Journal of Functional Analysis **171**, 15–33 (2000)
doi:10.1006/jfan.1999.3445, available online at http://www.idealibrary.com on IDEAL®

L^1-Theory of Scalar Conservation Law with Continuous Flux Function

Boris P. Andreianov

Lomonossov Moscow State University, Russia; and Université de Franche-Comté, Besançon, France

Philippe Bénilan

Université de Franche-Comté, Besançon, France

and

Stanislav N. Kruzhkov[1]

Lomonossov Moscow State University, Russia

Communicated by H. Brézis

Received July 19, 1998; revised March 11, 1999; accepted March 18, 1999

Uniqueness of a generalized entropy solution (g.e.s.) to the Cauchy problem for N-dimensional scalar conservation laws $u_t + \operatorname{div}_x \phi(u) = g$, $u(0, \cdot) = f$ with continuous flux function ϕ is still an open problem. For data (f, g) vanishing at infinity, we show that there exist a maximal and a minimal g.e.s. to the Cauchy problem and to the associated stationary problem $u + \operatorname{div}_x \phi(u) = f$. In the case of L^1 data, using the nonlinear semigroup theory, we prove that there is uniqueness for all data of a g.e.s. to the Cauchy problem if and only if there is uniqueness for all data of a g.e.s. to the related stationary problem. Applying this result and an induction argument on the dimension N, we extend uniqueness results of Bénilan, Kruzhkov (1996, *Nonlinear Differential Equations Appl.* **3**, 395–419) for flux having some monotonicity properties.

1. INTRODUCTION

We consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}_x \phi(u) = g & \text{on} \quad Q = \{(t, x);\, t \in (0, T),\, x \in \mathbb{R}^N\} \\ u(0, \cdot) = f & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{CP}$$

where $\phi\colon \mathbb{R} \mapsto \mathbb{R}^N$ is only assumed to be continuous and (f, g) satisfy

[1] Died in June 1997.

0022-1236/00 $35.00

$$\begin{cases} f = f_0 + c \quad \text{with} \quad c \in \mathbb{R}, \quad f_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \\ g \in L^1(Q), \quad g(t, \cdot) \in L^\infty(\mathbb{R}^N) \quad \text{for a.a.} \quad t \in (0, T) \\ \quad \text{and} \quad \int_0^T \|g(t, \cdot)\|_\infty \, dt < \infty \end{cases} \tag{1}$$

A solution of (CP) will be understood in the sense of the generalized entropy solution (g.e.s.) as introduced by S. N. Kruzhkov (cf. [K69a, K69b, K70]). In the case of a locally Lipschitz continuous flux function ϕ, there exists a unique bounded g.e.s.; this is actually true for any (f, g) satisfying

$$\begin{cases} f \in L^\infty(\mathbb{R}^N), \quad g \in L^1_{\text{loc}}(Q), \\ g(t, \cdot) \in L^\infty(\mathbb{R}^N) \quad \text{for a.a.} \quad t \in (0, T) \quad \text{and} \quad \int_0^T \|g(t, \cdot)\|_\infty \, dt < \infty. \end{cases} \tag{2}$$

For the general continuous flux function ϕ the situation is more delicate. Let consider the particular case $N = 2$, $\phi(u) = (|u|^{\alpha - 1} u/\alpha, |u|^{\beta - 1} u/\beta)$. It has been shown in [KP90] that if $\alpha \neq \beta$, $\alpha + \beta < 1$, then for some $f \in L^\infty(\mathbb{R}^2)$ the problem (CP), with $g = 0$, has a one-parameter family of different bounded g.e.s. On the other hand, it has been shown in [BK96] that if (f, g) satisfy (1), then for any $\alpha, \beta > 0$ there exists a unique bounded g.e.s. of (CP). In this paper we shall improve this last result, showing (cf. Theorem 3) that (CP) has a unique bounded g.e.s. for any (f, g) satisfying (1) according to whether the flux function ϕ satisfies

$$\begin{cases} \text{There exist orthonormal vectors } \xi_1, ..., \xi_{N-1} \text{ such that} \\ r \in \mathbb{R} \mapsto \xi_i \cdot \phi(r) \in \mathbb{R} \text{ is nondecreasing}, \quad i = 1, ..., N-1. \end{cases} \tag{3}$$

Actually, while we shall prove another uniqueness result (cf. Theorem 4), we still do not know whether there is or is not uniqueness of bounded g.e.s. for (f, g) satisfying (1) with any continuous flux function ϕ, but we shall prove that there always exist a maximum and a minimum bounded g.e.s. of (CP). More precisely, for any continuous flux function ϕ and (f, g) satisfying

$$\begin{cases} f = f_0 + c \text{ with } c \in \mathbb{R}, f_0 \in L^\infty_0(\mathbb{R}^N) \\ \quad = \{h \in L^\infty(\mathbb{R}^N);\ \lambda^N\{|h| > \delta\} < \infty \ \forall \delta > 0\} \\ g \in L^1_{\text{loc}}(Q), g(t, \cdot) \in L^\infty_0(\mathbb{R}^N) \text{ for a.a. } t \in (0, T) \\ \quad \text{and} \int_0^T \|g(t, \cdot)\|_\infty \, dt < \infty \end{cases} \tag{4}$$

(see Footnote[2]) for all $c \in \mathbb{R}$ there exist a maximum and a minimum bounded g.e.s. of (CP), which coincide except for a countable set of values of c depending on ϕ, f_0, and g (cf. Theorem 1 and Proposition 1).

[2] λ^N denote the N-dimensional Lebesgue measure; $\{|h| > \delta\}$ stands for $\{x \in \mathbb{R}^N;\ |h(x)| > \delta\}$ and so on.

As pointed out in [C72] and [B72], solutions of (CP) for (f, g) satisfying (1) can be constructed through the nonlinear semigroup theory from the solutions of the equation

$$u + \operatorname{div}_x \phi(u) = f \qquad \text{on } \mathbb{R}^N. \tag{E}$$

As was done in [BK96], we shall derive for the equation (E) the same properties as for the Cauchy problem (CP); actually, we shall prove (cf. Corollary 1), for ϕ and c given, that there is uniqueness of a bounded g.e.s. of (CP) for all (f, g) satisfying (1) if and only if there is uniqueness of a bounded g.e.s. of (E) for all $f = f_0 + c$ with $f_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

2. EXISTENCE OF MAXIMUM AND MINIMUM GENERALIZED ENTROPY SOLUTIONS

Throughout this paper ϕ: $\mathbb{R} \mapsto \mathbb{R}^N$ is a continuous function and we consider the Cauchy problem (CP) as well as the equation (E). Recall the following definition:

DEFINITION 1. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. A sub-g.e.s. (generalized entropy subsolution) (respectively, super-g.e.s.) of (E) is a function $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$\alpha \cdot (u-k) + \operatorname{div}_x\{\alpha \cdot (\phi(u) - \phi(k))\} \leqslant \alpha \cdot (f-k) \text{ in } \mathscr{D}'(\mathbb{R}^N) \text{ for any } k \in \mathbb{R},$$

where $\alpha = \operatorname{sign}^+(u-k)$ (respectively, $\operatorname{sign}^-(u-k)$) (see Footnote[3]). A function u is a generalized entropy solution (g.e.s.) of (E) if it is both sub- and super-g.e.s.

DEFINITION 2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $g \in L^1_{\text{loc}}(Q)$. A sub-g.e.s. (respectively, super-g.e.s.) of (CP) is a function $u \in L^\infty_{\text{loc}}(Q)$ satisfying

$$\frac{\partial}{\partial t}\{\alpha \cdot (u-k)\} + \operatorname{div}_x\{\alpha \cdot (\phi(u) - \phi(k))\} \leqslant \alpha \cdot g \text{ in } \mathscr{D}'(Q) \text{ for any } k \in \mathbb{R},$$

where $\alpha = \operatorname{sign}^+(u-k)$ (respectively, $\operatorname{sign}^-(u-k)$), and $(u(t, \cdot) - f)^+ \to 0$ (respectively, $(u(t, \cdot) - f)^- \to 0$) in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0$ essentially. A function u is a g.e.s. of (CP) if it is both sub- and super-g.e.s.

