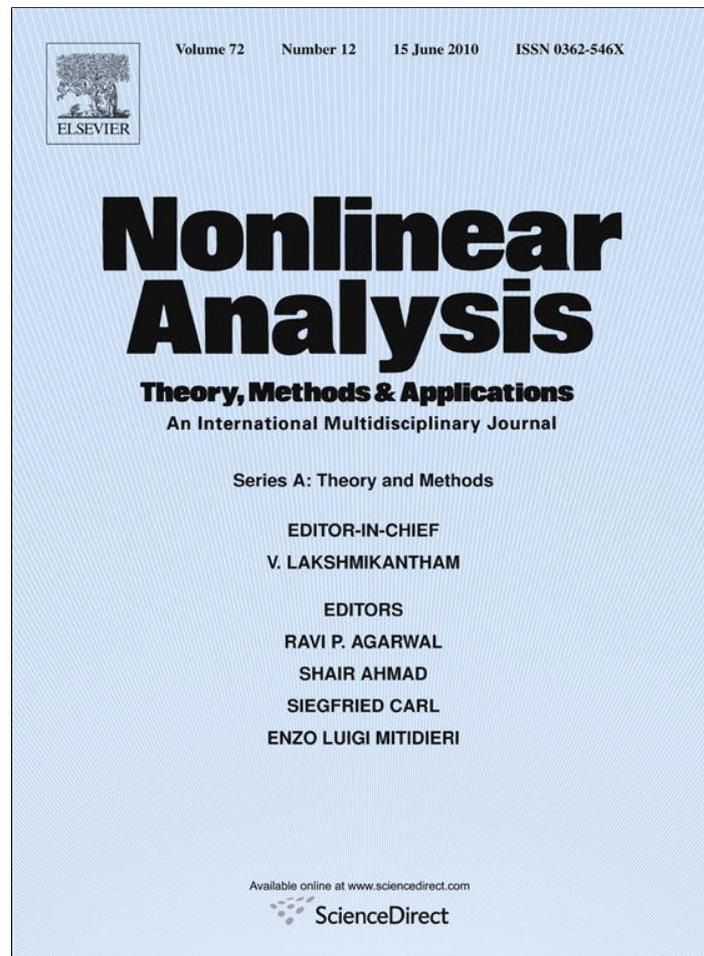


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## Nonlinear Analysis

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# Structural stability for variable exponent elliptic problems, II: The $p(u)$ -Laplacian and coupled problems

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## ABSTRACT

We study well-posedness for elliptic problems under the form

$$b(u) - \operatorname{div} a(x, u, \nabla u) = f,$$

where  $a$  satisfies the classical Leray–Lions assumptions with an exponent  $p$  that may depend both on the space variable  $x$  and on the unknown solution  $u$ . A prototype case is the equation  $u - \operatorname{div} \left( |\nabla u|^{p(u)-2} \nabla u \right) = f$ .

We have to assume that  $\inf_{x \in \Omega, z \in \mathbb{R}} p(x, z)$  is greater than the space dimension  $N$ . Then, under mild regularity assumptions on  $\Omega$  and on the nonlinearities, we show that the associated solution operator is an order-preserving contraction in  $L^1(\Omega)$ .

In addition, existence analysis for a sample coupled system for unknowns  $(u, v)$  involving the  $p(v)$ -Laplacian of  $u$  is carried out. Coupled elliptic systems with similar structure appear in applications, e.g. in modelling of stationary thermorheological fluids.

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## 1. Introduction

In the previous paper [1], we have studied the stability of solutions of the variable exponent problems of the  $p(x)$ -Laplacian kind problems under perturbation of the summability exponent  $p(x)$ . In the present paper, we study the closely related issue of convergence of approximations for variable exponent problems with dependency of  $p$  on the unknown solution  $u$  itself. The goal is to derive existence results for such  $p(x, u)$  variable exponent problems. We also investigate the uniqueness and structural stability issues.

We first consider the case where the dependency of  $p$  on  $u$  is local. Namely, we study the equation

$$b(u) - \operatorname{div} a(x, u, \nabla u) = f, \tag{1}$$

where  $b : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing, normalized by  $b(0) = 0$ . For the sake of simplicity, we supplement (1) with the homogeneous Dirichlet boundary condition:

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Problem (1), (2) fits into a generalized Leray–Lions framework under the assumption that  $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \rightarrow \mathbb{R}^N$  is a Carathéodory function with

$$a(x, z, 0) = 0 \quad \text{for all } z \in \mathbb{R} \text{ and a.e. } x \in \Omega \tag{3}$$

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satisfying, for a.e.  $x \in \Omega$ , for all  $z \in \mathbb{R}$ , the strict monotonicity assumption

$$(\alpha(x, z, \xi) - \alpha(x, z, \eta)) \cdot (\xi - \eta) > 0 \quad \text{for all } \xi, \eta \in \mathbb{R}^N, \xi \neq \eta, \tag{4}$$

as well as the growth and coercivity assumptions with variable exponent:

$$|\alpha(x, z, \xi)|^{p'(x,z)} \leq C (|\xi|^{p(x,z)} + \mathcal{M}(x)), \tag{5}$$

$$\alpha(x, z, \xi) \cdot \xi \geq \frac{1}{C} |\xi|^{p(x,z)}. \tag{6}$$

Here  $C$  is some positive constant,  $\mathcal{M} \in L^1(\Omega)$ ,

$$p : \Omega \times \mathbb{R} \longrightarrow [p_-, p_+] \text{ is Carathéodory, } 1 < p_- \leq p_+ < +\infty, \tag{7}$$

and  $p'(x, z) := \frac{p(x,z)}{p(x,z)-1}$  is the conjugate exponent of  $p(x, z)$ .

This set of assumptions is a straightforward generalization of the classical hypotheses of Leray and Lions [2]; yet the existence issue for problems (1)–(2) under the assumptions (3)–(7) remains open. Indeed, the possible non-denseness of regular functions in the variable exponent Sobolev spaces undermines the convergence analysis. In the present paper, we avoid this difficulty by requiring that  $p = p(x, u(x))$  is sufficiently regular. Therefore our analysis is reduced to the case where  $p_- > N$  ( $N$  being the dimension of the space domain) and where  $p(x, z)$  is, roughly speaking, Hölder continuous. Notice that the situation where the variable exponent  $p$  at the point  $x$  can depend on the unknown value  $u(x)$  (or on the whole set of unknown values  $(u(x))_{x \in \Omega}$ ) is non-standard, because problem (1), (2) cannot be written as equality in terms of duality in fixed Banach spaces. Nonetheless, under mild regularity assumptions on  $p$ ,  $\alpha$  and  $\Omega$  and under the key restriction  $p \geq p_- > N$ , we prove that problem (1)–(2) is well-posed in  $L^1(\Omega)$ . Following [3], we can interpret this result as the  $m - T$ -accretivity in  $L^1(\Omega)$  of the closure of the operator  $u \mapsto -\operatorname{div} \alpha(x, u, \nabla u)$ .

Although the  $p(x)$ -Laplacian kind problems have been extensively used in mathematical modelling in the last fifteen years (some of the references are listed in [1]), the authors are not aware of models with variable exponent  $p(x)$  explicitly depending on  $u(x)$ . Yet a related, although far more complicated, minimization problem with  $p = p(\nabla u)$  was suggested in [4], in the context of image processing.

On the contrary, the second class of problems that we study appears quite naturally in applications. Consider e.g. the toy problem

$$\begin{cases} u - \Delta_{p(x,v)} u = f(x, u, v) \\ v - \Delta v = g(x, u, v) \end{cases} \tag{8}$$

with homogeneous Dirichlet boundary conditions, under the assumption that  $f$  and  $g$  are bounded Carathéodory functions. In this case, because  $p = p(x, v(x))$  and because  $v$  is completely determined by  $u$ , we can consider that  $p$  depends on  $u$  in a non-local way (we denote such a dependency by  $p = p[u]$ ). For examples of models having similar structure, we refer to [5–9]. In order to demonstrate the applicability of the techniques of [1] for this kind of coupled variable exponent problems, in this paper we only justify the existence of solutions for the very simple problem (8). Notice that we heavily rely on the *a priori* regularity of  $p[u]$ , enforced by our assumptions.

