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| Acronyme/ Acronym | CoToCoLa |
| Titre de la proposition du projet | Thématiques actuelles en lois de conservation |
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# Table des matières

# 1 Proposal abstract

The proposal aims, firstly, at solving several concrete questions in the modern theory of conservation laws and related convection-diffusion problems. These questions (Tasks) are concerned with actual techniques of nonlinear analysis (some of the questions seemed not accessible only a few years ago) ; they originate from recently identified of from long-standing important applications. The Tasks are all relevant of the domains of expertise of one of the members of the project and at the same time, there is a strong connection to at least one of the other members and to our national and international collaborations.

The second goal is to make advance our understanding of the theory and the crucial tools (including the most modern ones) for analysis of conservation laws and related nonlinear PDEs. Indeed, the tasks we have selected are representative of several fundamental issues relevant to the problems in hand, such as : notions of solution and well-posedness ; analysis of non-local terms in conservation laws ; cooperation of semigroups in evolution equations ; convection-diffusion problems of mixed type ; boundary and interface problems for conservation laws ; (ir)regularity and qualitative behaviour of solutions ; converging and efficient numerical approximations.

Our third goal is, while advancing on the solution of these concrete questions, to form progressively an internationally recognized group in Besancon working on a wide spectrum of modern approaches to conservation laws problems.

Indeed, the proposal team consists of five MCF, all of us belong to the Laboratoire de Mathématiques de Besançon CNRS UMR 6623. For three of us, conservation laws or systems and related nonlinear PDEs constitute the main field of research. Yet at the present time, everyone focuses on a different domain and specific subjects : nonlocal (fractional) laws / degenerate parabolic equations, boundary and interface problems for scalar laws / systems and viscous limits. The project is born from our intention to bring our different cultures to interaction. These three persons share the main load of work on the project. The implication of the two other team members is a starting point for more ample collaboration in the future ; they joined the group because of their recent interest to the subject, and they contribute an expertise of different kind, the one on linear PDE techniques and the one on numerical analysis and scientific computing. We ask for funding for a post-doc to join the team during the second year of the project and bring an experience of his own in the quickly growing field of control of hyperbolic conservation laws.

The topics we treat were all developed in strong interaction between french, italian, american, russian, norwegian, polish, german, spanish, chinese researchers. We already have strong connections with some of the leading groups in the subjects we treat, and we are intended to maintain and develop such contacts.

The work on the Tasks will be complemented by three workshops or schools (one per year), plus a final meeting with participation of exterior experts and of many collaborators of the project ,on selected aspects of the modern theory of nonlinear PDEs including actual topics in control of systems of conservation laws.

# 2 Context, positioning and objectives of the proposal

## 2.1 Context

The proposal focuses on fundamental research in the domain of the analysis of nonlinear PDEs and their numerical approximation. To be specific, we are concerned with conservation laws and more general nonlinear PDEs mixing convection and diffusion features. One of the innovative and application-driven directions of this research concerns nonlocal problems (including memory terms, fractional diffusion and nonlocal advection terms).

The subject of hyperbolic conservation laws is a classical topic in nonlinear PDEs. The theory, although well developed, faced and still faces major restrictions and conceptual difficulties. The works on the scalar case (Hopf, Oleinik, Lax, Gelfand, Volpert) culminated in the celebrated theory of Kruzhkov's entropy solutions. The theory of hyperbolic systems of conservation laws (HSCL) in space dimension one was developed thanks to major contributions by Stokes, Riemann, Rayleigh, Lax, Oleinik, Glimm, Dafermos, Liu, DiPerna, Serre, and has achieved its degree of maturity with the works of Bressan's school in the 1990s and early 2000s (the Bianchini-Bressan theory). Since then, it has been realized that there is a dramatic lack of tools (starting from the lack of an appropriate functional framework) for understanding the theory of HSCL in multiple space dimensions.