The main result is the following theorem.

[3] We use the notation sign^+ for the Heaviside function, i.e, the characteristic function of $(0, +\infty)$, and $\operatorname{sign}^-(r) = -\operatorname{sign}^+(-r)$.

THEOREM 1. *Let* (f, g) *satisfy* (4). *Then there exist a maximum and a minimum bounded g.e.s. of* (E) *and of* (CP).

More precisely, considering the equation (E) and $f = f_0 + c$ with $c \in \mathbb{R}$, $f_0 \in L_0^\infty(\mathbb{R}^N)$, we shall prove that there exists a (unique) g.e.s $\bar{u} \in L^\infty(\mathbb{R}^N)$ such that $\bar{u} \geqslant u$ a.e. on $\mathbb{R}^N$ for any sub-g.e.s. $u \in L^\infty(\mathbb{R}^N)$ of (E).

This g.e.s. $\bar{u}$ will be obtained as the a.e. pointwise limit of a nonincreasing sequence $\{u_n\}$, where u_n is any bounded g.e.s. of (E) corresponding to $f = f_0 + c_n$ with a sequence $\{c_n\}$ in $\mathbb{R}$ decreasing to c.

The same corresponding results are valid for (CP) and minimum solutions.

The main new ingredient in the proof of Theorem 1 is the following lemma.

LEMMA 1. (a) *Let* u *and* $\hat{u}$ *be bounded sub- and super-g.e.s., respectively, of* (E), *corresponding to* f *and* $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, *respectively. Assume that*

$$\lambda^N\{x \in \mathbb{R}^N; u(x) > \hat{u}(x)\} < \infty, \tag{5}$$

then

$$\int (u - \hat{u})^+ + \int_{\{u > \hat{u}\}} (f - \hat{f})^- \leqslant \int_{\{u \geqslant \hat{u}\}} (f - \hat{f})^+, \tag{6}$$

and in particular, if $f \leqslant \hat{f}$ *a.e. on* $\{u \geqslant \hat{u}\}$, *then* $u \leqslant \hat{u}$ *a.e. on* $\mathbb{R}^N$.

(b) *Let* u *and* $\hat{u}$ *be bounded sub- and super-g.e.s., respectively, of* (CP), *corresponding to* (f, g) *and* $(\hat{f}, \hat{g}) \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \times L^1_{\mathrm{loc}}(Q)$, *respectively. Assume that*

$$\lambda^{N+1}\{(t, x) \in Q; u(t, x) > \hat{u}(t, x)\} < \infty, \tag{7}$$

then for a.a. $t \in (0, T)$,

$$\int (u(t, \cdot) - \hat{u}(t, \cdot))^+ + \int_0^t \int_{\{u > \hat{u}\}} (g - \hat{g})^- \leqslant \int (f - \hat{f})^+ + \int_0^t \int_{\{u \geqslant \hat{u}\}} (g - \hat{g})^+, \tag{8}$$

and in particular, if $f \leqslant \hat{f}$ *a.e. on* $\mathbb{R}^N$ *and* $g \leqslant \hat{g}$ *a.e. on* $\{u \geqslant \hat{u}\}$, *then* $u \leqslant \hat{u}$ *a.e. on* Q.

Proof. Applying Lemma 3.1(a) in [BK96] (cf. also [K70, B72, C72, Ba88]), we have in the case (a) that

$$\int (u-\hat{u})^+\zeta + \int_{\{u>\hat{u}\}} (f-\hat{f})^- \zeta$$

$$\leqslant \int_{\{u\geqslant \hat{u}\}} (f-\hat{f})^+ \zeta + \int |\phi(u)-\phi(\hat{u})|\, \chi_{\{u>\hat{u}\}}\, |D\zeta|$$

for all $\zeta \geqslant 0$, $\zeta \in \mathscr{D}(\mathbb{R}^N)$. By the assumption (5), $|\phi(u)-\phi(\hat{u})|\, \chi_{\{u>\hat{u}\}} \in L^1(\mathbb{R}^N)$, so that we may let ζ tend to 1 to obtain (6) at the limit and prove (a). The proof of (b) is identical using Lemma 3.1(b) in [BK96] and (7). ∎

We also need the following general existence result, partially contained in [B72, KH74], for which we give a complete proof in the Appendix.

LEMMA 2. *Let $f \in L^\infty(\mathbb{R}^N)$ (resp., and $g \in L^1_{\text{loc}}(Q)$ satisfying $g(t,\cdot) \in L^\infty(\mathbb{R}^N)$ for a.a. $t \in (0,T)$ and $\int_0^T \|g(t,\cdot)\|_\infty\, dt < \infty$). Then there exists a bounded g.e.s. of* (E) *(resp.,* (CP)*).*

Proof of Theorem 1. Let $\{c_n\}$ be a sequence in $\mathbb{R}$ decreasing to c and, for $n \in \mathbb{N}$, u_n be a bounded g.e.s of (E) corresponding to $f_n = f_0 + c_n$. Such a g.e.s. exists by Lemma 2.

Fix $n > m$. Set $h = (c_n + c_m)/2$ and take $0 < \delta < (c_m - c_n)/2$; $\delta = ((c_m - c_n)/2) - \alpha$ for some $\alpha > 0$. Using Theorem 2.2(a) in [BK96], we have

$$\int (u_n - u_m + 2\delta)^+ \leqslant \int (u_n - h + \delta)^+ + \int (h + \delta - u_m)^+$$

$$= \int ((u_n - c_n) - \alpha)^+ + \int ((u_m - c_m) + \alpha)^-$$

$$\leqslant \int (f_0 - \alpha)^+ + \int (f_0 + \alpha)^-$$

$$= \int (|f_0| - \alpha)^+ < \infty;$$

it follows that $|\{u_n > u_m\}| < \infty$ and thus we deduce from Lemma 1(a) that $u_n \leqslant u_m$.

Define $\bar{u} = \lim_{n\to\infty} u_n$; this is, clearly, a bounded g.e.s. of (E) corresponding to $f = f_0 + c$. Let now u be a bounded sub-g.e.s. of (E); with the same argument as above, $u \leqslant u_n$ a.e. for all n and thus $u \leqslant \bar{u}$ a.e. In other words, $\bar{u}$ is the maximum bounded g.e.s. of (E).

The proof of existence of the maximum bounded g.e.s. of (CP) is similar using Lemma 1(b); considering a bounded g.e.s. u_n of (CP) corresponding to (f_n, g), we only need to show that

$$\sup_{t\in[0,T]}\left[\int((u_n(t)-c_n)-\alpha)^+ + \int((u_m(t)-c_m)+\alpha)^-\right]<\infty. \tag{9}$$

To prove (9), for $M>0$ set $\kappa_M(t)=\inf\{\kappa; \int(|g(t)|-\kappa)^+ \leqslant M\}$. We have $\kappa_M(t)\leqslant \|g(t)\|_\infty$ and thus $\kappa_M\in L^1(0,T)$; on the other hand, for a.a. $t\in(0,T)$, since $g(t)\in L_0^\infty(\mathbb{R}^N)$, $\kappa_M(t)$ decrease to 0 as M increase to ∞. Thus there exists $M>0$ such that $\int_0^T \kappa_M(t)\,dt\leqslant \alpha/2$. Fix M such that this is satisfied, and set $k_M(t)=\alpha/2+\int_0^t \kappa_M(s)\,ds$ for $t\in[0,T]$. Applying Theorem 2.2(b) from [BK96], we get for all $t\in[0,T]$,

$$\begin{aligned}
&\int((u_n(t)-c_n)-\alpha)^+ + \int((u_m(t)-c_m)+\alpha)^- \\
&\quad\leqslant \int((u_n(t)-c_n)-k_M(t))^+ + \int((u_m(t)-c_m)+k_M(t))^- \\
&\quad\leqslant \int(|f_0|-\alpha/2)^+ + \int_0^t\int(|g(s)|-\kappa_M(s))^+\,ds \\
&\quad\leqslant \int(|f_0|-\alpha/2)^+ + MT.
\end{aligned}$$

The proof of existence of the minimal bounded g.e.s. for (E) and (CP) is identical. ∎

We actually do not know whether there is in general uniqueness of a bounded g.e.s. of (E) or (CP). However, we can prove the following result.