Existence results for the non-local case were already obtained by Zhikov [5,6,8,9] and Antontsev and Rodrigues [7] for different elliptic systems originating from the thermistor problem and from the modelling of thermorheological fluids. The existence proofs of [5–7] are based on the Schauder fixed-point theorem, and the regularity of  $p[u]$  is crucial also for this argument. We provide a complementary point of view, showing existence for the problem (8) through convergence of Galerkin approximations.

Convergence of numerical finite volume approximations for (1) and for (8) will be analyzed in the forthcoming paper [10], with essentially the same tools as used in [1] and in the present paper.

Let us briefly explain the techniques used in the proofs.

Sequences of approximate solutions are constructed either by regularization with the  $p_+$ -Laplacian, or by Galerkin approximations. Then the convergence analysis is carried out in terms of Young measures associated with a weakly convergent sequence of gradients of solutions, as in [1] (cf. [11,12] and references therein).

Our uniqueness proof for (1)–(2) uses the standard  $L^1$  techniques combined with a regularity and density hint (cf. [13]). Indeed, the basic uniqueness and continuous dependence theorem is valid for  $W^{1,\infty}$  solutions. This regularity is always true for the one-dimensional problem (cf. [14–17]); but in the case of several space directions, we need a Lipschitz regularity result for an  $L^1$ -dense set of right-hand sides  $f$ . Such results are indeed available, for  $f \in L^\infty(\Omega)$  (see [18,20–22]), under some restrictions on the regularity of  $\partial\Omega$  and for  $p_- > N$ . Therefore we are able to give a complete well-posedness result, provided  $\alpha(x, z, \xi)$  satisfies (6), (5) with a sufficiently regular exponent  $p(x, z)$ ,  $p \geq p_- > N$ .

The outline of the paper is the following. The functional spaces framework is set up in Section 2.1. Section 2.2 recalls the definitions given in [1] and states the results obtained for the local  $p(u)$ -Laplacian kind problems. Section 2.3 states the results for the case of the Dirichlet problem for the coupled system (8). The proofs are postponed to Sections 3 and 4, respectively.

## 2. Main definitions and results

### 2.1. Variable exponent Lebesgue and Sobolev spaces

The solutions to the Dirichlet problem (1), (2) are sought within the variable exponent and the variable exponent Sobolev spaces  $W_0^{1,\pi(\cdot)}(\Omega)$ ,  $\dot{E}^{\pi(\cdot)}(\Omega)$  defined below (notice that the exponent  $\pi$  itself is related to  $u$  by  $\pi(\cdot) = p(\cdot, u(\cdot))$ , so that solutions with different data will possess different integrability properties). For the sake of completeness, we also recall the definition of variable exponent Lebesgue spaces  $L^{\pi(\cdot)}(\Omega)$ .

**Definition 2.1.** Let  $\pi : \Omega \rightarrow [1, +\infty)$  be a measurable function.

•  $L^{\pi(\cdot)}(\Omega)$  is the space of all measurable functions  $f : \Omega \rightarrow \mathbb{R}$  such that the modular

$$\rho_{\pi(\cdot)}(f) := \int_{\Omega} |f|^{\pi(x)} < +\infty$$

is finite, equipped with the Luxembourgnorm

$$\|f\|_{L^{\pi(\cdot)}} := \inf \{ \lambda > 0 \mid \rho_{\pi(\cdot)}(f/\lambda) \leq 1 \}.$$

In the sequel, we will use the same notation  $L^{\pi(\cdot)}(\Omega)$  for the space  $(L^{\pi(\cdot)}(\Omega))^N$  of vector-valued functions.

•  $W^{1,\pi(\cdot)}(\Omega)$  is the space of all functions  $f \in L^{\pi(\cdot)}(\Omega)$  such that the gradient  $\nabla f$  of  $f$  (taken in the sense of distributions) belongs to  $L^{\pi(\cdot)}(\Omega)$ ; the space  $W^{1,\pi(\cdot)}(\Omega)$  is equipped with the norm

$$\|f\|_{W^{1,\pi(\cdot)}} := \|f\|_{L^{\pi(\cdot)}} + \|\nabla f\|_{L^{\pi(\cdot)}}.$$

Further,  $W_0^{1,\pi(\cdot)}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  in the norm of  $W^{1,\pi(\cdot)}(\Omega)$ .

• Finally,  $\dot{E}^{\pi(\cdot)}(\Omega)$  is the set of all  $f \in W_0^{1,\pi(\cdot)}(\Omega)$  such that  $\nabla f \in L^{\pi(\cdot)}(\Omega)$ . This space is equipped with the norm  $\|f\|_{\dot{E}^{\pi(\cdot)}} := \|\nabla f\|_{L^{\pi(\cdot)}}$ .

When  $1 < p_- \leq \pi(\cdot) \leq p_+ < \infty$ , all the above spaces are separable reflexive Banach spaces.

A difficulty in the interpretation and analysis of the variable exponent problems of the kind (1) lies in the fact that  $W_0^{1,\pi(\cdot)}(\Omega)$  can be a strict subspace of  $\dot{E}^{\pi(\cdot)}(\Omega)$ . Therefore there can be at least two different ways to interpret the homogeneous Dirichlet boundary condition (2). In this paper, we will always avoid the difficulty by ensuring that  $\pi(x) := p(x, u(x))$  satisfy the log-Hölder continuity assumption (9) below. Indeed, from the results of Zhikov and Fan we deduce

**Lemma 2.2** (See [1, Corollary 2.6]). Assume that  $\pi(\cdot) : \Omega \rightarrow [p_-, p_+]$  has a representative which can be extended into a function continuous up to the boundary  $\partial\Omega$  and satisfying the log-Hölder continuity assumption:

$$\exists L > 0 \forall x, y \in \overline{\Omega}, x \neq y, \quad -(\log|x-y|)|\pi(x) - \pi(y)| \leq L. \tag{9}$$

Then  $\mathcal{D}(\Omega)$  is dense in  $\dot{E}^{\pi(\cdot)}(\Omega)$ . In particular, the spaces  $\dot{E}^{\pi(\cdot)}(\Omega)$  and  $W_0^{1,\pi(\cdot)}(\Omega)$  are Lipschitz homeomorphic and therefore they can be identified.

### 2.2. Definitions and results: the $p(u)$ case

First, let us recall the different definitions of a weak solution to problem (1), (2).

**Definition 2.3** (See [1]; Cf. [23,24]). Let  $f \in L^1(\Omega)$ .

- (i) A function  $u$  satisfying  $u \in W_0^{1,\pi(\cdot)}(\Omega)$  for  $\pi(\cdot) = p(\cdot, u(\cdot))$  is called a narrow weak solution of problem (1), (2), if  $b(u) \in L^1(\Omega)$  and the equation  $b(u) - \operatorname{div} a(x, u, \nabla u) = f$  is fulfilled in  $\mathcal{D}'(\Omega)$ .
- (ii) A function  $u$  satisfying  $u \in \dot{E}^{\pi(\cdot)}(\Omega)$  for  $\pi(\cdot) = p(\cdot, u(\cdot))$  is called a broad weak solution of problem (1), (2), if  $b(u) \in L^1(\Omega)$  and for all  $\phi \in \dot{E}^{\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ ,

$$\int_{\Omega} b(u)\phi + a(x, u, \nabla u) \cdot \nabla \phi = \int_{\Omega} f\phi. \tag{10}$$

- (iii) A function  $u$  like in (ii) which satisfies (10) with test functions  $u \in W_0^{1,\pi(\cdot)}(\Omega)$  (or, equivalently, that satisfies  $b(u) - \operatorname{div} a(x, u, \nabla u) = f$  in  $\mathcal{D}'(\Omega)$ ) is called an incomplete weak solution of problem (1), (2).

