A Gradient Reconstruction Formula for Finite Volume Schemes and Discrete Duality

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ABSTRACT. *We point out a simple 2D formula to reconstruct the discrete gradient on a polygon from the values given at the vertices. Together with a finite volume kind divergence reconstruction, this discrete gradient can be used for discretization of various PDEs, such as fully nonlinear (or linear anisotropic) diffusion problems, starting from rather general meshes. Its key advantage is the discrete integration-by-parts formula, known as the discrete duality property. Our approach allows us to preserve the crucial properties of the continuous diffusion operators (such as the monotonicity, the coercivity, the variational structure) at the discrete level.*

Further, we apply the same formula in the context of 3D "double" schemes, in the spirit of [HER 98, HER 07] and [DOM 05]; we give the associated discrete duality formula. In the case of meshes with the orthogonality condition, we also give a discrete entropy dissipation formula. As an example, we obtain convergence of "double" finite volume discretizations to the entropy solution of a model doubly nonlinear hyperbolic-parabolic equation.

KEYWORDS: *Finite Volume Approximation, Discrete Gradient, Discrete Duality, Consistency, 3D*

1. Introduction

In recent years, considerable effort was directed to the construction of convenient finite volume kind approximations of anisotropic and full-tensor linear or quasi-linear diffusion problems on general (non-orthogonal, distorted, non-conformal) meshes. In this context, the classical two-point schemes do not apply. Different methods are proposed in [HER 98, COU 99, DOM 05, EYM 06, EGH 06]; see [HER 07] for an extensive list of references. Some of these approaches extend quite naturally to nonlinear problems: see e.g. [AND 04]. For further developments (in particular, for 3D schemes), see [PIE 05, HER 07] and a number of recent works including those by Angot, Boyer, Chainais, Coudière, Delcourte, Domelevo, Droniou, Eymard, Gallouët, Herbin, Hermeline, Hubert, Krell, Omnès and Pierre. In this note, we are interested in a 3D generalization of the "double" finite volume schemes proposed in [HER 98, DOM 05] (see also the earlier works by Nicolaides and Hu); but our main tool is a 2D gradient reconstruction formula. This 2D formula also yields a somewhat simpler alternative to the 2D "double" schemes described in § 2.1.

Section 2 starts with the gradient reconstruction formula. In § 2.1, a 2D meshing is described. The role of the discrete duality formulae, given in § 2.2, is outlined in the remark in § 2.2. In § 2.3, we briefly describe the 3D double meshes and state our main result (Proposition 2.5). Notice that "discrete entropy dissipation inequalities" are only shown for the restricted class of orthogonal meshes. These inequalities are important for nonlinear problems, like the one studied in section 3, and in the context of renormalized solutions for linear equations. In section 3, we describe the convergence result obtained for one model problem. The proofs will be published elsewhere.

Our 3D meshes and discrete divergence formula [8] are a particular case of those described and tested numerically in [HER 07]. Our contribution is the choice [7] of the discrete gradient; it allows for Proposition 2.5. For alternative 3D approaches with discrete duality, see [PIE 05] and a forthcoming work of Coudière and Hubert.

2. The gradient reconstruction formula and FV schemes

Although Lemma 2.1 below is a purely 2D property (see Figure 1), it is convenient to use the 3D vector calculus formalism. The "asterisk" superscripts allow us to make a unified 2D-3D presentation, but the reader can simply ignore them until § 2.3.

Let Π be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $\sigma \subset \Pi$ be a polygon with l vertices $x_1^*, \ldots, x_l^*$ numbered in the counter-clockwise sense with respect to the orientation of Π induced by $\vec{n}$. Let x_{l+1}^* stand for x_1^*. Introduce the barycentre (i.e., the middlepoint) $x_{i,i+1}^*$ of $[x_i^*, x_{i+1}^*]$. Let $x_\sigma^* \in \Pi$. Introduce the (signed) area

$$m_{i,i+1} = 0.5 \,\langle\, \vec{n}, \overrightarrow{x_\sigma^* x_{i,i+1}^*}, \overrightarrow{x_i^* x_{i+1}^*} \,\rangle \qquad [1]$$

of the triangle $x_i^* x_\sigma^* x_{i+1}^*$. Denote the area of σ by m; we have $m = \sum_{i=1}^{l} m_{i,i+1}$.