Proposition 1. *Let $f_0\in L_0^\infty(\mathbb{R}^N)$ (resp., and $g\in L^1_{\mathrm{loc}}(Q)$, $g(t,\cdot)\in L_0^\infty(\mathbb{R}^N)$ for a.a. $t\in(0,T)$ and $\int_0^T\|g(t,\cdot)\|_\infty\,dt<\infty$). Then there exists an at most countable set $\mathcal{N}$ in $\mathbb{R}$ such that for all $c\in\mathbb{R}/\mathcal{N}$ the equation* (E) *(resp., the problem* (CP)) *with $f=f_0+c$ has a unique bounded g.e.s.*

Proof. For $c\in\mathbb{R}$, denote by $\bar{u}(c)$ (resp., $\underline{u}(c)$) the maximum (resp. minimum) bounded g.e.s. By the proof above, we know that $c\mapsto\bar{u}(c)$ and $c\mapsto\underline{u}(c)$ are nondecreasing from $\mathbb{R}$ into L^∞ continuous from the right and the left, respectively, for the L^1_{loc} topology in L^∞; moreover, for $c_1<c_2$, $\underline{u}(c_1)\leqslant\bar{u}(c_1)\leqslant\underline{u}(c_2)$. Thus it follows that $\underline{u}(c)=\bar{u}(c)$ a.e. for any c except an at most countable set in $\mathbb{R}$.

3. THE L^1 SEMIGROUP APPROACH

In this section, using the nonlinear semigroup theory in L^1, we make the relation between the equation (E) and the problem (CP) under the assumption (1) on the data (f, g) clearer.

For simplicity we shall assume $c=0$.

For $\lambda>0$ and $f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, the equation

$$u+\operatorname{div}_x \lambda\phi(u)=f \qquad \text{in } \mathbb{R}^N \tag{10}$$

has a maximum bounded g.e.s. that we shall denote by $J_\lambda^+ f$; by Corollary 2.1 in [BK96] $J_\lambda^+ f\in L^1(\mathbb{R}^N)$. In other words, J_λ^+ maps $L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ into itself. Let us start with the following results.

PROPOSITION 2. *With the notation above, the following properties hold:*

(1) *for any $\lambda>0$, J_λ^+ is a T-contraction for the L^1-norm, i.e.,*

$$\int (J_\lambda^+ f-J_\lambda^+\hat f)^+\leqslant \int (f-\hat f)^+ \qquad \forall f, \hat f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N);$$

(2) *$\{J_\lambda^+\}_{\lambda>0}$ is a resolvent family, i.e.*

$$J_\lambda^+ f=J_\mu^+\left(\frac{\mu}{\lambda} f+\frac{\lambda-\mu}{\lambda} J_\lambda^+ f\right) \qquad \forall\lambda, \mu>0, \quad f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N);$$

(3) *the range $\mathcal{R}(J_\lambda^+)$, independent of λ by (2), is dense in $L^1(\mathbb{R}^N)$.*

Proof. For Part 1, let $f, \hat f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and, for $\delta>0$, denote by u^δ, $\hat u^\delta$ bounded g.e.s. of (E) corresponding to $f+\delta$, $\hat f+\delta$, respectively. As in the proof of Theorem 1, we have $\lambda^N\{u^\delta>\hat u^{2\delta}\}<\infty$ and then, by Lemma 1, $\int (u^\delta-\hat u^{2\delta})^+\leqslant\int (f-\hat f-\delta)^+\leqslant\int (f-\hat f)^+$. At the limit as $\delta\to 0$, $u^\delta\to J_\lambda^+ f$ and $\hat u^{2\delta}\to J_\lambda^+\hat f$ a.e., so that we get Part 1 by the Fatou Lemma.

For Part 2, let $f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and assume first that $\lambda>\mu>0$. Set $\bar u=J_\lambda^+ f$; it is a bounded g.e.s. of

$$v+\operatorname{div}_x \mu\phi(v)=\frac{\mu}{\lambda} f+\frac{\lambda-\mu}{\lambda}\bar u$$

and so $\bar u\leqslant\bar v=J_\mu^+((\mu/\lambda) f+((\lambda-\mu)/\lambda)\bar u)$. Then $(\mu/\lambda) f+((\lambda-\mu)/\lambda)\bar u\leqslant (\mu/\lambda) f+((\lambda-\mu)/\lambda)\bar v$ and $\bar v$ is a bounded sub-g.e.s. of $u+\operatorname{div}_x \lambda\phi(u)=f$. We deduce $\bar v\leqslant\bar u$ and thus $\bar v=\bar u$. To complete the proof of Part 2, we apply the abstract Lemma 3 below.

For the proof of Part 3, let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and, for $\lambda > 0$, set $u_\lambda = J_\lambda^+ f$. We have $u_\lambda \in \mathscr{R}(J_\lambda^+)$ and this set is clearly, by Part 2, independent of λ. Since $\|u_\lambda\|_\infty \leqslant \|f\|_\infty$ (see Corollary 2.1 in [BK96]), it follows immediately that $u_\lambda \to f$ in $\mathscr{D}'(\mathbb{R}^N)$ as $\lambda \to 0$; indeed, being a g.e.s., u_λ is also a solution of (10) in the sense of distributions. Now using translation invariance and Part 1, $\int |u_\lambda(x+h) - u_\lambda(x)|\, dx \leqslant \int |f(x+h) - f(x)|\, dx$, so that the set $\{u_\lambda\}_{\lambda > 0}$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^N)$ and $u_\lambda \to f$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. At last $\|u_\lambda\|_1 \leqslant \|f\|_1$ (cf. Corollary 2.1 in [BK96]) so that $u_\lambda \to f$ in $L^1(\mathbb{R}^N)$; indeed, for any compact set K in $\mathbb{R}^N$ we have

$$\limsup_{\lambda \to 0} \|u_\lambda - f\|_1 \leqslant \limsup_{\lambda \to 0} \left(\int_K |u_\lambda - f| + \int |u_\lambda| - \int_K |u_\lambda| + \int_{\mathbb{R}^N \setminus K} |f| \right)$$

$$\leqslant \int |f| - \int_K |f| + \int_{\mathbb{R}^N \setminus K} |f| = 2 \int_{\mathbb{R}^N \setminus K} |f|,$$

which can be made as small as we want. ∎

Lemma 3. *Let X_0 be a linear subspace of a Banach space X and $\{J_\lambda\}_{\lambda > 0}$ be a family of non-expansive mappings from X_0 into X_0. If the resolvent identity $J_\lambda = J_\mu((\mu/\lambda) I + ((\lambda - \mu)/\lambda) J_\lambda)$ holds for all $0 < \mu < \lambda$, then it still holds for any $\lambda, \mu > 0$.*

Proof. Following [BCP, Exercise E8.2], denote by A_λ the multivalued operator from X_0 into itself defined by

$$v \in A_\lambda u \quad \Leftrightarrow \quad u, v \in X_0,\ u = J_\lambda(u + \lambda v);$$

the graph of this operator is $\{(J_\lambda f, (f - J_\lambda f)/\lambda);\ f \in X_0\}$ and one has $(I + \lambda A_\lambda)^{-1} = J_\lambda$. For given $\lambda, \mu > 0$, the equality $J_\lambda = J_\mu((\mu/\lambda) I + ((\lambda - \mu)/\lambda) J_\lambda\}$ is equivalent to the inclusion $J_\lambda \subset J_\mu((\mu/\lambda) I + ((\lambda - \mu)/\lambda) J_\lambda)$ since these two maps are everywhere defined in the linear space X_0; so it is also equivalent to the inclusion $A_\lambda \subset A_\mu$.

