x, y) = r^{\frac{\sigma+p}{p-1}}$, where r is the distance to the centre of the domain. This solution has exactly one critical point and is ν-degenerate with $\nu = \frac{2(p-1)}{\sigma+1}$. We present in Table 4 the results obtained with $p = 4$ and $\sigma = 7$. We can see that, as in Barrett & Liu (1993), the theoretical error estimates given by Theorem 3.3 are pessimistic. Nevertheless, our result ensures a theoretical convergence rate greater than one in the $W^{1,1}$ norm whereas the results in Barrett & Liu (1993) only give a theoretical $\frac{11-\varepsilon}{16}$ rate. Our results in the $W^{1,p}$ and L^∞ norms are also sharper than the one in Barrett & Liu (1993) since they obtain, in this reference, the convergence rates of $\frac{1}{2} - \varepsilon$ and $\frac{9}{16} - \varepsilon$, respectively.

Note finally that the comparison between our results with the all-uniform finite-volume method and the finite-element schemes used in Barrett & Liu (1993) is fair. Indeed, up to the choice of discretisation of the source term, the finite-element scheme studied in the reference above coincides with a slight modification of our all-uniform finite-volume scheme, where the discrete gradients are reconstructed by

TABLE 1 *Non-degenerate case,* $p = 4$

	$W^{1,2}$ error	$W^{1,p}$ error	L^∞ error
Theoretical order	2	1	2
Numerical order	2.0	2.0	2.0

TABLE 2 *Non-degenerate case,* $p = 1.5$

	$W^{1,p}$ error	L^p error	L^∞ error
Theoretical order	2	2	$\frac{5}{3} \approx 1.66$
Numerical order	2.0	2.0	2.0

TABLE 3 *Sinusoidal case,* $p = 4$, $\nu = 2 - \varepsilon$

	$W^{1,p}$ error	$W^{1,1}$ error	L^p error	L^∞ error
Theoretical order	$\frac{3}{4} - \varepsilon \approx 0.75$	$\frac{3}{2} - \varepsilon \approx 1.5$	$\frac{5}{4} - \varepsilon \approx 1.25$	1
Numerical order	1.15	1.8	1.8	1.5

TABLE 4 *Radial case,* $\sigma = 7$, $p = 4$, $\nu = \frac{3}{4}$

	$W^{1,p}$ error	$W^{1,1}$ error	L^p error	L^∞ error
Theoretical order	$\frac{19}{27} \approx 0.70$	$\frac{28}{27} \approx 1.04$	$\frac{25}{27} \approx 0.93$	$\frac{22}{27} \approx 0.81$
Numerical order	2.0	2.0	2.0	2.0

affine interpolation in triangles obtained by bisection of our dual control volumes $\mathcal{K}^*$. Moreover, if all the bisections are performed in the same direction, as it is the case for the numerical results presented in Barrett & Liu (1993), it can be shown that these schemes also possess the higher-order consistency properties of Proposition 3.1. Hence, the proof of Theorem 3.3 can be easily adapted to the finite-element method and we expect the two schemes to have the same behaviour on uniform grids.

5. Conclusions

We presented in this paper the convergence analysis of a family of nine-point finite-volume schemes for sufficiently smooth solutions to the p-Laplace equation on uniform Cartesian meshes. By taking advantage of the symmetries of the scheme, we have been able to improve the error estimates in the energy space $W^{1,p}$ obtained previously for the same schemes. Moreover, adapting to the finite-volume scheme the quasi-norm estimate method introduced in Barrett & Liu (1993), we have given further improved convergence rates under the assumption that the exact solution is either non-degenerate or weakly degenerate. In particular, for non-degenerate smooth solutions we obtain, in the case $p \geqslant 2$, a sharp second-order estimate in the L^∞ norm. Finally, numerical examples have been presented which show that our results are sharp in some but not all situations. We have also provided comparisons with previously published convergence results for P^1 finite-element schemes closely related to our finite-volume schemes.

Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions.

REFERENCES

AINSWORTH, M. & KAY, D. (2000) Approximation theory for the hp-finite element method and application to the non-linear Laplacian. *Appl. Numer. Math.*, **34**, 329–344.

ANDREIANOV, B., BOYER, F. & HUBERT, F. (2004a) Finite volume schemes for the p-Laplacian on cartesian meshes. *Math. Model. Numer. Anal.*, **38**, 931–959.

ANDREIANOV, B., BOYER, F. & HUBERT, F. (2005) Besov regularity and new error estimates for finite volume approximations of the p-Laplacian. *Numer. Math.*, **100**, 565–592.

ANDREIANOV, B., GUTNIC, M. & WITTBOLD, P. (2004b) Convergence of finite volume approximations for a nonlinear elliptic-parabolic problem: a “continuous approach.” *SIAM J. Numer. Anal.*, **42**, 228–251.

BARRETT, J. W., & LIU, W. B. (1993) Finite element approximation of the p-Laplacian. *Math. Comput.*, **61**, 523–537.

CHOW, S. (1989) Finite element error estimates for non-linear elliptic equations of monotone type. *Numer. Math.*, **54**, 373–393.

COUDIÈRE, Y., GALLOUËT, T. & HERBIN, R. (2001) Discrete Sobolev inequalities and L^p error estimates for approximate finite volume solutions of convection diffusion equations. *Math. Model. Numer. Anal.*, **35**, 767–778.

EBMEYER, C. & LIU, W. B. (2005) Quasi-norm interpolation error estimates for the piecewise linear finite element approximation of p-Laplacian problems. *Numer. Math.*, **100**, 233–258.

EBMEYER, C., LIU, W. B. & STEINHAUER, M. (2005) Global regularity in fractional order Sobolev spaces for the p-Laplace equation on polyhedral domains. *Z. Anal. Anwend.*, **24**, 353–374.

EYMARD, R., GALLOUËT, T. & HERBIN, R. (2000) Finite volume methods. *Handbook of Numerical Analysis*, vol. VII (P. G. Ciarlet & J. L. Lions eds). Amsterdam, North-Holland.

GLOWINSKI, R. & MARROCCO, A. (1975) Sur l’approximation par éléments finis d’ordre un, et la résolution, par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires. *J. Math. Anal. Appl. RAIRO Anal. Num.*, **2**, 41–64.

GRISVARD, P. (1985) *Elliptic Problems on Nonsmooth Domains*. Monographs and Studies in Mathematics **24**, Pitman, Boston.

SAMARSKII, A. A. & ANDRÉEV, V. B. (1978) *Méthodes aux différences pour équations elliptiques*. Moscou: Mir.

SIMON, J. (1978) Caractérisation d’espaces fonctionnels. *Boll. Unione Mat. Ital.*, **15**, 687–714.