LEMMA 2.1 *For all vector $\vec{r}$ parallel to Π, $\vec{r} = \frac{1}{m} \sum_{i=1}^{l} (\vec{r} \cdot \overrightarrow{x_i^* x_{i+1}^*}) \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right]$.*

The proof combines the formulae of [EYM 06, Lemma 6.1], [AND 04, Lemma 2.4].

COROLLARY 2.2 *Take $(w_i^*)_{i=1}^l \subset \mathbb{R}$, $w_{l+1}^* := w_1^*$. Consider the expression*

$$\Big(\sum_{i=1}^{l} m_{i,i+1}\Big)^{-1} \sum_{i=1}^{l} (w_{i+1}^* - w_i^*) \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right]. \qquad [2]$$

If w_i^* are the values of an affine function w at the vertices x_i^* of the polygon σ, expression [2] *gives the projection of ∇w on the plane Π (see Figure 1).*

REMARK. — We guess that affine interpolation formula [2] is well known. Unless $l = 3$, formula [2] is one among infinitely many linear forms in $\left(w_i^*\right)_{i=1}^{l}$ which share the consistency property of Corollary 2.2. Our choice of [2] is motivated by the calculation that leads to the discrete duality property [4]. If $l = 3$, then [2] is equivalent to any of the known formulae for three-point affine interpolation.

2.1. *2D meshes and the associated gradient and divergence operators*

The terminology and notation we use differ slightly from the commonly used ones; our choice is motivated by the analogy with the "double" finite volume meshing as in [AND 04] and in § 2.3 below. The construction and notations are depicted in Figure 1.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. A *partition* of Ω is a finite set of disjoint open polygons such that Ω is their union, up to a set of measure zero. The mesh we consider is actually the couple $(\mathfrak{D}, \overline{\mathfrak{M}^*})$, which we denote by $\mathfrak{T}$. We take $\mathfrak{D}$ a partition of Ω; each element of $\mathfrak{D}$ is denoted by D and called a *diamond cell*. Each $D \in \mathfrak{D}$ is supplied with a centre x_D^*; for the sake of simplicity, one may assume that $x_D^* \in D$ and D is convex. In practice, $\mathfrak{D}$ can be a triangulation of Ω. For each $D \in \mathfrak{D}$, we fix a counter-clockwise numbering of its vertices by $x_1^*, \ldots, x_l^*$ ($l \geq 3$), letting $l{+}1 := 1$. We set $x_{i,i+1}^* = 0.5\,(x_i^* + x_{i+1}^*)$ (the middlepoint of the segment $[x_i^*, x_{i+1}^*]$).