By assumption $A_\lambda \subset A_\mu$ for $0 < \mu < \lambda$. We deduce that for any $\lambda > 0$, A_λ is an accretive operator; indeed, for $\mu > 0$ small enough ($\mu < \lambda$), $(I + \mu A_\lambda)^{-1}$ is a non-expansive mapping since it is contained in J_μ. Thus, for $0 < \lambda < \mu$, $(I + \mu A_\lambda)^{-1}$ is a single-valued operator in X_0 containing $(I + \mu A_\mu)^{-1} = J_\mu$, which is everywhere defined in X_0; so $(I + \mu A_\lambda)^{-1} = (I + \mu A_\mu)^{-1}$ and $A_\lambda = A_\mu$. ∎

As we have seen in the proof above, there exists a multivalued operator A^+ in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $J_\lambda^+ = (I + \lambda A^+)^{-1}$ for any $\lambda > 0$. This operator is accretive densely defined in $L^1(\mathbb{R}^N)$ and $\mathscr{R}(I+\lambda A^+)=D(J_\lambda^+)=L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$ for any $\lambda>0$. This operator A^+ is exactly defined by

$$v\in A^+u \Leftrightarrow \exists f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N) \text{ such that}$$
$$u \text{ is the maximum bounded g.e.s. of (E) and } v=f-u;$$

it follows that A^+ is actually single-valued since for $v\in A^+u$ one has $v=\operatorname{div}_x \phi(u)$ in $\mathscr{D}'(\mathbb{R}^N)$. By the Crandall–Liggett theorem (cf. [CL71, B72, C76, BCP]), for any $(f, g)\in L^1(\mathbb{R}^N)\times L^1(Q)$ there exists a unique mild (or integral) solution $u\in C([0, T]; L^1(\mathbb{R}^N))$ of

$$\frac{du}{dt}+A^+u=g \qquad \text{on } (0, T), \quad u(0)=f. \tag{11}$$

THEOREM 2. *With the notations above, for* (f, g) *satisfying* (1) *with* $c=0$, *the mild solution of* (11) *is the maximum bounded g.e.s. of* (*CP*).

Proof. With the same argument as that in [C72, B72], it is clear that, under the assumptions (1), the mild solution $u\in C([0, T]; L^1(\mathbb{R}^N))$ of (11) is in $L^\infty(Q)$ and a g.e.s. of (CP). Therefore $u\leqslant\bar{u}$ a.e. on Q.

Now we prove that $\bar{u}$ satisfies

$$\int (\bar{u}(t)-w)^+ \leqslant \int (f-w)^+ + \int_0^t [\bar{u}(\tau)-w, g(\tau)-A^+w]_+ \, d\tau \tag{12}$$

for a.a. $t\in(0, T)$ and for all $w\in D(A^+)$, where for $u, f\in L^1(\mathbb{R}^N)$

$$[u, f]_+ = \int_{\{u>0\}} f + \int_{\{u=0\}} f^+ \equiv \inf_{\mu>0} \frac{\int (u+\mu f)^+ - \int u^+}{\mu}.$$

Using translation invariance in time, we shall have $(d/dt)\int (\bar{u}(t)-w)^+ \leqslant [\bar{u}(t)-w, g(t)-A^+w]_+$ in $\mathscr{D}'((0, T))$. Applying the results of [BB92], we shall conclude that $\bar{u}\leqslant u$ a.e. on Q and this will end the proof.

Let $w\in D(A^+)$, $\delta>0$. By definition $w=J_1^+h$, $A^+w=h-J_1^+h$ with some $h\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Consider w^δ a bounded g.e.s. of $w+\operatorname{div}_x \phi(w)=h+\delta$. Take w^δ as a stationary bounded g.e.s. of the corresponding (CP); since $w^\delta-\delta\in L^\infty(0, T; L^1(\mathbb{R}^N))$ and $\bar{u}\in L^\infty(0, T; L^1(\mathbb{R}^N))$ (cf. Corollary 2.1 in [BK96]), we have $\lambda^{N+1}(\{\bar{u}>w^\delta\})<\infty$ with the same argument as in the proof of Theorem 1. Thus Lemma 1(b) yields

$$\int (\bar{u}(t) - w^\delta)^+ \leqslant \int (f - w^\delta)^+ + \int_0^t [\bar{u}(\tau) - w^\delta, g(\tau) - (h + \delta - w^\delta)]_+ \, d\tau$$

$$\leqslant \int (f - w^\delta)^+ + \int_0^t \frac{1}{\mu} \left\{ \int (\bar{u}(\tau) - w^\delta + \mu(g(\tau) - A^+ w \right.$$

$$\left. + w^\delta - w - \delta))^+ - \int (\bar{u}(\tau) - w^\delta)^+ \right\} d\tau \tag{13}$$

for any $\mu > 0$. As δ decreases to 0, w^δ decreases to w; moreover for $0 < \mu \leqslant 1$, $(\bar{u}(\tau) - w^\delta + \mu(g(\tau) - A^+ w + w^\delta - w - \delta))^+$ increases to $(\bar{u}(\tau) - w + \mu(g(\tau) - A^+ w))^+$. So we may pass to the limit in (13) and obtain

$$\int (\bar{u}(t) - w)^+ \leqslant \int (f - w)^+ + \int_0^t \frac{1}{\mu} \left\{ \int (\bar{u}(\tau) - w + \mu(g(\tau) - A^+ w))^+ \right.$$

$$\left. - \int (\bar{u}(\tau) - w)^+ \right\} d\tau$$

for any $0 < \mu \leqslant 1$. Letting $\mu \to 0$ yields (12). ∎

Remark 1. Of course, one may consider minimum bounded g.e.s. of (10), define the corresponding operator A^-, and prove the following result analogous to Theorem 2:

The mild solution of (11) *with* A^- *in place of* A^+ *is exactly the minimum bounded g.e.s. of* (CP).

COROLLARY 1. *For a given continuous flux function* ϕ *and* $c \in \mathbb{R}$, *the following assertions are equivalent*:

(i) *for all* $f = f_0 + c$ *with* $f_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *there exists a unique bounded g.e.s. of* (E)

(ii) *for all* (f, g) *satisfying* (1) *there exists a unique bounded g.e.s. of* (CP).

Proof. Replacing $\phi(r)$ by $\phi(r + c)$, we may assume $c = 0$.

If (i) holds, the operators A^+ and A^- coincide and then, by Theorem 2 (see also Remark 1), for any (f, g) satisfying (1) the maximum and minimum bounded g.e.s. of (CP) coincide, so that (ii) holds.

Conversely, assume that (ii) holds and for $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ let u, $\hat{u}$ be two bounded g.e.s. of (E). Then $u(t) \equiv u$ is a bounded g.e.s. of (CP) corresponding to $(u, g(t) \equiv f - u)$ and so, by uniqueness, the maximum bounded g.e.s. and then, by Theorem 2, the unique mild solution of the corresponding evolution problem (11). In the same way $\hat{u}(t) \equiv \hat{u}$ is the unique mild solution of (11) corresponding to $(\hat{u},\ g(t) \equiv f - \hat{u})$. Then by the integral inequality (see [B72, BCP, BW94])

$$-\int |u - \hat{u}| = [u(t) - \hat{u}(t), (f - u) - (f - \hat{u})] \geqslant \frac{d}{dt} |u(t) - \hat{u}(t)| = 0,$$

where $[\,\cdot, \cdot\,]$ stands for the bracket associated with the standard norm in L^1, i.e., for all u, $f \in L^1(\mathbb{R}^N)$, $[u, f] = \int_{\{u \neq 0\}} f \operatorname{sign} u + \int_{\{u=0\}} |f|$. It follows that $u = \hat{u}$ a.e. in $\mathbb{R}^N$ so that (i) holds.