Notice that, under the growth assumption (5),  $a(x, u, \nabla u)$  belongs to  $L^1(\Omega)$  and even to  $L^{p'(\cdot, u(\cdot))}(\Omega)$ , so the formulations (i)–(iii) make sense.

**Remark 2.4.** Narrow and broad solutions are also incomplete. Note that uniqueness of incomplete solutions cannot be expected, unless the notions of broad and narrow solutions coincide. In this paper, we are not interested in incomplete solutions.

In [1], we have shown that both narrow and broad weak solutions defined below appear as natural notions of solutions for the case where  $p$  only depends on  $x$ ; in particular, each of these notions is stable under the *ad hoc* monotone approximation of  $p(x)$  by a sequence  $p_n(x)$ .

The general problem (1), (2) with  $(x, u(x))$ -dependent  $p$  is not yet well understood; we now turn to partial well-posedness results for this framework. In the present paper, we need a framework which ensures that  $W_0^{1,p(\cdot, u(\cdot))}(\Omega)$  and  $\dot{E}^{p(\cdot, u(\cdot))}(\Omega)$  coincide *a priori*, i.e. without any additional information on the function  $u(\cdot) \in \dot{E}^{p(\cdot, u(\cdot))}$ . Clearly, narrow and broad weak solutions would coincide in this case. A sufficient condition is

$$\left\{ \begin{array}{l} p : \overline{\Omega} \times \mathbb{R} \longrightarrow [p_-, p_+] \text{ with } \inf p > N, \text{ and for all } M > 0, \\ p \text{ is log-H\"older continuous in } (x, z) \text{ uniformly on } \overline{\Omega} \times [-M, M]. \end{array} \right. \quad (11)$$

Notice that assumption

$$N < \inf p$$

in (11) ensures that  $L^1(\Omega) \subset (\dot{E}^{p(\cdot, u(\cdot))}(\Omega))^*$ . Therefore the notion of a weak solution is sufficient (cf. [1], where we also consider renormalized solutions of problem (1), (2)).

**Remark 2.5.** Let us stress that existence of an incomplete solution is a simpler problem; we guess that restriction (11) can be dropped in this context. The technique to be used is the one of Zhikov [9, Theorem 3]. Another useful approach is the one of Dolzmann, Hungerbühler and Müller [25].

The role of assumption (11) for the existence theory is explained in Remark 3.2 in Section 3.

We prove the following existence result, which can be viewed as a SOLA-kind<sup>1</sup> well-posedness result for (1), (2).

**Theorem 2.6.** Assume  $\alpha = \alpha(x, z, \xi)$  satisfies (3)–(6), and  $p$  satisfies (11). Then there exists a map  $f \in L^1(\Omega) \mapsto u_f \in C(\overline{\Omega})$ , such that  $u_f$  is (both narrow and broad) weak solution to (1), (2); moreover, for all  $f, \hat{f} \in L^1(\Omega)$ ,

$$\int_{\Omega} (b(u_f) - b(u_{\hat{f}}))^+ \leq \int_{\Omega} (f - \hat{f}) \text{sign}^+(u_f - u_{\hat{f}}) + \int_{[u_f = u_{\hat{f}}]} (f - \hat{f})^+. \quad (12)$$

In the other words, there exists an  $m - T$ -accretive operator  $\mathcal{A}$  on  $L^1(\Omega)$  such that  $w \in L^1(\Omega)$  fulfills  $w + \mathcal{A}(w) = f$  if and only  $w = b(u_f)$  and  $u_f$  is the weak solution of (1), (2) constructed in Theorem 2.6 (see e.g. [3] for information on accretive operators in Banach spaces).

The following structural stability result analogous to the one of [1, Theorem 3.8] holds; notice that the regularity assumption (11) is only needed at the limit.

**Theorem 2.7.** Assume  $\alpha = \alpha(x, z, \xi)$  satisfies (3)–(6), and  $p$  satisfies (11).

Assume  $(\alpha_n)_n$  is a sequence of diffusion flux functions of the form  $\alpha_n(x, z, \xi)$  such that (3), (4) hold for all  $n$ . Assume (5), (6) hold with  $C, p_{\pm}$  independent of  $n$ , and with a sequence  $(\mathcal{M}_n)_n$  equi-integrable on  $\Omega$ . Assume that the associated exponents  $p_n$  featuring in assumptions (5), (6) satisfy (7). Assume

$$\left\{ \begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R} \times \mathbb{R}^N, \\ \sup_{(z, \xi) \in K} |\alpha_n(\cdot, z, \xi) - \alpha(\cdot, z, \xi)| \text{ converges to zero in measure on } \Omega, \end{array} \right. \quad (13)$$

$$\left\{ \begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R}, \\ \sup_{z \in K} |p_n(\cdot, z) - p(\cdot, z)| \text{ converges to zero in measure on } \Omega. \end{array} \right. \quad (14)$$

Finally, assume  $(f_n)_n$  is a sequence of data weakly convergent to  $f$  in  $L^1(\Omega)$ .

Denote by (1)<sub>n</sub>, (2)<sub>n</sub> the problem associated with  $\alpha_n, f_n$ . Assume  $(u_n)_n$  is a sequence of (broad or narrow) weak solutions to problems (1)<sub>n</sub>, (2)<sub>n</sub>.

Then there exists  $u \in C(\overline{\Omega})$  and a subsequence  $(n_k)_k$  such that  $u_{n_k}, \nabla u_{n_k}$  converge to  $u, \nabla u$ , respectively, a.e. on  $\Omega$ , and  $u$  is a weak (broad and narrow) solution of the limit problem (1), (2).

Clearly, extraction of a subsequence is not needed when the solution to the limit problem is unique. Concerning the uniqueness of solutions to (1), (2), we are able to prove the following conditional result, which does not rely directly on assumption (11) but depends on *a priori* regularity<sup>2</sup> of bounded weak solutions of (1), (2).

<sup>1</sup> SOLA = “Solution Obtained as Limit of Approximation”, see [26] and Remark 3.1.

<sup>2</sup> Regularity that probably relies on assumption (11): see [22]!

**Theorem 2.8.** Assume  $b$  is strictly increasing. Assume that  $a = a(x, z, \xi)$  satisfies (3)–(6), and the function  $\mathcal{M}$  in (5) can be taken constant. Assume in addition that  $a$  satisfies

$$\left\{ \begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R} \times \mathbb{R}^N \text{ there exists a constant } C(K) \\ \text{such that for a.e. } x \in \Omega, \text{ for all } (z, \xi), (\hat{z}, \hat{\xi}) \in K, \\ |a(x, z, \xi) - a(x, \hat{z}, \hat{\xi})| \leq C(K)|z - \hat{z}|. \end{array} \right. \quad (15)$$

Finally, suppose the following regularity property for weak solutions holds true:

$$\left\{ \begin{array}{l} \text{there exists a dense set } \mathcal{F} \text{ in } L^1(\Omega) \text{ such that for all } f \in \mathcal{F}, \\ \text{there exists a weak solution}^3 \text{ of (1), (2) which is Lipschitz continuous on } \overline{\Omega}. \end{array} \right. \quad (16)$$

Then for all  $f \in L^1(\Omega)$  there exists at most one function  $u$  such that  $u$  is a narrow weak solution of (1), (2) or a broad weak solution of (1), (2).

To be precise, in the conclusion of Theorem 2.8 we mean that there could exist at most one narrow solution, at most one broad solution, and if both exist, then they coincide.