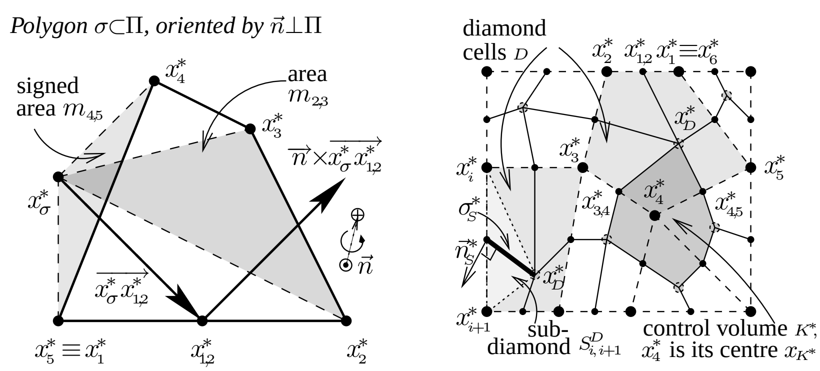

Figure 1. *2D Reconstruction property in σ. 2D diamond mesh and its dual mesh.*

A generic vertex of $\mathfrak{D}$ is denoted by x_{K^*}. Each x_{K^*} is the centre of a *control volume* K^*, as shown in Figure 1. The mesh $\overline{\mathfrak{M}^*} = \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ is the *median dual mesh* of $\mathfrak{D}$ (see [HER 07]). In the case $x_{K^*} \in \partial\Omega$, we write $K^* \in \partial\mathfrak{M}^*$; if $x_{K^*} \in \Omega$, we write $K^* \in \mathfrak{M}^*$. In the case each D is an inscriptible polygon and x_D^* is the centre of its circumscribed circle, the median dual mesh $\overline{\mathfrak{M}^*}$ coincides with the Voronoï dual mesh of $\mathfrak{D}$. Each diamond $D \in \mathfrak{D}$ is a polygon with vertices $x_1^* = x_{K_1^*}, \ldots, x_l^* = x_{K_l^*}$; it is split into l *subdiamonds* $S_{i,i+1}^D$ which are the triangles with vertices x_D^*, x_i^*, x_{i+1}^*. For $K^* \in \mathfrak{M}^*$, $\mathcal{V}^*(K^*)$ is the set of all subdiamonds having x_{K^*} for a vertex. The set of all subdiamonds is denoted by $\mathfrak{S}$. In a subdiamond $S = S^D_{i,i+1}$, we denote by σ^*_S the part of $\partial K^*_i \cap \partial K^*_{i+1}$ included into S; we denote its length by m^*_S. We have $\sigma^*_S \equiv [x^*_D, x^*_{i,i+1}]$; denote by $\vec{n}^*_S$ its unit normal vector such that[1] $\vec{n}^*_S = \vec{n} \times \overrightarrow{x^*_D x^*_{i,i+1}}/m^*_S$ (if $m^*_S \equiv \|\overrightarrow{x^*_D x^*_{i,i+1}}\| = 0$, $\vec{n}^*_S$ is arbitrary). Finally, for $K^* \in \mathfrak{M}^*$ and $S \in \mathcal{V}^*(K^*)$, set $\epsilon^{K^*}_S := 0$ if $K^* = K^*_i$, and $\epsilon^{K^*}_S := 1$ if $K^* = K^*_{i+1}$.

Diamonds, resp. subdiamonds, serve to define the gradient, resp. divergence, operators between the spaces of discrete functions and discrete fields defined below. By *Vol*(S) we denote the measure of $S \subset \mathbb{R}^d$ (here, $d = 2$). A *discrete function on* Ω is a set $w^{\mathfrak{T}} = \Big(w_{K^*}\Big)_{K^* \in \mathfrak{M}^*}$ of real values. The set of all such functions is denoted by $\mathbb{R}^{\mathfrak{T}}$. Whenever convenient, a discrete function $w^{\mathfrak{T}}$ is identified with the function $x \in \Omega \mapsto \sum_{K^* \in \mathfrak{M}^*} w_{K^*} 1\!\!1_{K^*}(x)$. Similarly, a *discrete function on* $\overline{\Omega}$ is a set $w^{\overline{\mathfrak{T}}} = \Big(w_{K^*}\Big)_{K^* \in \overline{\mathfrak{M}^*}}$. The set of all such functions is denoted by $\mathbb{R}^{\overline{\mathfrak{T}}}$. In case all the components of w_{K^*} with $K^* \in \partial\mathfrak{M}^*$ are zero, we write $w^{\overline{\mathfrak{T}}} \in \mathbb{R}^{\overline{\mathfrak{T}}}_0$. On $\mathbb{R}^{\mathfrak{T}}$, we define the scalar product $\Big[\!\!\Big[w^{\mathfrak{T}}, v^{\mathfrak{T}} \Big]\!\!\Big] = \sum_{K^* \in \mathfrak{M}^*} Vol(K^*)\; w_{K^*} v_{K^*}$. A *discrete* (resp., scalar) *field on* Ω is a set $\vec{\mathcal{F}}^{\mathfrak{T}} = \Big(\vec{\mathcal{F}}_D\Big)_{D \in \mathfrak{D}}$ in $\mathbb{R}^2$ (resp., in $\mathbb{R}$). If a subdiamond S is included into D, we set $\vec{\mathcal{F}}_S := \vec{\mathcal{F}}_D$. The set of all discrete fields is denoted by $(\mathbb{R}^2)^{\mathfrak{D}}$. We identify $\vec{\mathcal{F}}^{\mathfrak{T}}$ with the function $x \in \Omega \mapsto \sum_{D \in \mathfrak{D}} \vec{\mathcal{F}}_D 1\!\!1_D(x)$, and define on $(\mathbb{R}^2)^{\mathfrak{D}}$ the scalar product $\Big\{\!\!\Big\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \Big\}\!\!\Big\} = \sum_{D \in \mathfrak{D}} Vol(D)\, \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D$.