Notwithstanding the apparent impossibility with the existing notions and tools to understand the multi-dimensional hyperbolic systems and related convection-diffusion problems (including concrete systems as important as the Euler and Navier-Stokes systems in fluid mechanics), the world-wide community of researchers is continuing fruitful work on particular aspects of the theory and numerics for conservation laws and systems. Moreover, there is a growing interaction with research on other types of nonlinear PDEs. Parabolic or degenerate parabolic convection-diffusion problems (including linear problems treated by nonlinear methods) form a wide class of problems where some of the "hyperbolic" techniques brought a new insight. There has been a very important progress on this kind of problems in the last fifteen years, in relation with Carillo's work and with entropy, renormalized and kinetic solution techniques.

New centers of interest emerge, stimulated either by problems coming from applied sciences (such as conservation laws with discontinuous fluxes, problems with non-local diffusion terms or non-local advection, numerics on anisotropic diffusion problems and distorted meshes) or by the inner development of mathematics of nonlinear PDEs (such as the aforementioned kinetic, entropy and renormalized solutions ; compensated compactness features ; fine properties of $BV$ and Sobolev functions ; optimal transportation and gradient flow approaches to diffusive PDEs ; study of singularities of evolution equations).

## 2.2 State of the art and positioning

**1.** Fundamental issues, a state of the art

**a.** The classical theory of the first-order scalar conservation law $(SCL)$ $u_t + \operatorname{div} f(u) = 0$ concerns the Cauchy problem in the whole space and the Dirichlet initial-boundary value problem with $L^\infty$ data, in the setting of Kruzhkov's entropy solutions. This notion of generalized solutions incorporates dissipation inequalities $\eta(u)_t + \operatorname{div} q(u) \leq 0$ in the sense of distributions for a wide class of "entropy-flux pairs" $(\eta, q)$. This notion of solution can be seen as the singular limit of the the "physical" approximation of (SCL) by vanishing viscosity : $(VV{-}SCL)$ $u_t + \operatorname{div} f(u) = \varepsilon\Delta u$.

A key feature of the entropy solutions is the $L^1$ contraction property, closely linked to the monotonicity and the order preservation.

The situation of hyperbolic systems of conservation laws $(HSCL)$ $U_t + F(U)_x = 0$ in one space dimension (hyperbolicity means that the jacobian $F'(U)$ is assumed to have simple real eigenvalues) is much more involved. Here, only few classes of systems (Temple systems, $p$-systems, Keyfitz-Kranzer systems ...) were shown to possess global entropy-compatible solutions in $L^\infty$, essentially using invariant regions, suitable families of entropies and compensated compactness. The issue of uniqueness for these solutions is extremely delicate. In general (and, for the time being, under structural restrictions on the nonlinearity of the characteristic fields) the well-posedness class for $(HSCL)$ constructed in the late 90s by Bressan et al. is given by $BV$ solutions with small variation, the solutions being defined by a local matching with Riemann solvers and with smooth solutions to the linearized systems. These solutions form the "standard Riemann semigroup" that is contractive w.r.t. a highly non-trivial metric locally equivalent to the $L^1$ metric on the set of solutions with small total variation. This semigroup is the limit of the associated "artificial" vanishing viscosity approximation $(AV{-}HSCL)$ $U_t + F(U)_x = \varepsilon U_{xx}$ (the Bianchini-Bressan theory) ; it is therefore compatible with entropy dissipation (physically important systems are usually endowed with one or several entropy-flux pairs). Physical viscosities may take the form $(B(U)U_x)_x$ where $B$ is a degenerate non-negative matrix, and the singular limit of the associated problems is not yet fully understood.

It should be stressed that many important contributions to the theories of $(SCL)$ and $(HSCL)$ started from considering particular equations or issues (Hopf-Burgers equation, $p$-system, Keyfitz-Kranzer system, fractional Burgers equation... ; Oleinik condition, self-similar solutions, travelling-wave profiles, Riemann problem, wave interaction patterns, wave-front tracking,...) and generalizations only came out progressively.

**b.** The key point is that no well-posedness theory is available for multi-dimensional nonlinear systems of hyperbolic conservation laws. Moreover, it has been realized that the $BV$ tools are not appropriate in the multi-D setting. In recent years, much effort was directed to refine of the linear theory of transport equations driven by low regularity coefficients (Bouchut, Ambrosio, DeLellis...), which resulted in well-posedness results for very particular systems. On the other hand, an extensive numerical experience, often based on one-dimensional Riemann solvers, has been accumulated for many concrete systems, in view of the utmost importance of these systems of hyperbolic conservation laws in fluid mechanics.