Remark 2. For $f = f_0 + c$ with $f_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, any bounded g.e.s. of (E) is in $c + L^1(\mathbb{R}^N)$ (cf. Corollary 2.1 in [BK96]); so there is uniqueness of a bounded g.e.s. to (E) if and only if $\int (\bar{u}(f) - \underline{u}(f)) = 0$, where $\bar{u}(f)$ and $\underline{u}(f)$ are the maximum and the minimum bounded g.e.s. of (E), respectively.

By Part 1 of Proposition 2, for given $c \in \mathbb{R}$ the map $f_0 \mapsto \bar{u}(f_0 + c) - c$ is a contraction for the L^1-norm; the same holds for $f_0 \mapsto \underline{u}(f_0 + c) - c$ so that $f_0 \mapsto \int (\bar{u}(f_0 + c) - \underline{u}(f_0 + c))$ is continuous for the L^1-topology. It follows that (i) of Corollary 1 is equivalent to the uniqueness of a bounded g.e.s. of (E) for all f_0 in some L^1-dense subset of $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Consequently, since the L^1-topology in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is separable, Proposition 1 can be improved as follows.

PROPOSITION 3. *There exists an at most countable set $\mathcal{N}$ in $\mathbb{R}$ such that, for all $c \in \mathbb{R} \backslash \mathcal{N}$, the two properties* (i) *and* (ii) *of Corollary* 1 *hold.*

4. SOME UNIQUENESS RESULTS IN $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

As noted in the introduction, we still do not know if, for any continuous flux function ϕ, there is uniqueness of a bounded g.e.s. to (CP) under assumption (1) or to (E) for all $f = f_0 + c$, $f_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $c \in \mathbb{R}$. In this section we shall improve some uniqueness results shown in [BK96].

THEOREM 3. *Assume there exist orthonormal vectors $\xi_1, ..., \xi_{N-1}$ and $C: \mathbb{R} \to [0, +\infty)$ continuous such that*

$$\frac{d}{dr} \xi_i \cdot \phi(r) \leqslant C(r) \qquad \text{in } \mathscr{D}'(\mathbb{R}) \qquad \text{for} \quad i = 1, ..., N-1. \tag{14}$$

Then for any $c \in \mathbb{R}$ the two properties of Corollary 1 *hold.*

We shall need the following lemma.

LEMMA 4. *Let $\xi \in \mathbb{R}^N$, $\xi \neq 0$, such that $r \in \mathbb{R} \mapsto \xi \cdot \phi(r)$ is nondecreasing. Let $\alpha \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R}^N)$ with support contained in $H = \{x \in \mathbb{R}^N;\ \xi \cdot x \geqslant \alpha\}$. Assume that one of the following conditions holds*:

(a) *there exists a unique bounded g.e.s. of* (E);

(b) $f \in L_0^\infty(\mathbb{R}^N)$.

Then for all bounded g.e.s. u of (E) *the support of u is also contained in H.*

Proof. (a) This is clearly true if ϕ is locally Lipschitz continuous. Indeed, by the definition of g.e.s.,

$$\begin{aligned}
&\int |u(x)|\, \rho\left(\frac{\alpha - \xi \cdot x}{\varepsilon}\right) \zeta(x)\, dx \\
&\qquad \leqslant \int |f(x)|\, \rho\left(\frac{\alpha - \xi \cdot x}{\varepsilon}\right) \zeta(x)\, dx + \int \operatorname{sign} u(x)(\phi(u(x)) - \phi(0)) \\
&\qquad \times \left\{ -\frac{\xi}{\varepsilon} \rho'\left(\frac{\alpha - \xi \cdot x}{\varepsilon}\right) \zeta(x) + \rho\left(\frac{\alpha - \xi \cdot x}{\varepsilon}\right) D\zeta(x) \right\} dx
\end{aligned}$$

for $\zeta \in \mathcal{D}(\mathbb{R}^N)$, $\zeta \geqslant 0$, $\rho \in C^\infty(\mathbb{R})$ with $\rho' \geqslant 0$, $\rho = 0$ on $(-\infty, 0]$, $\rho = 1$ on $[1, +\infty)$, and $\varepsilon \geqslant 0$. Since $\operatorname{sign} u(x)(\phi(u(x)) - \phi(0)) \cdot \xi \geqslant 0$ and $f(x)\, \rho((\alpha - \xi \cdot x)/\varepsilon) \equiv 0$, using the Lipschitz continuity of ϕ we get $\int |u(x)|\, \rho((\alpha - \xi \cdot x)/\varepsilon)\, \zeta(x)\, dx \leqslant C \int |u(x)|\, \rho((\alpha - \xi \cdot x)/\varepsilon)\, |D\zeta(x)|\, dx$. Let $\varepsilon \to 0$ and $\zeta \to 1$; it follows that $u = 0$ a.e. in $\mathbb{R}^N \backslash H$ (see, for instance, Lemma 1.1 in [BK96]).

For the general case, let $\phi_n = \phi * \rho_n$, where $\{\rho_n\}$ is a sequence of mollifiers, and let u_n be the bounded g.e.s. of (E) corresponding to the flux ϕ_n. Using the contraction property and translation invariance, we see that the sequence $\{u_n\}$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^N)$; clearly any limit point is a bounded g.e.s. of (E) and then by the uniqueness assumption u is the limit in $L^1_{\text{loc}}(\mathbb{R}^N)$ of the sequence $\{u_n\}$. Note that $r \in \mathbb{R} \mapsto \xi \cdot \phi_n(r)$ is nondecreasing for all n; thus by the argument above $\operatorname{supp} u_n \subset H$ and the same is true at the limit.

(b) Let $f \in L_0^\infty(\mathbb{R}^N)$. By Theorem 1 the equation (E) has a maximum bounded g.e.s. $\bar{u}$, which is the limit of any sequence (u_n) of bounded g.e.s. of (E) corresponding to $f_n = f + c_n$ with $c_n \downarrow 0$. Moreover, by Proposition 1 we may choose c_n so that there is uniqueness of bounded g.e.s. of (E) corresponding to f_n. By the first part of Lemma 4, $\operatorname{supp}(u_n - c_n) \subset H$, therefore $\operatorname{supp} \bar{u} \subset H$. Using the same argument for the minimum bounded g.e.s. $\underline{u}$ of (E), we see that the conclusion of Lemma 4 still holds. ∎

Proof of Theorem 3. Replacing $\phi(r)$ by $\phi(r + c) - \phi(c)$, we may assume $c = 0$ and $\phi(0) = 0$.

Since we are working with bounded solutions, we may also assume that $C(r)$ is constant. By replacing ξ_i with $-\xi_i$, it is then equivalent to assume instead of (14) that

$$\frac{d}{dr}\xi_i \cdot \phi(r) + C \geqslant 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^N) \qquad \text{for} \quad i = 1, ..., N-1.$$

Now we notice that for $\eta \in \mathbb{R}^N$, u is a bounded g.e.s. of (CP) corresponding to ϕ, f, g if and only if $\tilde{u}(t, x) = u(t, x - t\eta)$ is a bounded g.e.s. of (CP) corresponding to $\tilde{\phi}(r) = \phi(r) + r\eta$, $\tilde{f}(x) = f(x)$, and $\tilde{g}(t, x) = g(t, x - t\eta)$. Then, according to Corollary 1, the conclusion of Theorem 3 holds for ϕ if and only if it holds for the flux function $\phi(r) + r\eta$. Choosing $\eta \in \mathbb{R}^N$ such that $\eta \cdot \xi_i > C$ for $i = 1, ..., N-1$, which is always possible since the vectors are linearly independent, we may assume that

$$r \in \mathbb{R} \mapsto \xi_i \cdot \phi(r) \in \mathbb{R} \text{ is an increasing homeomorphism for } i = 1, ..., N-1, \tag{15}$$

a slightly strengthened version of (3).