We define the *discrete gradient* operator $\nabla^{\mathfrak{T}} : w^{\overline{\mathfrak{T}}} \in \mathbb{R}^{\overline{\mathfrak{T}}} \mapsto \Big(\nabla_D w^{\overline{\mathfrak{T}}}\Big)_{D \in \mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$. The value $\nabla_D w^{\overline{\mathfrak{T}}}$ is reconstructed by formulae [2], [1] from the values $w^*_1 = w_{K^*_1}, \ldots,$ $w^*_l = w_{K^*_l}$ of $w^{\overline{\mathfrak{T}}}$ at the vertices $x^*_1 = x_{K^*_1}, \ldots, x^*_l = x_{K^*_l}$ of $\sigma = D$, with $x^*_\sigma = x^*_D$. Then we define the *discrete divergence* operator $\mathrm{div}^{\mathfrak{T}} : \vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto v^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}$, where $v^{\mathfrak{T}} = (v_{K^*})_{K^* \in \mathfrak{M}^*}$ is the discrete function on Ω with the entries v_{K^*} given by

$$\frac{1}{Vol(K^*)} \sum_{S \in \mathcal{V}^*(K^*)} m^*_S \vec{\mathcal{F}}_S \cdot (-1)^{\epsilon^{K^*}_S}\, \vec{n}^*_S \equiv \frac{1}{Vol(K^*)} \sum_{S \in \mathcal{V}^*(K^*)} (-1)^{\epsilon^{K^*}_S} \langle\, \vec{\mathcal{F}}_S, \vec{n}, \overrightarrow{x^*_D x^*_{i,i+1}} \,\rangle. \quad [3]$$

Here we mean that each S in $\mathcal{V}^*(K^*)$ is of the form $S^D_{i,i+1}$; the notation $\epsilon^{K^*}_S$, $x^*_D, x^*_{i,i+1}$ under the sign "$\sum$" refers to $S^D_{i,i+1}$. This formula corresponds to the standard finite volume (i.e. based upon Stokes' formula) discretization procedure on $\mathfrak{M}^*$. The value *Vol*$(K^*)\, v_{K^*}$ is the flux of the vector field $\vec{\mathcal{F}}^{\mathfrak{T}}$ through the boundary ∂K^*, thus it represents $\int_{K^*} \mathrm{div}\, \vec{\mathcal{F}}^{\mathfrak{T}}$. Indeed, thanks to the constraint $x^*_D \in D$, whenever x_{K^*} is a vertex of $D \supset S$, the vector $(-1)^{\epsilon^{K^*}_S}\, \vec{n}^*_S$ is the unit normal vector to $\sigma_S \subset \partial K^*$ exterior to K^*.

Note that we can use the same formulae for $\nabla^{\mathfrak{T}}, \mathrm{div}^{\mathfrak{T}}$ also when $x^*_D \notin D$; in this case we consider subdiamonds of signed area, as in [1]. This generalization allows us to consider e.g. a Delaunay triangulation $\mathfrak{D}$ together with its Voronoï dual mesh $\overline{\mathfrak{M}^*}$.

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1. In what follows, we identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, set $\vec{n} = \overrightarrow{(0,0,1)}$ and use the 3D formalism: $\|\vec{a}\|$ denotes the euclidean norm of $\vec{a} \in \mathbb{R}^3$; $\vec{a} \cdot \vec{b}$ (respectively, $\vec{a} \times \vec{b}$ and $\langle\, \vec{a}, \vec{b}, \vec{c}\,\rangle$) denotes the scalar product (respectively, the vector product and the mixed one), for $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.

2.2. ***The discrete duality and entropy dissipation properties and their consequences***

PROPOSITION 2.3 *($d=2$) The discrete divergence and gradient operators* $\mathrm{div}^{\mathfrak{T}}$, $\nabla^{\mathfrak{T}}$ *defined in § 2.1 are linked by the following duality property:*

$$\forall\, w^{\overline{\mathfrak{T}}}\in\mathbb{R}_0^{\overline{\mathfrak{T}}} \quad \forall\,\vec{\mathcal{F}}^{\mathfrak{T}}\in(\mathbb{R}^d)^{\mathfrak{D}} \qquad \left[\!\!\left[\, -\mathrm{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}]\,,\, w^{\mathfrak{T}} \right]\!\!\right] = \left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}\,,\ \nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}} \right\}\!\!\right\}. \quad [4]$$

REMARKS. — In conjunction with discrete Poincaré ineqalities and consistency properties of the discrete gradient operator (the latter come from Corollary 2.2), the discrete duality formula [4] is one of the crucial tools of the "discrete calculus" for finite volume schemes. It allows us in particular to discretize coercive and monotone diffusion operators with the help of coercive and monotone finite volume schemes.