Condition (16) goes back to an idea of [13]. In practice,  $L^\infty(\Omega)$  is a good candidate for being  $\mathcal{F}$  in the above statement. Indeed, the results of Fan [22] (see also [18–21]) can be applied, provided  $\partial\Omega$  is Hölder regular and, moreover, the log-Hölder regularity of  $p(\cdot, z)$  in assumption (11) is upgraded to the Hölder continuity with some non-zero exponent  $\alpha$ . In this way, we deduce the following well-posedness result, which applies for instance to the problem

$$u - \Delta_{p(x,u)}u = f, \quad u|_{\partial\Omega} = 0, \quad (17)$$

assuming that  $p$  is locally Lipschitz continuous on  $\overline{\Omega} \times \mathbb{R}$  and  $p(x, z) > N$  for all  $(x, z)$ .

Assumption (19) below is a combination of (15) and of the hypothesis made by Fan [22]; this assumption is natural when  $a(x, z, \xi)$  grows as  $|\xi|^{p(x,z)}$  and  $p$  satisfies (18), in view of the fact that, for  $p, \hat{p} \geq p_- > 1$ ,

$$\left| |\xi|^{p-1} - |\xi|^{\hat{p}-1} \right| \leq |\xi|^{\max\{p, \hat{p}\}-1} \ln(|\xi|) |p - \hat{p}|.$$

**Theorem 2.9.** Assume that  $b$  is strictly increasing. Assume that  $a = a(x, z, \xi)$  satisfies (3)–(6), and the function  $\mathcal{M}$  in (5) can be taken constant. Assume that  $\text{ess inf } p > N$  and that

$$\left\{ \begin{array}{l} \text{there exist } \alpha > 0 \text{ such that for all bounded subset } K \text{ of } \mathbb{R}, \\ |p(x, z) - p(\hat{x}, \hat{z})| \leq C(K) (|x - \hat{x}|^\alpha + |z - \hat{z}|) \\ \text{for all } x, \hat{x} \in \overline{\Omega} \text{ and all } z, \hat{z} \in K, \end{array} \right. \quad (18)$$

where  $p = p(x, z)$  is the variable exponent in (5), (6).

In addition, assume that  $\partial\Omega$  belongs to some Hölder class  $C^{0,\alpha}$ , that  $a(x, z, \xi)$  is continuously differentiable in  $\xi$  on  $\mathbb{R}^N \setminus \{0\}$ , and that  $a$  satisfies the assumption

$$\left\{ \begin{array}{l} \text{there exist } \alpha > 0 \text{ such that for all } \delta > 0 \text{ and all bounded } K \subset \mathbb{R}, \\ |a(x, z, \xi) - a(\hat{x}, \hat{z}, \xi)| \leq C(\delta, K) (|x - \hat{x}|^\alpha + |z - \hat{z}|) \left( 1 + |\xi|^{\max\{p(x,z), p(\hat{x}, \hat{z})\}-1+\delta} \right) \\ \text{for all } x, \hat{x} \in \overline{\Omega}, \text{ for all } z, \hat{z} \in K \text{ and all } \xi \in \mathbb{R}^N. \end{array} \right. \quad (19)$$

Then for all  $f \in L^1(\Omega)$ , there exists one and only one weak broad solution  $u_f$  to problem (1), (2) (it is also the unique narrow weak solution of the problem). Moreover, the solution  $u_f$  depends continuously on the datum  $f$  in the sense (12).

**Remark 2.10.** A very particular case where the assumptions of Theorems 2.6–2.8 can be simplified is the case  $N = 1$ . We do not formulate the exact assumptions, because the framework of growth and coercivity assumptions of the kind (5), (6) is too restrictive. Indeed, for  $N = 1$ , any weak solution is automatically bounded; moreover, any weak solution is Lipschitz continuous under a mild uniform boundedness assumption on  $a^{-1}(x, z, \cdot)$ . In addition, in the one-dimensional case,  $\mathcal{D}(\Omega)$  is dense in  $\dot{E}^{p(\cdot)}(\Omega)$  for all  $p : \Omega \rightarrow [p_-, p_+]$  measurable. Therefore the one-dimensional problem (17) is well-posed also for a discontinuous in  $x$  exponent  $p$  satisfying (7), provided that  $p$  is locally Lipschitz continuous in  $z$  uniformly in  $x$ .

The case  $N = 1$  has been investigated, under fairly general coercivity and growth conditions on  $a$ , in the works of Bénilan and Touré [14], Ouaro and Touré [17], and Ouaro [15,16].

<sup>3</sup> Under the assumptions of Theorem 2.8, broad and narrow weak solutions that are Lipschitz continuous coincide.

2.3. The  $p[u]$  case

For system (8) endowed with the homogeneous Dirichlet boundary conditions, we can simply seek for  $u \in W_0^{1,p(u)}(\Omega)$  and  $v \in H_0^1(\Omega)$ . The assumptions we make on the reaction terms  $f, g$  and on  $p(\cdot, \cdot)$  entail a Hölder regularity of these solutions. We prove

**Theorem 2.11.** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$  with  $C^{0,\alpha}$  boundary, with some  $\alpha > 0$ . Let  $g, h : \Omega \times (\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$  be globally bounded Carathéodory functions. Assume  $p$  is a locally  $C^{0,\alpha}$  continuous function on  $\overline{\Omega} \times \mathbb{R}$  taking values in  $[p_-, p_+] \subset (1, +\infty)$ .*

*Then there exists a couple of functions  $u, v : \Omega \rightarrow \mathbb{R}$  such that  $v \in H_0^1(\Omega)$ ,  $u \in W_0^{1,p(v)}(\Omega)$ , and  $\begin{cases} u - \Delta_{p(x,v)} u = f(x, u, v) \\ v - \Delta v = g(x, u, v) \end{cases}$  is fulfilled in  $\mathcal{D}'(\Omega)$ . Moreover, we have  $u, v \in C^{0,\beta}(\overline{\Omega})$  for some  $\beta \in (0, 1)$ .*

This theorem is obtained from the fact that the suitably constructed Galerkin approximations of the homogeneous Dirichlet problem (8) converge. In a similar manner, convergence of numerical methods for (8) can be justified (see [10]).

3. Well-posedness results for (1), (2) with  $p = p(u)$

In the subsequent proofs, a number of references is made to the proof of [1, Theorems 3.8, 3.11]. We also use without proofs the properties of Young measures stated in [27] and in [1, Theorem 2.10].

3.1. Existence of weak solutions

Recall that we need to assume (11), therefore we are in the case where  $\mathcal{D}(\Omega)$  is dense in the space where weak solutions belong to. In this framework, existence can be shown by the Galerkin method, as in the work [28] of Antontsev and Shmarev. We give another proof, using a regularization technique (cf. [24]). Then in Remark 3.2, we isolate and discuss the point where the possible discrepancy between  $W_0^{1,p(\cdot, u(\cdot))}(\Omega)$  and  $\dot{E}^{p(\cdot, u(\cdot))}(\Omega)$  becomes the obstacle for proving the general existence result.

**Proof of Theorem 2.6.**

• Step 1. For  $n \in \mathbb{N}$ , introduce  $\alpha_n(x, z, \xi) := \alpha(x, z, \xi) + \frac{1}{n}|\xi|^{p_+ - 2}\xi$ . Notice that  $\alpha_n$  verifies the assumptions (3)–(6) with  $p(x, z)$  replaced by the constant exponent  $p_+$ , and with  $C, \mathcal{M}$  that depend on  $n$ . Denote by  $(1)_n$  the equation of the form (1) associated with the diffusive flux  $\alpha_n$ .