**2.** Starting point for the project

Although seemingly complete since 1970, the Kruzhkov theory of scalar laws was enriched by a number of remarkable results an techniques. Different extensions of the theory applicable to scalar conservation laws are instrumental for our project. The notions of kinetic solutions (Brenier, Lions, Perthame, Tadmor) and renormalized solutions (Bénilan, Carrillo, Wittbold) allowed for extending the theory to the $L^1$ framework in a pure PDE perspective (this gave a new interpretation to the nonlinear semigroup theory results by Crandall and Bénilan from the early 70s). Tools of kinetic solutions and parametrized $H$-measures (Panov) allowed for a deeper study of fine properties of solutions, including compactification and strong boundary traces properties. The use of boundary entropies and weak/strong traces (Otto, Chen, Frid, Vasseur, Panov) fully extended the Kruzhkov theory to initial-boundary value problems. Conservation laws with $x$-discontinuous fluxes $f(x;u)$ were thoroughly investigated, and a general approach and theory was recently formulated (Andreianov, Karlsen and Risebro). The tools of generalized characteristics or new interaction functionals led to original $SBV$ regularity results for solutions of one-dimensional genuinely nonlinear conservation laws (DeLellis et al.). Continuous dependence results and convergence orders for different approximations were obtained (Kuznetsov, Bouchut, Perthame, Karlsen, Risebro, Vovelle, Droniou). The entropy solution definition was adapted to some non-local monotone PDEs (Wittbold, Alibaud).

Degenerate or fractional parabolic problems $(P-SCL)$ $u_t + \operatorname{div} f(u) + \mathcal{L}[\varphi(u)] = 0$, where $\mathcal{L}$ is a linear diffusion operator (a possibly fractional laplacian) or a nonlinear diffusion operator ($p$-laplacian, Leray-Lions operator) can be treated using the entropy solution paradigm, with the techniques of Carrillo and Wittbold, of Bendahmane and Karlsen in the local case, of Alibaud and Karlsen et al. in the non-local case. Splitting techniques and the issue of cooperation of dissipative semigroups are instrumental both for the theory and for the numerical approximation of such equations. Singular parabolic problems (including one-laplacian and infinity-laplacian) and some non-local problems were treated with the help of nonlinear semigroup theory techniques, convex analysis tools and optimal transportation theory, using a different notion of entropy solution due to Bénilan et al. (Crandall, Evans, Andreu, Caselles, Mazon, Rossi, Igdida).

Also, the theory of systems, far more incomplete than the scalar theory, continued to be developed in various directions. Let us mention, as relevant to our project, the issue of viscous approximation of shock profiles (Goodman and Xin) ; systems with source terms (Amadori, Dafermos, Hsiao, Christoforou) ; systems with memory terms (Dafermos, McCamy, Rogers, Nohel, Tzavaras, Feireisl, Christoforou) ; interface coupling of systems (Coquel, Godlewski et al.) ; SBV regularity for systems (DeLellis et al., Bianchini et al.).

**3.** Positioning

All the aforementioned contributions show the vitality of the subject. A considerable impact on the general theory came from solving questions that first appeared as particular ones. Our goal is to explore several aspects of the theory of conservation laws and related nonlinear PDEs, as well as properties of their numerical approximation, via solution of particular questions. The results mentioned above lay a basis for the development of the eight Tasks of our project.

Tasks 1-7 start with exploring the scalar case for different equations. We set out to study : fractional conservation laws with degenerate non-local diffusion, conservation laws with non-local advection, the so-called relativistic heat equation, a one-dimensional model of fluid-particle interaction ; the coupling of scalar laws by an interface, the issue of structural stability (dependence upon the coefficients and more generally upon the operators involved) ; the comparative behavior of profiles for different (local and non local) approximations of scalar laws ; the $BV$ and $SBV$ regularizing effects for scalar multi-D equations.