Furthermore, [4] ensures that the variational character of a diffusion operator is preserved at the discrete level. If $\vec{\mathfrak{a}}(\cdot)$ is the gradient of a convex functional $\Phi(\cdot)$, so that the diffusion operator $-\mathrm{div}\,\vec{\mathfrak{a}}(\,\nabla w)$ derives from the functional $w\mapsto\int_\Omega\Phi(\,\nabla w)$, then the discrete diffusion operator $-\mathrm{div}^{\mathfrak{T}}\vec{\mathfrak{a}}(\,\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}})$ derives from the discrete functional $w^{\mathfrak{T}}\ \mapsto\ \sum_{D\in\mathfrak{D}} \mathit{Vol}(D)\,\Phi(\,\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}})$ (see e.g. [AND 04]). This allows us to calculate discrete solutions by minimization algorithms such as the Polak-Ribière method.

Note that discrete Poincaré inequality $\|w^{\mathfrak{T}}\|_{L^p(\Omega)}\le C\mathrm{diam}\,(\Omega)\|\,\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}}\|_{L^p(\Omega)}$ for all $w^{\overline{\mathfrak{T}}}\in\mathbb{R}_0^{\overline{\mathfrak{T}}}$ holds (uniformly in $\mathfrak{T}$, under a mild regularity constraint on $\mathfrak{T}$), if $\mathfrak{D}$ is a triangulation of Ω. The inequality fails if the number l of vertices of D is arbitrary.

Property [4] is well suited for the study of diffusion problems by variational techniques. In the entropy or renormalized solutions setting, one uses nonlinear test functions of the unknown solution. One application is outlined in section 3. In this context, we need the "entropy dissipation" inequality [5] stated below. We are able to show it if $\overline{\mathfrak{M}^*}$ is "orthogonal": more exactly, we ask that each diamond cell D admits a circumcircle, and $x_D^*\in D$ is its centre (thus $\overline{\mathfrak{M}^*}$ is the Voronoï dual mesh of $\mathfrak{D}$).

For $A:\mathbb{R}\mapsto\mathbb{R}$ and a discrete function $u^{\overline{\mathfrak{T}}}$ with entries u_{K^*}, denote by $A(u^{\overline{\mathfrak{T}}})$ the discrete function with the entries $A(u_{K^*})$. Let $S':\mathbb{R}\mapsto\mathbb{R}$ be a bounded non-decreasing function, and $S(r)=\int_0^r S'(s)\,ds$. For $\psi\in L^1(\Omega)$, denote by $\psi^{\overline{\mathfrak{T}}}$ the discrete function on $\overline{\Omega}$ with the entries $\psi_{K^*}=\frac{1}{\mathit{Vol}(K^*)}\int_{K^*}\psi$, $K^*\in\overline{\mathfrak{M}^*}$.

PROPOSITION 2.4 *(Entropy dissipation inequality) Let $\overline{\mathfrak{M}^*}$ be an orthogonal mesh. Let $u^{\overline{\mathfrak{T}}}\in\mathbb{R}_0^{\overline{\mathfrak{T}}}$ and $\psi\in\mathcal{D}(\overline{\Omega})$, $\psi\ge 0$. Let $A:\mathbb{R}\to\mathbb{R}$ be a non-decreasing continuous function; set $A_{S'}(r)=\int_0^r S'(s)\,dA(s)$. Assume that either $S'(0)=0$, or $\psi\in\mathcal{D}(\Omega)$ and $\max_{K^*\in\partial\mathfrak{M}^*}\mathrm{diam}\,(K^*)$ is small enough. Let $\mathcal{G}^{\mathfrak{T}}\in(\mathbb{R}^1)^{\mathfrak{D}}$ be a scalar field. Then*

$$\left[\!\!\left[-\mathrm{div}^{\,\mathfrak{T}}\left(\,|\mathcal{G}^{\mathfrak{T}}|\ \nabla^{\mathfrak{T}} A(u^{\overline{\mathfrak{T}}})\right),\ S'(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\!\right] \ge \left\{\!\!\left\{\ |\mathcal{G}^{\mathfrak{T}}|\ \nabla^{\mathfrak{T}} A_{S'}(u^{\overline{\mathfrak{T}}})\,,\ \nabla^{\mathfrak{T}}\psi^{\overline{\mathfrak{T}}} \right\}\!\!\right\}. \ [5]$$

2.3. ***Discrete gradient and divergence operators for 3D "double" meshes***

Let Ω be a polyhedral open bounded subset of $\mathbb{R}^3$. We briefly describe the meshes and discrete spaces and operators, referring to Figure 2 and to the notation of § 2.1.