Because  $p_+ \geq p_- > N, L^1(\Omega) \subset W_0^{-1,p_+}(\Omega)$ ; therefore for all  $f \in L^1(\Omega)$ , there exists a weak solution  $u_f^n$  of  $(1)_n, (2)$  in  $W_0^{1,p_+}(\Omega)$  (see e.g. [1, Theorem 3.11]). Moreover, without loss of generality, we can assume that the  $T$ -contraction property in  $L^1(\Omega)$  holds for all  $n$ :

$$\int_{\Omega} (b(u_f^n) - b(u_{\hat{f}}^n))^+ \leq \int_{\Omega} (f - \hat{f}) \text{sign}^+(u_f^n - u_{\hat{f}}^n) + \int_{[u_f^n = u_{\hat{f}}^n]} (f - \hat{f})^+. \tag{20}$$

It is easy to deduce (20) by the technique on the proof of Theorem 2.8 below, provided  $\alpha_n$  is Lipschitz continuous in  $z$ , more exactly,

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^N \sup_{|z|, |\hat{z}| \leq L} |\alpha_n(x, z, \xi) - \alpha_n(x, \hat{z}, \xi)| \leq C(\alpha_n, L)|\xi|^{p_+ - 1}|z - \hat{z}|. \tag{21}$$

In order to get rid of the regularity assumption (21), we approximate  $\alpha$  by a sequence of regular, in the sense (21), diffusion fluxes constructed by convolution in  $(x, z)$  and normalized as in (3). The so constructed sequence of fluxes verifies the standard coercivity and growth assumptions with the constant and fixed exponent  $p_+$  and with common  $C > 0, \mathcal{M} \in L^1(\Omega)$ . Thus we can use the classical Minty–Browder argument (or, alternatively, the appropriately simplified arguments of Step 2 below) to pass to the limit. As a result, we justify inequalities (20), at least for  $f, \hat{f}$  in some countable subset  $\mathcal{F}$  of  $L^1(\Omega)$ .

• Step 2. Now, fix a countable family  $\mathcal{F}$  of right-hand sides  $f$  such that  $\mathcal{F}$  is dense in  $L^1(\Omega)$ . In the sequel, we will write  $u_n$  for  $u_f^n$ , meaning that  $f \in \mathcal{F}$ ; moreover, extracting subsequences, we will do it simultaneously for all  $f \in \mathcal{F}$ , using the standard diagonal procedure.

In this way, we now pass to the limit in (a subsequence of)  $(u_n)_n$ , as  $n \rightarrow +\infty$ . We cannot apply [1, Theorem 3.8] directly, but we adapt its proof to the present case. Let us set

$$\tilde{\alpha}_n(x, \xi) := \alpha(x, u_n(x), \xi);$$

then

$$p_n(x) := p(x, u_n(x))$$

is the corresponding variable exponent, which we now consider as a function of  $x$  alone. Let us repeat the itinerary of the proof of [1, Theorem 3.11], that we now sketch. Firstly, we show that  $u_n$  converges a.e. on  $\Omega$  to some function  $u$ . Secondly,

we deduce that  $p_n, \tilde{a}_n$  converge to  $p_\infty, \tilde{a}$  in the sense

$$p_n \text{ converges to } p_\infty \text{ in measure on } \Omega, \tag{22}$$

$$\left\{ \begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R}^N, \\ \sup_{\xi \in K} |\tilde{a}_n(\cdot, \xi) - \tilde{a}(\cdot, \xi)| \text{ converges to zero in measure on } \Omega. \end{array} \right. \tag{23}$$

Here and in the sequel of the proof, we will write  $p_\infty(x)$  for  $p(x, u(x))$ ; and  $\tilde{a}(x, \xi)$  denotes  $\alpha(x, u(x), \xi)$ . Finally, we apply the passage-to-the limit arguments from the proof of [1, Theorem 3.8], taking a particular care of Claim 9 of the proof.

Let us give the key details. Thanks to (6) and the definition of  $a_n$ , the sequence  $(u_n)_n$  verifies the standard estimates

$$\int_{\Omega} \left( b(u_n)u_n + |\nabla u_n|^{p(x, u_n(x))} + \frac{1}{n} |\nabla u_n|^{p^+} \right) \leq C,$$

with  $C$  that depends on  $f$  but not on  $n$ . Thus we deduce that  $\frac{1}{n} |\nabla u_n|^{p^+ - 2} \nabla u_n$  converges to zero in  $L^1(\Omega)$ ; moreover,  $\int_{\Omega} |\nabla u_n|^{p_n(x)} \leq C$ . Thus, up to extraction of a subsequence,  $u_n$  converges a.e. on  $\Omega$  (and also weakly in  $W_0^{1, p^+}(\Omega)$ ) to a limit  $u$ ; using the representation of weakly convergent sequences in  $L^1(\Omega)$  in terms of Young measures, we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda \, d\nu_x(\lambda). \tag{24}$$

Further, we have shown in the proof of [1, Theorem 3.11] that, because  $u_n$  converges strongly to  $u$ , (23) is true. Similarly, (22) holds (the proof of (22) under assumption (11) is immediate, because  $p$  is continuous on  $\bar{\Omega} \times \mathbb{R}$ , and thus locally uniformly continuous; and  $u_n$  converges to  $u$  in  $C(\bar{\Omega})$ , by the standard embedding argument). Now we can apply Claim 4 of the proof of [1, Theorem 3.8], where we formally put  $\gamma = \infty$ , and get  $u \in \dot{E}^{p_\infty(\cdot)}(\Omega)$ . We also apply Claims 6, 7 to  $\chi_n := \tilde{a}_n(x, \nabla u_n)$  and conclude that  $\chi_n$  converge weakly in  $L^1(\Omega)$  to the limit  $\chi$  given by

$$\chi(x) = \int_{\mathbb{R}^N} \tilde{a}(x, \lambda) \, d\nu_x(\lambda). \tag{25}$$

Now, let us concentrate on deducing the key inequality

$$\int_{\Omega} \chi \cdot \nabla u \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n \cdot \nabla u_n. \tag{26}$$

For all  $e \in \mathcal{D}(\Omega)$ , we have

$$\int_{\Omega} \left( b(u_n)e + \tilde{a}_n(x, \nabla u_n) \cdot \nabla e + \frac{1}{n} |\nabla u_n|^{p^+ - 2} \nabla u_n \cdot \nabla e \right) = \int_{\Omega} f e. \tag{27}$$

We can pass to the limit as  $n \rightarrow \infty$  and infer

$$\int_{\Omega} (b(u)e + \chi \cdot \nabla e) = \int_{\Omega} f e \tag{28}$$

for all  $e \in \mathcal{D}(\Omega)$ ; here  $\chi$  is given by (25). Because for all  $n, \nabla u_n \in L^{p^+}(\Omega)$  by construction, by the density argument we can replace  $e$  with  $u_n$  in (27). Further, because we have assumed (11),  $u \in W_0^{1, p_\infty(\cdot)}(\Omega) \subset W_0^{1, p^+}(\Omega)$  is Hölder continuous and the exponent  $p_\infty, p_\infty(x) = p_\infty(x, u(x))$ , verifies (9). Thus  $\mathcal{D}(\Omega)$  is dense in  $\dot{E}^{p_\infty(\cdot)}(\Omega)$ , therefore we can pick  $u \in \dot{E}^{p_\infty(\cdot)}(\Omega)$  for the test function in (28). Passing to the limit as  $n \rightarrow \infty$ , we infer the desired inequality

$$\int_{\Omega} \chi \cdot \nabla u \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \left( \chi_n \cdot \nabla u_n + \frac{1}{n} |\nabla u_n|^{p^+} \right) \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n \cdot \nabla u_n. \tag{29}$$

Continuing as in the proof of [1, Theorem 3.8] (but with  $\gamma = \infty$ , thus avoiding the technicalities of the renormalized formulation), from the representation formulas (24), (25), the inequality (26) and the monotonicity (4) of  $\tilde{a}$  we infer that a.e. on  $\Omega$ , the measure  $\nu_x$  reduces to the Dirac measure  $\delta_{\nabla u(x)}$ . Therefore we identify  $\chi(x)$  with  $\tilde{a}(x, \nabla u(x))$ .