Tasks 1,3,5 aim mainly at particular one-dimensional systems (viscoelasticity with memory and related systems with weak damping in bounded domains ; triangular systems ; systems with non-local regularization terms).

Although extremely difficult, the study of multi-dimensional systems of hyperbolic conservation laws is among the main objectives of our project (Tasks 1,4,6). We will start by looking at the particular class of triangular systems and at systems related to the Euler equations of gas dynamics (quasi-geostrophic equations, fluid-particle interaction models). We hope to further the understanding of the zoology of cases, of the functional properties and qualitative behavior of possible solutions.

### 2.3 Objectives, originality, novelty

Our project is concerned with selected questions of the theory, mainly related to particular equations or systems. Our interest to these questions is an occasion to better understand :

A. different approaches to defining solutions and the associated well-posedness techniques ;
B. treatment of non-local fluxes and diffusion operators in conservation laws ;
C. the issue of cooperation of different semigroups ;
D. boundary and interface problems in conservation laws ;
E. quantitative behaviour of solutions and new tools for their analysis ;
F. adequate numerical schemes for complex problems ;
G. different approaches to the control of HSCL.

Each of the Tasks presented in the next section is directly concerned with at least two of the above questions. We address the issues [A]-[F] through solving the Tasks 1-8, hoping for emergence of more general points of view as a contribution to the general theory.

The latter direction [G] is of a different nature : we do not have yet a concrete project in this direction other than to acquire a fundamental culture in the mathematics of control in PDEs. Indeed, there exists already a group in control of PDEs in Besançon (actively collaborating with colleagues in Marseille, Orleans, Sevilla), and we intend to prospect the possibilities of interaction in the field of control of nonlinear HSCL, appealing to our expertise in conservation laws. At the same time, the activities of the European networks ERC Starting Grant ConLaw, GDRE CONEDP around this topic offer us an opportunity to learn the existing approaches and to meet the world experts in the subject of control of HSCL.

Thus, the main objectives are three-fold :

- firstly, solve or advance considerably on the Tasks presented in the project ;
- secondly, extend our understanding of fundamental issues [A-F] in conservation laws, hyperbolic systems and related equations ; acquire mathematical culture on the issue [G] of control of conservation laws ;
- finally, bring into play our different cultures and know-how, in order to form, by the end of the project, an autonomous internationally recognized team on conservation laws and convection-diffusion problems with a wide spectrum of competence.

It is easily seen that the Tasks are rather diversified, which corresponds to the diversity of the “initial conditions” on the members of our future team. It should be stressed that advancing more or less simultaneously on these diverse Tasks would be facilitated by the exterior (national and international) collaborations we engaged. The funding of missions and invitations will be essential for a quick realization of the objectives. An original feature is that, while the project team is fully localized in Besançon, the project itself requires interaction and brings visibility far beyond the walls of our Laboratory.

Further originality and novelty features of scientific order require a more detailed discussion. Because the Tasks are much diversified, these features will be presented per Task in the next section. Here we can simply state that members of our group possess the command and experience in using several very modern technical tools : the Bianchini-Bressan techniques for viscous conservation laws ; the techniques of entropy solutions for non-local diffusion operators ; the general trace techniques and a unifying framework for boundary and interface conditions on conservation laws ; the whole spectrum of the recent solution notions (entropy, viscosity, renormalized, semigroup, kinetic solutions) and methods on monotone nonlinear PDEs ; design and analysis of finite volume methods for these problems. New ideas on how these tools can be utilized (often on problems that have remained open for years, e.g. uniqueness for triangular systems, existence in BV for the viscoelasticity system with memory, fluid-particle interaction via a drag force) brought some of the Tasks to our attention. Other Tasks are related to original applications in natural and social sciences (fractional PDEs appearing as phenomenological or averaged stochastic models in physics and finance ; simplified fluid-particle interaction models giving access to deeper analysis ; relativistic heat equations featuring diffusion with limited speed of propagation ; conservation laws with non-local flux occuring in biology and physics). Thus, the study of well-posedness, of numerical approximation, and of qualitative properties of their solutions may have an impact on mathematical modelling of the underlying phenomena.