A "double" finite volume mesh of Ω is a triple $\mathfrak{T} = \left(\overline{\mathfrak{M}}, \overline{\mathfrak{M}^*}, \mathfrak{D}\right)$. Here $\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}$, $\mathfrak{M}$ is a partition of Ω in tetrahedra K, called *primal volumes*. We call $\partial\mathfrak{M}$ the set of all faces of volumes that are included in $\partial\Omega$. These faces are considered as *boundary volumes*. For $K \in \partial\mathfrak{M}$, we choose a centre $x_K \in K$. The set of vertices of all volumes $K \in \overline{\mathfrak{M}}$ is denoted by $(x_{K^*})_{K^* \in \overline{\mathfrak{M}^*}}$; these are the centres of *dual volumes*. If $K_\odot, K_\oplus \in \overline{\mathfrak{M}}$ have a common face, we denote it by $K_\odot|K_\oplus$. Each face is supplied with a centre $x_{K_\odot|K_\oplus}$, assumed, for the sake of simplicity, to belong to $K_\odot|K_\oplus$. If x_{K^*}, x_{L^*} are neighbour vertices of $K_\odot|K_\oplus$, then $x_{K^*|L^*}$ is the middlepoint of $[x_{K^*}, x_{L^*}]$ ($K^*|L^*$ refers to the common boundary of the dual volumes with centres x_{K^*}, x_{L^*}). The mesh $\overline{\mathfrak{M}^*}$ is the dual mesh of $\overline{\mathfrak{M}}$ such that the dual volume K^* with centre x_{K^*} has its vertices in the set $(x_K)_K \cup (x_{K|L})_{K|L} \cup (x_{K^*|L^*})_{K^*|L^*}$ (the precise construction is similar to that of § 2.1). We write $x_{K^*} \in \partial\Omega$ if $K^* \in \partial\mathfrak{M}^*$, and $K^* \in \mathfrak{M}^*$ otherwise. A couple

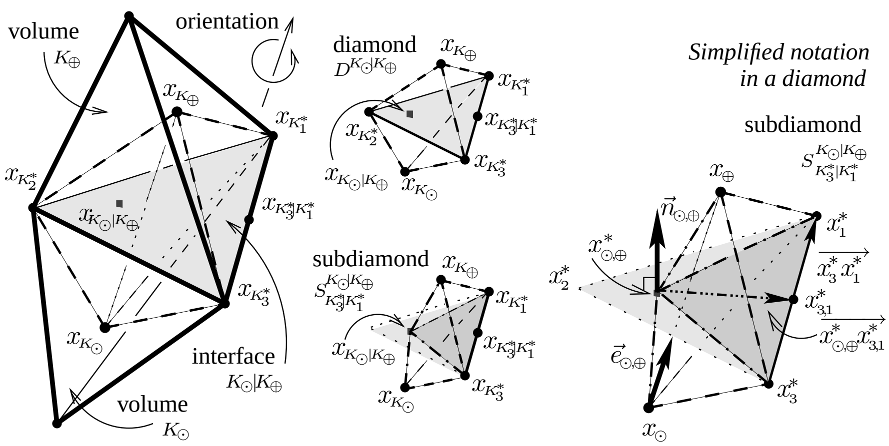

Figure 2. *3D neighbour volumes, diamond, subdiamond. Zoom on a subdiamond.*

of neighbours $\{K_\odot, K_\oplus\}$ defines an (oriented) *diamond* $D = D^{K_\odot|K_\oplus}$; it is the convex hull of $x_{K_\odot}$, $x_{K_\oplus}$ and $K_\odot|K_\oplus$. The set of all diamonds is denoted by $\mathfrak{D}$. In an oriented diamond $D^{K_\odot|K_\oplus}$, $\vec{e}_{K_\odot,K_\oplus}$ is the unit vector pointing from $x_\odot$ to $x_\oplus$; $\vec{n}_{K_\odot|K_\oplus}$ is the unit normal vector to $K_\odot|K_\oplus$ such that $\vec{n}_{K_\odot|K_\oplus} \cdot \vec{e}_{K_\odot,K_\oplus} > 0$. This induces an orientation in the triangle $K_\odot|K_\oplus$ (see Figure 2). We denote by Proj_D, Proj_D^* the orthogonal projectors of $\mathbb{R}^3$ on the line $<\vec{e}_{K_\odot,K_\oplus}>$, resp. on the plane $K_\odot|K_\oplus$. As in § 2.1, each diamond $D^{K_\odot|K_\oplus}$ is split into $l = 3$ subdiamonds; a generic subdiamond $S = S^{K_\odot|K_\oplus}_{K_i^*|K_{i+1}^*}$ is the convex hull of $x_{K_\odot}, x_{K_\oplus}, x_{K_\odot|K_\oplus}$ and of the neighbour vertices $x_{K_i^*}, x_{K_{i+1}^*}$ of $K_\odot|K_\oplus$. Whenever a diamond $D^{K_\odot|K_\oplus}$ is fixed, we use a simplified notation, as shown in Figure 2. Notations $\mathcal{V}(K)$, resp. $\mathcal{V}^*(K^*)$ stand for the sets of all subdiamonds intersecting K, resp., K^*. Finally, for $S = S^{K_\odot|K_\oplus}_{K_i^*|K_{i+1}^*}$ we set $\epsilon_S^K := \begin{cases} 0, \text{ if } K = K_\odot \\ 1, \text{ if } K = K_\oplus \end{cases}$, $\epsilon_S^{K^*} := \begin{cases} 0, \text{ if } K^* = K_i^* \\ 1, \text{ if } K^* = K_{i+1}^*. \end{cases}$