We conclude that  $u$  is a weak solution of (1), (2) with  $f \in \mathcal{F}$ . Fixing arbitrarily the extracted subsequence, we denote by  $u_f$  the solution  $u$  obtained in Step 2.

• Step 3. Note that we also have  $u_f^n \rightarrow u_f$  in  $L^1(\Omega)$ . Therefore we can pass to the limit in (20) and infer (12) with  $f, \hat{f} \in \mathcal{F}$ . Indeed, the right-hand side of (12) is the so-called “ $L^1$  bracket”  $[u, f]_+ = \int_{\Omega} f \text{sign}^+ u + \int f^+ \mathbb{1}_{[u=0]}$ , which is known to be upper-semicontinuous in  $L^1$  (see e.g. [3]).

• Step 4. Now we approximate  $f, \hat{f} \in L^1(\Omega)$  by sequences  $(f_i)_i, (\hat{f}_i)_i \in \mathcal{F}$ . The arguments of the above Step 2 permit to deduce that the corresponding solutions  $u_{f_i}$  converge to a weak solution  $u_f$  of (1), (2), up to extraction of a subsequence. In order to define correctly the map  $f \mapsto u_f$ , we can simply fix, for all  $f$ , the corresponding approximating sequence  $(f_i)_i$ .

Inequalities (12) can now be extended to all  $f, \hat{f} \in L^1(\Omega)$ , by the same argument as in Step 3.  $\square$

**Remark 3.1.** In the above proof, we have justified the convergence of a particular approximation  $u_f^n$  to  $u_f$  only for  $f$  in a dense countable subset of  $L^1(\Omega)$ . Notice that under the additional assumption that  $b$  is strictly increasing, this convergence remains true for all  $f \in L^1(\Omega)$ , thanks to (20) and (12). Thus for a strictly increasing  $b$ , the image of the map  $f \mapsto u_f$  consists of the limits, as  $n \rightarrow \infty$ , of approximate solutions  $u_f^n$  obtained by the approximation procedure employed in the proof of Theorem 2.6. Thus  $u_f$  can be seen as a SOLA in the sense of Dall’Aglio [26].

**Proof of Theorem 2.7 (Sketched).** This is a straightforward combination of the arguments of the proofs of [1, Theorems 3.8, 3.11] and of Theorem 2.6. We only notice that, upon writing

$$|p_n(x, u_n(x)) - p(x, u(x))| \leq |p_n(x, u_n(x)) - p(x, u_n(x))| + |p(x, u_n(x)) - p(x, u(x))|,$$

from the a.e. convergence of  $u_n$  to  $u$ , from assumption (14) and from the Lusin Theorem applied to the map

$$p : \Omega \mapsto p(x, \cdot) \in C(\mathbb{R})$$

we deduce that  $p_n(\cdot, u_n(\cdot))$  converges to  $p(\cdot, u(\cdot))$  in measure on  $\Omega$ . In the same way, we convert assumption (13) into assumption (23) for the nonlinearities  $\tilde{a}_n(x, \xi) := a_n(x, u_n(x), \xi)$  and  $\tilde{a}(x, \xi) := a(x, u(x), \xi)$ .  $\square$

**Remark 3.2.** Let us clarify the role of the restriction  $\inf p > N$  for the existence result of Theorem 2.6. It is only needed in order to ensure that

$$\mathcal{D}(\Omega) \text{ is dense in } \dot{E}^{p(\cdot, u(\cdot))}(\Omega), \quad \text{i.e., } \dot{E}^{p(\cdot, u(\cdot))}(\Omega) = W_0^{1, p(\cdot, u(\cdot))}(\Omega). \tag{30}$$

Indeed, in the beginning of the existence proof we are able to show that  $u \in \dot{E}^{p(\cdot, u(\cdot))}(\Omega)$ ; and we need that  $u$  belong to  $W_0^{1, p(\cdot, u(\cdot))}(\Omega)$  in order to conclude the proof.<sup>4</sup>

The difficulty stems from the fact that, in the above proofs, we have to justify that  $u$  ( $u$  being the accumulation point of a sequence  $(u_n)_n$  of suitably constructed approximate solutions) is an admissible test function in the formulation (28) obtained by the passage to a weak limit in (27). In this way we infer (29), which is the starting point for the monotonicity-based identification argument.

To make the difficulty apparent, let us ask the following simple question: the set of weak (broad, or narrow) solutions of (1), (2) is it closed, in the sense of the a.e. convergence of the solutions and gradients? This is a very particular case of Theorem 2.7. In this context, we omit the regularization term in (27), (29) and refer to the passage from (27), (28) to (29). Then we are able to prove that:

- any accumulation point  $u$  belongs to the “broad” space  $\dot{E}^{p(\cdot, u(\cdot))}(\Omega)$ ;
- the passage from (27) to (28) is possible with test functions  $e$  such that for  $n$  large enough,  $e \in W_0^{1, p_n(\cdot)}(\Omega)$  (if  $u_n$  are narrow solutions) or  $e \in \dot{E}^{p_n(\cdot)}(\Omega)$  (if  $u_n$  are the broad ones).

Functions  $e \in \mathcal{D}(\Omega)$  are always suitable; by the density argument, we are able to pick any function from the “narrow” space  $W_0^{1, p(\cdot, u(\cdot))}(\Omega)$  as test function in (28). This is how condition (30) arises.

### 3.2. Uniqueness of weak solutions

#### Proof of Theorem 2.8.

• Step 1. Let  $u$  be a Lipschitz continuous (broad or narrow) weak solution of (1), (2) with  $f \in \mathcal{F}$ , and  $\hat{u}$  be a weak solution in the same sense with a source term  $\hat{f} \in L^1(\Omega)$ . For  $\gamma > 0$ , set

$$T_\gamma : z \in \mathbb{R} \mapsto T_\gamma(z) = \max\{\min\{z, \gamma\}, -\gamma\}, \quad \gamma > 0.$$

Then the test function  $\frac{1}{\gamma} T_\gamma(u - \hat{u})$  is admissible in the weak formulations for both  $u, \hat{u}$ .

Indeed, because  $u$  is bounded,

$$\phi := \frac{1}{\gamma} T_\gamma(u - \hat{u}) = \frac{1}{\gamma} T_\gamma(u - T_{\gamma + \|u\|_{L^\infty}}(\hat{u})). \tag{31}$$

Therefore  $\phi$  belongs to  $W_0^{1,1}(\Omega)$  and we have

$$\nabla \phi = \frac{1}{\gamma} (\nabla u - \nabla T_{\gamma + \|u\|_{L^\infty}}(\hat{u})) \mathbb{1}_{|u - \hat{u}| < \gamma} = (\nabla u - \nabla \hat{u}) \mathbb{1}_{|u - \hat{u}| < \gamma}. \tag{32}$$

Now we use the fact that  $\nabla u$  is bounded. By the assumptions of the theorem,  $|a(x, u, \nabla u)| \leq C(|\nabla u|^{p(x, u(x))} + 1) \in L^\infty(\Omega)$ . This readily implies that  $\phi \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$  is admissible as a test function in the weak formulation for the solution  $u$ . Because  $u \in W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$ , this also allows to assert that  $u$  is necessarily both narrow and broad solution to (1), (2).

<sup>4</sup> Notice that the aforementioned difficulty does not arise in the identification argument of Dolzmann, Hungerbühler and Müller [25], nor in the proof of Zhikov [9]. With the help of any of these arguments, one can study the existence of incomplete solutions of (1), (2) without imposing the restriction (11).