# 3 Scientific and technical program, proposal organization

## 3.1 Scientific program, proposal structure

The Tasks are relatively independent in a large extent. The coordination between the tasks mainly concerns the repartition of our time between them. Thus the appropriate description will be Task per Task (see next subsection).

We just point out here that the technical aspects of Task 2 (both theoretical and numerical) can bring additional insight into Task 4, and that Tasks 1 and 3 provide particular examples for profile studies in Task 5. Task 8, which is a reconversion task, may find its first application in the concrete problem of Task 3.

## 3.2 Description of tasks

Task 0 Project coordination and common activities.

Starting from the fall 2011, we will organize a weekly working group on the project topics. This format will include, alternatively, discussion on the advancement in the Tasks, exposition of key cultural results and important techniques to be shared between the group participants, and invitation of exterior speakers on selected subjects where local participants lack experience. Two or three longer-stay invitations (three to ten weeks) of exterior experts on the selected topics of the project will contribute to these working sessions and stimulate our progression.

Three meetings (two or three day long each) will be organized for the members of the group and a limited (five to fifteen) number of exterior participants, under the name of "Journées d'analyse non linéaire". These meetings may mix the format of a school (with one or two mini-courses of four to six hours) and of a workshop. The 2011/12 meeting will be devoted to hyperbolic systems ; the 2012/13 edition will be devoted to aspects of control of conservation laws and degenerate diffusion equations ; the 2013/14 edition will concentrate on nonlocal problems. In addition, a final meeting will be organized in the summer or fall 2014, in Besançon or in Luminy (CIRM), which will take the form of a conference (30 to 50 participants) which will bring together our collaborators and some of other leading experts in the domain of the project. This conference, under the same name as the present proposal, is the key valorization activity of the project.

Beyond the global project management decided collectively and conducted by Boris Andreianov, each task will have an organization of its own directed by one of the three most involved members of the group. The Task 8 (control) is particular : it will be led by C. Donadello, B. Andreianov and F. Ammar-Khodja, the senior local collaborator.

Task 1 Triangular systems of conservation laws (B.A.+C.D.).

This task is concerned with well-posedness, in the framework of the appropriately defined entropy solutions, of the $2 \times 2$ hyperbolic systems of the form

$$\begin{cases} u_t + \operatorname{div} f(u) = 0 \\ v_t + \operatorname{div} g(u, v) = 0. \end{cases} \tag{1}$$

Existence (of vanishing viscosity limits) has already been studied by Risebro et al. [KaMRi, CKaMRi, CMRi] but the issue of uniqueness remains inexplored in most cases ; [CKaMRi] studies one application in porous medium and prove both convergence of viscosity regularizations and the uniquenes of the entropy solution at the limit.

Uniqueness for such systems is our main interest. Indeed, uniqueness results for systems (especially for solutions with large variation or even with merely $L^\infty$ data) are very few, and in the multi-dimensional setting, almost nothing is known. While the one-D case is the main object of our study where we hope for fairly interesting results, exploration of the multi-D case os an ambitious project for which strong restrictions may appear.