Under the assumption (15), we prove the result by induction in the dimension N. The result is true for $N = 1$ (see [B72]). Assuming that it is true for $N-1$, we prove it for $N \geqslant 2$. Changing coordinates, we may assume from (15) that $\phi(r) = (\phi_1(r), ..., \phi_N(r))$ with $\phi_i(\cdot)$ increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ for $i = 1, ..., N-1$. We shall prove that the equation (E) has a unique bounded g.e.s. for any $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. According to Corollary 1, this will end the proof of the theorem.

So let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and u be a bounded g.e.s. of (E); one has $u \in L^1(\mathbb{R}^N)$ (see Corollary 2.1 in [BK96]). Set $x = (x_1, x')$ with $x' = (x_2, ..., x_N)$, $w(x_1, x') = \phi_1(u(x_1, x'))$, $\beta = \phi_1^{-1}$, $\psi(r) = (\phi_2(\beta(r)), ..., \phi_N(\beta(r)))$. Suppose that $w(x_1 + t, \cdot) \to w(x_1, \cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{N-1})$ as $t \to 0$ for some $x_1 \in \mathbb{R}$; then for every $T > 0$ the function $v\colon (t, x') \in Q' = (0, T) \times \mathbb{R}^{N-1} \mapsto w(x_1 + t, x')$ is a bounded g.e.s. of the Cauchy problem

$$\frac{\partial v}{\partial t} + \operatorname{div}_{x'} \psi(v) = g \quad \text{on } Q', \qquad v(0, \cdot) = v_0(\cdot) \quad \text{on } \mathbb{R}^{N-1}, \tag{16}$$

where $v_0(x') = w(x_1, x')$ and $g(t, x') = f(x_1 + t, x') - \beta(w(x_1 + t, x'))$; $g \in L^1(Q') \cap L^\infty(Q')$ since f and $\beta(w) = u$ are in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

According to Remark 2, it suffices to prove the uniqueness of a bounded g.e.s. of (E) corresponding to compactly supported f. So assume supp $f \subset H = \{x_1 \geqslant \alpha_0\}$ and suppose there exist u, $\hat{u}$ two bounded g.e.s. of (E). By Lemma 4, supp $u \subset H$, supp $\hat{u} \subset H$. Take $x_1 = \alpha > \alpha_0$; consider $v(t, x') = \phi_1(u(t + \alpha, x'))$, $\hat{v}(t, x') = \phi_1(\hat{u}(t + \alpha, x'))$. The functions v, $\hat{v}$ are bounded g.e.s. of (16) corresponding to $(v_0(\cdot) \equiv 0,\ g(t,\cdot) = f(t+\alpha,\cdot) - \beta(v(t,\cdot)))$ and $(\hat{v}_0(\cdot) \equiv 0,\ \hat{g}(t,\cdot) = f(t+\alpha,\cdot) - \beta(\hat{v}(t,\cdot)))$, respectively.

By the inductive assumption the Cauchy problem (16) has a unique bounded g.e.s., which is in $L^1(\mathbb{R}^{N-1})$ for a.a. $t \in (0,T)$; as in the proof of Corollary 1, it follows that the integral inequality holds:

$$\frac{d}{dt}\int |v(t)-\hat{v}(t)| \leqslant [v(t)-\hat{v}(t),\ g(t)-\hat{g}(t)]$$

$$= -\int |\beta(v(t)) - \beta(\hat{v}(t))| \leqslant 0.$$

Hence $\int |v(t)-\hat{v}(t)| \leqslant \int |v_0 - \hat{v}_0| = 0$, so that $v = \hat{v}$ a.e. in Q'. Thus $u = \hat{u}$ a.e. in H, which proves the theorem. ∎

In [B72] it has been proved that for any $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there is uniqueness of $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ g.e.s. of (E) under the isotropic assumption $\lim_{r\to 0} \|\phi(r)\|/r^{1-1/N} = 0$. In the next theorem we shall prove the uniqueness under the anisotropic assumption introduced in [KP90, BK96].

THEOREM 4. *Let $c \in \mathbb{R}$ and $\omega_1, ..., \omega_N$ be moduli of continuity, i.e., increasing sub-additive continuous functions from $[0,\delta]$ into $[0,+\infty)$, $\delta > 0$, with $\omega_i(0) = 0$, $i = 1, ..., N$; assume that*

$$\liminf_{r\to 0} \frac{1}{r^{N-1}} \prod_{i=1}^{N} \omega_i(r) < \infty. \tag{17}$$

Assume that there exist orthonormal vectors $\xi_1, ..., \xi_N$ such that $|\xi_i \cdot \phi(c+r) - \xi_i \cdot \phi(c)| \leqslant \omega_i(|r|)$ for all $r \in [-\delta,\delta]$, $i = 1, ..., N$. Then the two assertions of Corollary 1 hold.

Proof. We may assume that $c = 0$, $\phi(0) = 0$, and $\phi = (\phi_1, ..., \phi_N)$ with $|\phi_i(r)| \leqslant \omega_i(|r|)$ for $r \in [-\delta,\delta]$, $i = 1, ..., N$. Recalling Remark 2 and Corollary 1, we only need to prove for $f, u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with u g.e.s. of (E) that $\int u = \int f$. Replacing $\omega_i(r)$, ϕ_i, f, and u by $\omega_i(Mr)/M$, $\phi(Mr)/M$, f/M, and u/M, respectively, we may assume that $\|u\|_\infty \leqslant \delta$.

Clearly, it suffices to show that for all $\mu > 0$, $R > 0$ there exists a function ζ such that

$$0 \leqslant \zeta \leqslant 1 \text{ on } \mathbb{R}^N, \quad \zeta(x) = 1 \text{ for all } x \in [-R,R]^N, \quad \text{and}$$

$$\left|\int (u-f)\,\zeta\right| < \mu; \tag{18}$$

for this we follow the proof of Lemma 1.1 in [BK96].

For $r>0$ set $\lambda_i(r)=\omega_i(r)/r$. If all λ_i are bounded, then ϕ is Lipschitz continuous and the result is well known (see the Introduction). Without loss of generality we may assume that $\lim_{r\to 0}\lambda_i(r)=+\infty$ for $i=1, ..., l$ and $\lambda_i(r)\leqslant\lambda$ for $i=l+1, ..., N$ with some $l\in\{1, ..., N\}$. Since ω_i are sub-additive and positive for $r>0$, $\omega_i(r)\geqslant\lambda_0 r$ for some $\lambda_0>0$, so that it is equivalent to assume instead of (17) that $\liminf_{r\to 0} C(r)=C<\infty$, where $C(r)=r\lambda^{N-l}\prod_{i=1}^{l}\lambda_i(r)$. Note that if $l=1$, then clearly $C=0$.

For all u bounded g.e.s. of (E), for all $\zeta\in\mathscr{D}(\mathbb{R}^N)$ we have

$$\int u\zeta=\int\phi(u)\cdot D\zeta+\int f\zeta. \tag{19}$$

Moreover, since f and u are bounded, (19) is also valid for ζ given by

$$\zeta(x_1, ..., x_N)=\prod_{i=1}^{N}\exp\left(-\left(\frac{|x_i|}{R_i}-1\right)^+\right)$$

with arbitrary positive R_i. Take ζ corresponding to $R_i=\lambda_i(\varepsilon)/\eta$ for $i=1, ..., l$ and $R_i=\lambda/\alpha$ for $i=l+1, ..., N$; positive numbers α, η, ε will be chosen later. We note that $0\leqslant\zeta\leqslant 1$, $\zeta(x)\equiv 1$ on $\prod_{i=1}^{N}[-R_i, R_i]$, $\int\zeta=(2^{2N}/(\eta^l\alpha^{N-l}))\,\lambda^{N-l}\prod_{i=1}^{l}\lambda_i(\varepsilon)$, and $|D_i\zeta|=(\eta\zeta/\lambda_i(\varepsilon))\,\chi_{\{|x_i|>R_i\}}$ for $i=1, ..., l$, $|D_i\zeta|=(\alpha\zeta/\lambda)\,\chi_{\{|x_i|>R_i\}}$ for $i=l+1, ..., N$.