Further, if  $\hat{u} \in \dot{E}^{p(\cdot, \hat{u}(\cdot))}(\Omega)$ , then by (32),  $\phi \in \dot{E}^{p(\cdot, \hat{u}(\cdot))}(\Omega) \cap L^\infty(\Omega)$  and thus it is an admissible test function in the broad weak formulation for the solution  $\hat{u}$ . If  $\hat{u} \in W_0^{1, p(\cdot, \hat{u}(\cdot))}(\Omega)$ , then by (31), by [1, Lemma 2.9] and because  $u \in W_0^{1, p^+}(\Omega) \subset W_0^{1, p(\cdot, \hat{u}(\cdot))}(\Omega)$ , we have  $\phi \in W_0^{1, p(\cdot, \hat{u}(\cdot))}(\Omega) \cap L^\infty(\Omega)$ ; we conclude that  $\phi$  is an admissible test function in the narrow weak formulation for  $\hat{u}$ .

• Step 2. With the test function of Step 1, we deduce

$$\begin{aligned} & \int_{\Omega} \frac{1}{\gamma} T_{\gamma}(u - \hat{u})(b(u) - b(\hat{u})) + \frac{1}{\gamma} \int_{[0 < |u - \hat{u}| < \gamma]} (a(x, u, \nabla u) - a(x, \hat{u}, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) \\ &= \int_{\Omega} \frac{1}{\gamma} T_{\gamma}(u - \hat{u})(f - \hat{f}). \end{aligned} \tag{33}$$

We add and subtract the term  $a(x, \hat{u}, \nabla u)$ . As  $\gamma \rightarrow 0$ , by the monotonicity of  $a$  we infer

$$\int_{\Omega} |b(u) - b(\hat{u})| \leq \int_{\Omega} |f - \hat{f}| + \liminf_{\gamma \downarrow 0} \frac{1}{\gamma} \int_{[0 < |u - \hat{u}| < \gamma]} |a_n(x, u, \nabla u) - a(x, \hat{u}, \nabla u)| |\nabla u - \nabla \hat{u}|. \tag{34}$$

Denote the lim inf term in (33) by  $R$ . Since  $u$  is bounded, also  $\hat{u}$  is bounded on the set  $[0 < |u - \hat{u}| < \gamma]$ ; thus we can use the Lipschitz continuity assumption (15) to get

$$\begin{aligned} R &\leq C(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}) \liminf_{\gamma \downarrow 0} \frac{1}{\gamma} \int_{[0 < |u - \hat{u}| < \gamma]} |u - \hat{u}| |\nabla u - \nabla \hat{u}| \\ &\leq C \lim_{\gamma \downarrow 0} \int_{E_{\gamma}} |\nabla u - \nabla \hat{u}|, \end{aligned}$$

where  $E_{\gamma} := [0 < |u - \hat{u}| < \gamma]$ . Because  $\nabla u, \nabla \hat{u} \in L^1(\Omega)$  and  $\text{meas}(E_{\gamma})$  tends to zero as  $\gamma \rightarrow \infty$ , we deduce that the last term in (34) is zero.

• Step 3. Now assume  $u, \hat{u}$  are two weak solutions of (1), (2); either of them can be a narrow or a broad weak solution. Take a sequence  $(g_i)_i \subset \mathcal{F}$ , and let  $(\hat{u}_i)_i$  be the corresponding sequence of Lipschitz continuous weak solutions. By the result of Step 2, we have

$$\int_{\Omega} |b(u) - b(\hat{u})| \leq \int_{\Omega} (|b(u) - b(\hat{u}_i)| + |b(\hat{u}) - b(\hat{u}_i)|) \leq \int_{\Omega} (|f - \hat{f}_i| + |\hat{f} - \hat{f}_i|),$$

so that at the limit as  $i \rightarrow \infty$  we infer that  $b(u) = b(\hat{u})$ . Because  $b$  is assumed to be strictly increasing, we conclude that  $u$  and  $\hat{u}$  coincide.  $\square$

**Proof of Theorem 2.9.** We only need to justify the uniqueness of a weak solution; existence and continuous dependence then follow by Theorem 2.6.

Uniqueness is obtained from Theorem 2.8, with the help of the global (up-to-the-boundary) regularity result of [22] (see also [18–21]). This result applies to solutions which are *a priori* bounded, and in the case the source term  $f$  is in  $L^\infty(\Omega)$ . The boundedness is trivial because  $u \in W_0^{1, p^-}(\Omega)$  and we have assumed that  $p_- > N$ . Notice that, more generally, the boundedness of  $u$  is guaranteed by the maximum principle, for the  $L^\infty$  source terms. The maximum principle for (1), (2) is easily obtained with the technique of the proof of Theorem 2.8, Step 2, thanks to assumption (3).  $\square$

#### 4. Non-local dependence of $p$ on $u$ : existence of weak solutions

**Proof of Theorem 2.11.** • Step 1. Let us construct a sequence of approximate solutions.

Pick a countable set  $(w_i)_i \subset \mathcal{D}(\Omega)$  which is dense, e.g., in the weak topology of  $W^{1, \infty}(\Omega)$ . Take  $n \in \mathbb{N}$ . Consider the nonlinear algebraic system of  $2n$  equations with the  $2n$  unknowns  $(c_i^n)_{i=1}^n, (d_i^n)_{i=1}^n$ :

$$\begin{cases} u_n(x) := \sum_{i=1}^n c_i^n w_i(x), & \int_{\Omega} (u_n w_i + |\nabla u_n|^{p(x, v_n)-2} \nabla u_n \cdot \nabla w_i) = \int_{\Omega} f(x, u_n, v_n) w_i, \\ v_n(x) := \sum_{i=1}^n d_i^n w_i(x), & \int_{\Omega} (v_n w_i + \nabla v_n \cdot \nabla w_i) = \int_{\Omega} g(x, u_n, v_n) w_i, \\ i = 1, \dots, n. \end{cases} \tag{35}$$

It is easy to check the coercivity condition of [29, Ch. I, Lemma 4.3]; therefore existence of a solution to system (35) follows from the Brouwer fixed-point theorem.

• Step 2. The functions  $u_n, v_n$  constructed in Step 1 verify the uniform estimate

$$\int_{\Omega} (u_n^2 + |\nabla u_n|^{p(x, v_n)} + v_n^2 + |\nabla v_n|^2) \leq C(\|f\|_{L^\infty}, \|g\|_{L^\infty}, \Omega). \tag{36}$$

This estimate is standard; it permits to assert that, upon extracting a (not relabelled) subsequence,  $v_n \rightharpoonup v$  in  $H_0^1(\Omega)$  weakly and a.e. on  $\Omega$ , that  $u_n \rightharpoonup u$  in  $W^{1,p^-}(\Omega) \cap L^2(\Omega)$  weakly and also a.e. on  $\Omega$ , and, moreover, that

$$\nabla u(x) = \int_{\mathbb{R}^N} \lambda \, dv_x(\lambda), \tag{37}$$

where  $(v_x)_x$  is the family of Young measures associated with the weakly convergent in  $L^1(\Omega)$  sequence  $(\nabla u_n)_n$  (see e.g. [27] or [1, Theorem 2.10]).

• Step 3. Estimate (36) implies that  $\nabla u \in L^{p(x,v(x))}$ .

This is made by showing that  $|\lambda|^{p(x,v(x))}$  is summable wrt measure  $dv_x(\lambda) \, dx$  on  $\mathbb{R}^N \times \Omega$  (then the Jensen inequality is applied). To this end, let us first point out that  $p_n(\cdot) := p(\cdot, v_n(\cdot))$  converge to  $p_\infty(\cdot) := p(\cdot, v(\cdot))$  a.e. on  $\Omega$ , because  $p$  is uniformly continuous on  $\Omega \times \mathbb{R}$  and  $v_n$  converge pointwise. In particular, by the results stated in [27] and recalled in [1, Theorem 2.10(iii)], the Young measure  $(\mu_x)_x$  on  $\mathbb{R} \times \mathbb{R}^N$  associated with (an extracted subsequence of) the sequence  $(v_n, \nabla u_n)_n$  is equal to  $\delta_{v(x)} \otimes \nu_x$ .