Let us explain the essence of our approach. It is clear that there is well-posedness for the first equation, decoupled from the second one, within the class of Kruzhkov entropy solutions. Using the framework provided by [AKR], at least for a sufficiently regular $u$ we can define entropy solutions $v$ for the equation $v_t + \operatorname{div} g(u(t,x),v) = 0$ compatible with the vanishing viscosity approximation (see [AKR10] for the simplest case where $u$ is a simple shock). Uniqueness is inherent to this notion of solution. The main issue of the project is, how far can the technique be pushed, regarding the regularity of $u$ ? The method will work for piecewise $C^1$ solution $u$ with Lipschitz jump curves, provided strong trace results (see Panov [Pa] and references therein) are extended to non-autonomous regular fluxes. An alternative is to recast the theory of [AKR] using weak trace tools of Otto [Ot]. It should be also be mentioned that, according to an old result of D. Schaeffer [Sch] (see also more recent results in [Ji]), generic (in the ad hoc sense) solutions to a one-dimensional convex scalar law are piecewise smooth ; thus at least for generic data $u_0$, we have good hope to establish uniqueness. For $BV$ data $u_0$, the solution $u$ is $BV$ ; yet we are not certain that merely $BV$ function $u(t,x)$ can be treated within the flux $g(u(t,x),\cdot)$. It should be noticed that already for a linear flux $g(u(,t,x),v) = A(u(t,x))\,v$, the question of uniqueness of solutions to this transport equation is very delicate and it was solved in the difficult work by Ambrosio [Amb]. Yet we look at entropy solutions, which brings more restrictions for existence and less, for uniqueness ; cf. the recent work of Ambrosio, Crippa et al. [ACFS]. We have even more hope for $SBV$ solutions $u$ ($SBV$ being the subclass of $BV$ functions which derivatives lack the Cantor measure part, thus keeping only the absolutely continuous and the pure jump parts). Fortunately, the work of Ambrosio and De Lellis [ADL] and several subsequent works (Robyr, Bianchini [Rob, BDR]) do establish the $SBV$ regularity in the one-dimensional case for solutions of $u_t + f(u)_x = 0$. We ask the same question for the multi-D conservation laws. Finally, the question of continuous dependence of solutions $v$ on the discontinuous coefficient $u$ will be considered, starting from the simplest examples (one-shock solution $u$, shifted in a given direction). For this study, some of the ideas of the Bressan theory [Br] can be instrumental. And, depending on the regularity of $v$ we manage to prove, higher-order ($3\times 3$, etc.) triangular systems can be considered.

Thus, Task 1 will contain the following stages :

- the on-going work of B.A. with Karlsen and Risebro on the general multi-dimensional scalar equation with discontinuous flux, B.A. ;
- study of the dependence on the discontinuous coefficient $u$ (and integration with the known results, see e.g. Bouchut, Perthame [BP] and Karlsen, Risebro [KaRi], on continuous dependence on a smooth coefficient), B.A.+C.D. ;
- study of the $2\times 2$ triangular systems with (generic) piecewise regular $u$, in one dimension than in multi-D, B.A. ;

- deep technical refinements needed to extend the method to $SBV$ (or, hopefully, $BV$) coefficients, B.A.+C.D. ;
- study of the $SBV$ regularity for multi-dimensional scalar conservation laws, C.D.+B.A. ;
- study of higher-order triangular systems (at least, for particular solutions), B.A.+C.D.

Further questions, such as the continuous dependence of system solutions on the data, would follow, but a three years duration is definitely too short to include these questions into Task 1.

Exterior collaborators for this task are K.H. Karlsen and N.H. Risebro (U. Oslo) ; potential collaborators are N. Seguin (Paris VI), G. Vallet (Pau), S. Mishra (ETH Zürich and U. Oslo) and Adimurthi (U. Bangalore).

The fundamental questions [A], [D], [E] (and, from a further perspective, [F]) are addressed in this Task.

Task 2 Fractional conservation laws and convection-diffusion equations (N.A.+B.A+M.B+U.R+C.D.).

The equation takes the form

$$u_t + \mathrm{div}\,(F(u) - a(\nabla A(u))) + \mathcal{L}[B(u)] = 0, \tag{2}$$

where $-\mathrm{div}\, a(\nabla\cdot)$ is either zero (fractional conservation laws case) or a Leray-Lions operator (e.g., the Laplacian), and $\mathcal{L}$ is a non-local diffusion operator (e.g., a fractional Laplacian, $\mathcal{L} = (-\Delta)^\lambda$ with $\lambda \in (0, 2)$). Both $A$ and $B$ are continuous non-decreasing scalar functions (anisotropic versions can be considered, using in particular kinetic solution techniques). When $B$ is zero (i.e., the problem is local), the problem in the whole space and the Dirichlet problem are throughly investigated using the methods of entropy solutions. Work on the Neumann boundary conditions continues, in particular, in the PhD thesis of M. Gazibo under the supervision of B.A..