The space $(\mathbb{R}^3)^{\mathfrak{D}}$ of discrete fields is defined as in § 2.1. The space $\mathbb{R}^{\mathfrak{T}}$ is the one of all discrete functions $w^{\mathfrak{T}}$, which are collections of values $\Big((w_K)_{K\in\mathfrak{M}}, (w_{K^*})_{K^*\in\mathfrak{M}^*}\Big)$;

$$\Big[\!\!\Big[w^{\mathfrak{T}},\, v^{\mathfrak{T}} \Big]\!\!\Big] := \frac{1}{3}\sum_{K\in\mathfrak{M}} \mathit{Vol}(K)\; w_K v_K + \frac{2}{3}\sum_{K^*\in\mathfrak{M}^*} \mathit{Vol}(K^*)\; w_{K^*} v_{K^*} \qquad [6]$$

is the scalar product on $\mathbb{R}^{\mathfrak{T}}$. The choice of weights $\frac{1}{3}, \frac{2}{3}$ is needed to ensure Proposition 2.5. The definitions of $\mathbb{R}^{\overline{\mathfrak{T}}}, \mathbb{R}_0^{\overline{\mathfrak{T}}}$ and of $(\mathbb{R}^3)^{\mathfrak{D}}, \Big\{\!\!\Big\{ \cdot\,,\,\cdot \Big\}\!\!\Big\}$ mimic the ones in § 2.1.

The discrete gradient $\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}}} \in (\mathbb{R}^3)^{\mathfrak{D}}$ is defined by its entries: for $D = D^{K_\odot|K_\oplus}$,

$$\nabla_D w^{\overline{\mathfrak{T}}} \text{ is s.t. } \begin{cases} \mathrm{Proj}_D(\nabla_D w^{\overline{\mathfrak{T}}}) = \frac{w_\oplus - w_\odot}{d_{\odot,\oplus}}\, \vec{e}_{\odot,\oplus}, \text{ with } w_\odot = w_{K_\odot},\ w_\oplus = w_{K_\oplus}; \\ \mathrm{Proj}^*_D(\nabla_D w^{\overline{\mathfrak{T}}}) \qquad \text{is the vector defined by formulae [2], [1]} \\ \qquad\qquad\qquad\qquad \text{with } w^*_i = w_{K^*_i}, \vec{n} = \vec{n}_{\odot,\oplus}, x^*_\sigma = x^*_{\odot\oplus}. \end{cases} \qquad [7]$$

Thus, the primal (resp., dual) mesh yields $\frac{1}{3}$ (resp., $\frac{2}{3}$) of the components of $\nabla_D w^{\overline{\mathfrak{T}}}$.

The discrete divergence $\mathrm{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}$ is defined by its entries as follows:

$$\begin{aligned} v_K &= \frac{1}{2\mathit{Vol}(K)} \sum\nolimits_{S\in\mathcal{V}(K)} (-1)^{\epsilon^K_S} \langle\, \vec{\mathcal{F}}_S, \overrightarrow{x^*_{\odot\oplus} x^*_{i,i+1}}, \overrightarrow{x^*_i x^*_{i+1}} \,\rangle, \\ v_{K^*} &= \frac{1}{2\mathit{Vol}(K^*)} \sum\nolimits_{S\in\mathcal{V}^*(K^*)} (-1)^{\epsilon^{K^*}_S} \langle\, \vec{\mathcal{F}}_S, \overrightarrow{x_\odot x_\oplus}, \overrightarrow{x^*_{\odot\oplus} x^*_{i,i+1}} \,\rangle. \end{aligned} \qquad [8]$$

Formulae [8] come from the finite volume (Stokes'-formula) discretization on $\mathfrak{M}$,$\mathfrak{M}^*$.