Then we apply the nonlinear weak-\* convergence property of [1, Theorem 2.10(i)] to the function

$$F : (x, (\lambda_0, \lambda)) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N) \mapsto |h_m(\lambda)|^{p(x,\lambda_0)},$$

where  $(h_m)_m$  is the sequence of truncations defined by

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda, & |\lambda| \leq m \\ m \frac{\lambda}{|\lambda|}, & |\lambda| > m, \end{cases} \tag{38}$$

$m > 0$ . Hence

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^N} |h_m(\lambda)|^{p(x,v(x))} \, dv_x(\lambda) \, dx &= \int_{\Omega \times (\mathbb{R} \times \mathbb{R}^N)} |h_m(\lambda)|^{p(x,\lambda_0)} \, d\mu_x(\lambda_0, \lambda) \, dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} |h_m(\nabla u_n)|^{p(x,v_n(x))} \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p_n(x)} \leq C. \end{aligned}$$

As  $m \uparrow \infty$ , from the monotone convergence theorem we deduce our claim.

• Step 4. The sequence  $(\chi_n)_n$ ,  $\chi_n := |\nabla u_n|^{p(x,v_n(x))-2} \nabla u_n$ , is relatively weakly compact in  $L^1(\Omega)$ , the weak  $L^1$  limit  $\chi$  of (an extracted subsequence of)  $(\chi_n)_n$  belongs to  $L^{p_\infty(\cdot)}(\Omega)$ , and we have for a.e.  $x \in \Omega$ ,

$$\chi(x) = \int_{\mathbb{R}^N} |\lambda|^{p_\infty(x)-2} \lambda \, dv_x(\lambda). \tag{39}$$

The claim follows by [1, Theorem 2.10(i)] applied to the function  $F(x, (\lambda_0, \lambda)) = |\lambda|^{p(x,\lambda_0)-2} \lambda$  and to the Young measure  $(\mu_x)_x = \delta_{v(x)} \otimes \nu_x$  introduced in Step 3. We only have to show that  $(\chi_n)_n$  is equi-integrable on  $\Omega$ . The proof of the equi-integrability of  $(\chi_n)_n$ , based on estimate (36) and on the generalized Hölder inequality for variable exponent spaces, is technical. It is detailed in Claim 6 of the proof of [1, Theorem 3.8].

• Step 5. For all fixed  $i$ , we can pass to the limit in (35) with the test function  $w_i$ . By the density of the family  $(w_i)_i$  in  $\mathcal{D}(\Omega)$  supplied with the weak  $W^{1,\infty}(\Omega)$  topology, we infer that

$$\begin{cases} u - \operatorname{div} \chi = f(x, u, v) \\ v - \Delta v = g(x, u, v) \end{cases} \tag{40}$$

in  $\mathcal{D}'(\Omega)$ . Thus  $v \in H_0^1(\Omega)$  is a variational solution of the second equation in (40). Because  $g \in L^\infty(\Omega)$  and  $\partial\Omega$  is assumed to be sufficiently regular, by the classical regularity result we conclude that  $v \in C^{0,\beta}(\Omega)$  for some  $\beta > 0$ .

• Step 6. We deduce that  $u \in W_0^{1,p_\infty(\cdot)}(\Omega)$ . Indeed, we already know that  $u \in W_0^{1,p^-}(\Omega)$  and that  $\nabla u \in L^{p_\infty(\cdot)}(\Omega)$ . Thus  $u$  belongs to the space  $\dot{E}^{p_\infty(\cdot)}(\Omega)$  introduced in Definition 2.1.

Moreover, by the Hölder regularity of  $p$  and  $v$  (see the assumptions of the theorem and Step 5, respectively),  $p_\infty(\cdot) = p(\cdot, v(\cdot))$  is also Hölder continuous of some order  $\gamma > 0$ . Thus  $p_\infty$  also satisfies the weaker log-Hölder continuity condition (9), and  $\dot{E}^{p_\infty(\cdot)}(\Omega) = W_0^{1,p_\infty(\cdot)}(\Omega)$  by Lemma 2.2.

• Step 7. The “div-curl” inequality holds:

$$\int_{\Omega \times \mathbb{R}^N} (|\lambda|^{p_\infty(x)-2} \lambda - |\nabla u|^{p_\infty(x)-2} \nabla u) \cdot (\lambda - \nabla u) \, dv_x(\lambda) \, dx \leq 0. \tag{41}$$

In order to justify this claim, we use the first equation of (8) with the test function  $u_n$ , and the first equation of (40) with the test function  $u$ . Let us stress that  $u$  is an admissible test function in this equation. Indeed,  $u \in W_0^{1,p_\infty(\cdot)}(\Omega) \cap L^2(\Omega)$  (see the above Steps 2, 6),  $\chi \in L^{p_\infty(\cdot)}(\Omega)$  (see Step 4), and  $\mathcal{D}(\Omega)$  is dense in  $W_0^{1,p_\infty(\cdot)}(\Omega) \cap L^2(\Omega)$ .

By the dominated convergence theorem,  $f(x, u_n, v_n)$  converges to  $f(x, u, v)$  in  $L^2(\Omega)$  strongly; because  $u_n$  converges to  $u$  in  $L^2(\Omega)$  weakly, we infer that

$$\begin{aligned} \int_{\Omega} (u^2 + \chi \cdot \nabla u) &= \int_{\Omega} f(x, u, v)u = \lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, v_n)u_n \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} (u_n^2 + \chi_n \cdot \nabla u_n). \end{aligned}$$

By the Fatou Lemma, we deduce the inequality

$$\int_{\Omega} \chi \cdot \nabla u \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n \cdot \nabla u_n. \tag{42}$$

Now (42), (37) and (39) lead to the desired inequality (41). Indeed, with the help of the truncations (38) we find that the right-hand side of (42) is lower bounded by  $\int_{\Omega \times \mathbb{R}^N} |\lambda|^{p_{\infty}(x)} d\nu_x(\lambda) dx$ ; then we perform easy algebraic manipulations using the fact that  $(\nu_x)_x$  are probability measures. The details of the argument are given in the Claim 10 of the proof of [1, Theorem 3.8].

• Step 8. By the strict monotonicity of the map  $\alpha : (x, \xi) \mapsto |\xi|^{p_{\infty}(\cdot)-2}\xi$  in the sense (4), from (41) we deduce that for a.e.  $x \in \Omega$ , the support of the measure  $\nu_x$  is reduced to the singleton  $\{\nabla u(x)\}$ . In other words,  $(\nu_x)$  is a Dirac Young measure which can be identified with the function  $\nabla u$  on  $\Omega$ . By (39), we deduce that  $\chi = |\nabla u|^{p_{\infty}(x)-2}\nabla u = |\nabla u|^{p(x, v(x))-2}\nabla u$  a.e. on  $\Omega$ . Therefore (40) is exactly the  $\mathcal{D}'$  formulation of system (8).

• Step 9. Finally, the function  $u$  is also Hölder continuous, by a straightforward application of the regularity result [22]. Indeed, the right-hand side  $f(x, u(x), v(x))$  is bounded, the exponent  $p_{\infty}$  is Hölder continuous, and  $\alpha : (x, \xi) \mapsto |\xi|^{p_{\infty}(\cdot)-2}\xi$  verifies the assumptions of [22].

This ends the proof of the theorem. Notice that it can be deduced from the above results that  $v_n$  converges to  $v$  strongly in  $H_0^1(\Omega)$ ,  $\int_{\Omega} |\nabla u_n|^{p(x, v_n(x))}$  tends to  $\int_{\Omega} |\nabla u|^{p(x, v(x))}$ , and  $\nabla u_n$  converges to  $\nabla u$  a.e. on  $\Omega$ , as  $n \rightarrow \infty$ , up to extraction of a subsequence.  $\square$

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