The non-local problem in the whole space was investigated by many authors, firstly in the framework of more or less regular solutions : see e.g. Biler et al. [BiFW], Droniou, Gallouët et Vovelle [DrGVo] and references therein. Different methods of Sobolev spaces and functional analysis (Duhamel formula, etc.) were used, particularly successful in the "subcritical case" $\lambda > 1$. The study of general $L^\infty$ solutions was firmly assessed with the definition of entropy solutions by Alibaud [Al] for the case $A \equiv 0$ ; this was generalized by Karlsen et al. [KaU, CiKaJ] for the general case (2). It was shown that indeed, discontinuities do form ([AlDV, DoDuLi]) and then, it was also demonstrated that they may be regularized in finite time, Sylvestre et al. [ChCzS] (see also Droniou [Dr] for numerical evidence and convergence analysis).

A first goal in this task is to establish optimal continuous dependence estimates in the framework of [CiKaJ]. The next goal is to treat the Dirichlet problem in bounded domains. Here, there are different approaches to taking boundary conditions into account in nonlocal terms ; they correspond to different interpretations of the underlying stochastic Lévy process when it jumps out of the domain. Our first approach is via functional-analytic definition of the nonlocal laplacian ; the second one is a pointwise definition combining the ideas of the Bardos-LeRoux-Nédélec [BLN] and Carrillo [Ca] definitions for the conservation law with techniques specific to the nonlocal terms.

Further, we are intended to investigate different notions of solution to the non-local problems. For fractional conservation laws in the $L^\infty$ framework, it was shown ([Al]) that the entropy solutions are appropriate, and that the weak solutions are not appropriate ([AlAn]) for $\lambda < 1$. Gradient flow approaches to diffusion problems (using in particular the Wasserstein metric) is a recent, beautiful and powerful tool of study of PDEs, actively developed by Otto, Ambrosio, Savaré, J.A. Carrillo et al.. We are intended to prospect the possibility of writing gradient flow formulations for fractional diffusions. A notion of duality solution going back to Stampaccia was developed in [KaPU], and a definition of renormalized solution for a simplest non-local operator (allowing for a pure $L^1$ theory) was proposed by N.A., B.A. and Bendahmane in [AlAnBe]. A related (equivalent, in most of the known cases) notion is "entropy solution" in the sence of Bénilan et al. [B–V], cf. [AVCM], which we will also generalize to non-local diffusions. We are intended to continue this research, integrating the local convection and diffusion terms in (2), and to develop a kinetic approach to the problem with quasilinear diffusion.

A related problem, with a time-nonlocal term, is the study of second-order evolution equations of the form

$$u_{tt} + \mathcal{A}u_t + \mathcal{B}u = 0, \tag{3}$$

where $A$ is a Leray-Lions operator and $B$ is a linear bounded operator, see Lions and Strauss [LS]. The point is to use a non-local formulation on $u_t$ (the term $\mathcal{B}u$ becomes non-local in time). This technique was recently revisited by E. Emmrich and M. Thalhammer [ET] (see [G]), for the sake of establishing well-posedness of energy solutions. The non-local form of the problem gives good hope for defining renormalized solutions, which has not yet been done for order two evolution equations (here, the presence of $\mathcal{A}$ is essential).

Thus, Task 2 will contain the following stages :

- the on-going work of N.A. and M.B. on Dirichlet problem, N.A.+M.B. ;
- the on-going work of N.A. and B.A. on renormalized solutions, N.A.+B.A. ;
- the work of with a master student in Hat Yai, Thailand, on characterization of entropy solutions by an Oleinik condition, N.A. ;
- optimal continuous dependence and error estimation for (2), N.A. ;
- a different approach to boundary-value problems relying on extension of "local" techniques, B.A.+N.A. ;
- $L^1$ theory by renormalized, kinetic and entropy approaches, B.A.+N.A.+M.B. ;
- study of asymptotic behavior of solutions in a bounded domain, study of solution profiles (intersection with Task 5), C.D.+N.A.+B.A. ;
- gradient flow approach to nonlocal heat equation, B.A. ;
- renormalized solutions of Lions-Strauss second-order evolution equations (3), B.A..