PROPOSITION 2.5 *($d = 3$) With the above definitions,* [4] *holds. Further, if x_K is the centre of the circumscribed ball of K and $x_{K_\odot|K_\oplus}$ is that of the circumscribed circle of $K_\odot|K_\oplus$ (in which case $\overline{\mathfrak{M}^*}$ is the Voronoï mesh dual to $\mathfrak{M}$), then* [5] *holds.*

3. Application to a degenerate nonlinear parabolic equation

Let $\Omega \subset \mathbb{R}^3$ is a bounded polygonal domain, and $T > 0$. Consider the problem

$$\begin{cases} \partial_t u + \mathrm{div}\, \vec{\mathfrak{f}}(u) - \mathrm{div}\, \vec{\mathfrak{a}}(\nabla w) = f,\ w = A(u) \quad \text{in } Q = (0,T)\times\Omega, \\ u|_{t=0} = u_0 \text{ in } \Omega, \quad u = 0 \text{ on } \Sigma = (0,T)\times\partial\Omega, \end{cases} \qquad [9]$$

with bounded data u_0, f. The function $\vec{\mathfrak{a}} : \mathbb{R}^N \to \mathbb{R}^N$, taken under the form $\vec{\mathfrak{a}}(\xi) = k(|\xi|)\xi$, is assumed continuous and strictly monotone such that the associated operator $w \mapsto -\mathrm{div}\,\Big(k(|\nabla w|)\nabla w\Big)$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ to $W^{-1,\frac{p}{p-1}}(\Omega)$. We assume $A(\cdot)$ to be continuous non-decreasing, with $A(0) = 0$. The convective flux function $\vec{\mathfrak{f}} : \mathbb{R} \to \mathbb{R}^3$ is assumed continuous. We use the notion of "entropy solution" of [9] due to J. Carrillo (see e.g. [EYM 02]).

We combine a finite volume discretization in space on the orthogonal "double" mesh with the time-implicit discretization in time with step Δt. The diffusion term at the time $n\Delta t$ is discretized under the form $-\mathrm{div}^{\mathfrak{T}}\vec{\mathfrak{a}}(\nabla^{\mathfrak{T}} w^{n,\overline{\mathfrak{T}}})$, with the help of formulae [8], [7]; here $u^{n,\overline{\mathfrak{T}}} \in \mathbb{R}_0^{\overline{\mathfrak{T}}}$ is the unknown solution at time $n\Delta t$, and $w^{n,\overline{\mathfrak{T}}} = A(u^{n,\overline{\mathfrak{T}}})$. A term penalizing the differences $(w_K - w_{K^*})$ for $K \cap K^* \neq \emptyset$ is added in order to enforce the convergence of the functions $w^{n,\mathfrak{M}} = \sum_{K\in\mathfrak{M}} A(u_K^n)1\!\!1_K$, $w^{n,\mathfrak{M}^*} = \sum_{K^*\in\mathfrak{M}^*} A(u_{K^*}^n)1\!\!1_{K^*}$ to the same limit. The convection term is discretized with the help of a discrete convection operator built from bi-monotone numerical flux functions $g_{K|L}, g_{K^*|L^*} : \mathbb{R}^2 \to \mathbb{R}$ (see [EYM 02] and references therein). We adapt the associated "weak BV" techniques to the non-Lipschitz case, and obtain an analogue of the entropy dissipation property of Proposition 2.4 for the convection terms.

Let $\left\{\mathfrak{T}_h,\, \Delta t_h\right\}_h$ be a family with discretization size parameter h going to zero, subject to uniform in h regularity constraints. Existence, it a priori estimates, approximate entropy inequalities, compactness and convergence of the discrete solutions come from the discrete calculus formulae [4], [5], the tools of [EYM 02, AND 04] and references therein, and the Minty-Browder argument. The context of "double" schemes makes us introduce the technical device of "double entropy-process solutions" to [9]; as in [EYM 02], this kind of solutions eventually reduce to entropy solutions in the sense of J. Carrillo. We thus prove that the unique entropy solution of [9] can be approximated by finite volume schemes on orthogonal "double" meshes.

4. References

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