From (19) we get $|\int(u-f)\,\zeta|\leqslant\sum_{i=1}^{N}\int\omega_i(|u|)\,|D_i\zeta|$.

Now, by the sub-additivity of ω_i, for $i=1, ..., l$ we have $\omega_i(r)\leqslant r\omega_i(\varepsilon)/\varepsilon+\omega_i(\varepsilon)=r\lambda_i(\varepsilon)+\varepsilon\lambda_i(\varepsilon)$ for all $\varepsilon>0$; for $i=l+1, ..., N$ we have $\omega_i(r)\leqslant r\lambda$. Hence by substituting into the last estimate the expressions above for $|D_i\zeta|$ and $\int\zeta$, we get

$$\begin{aligned}\left|\int(u-f)\,\zeta\right|&\leqslant\sum_{i=1}^{l}\eta\int_{\{|x_i|>R_i\}}|u|\,\zeta+\sum_{i=1}^{l}\varepsilon\eta\int_{\{|x_i|>R_i\}}\zeta\\&\quad+\sum_{i=l+1}^{N}\alpha\int_{\{|x_i|>R_i\}}|u|\,\zeta\\&\leqslant\eta\sum_{i=1}^{l}\int_{\{|x_i|>R_i\}}|u|+\frac{l2^{2N}}{\eta^{l-1}\alpha^{N-l}}\\&\quad\times\varepsilon\lambda^{N-l}\prod_{i=1}^{l}\lambda_i(\varepsilon)+\alpha(N-l)\int|u|.\end{aligned}$$

Take $\mu>0$, $R>0$. Choose $\alpha_0>0$ such that $\lambda/\alpha_0>R$ and $\alpha_0\cdot(N-l)\,\|u\|_1<\mu/3$. Choose $\eta_0>0$ such that $(1/\eta_0^{l-1})\cdot(l2^{2N}/\alpha_0^{N-l})\cdot 2C<\mu/6$; note that if $l=1$ then $C=0$ and whatever η_0 is good. Finally, since $\lambda_i(\varepsilon)\to\infty$ as $\varepsilon\to 0$ for $i=1, ..., l$ and $u\in L^1(\mathbb{R}^N)$, by the definition of C there exists $\varepsilon_0>0$ satisfying

$\varepsilon_0 \lambda^{N-l} \prod_{i=1}^{l} \lambda_i(\varepsilon_0) < 2C + (\mu/6) \cdot (\alpha_0^{N-l} \eta_0^{l-1})/(l\, 2^{2N})$ such that $R_i = \lambda_i(\varepsilon_0)/\eta_0 > R$, $\sum_{i=1}^{l} \int_{\{|x_i| > R_i\}} |u| < \mu/3\eta_0$. It follows that (18) holds for ζ constructed with $\alpha_0, \eta_0, \varepsilon_0$. ∎

Remark 3. Introducing $\xi_1, ..., \xi_N$ in the condition (17) is not superfluous. Indeed, take $N=2$ and let $\phi = (u, u/|u|^{2/3})$ in some orthonormal basis ξ_1, ξ_2; here (17) holds. Changing coordinates by rotation by any angle θ such that $\theta \neq \pi k/2$, $k \in \mathbb{Z}$, we see that condition (17) fails in the new basis.

APPENDIX: PROOF OF LEMMA 2

We give here the complete proof of Lemma 2. More precisely, we shall prove the following result:

THEOREM 5. *Let* $\phi: \mathbb{R}^N \mapsto \mathbb{R}$ *be a continuous function.*

(a) *There exists a map* $G: L^\infty(\mathbb{R}^N) \mapsto L^\infty(\mathbb{R}^N)$ *satisfying*:

(i) *for any* $f \in L^\infty(\mathbb{R}^N)$, $u = Gf$ *is a g.e.s. of* (E);

(ii) *G is a T-contraction for the* L^1*-norm, i.e., for any* $f, \hat{f} \in L^\infty(\mathbb{R}^N)$,

$$\int (Gf - G\hat{f})^+ \leqslant \int (f - \hat{f})^+.$$

(b) *Set* $X = \{(f, g); (f, g)$ *satisfies* (2)$\}$; *there exists a map* $U: X \mapsto L^\infty(Q) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *satisfying*:

(i) *for any* $(f, g) \in X$, $u = U(f, g)$ *is a g.e.s. of* (CP);

(ii) *any* $(f, g), (\hat{f}, \hat{g}) \in X$, *the T-contraction property holds*:

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^N} (U(f, g)(t) - U(\hat{f}, \hat{g})(t))^+ \leqslant \int_{\mathbb{R}^N} (f - \hat{f})^+ + \iint_Q (g - \hat{g})^+.$$

Proof of (*a*). First, let $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For $\varepsilon > 0$, take $\phi^\varepsilon: \mathbb{R} \to \mathbb{R}^N$ Lipschitz continuous functions such that ϕ^ε converge to ϕ uniformly on compact sets in $\mathbb{R}$, as $\varepsilon \to 0$. It is well-known that there exists a unique solution u^ε to the equation

$$u^\varepsilon + \mathrm{div}_x\, \phi^\varepsilon(u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon + f \qquad \text{on } \mathbb{R}^N;$$

moreover, the map $G_\varepsilon: f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \mapsto u^\varepsilon \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a T-contraction for the L^1-norm, the maximum principle ($\|u^\varepsilon\|_\infty \leqslant \|f\|_\infty$) holds and there is translation invariance in x. Thus the family $\{G_\varepsilon f\}_{\varepsilon > 0}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Take a countable L^1-dense set M in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; by the diagonal process, there exist $\varepsilon_n \to 0$ such that $G_{\varepsilon_n} f \to u =: G_0 f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for all $f \in M$. It is clear that u is a g.e.s. of (E) and $G_0\colon M \mapsto L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a T-contraction for the L^1-norm. Thus G_0 can be extended to the whole of $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ so that G_0 is a T-contraction for the L^1-norm, $G_0 f$ is a g.e.s. of (E) and the maximum principle holds.

Now for the general case $f \in L^\infty(\mathbb{R}^N)$, set $f_{n,m} = f^+ \chi_{\{|x| \leqslant n\}} - f^- \chi_{\{|x| \leqslant m\}} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. As $n \to \infty$, $G_0 f_{n,m} \uparrow u_m \in L^\infty(\mathbb{R}^N)$; further, as $m \to \infty$, $u_m \downarrow u =: Gf$. It is clear that u is a bounded g.e.s. of (E); by the Fatou Lemma, it follows that $\int (Gf - G\hat{f})^+ \leqslant \liminf_{n \to \infty, m \to \infty} \int (f_{n,m} - \hat{f}_{n,m})^+$ for $f, \hat{f} \in L^\infty(\mathbb{R}^N)$. It is easy to check that $\lim_{m \to \infty} \lim_{n \to \infty} \int (f_{n,m} - \hat{f}_{n,m})^+ = \int (f - \hat{f})^+ \in [0, +\infty]$, so that (ii) also holds. ∎

Proof of (b). First, set $X_0 = [L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)] \times [L^1(Q) \cap L^\infty(Q)]$ and let $(f, g) \in X_0$. For $\varepsilon > 0$, take ϕ^ε as in the proof of (a); there exists a unique solution u^ε to the Cauchy problem

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \operatorname{div}_x \phi^\varepsilon(u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon + g & \text{on } Q \\ u^\varepsilon(0, \cdot) = f & \text{on } \mathbb{R}^N; \end{cases}$$

moreover, the map $U_\varepsilon\colon (f, g) \in X_0 \mapsto u^\varepsilon \in L^1(Q) \cap L^\infty(Q) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ satisfies the maximum principle ($\|u^\varepsilon\|_\infty \leqslant \|f\|_\infty + \int_0^T \|g(\tau, \cdot)\|_\infty \, d\tau$), the T-contraction property holds, and there is translation invariance in x. Hence there exists a modulus of continuity $\omega_{f,g}$ such that

$$\int |U_\varepsilon(f, g)(t, x + \Delta x) - U_\varepsilon(f, g)(t, x)| \, dx \leqslant \omega_{f,g}(\Delta x)$$

uniformly in $\varepsilon > 0$ and $t \in [0, T]$. By Theorem 2 in [K69a], it follows that for any compact set $K \subset \mathbb{R}^N$

$$\int_K |U_\varepsilon(f, g)(t + \Delta t, x) - U_\varepsilon(f, g)(t, x)| \, dx \leqslant \omega_{f,g,K}(\Delta t)$$

uniformly in $\varepsilon > 0$ and $t \in [0, T]$, where $\omega_{f,g,K}$ is a modulus of continuity. Take a countable set M dense in X_0 for the $L^1(\mathbb{R}^N) \times L^1(Q)$-topology. By the diagonal process, there exist $\varepsilon_n \to 0$ such that $U_{\varepsilon_n}(f, g) \to u =: U_0(f, g)$ in $L^1_{\mathrm{loc}}(Q)$ for all $(f, g) \in M$; u is a g.e.s. of (CP), and the maximum principle and the T-contraction property hold for $U_0\colon M \mapsto L^1(Q) \cap L^\infty(Q) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. Thus U_0 can be extended to the whole of X_0, so that $U_0(f, g)$ is a g.e.s. of (CP), the T-contraction property holds, and there is translation invariance in x and the maximum principle holds.

Now for the general case $(f, g) \in X$, set $f_{n,m} = f^+ \chi_{\{|x| \leqslant n\}} - f^- \chi_{\{|x| \leqslant m\}}$ and $g_{n,m} = \min\{n, g^+\} \chi_{\{|x| \leqslant n\}} - \min\{m, g^-\} \chi_{\{|x| \leqslant m\}}$, so that we have

$(f_{n,m}, g_{n,m}) \in X_0$. As $n \to \infty$, $U_0(f_{n,m}, g_{n,m}) \uparrow u_m \in L^\infty(Q)$; further, as $m \to \infty$, $u_m \downarrow u =: U(f, g) \in L^\infty(Q)$. By the Fatou Lemma, it follows that (ii) holds.

We now show that $u = U(f, g)$ is in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and a g.e.s. of (CP). Indeed, there exists an increasing sequence $\{(n_i, m_i)\}_{i \in \mathbb{N}}$ in $\mathbb{N}^2$ such that $u_i = U_0(f_i, g_i) \to u$ in $L^1_{\mathrm{loc}}(Q)$ as $i \to \infty$, where $f_i = f_{n_i, m_i}$, $g_i = g_{n_i, m_i}$. By Lemma 3.1 in [BK96] and translation invariance, for $\zeta \in \mathscr{D}(\mathbb{R}^N)$,

$$\sup_{t \in [0, T]} \int |u_i(x + \Delta x, t) - u_i(x, t)| \, \zeta(x) \, dx$$

$$\leqslant \int |f_i(x + \Delta x) - f_i(x)| \, \zeta(x) \, dx$$

$$+ \int_0^T \int |g_i(s, x + \Delta x) - g_i(s, x)| \, \zeta(x) \, dx \, ds$$

$$+ \int_0^T \int |\phi(u_i(s, x + \Delta x)) - \phi(u_i(s, x))| \, |D\zeta(x)| \, dx \, ds.$$

The last term tends to $\int_0^T \int |\phi(u(s, x + \Delta x)) - \phi(u(s, x))| \, |D\zeta(x)| \, dx \, ds$ as $i \to \infty$, therefore for any compact set $K \subset \mathbb{R}^N$

$$\int_K |u_i(t, x + \Delta x) - u_i(t, x)| \, dx \leqslant \omega_{f, g, K}(\Delta x)$$

uniformly in $i \in \mathbb{N}$ and $t \in [0, T]$, where $\omega_{f, g, K}$ is a modulus of continuity. Hence, again by Theorem 2 in [K69a], the family $\{u_i(t, \cdot)\}_{i \in \mathbb{N}}$ is equicontinuous from $[0, T]$ to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Thus $u \in C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and $u(0, \cdot) = \lim_{i \to \infty} f_i = f$, so that u is a bounded g.e.s. of (CP). ∎

ACKNOWLEDGMENT

The work on this paper was partially supported by INTAS Grant INTAS-94-2187.

REFERENCES

[Ba88] L. Barthélemy, Problème d'obstacle pour une équation quasilinéaire du premier ordre, *Ann. Fac. Sci. Toulouse* **9**, No. 2 (1988), 137–159.

[BB92] L. Barthélemy and Ph. Bénilan, Subsolutions for abstract evolution equations, *Potential Anal.* **1** (1992), 93–113.

[B72] Ph. Bénilan, "Equations d'évolution dans un espace de Banach quelconque et applications," doctoral thesis, Université Paris Orsay, 1972.

[BCP] Ph. Bénilan, M. G. Crandall, and A. Pazy, "Nonlinear Evolution Equations in Banach Spaces," to appear.

[BK96] Ph. Bénilan and S. N. Kruzhkov, Conservation laws with continuous flux function, *Nonlinear Differential Equations Appl.* **3** (1996), 395–419.

[BW94] Ph. Bénilan and P. Wittbold, Nonlinear evolution equations in Banach spaces: Basic results and open problems, *in* "Functional Analysis: Proceedings of the Essen Conference, 1991" (Bierstedt *et al.*, Eds.), Lecture Notes in Pure and Applied Mathematics, Vol. 150, pp. 1–32, Dekker, New York, 1994.

[C72] M. G. Crandall, The semigroup approach to first order quasilinear equations in several space variables, *Israel J. Math.* **12** (1972), 108–122.

[C76] M. G. Crandall, An introduction to evolution governed by accretive operators, *in* "Dynamics Systems 1, International Symposium, Providence, RI, 1974," pp. 131–165, 1976.

[CL71] M. G. Crandall and T. Liggett, Generation of semi-groups of nonlinear transformations in general Banach spaces, *Amer. J. Math.* **93** (1971), 265–298.

[K69a] S. N. Kruzhkov, Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications, *Mat. Zametki* **6**, No. 1 (1969), 97–108; *Mat. Notes* **6**, No. 1 (1969), 517–523 [Engl. transl.].

[K69b] S. N. Kruzhkov, Generalized solutions of the Cauchy problem in the large for first-order nonlinear equations, *Dokl. Akad. Nauk. URSS* **187** (1969), 29–32; *Soviet Math. Dokl.* **10** (1969) [Engl. transl.].

[K70] S. N. Kruzhkov, First-order quasilinear equations in several independent variables, *Mat. Sb.* **81**, No. 2 (1970), 228–255; *Mat. USSR Sb.* **10** (1970) [Engl. transl.].

[KH74] S. N. Kruzhkov and F. Hildebrand, The Cauchy problem for quasilinear first-order equations in the case the domain of dependence on initial data is infinite, *Vest. Mosc. Univ.* **1** (1974), 93–100; *Moscow Univ. Math. Bull.* **29** (1974) [Engl. transl.].

[KP90] S. N. Kruzhkov and E. Yu. Panov, First-order quasilinear conservation laws with infinite initial data dependence area, *Dokl. Akad. Nauk. URSS* **314**, No. 1 (1990), 79–84; *Soviet Math. Dokl.* **42**, No. 2 (1991), 316–321 [Engl. transl.].