Exterior collaborators for this Task are G. Karch (U. Wrocław), E. Jakobsen and S. Cifani (Trondheim), K.H. Karlsen (Oslo), M. Bendahmane (Bordeaux), A. Ouédraogo (Burkina-Faso), E. Emmrich (Bielefeld) ; potential collaborators are N. Igbida (Limoges), P. Wittbold (Essen), G. Vallet (Pau), C. Imbert (Paris-Dauphine), M. Kassmann (Bielefeld), J. Vovelle (Lyon).

The fundamental questions [A], [B], [C], [D], [E], [F] are all addressed in this Task.

<u>Task 3</u> Viscoelasticity system with memory (C.D.+B.A.).

This is a concrete physical model that writes :

$$\begin{cases} w_t = v_x \\ v_t = p(w)_x + \int_0^t k(t-s)p(w(s,x))_x\,ds, \end{cases} \tag{4}$$

where $k$ is a non-increasing, non singular function on $\mathbb{R}^+$ of which a decreasing exponential is a typical representative. The first result we intend to obtain concerns precisely this exponential case $k(t) = c \exp -\gamma t$ (and the case of a convex combination of a finite family of decreasing exponentials is entirely similar). The transformation due to Ammar-Khodja, Benabdallah et al. in [AmBeMuRa] (in the linear case) transforms the system into a system of conservation laws with two genuinely nonlinear fields (those of the $p$-system) and all the remaining fields are linearly degenerate. A work of Benzoni and Serre [BS] allows to use compensated compactness arguments on this system ; through this technique we would obtain an alternative proof of the result of Nohel, Rogers, Tzavaras [NRTz1, NRTz2] for $L^\infty$ solutions of (4), at least for particular kernels $k$.

Our main goal in this Task is to address the issue of existence (and uniqueness, using the Bressan theory) of $BV$ solutions in the framework in which a Standard Riemann Solver can be defined and the flux splitting techniques apply. The techniques of Dafermos and Hsiao [D1, DHs, D2] for the Glimm scheme brought several key contributions to the field of hyperbolic systems with (strong or weak) damping ; see also Christoforou [Ch], Amadori and Guerra [AG]. Our focus will be, in particular, on the case of a bounded domain (we are not aware of well-posedness results in this case), where the presence of a quadratic entropy allows for the $L^1$ estimate, essential for the arguments of Dafermos [D2]. Yet the interaction estimates are much more involved in this case because of repeated reflexion from the boundaries of the domain. A further key stage of our investigation focuses on the robustness of the arguments as the number of exponential modes (used to approximate a general kernel $k(\cdot)$) goes to infinity. Finally, the question of asymptotic decay of the solutions should be studied.

Thus, Task 3 will contain the following stages :

- well-posedness of the system (4) in $L^\infty$ through change of variables and compensated compactness method, B.A. ;
- well-posedness for small $BV$ data in a bounded domain through exploitation of weak damping, for memory terms governed by a convex combination of decreasing exponentials, B.A.+C.D. ;
- study of the two previous questions for an unlimited number of exponential modes (thus for general memory terms), B.A.+C.D. ;
- investigation of well-posedness in $BV$ in the whole space (this is a very difficult question), B.A.+C.D. ;
- study of the asymptotic decay for the problem, B.A.+C.D..

The next important question is on control of system (4). It should be stressed that, as for the damping effect of the memory term, the difficulty lies in particular in the fact that the control only acts on the second equation of the system. This question will probably constitute the first research experience for our perspective [G] of research.

The essential exterior collaborator for this Task is A. Benabdallah (Marseilles) ; potential collaborators are M. Gisclon (Chambery), C. Christoforou (Evanston), R. Colombo (Brescia), A. Tzavaras (U. Crete).

The fundamental questions [A], [B], [C], [D], [E], [F], [G] are all addressed in this Task.

<u>Task 4</u> Conservation laws with non-local fluxes (N.A.+B.A.+M.B.+U.R.+C.D).

In recent years, more and more attention has been drawn to non-locality aspects in partial differential equations. In contrast to Task 2 (non-local diffusion) and Task 3 (non-local in time memory term), we think here of different situations where non-local fluxes of advection type occur. It is certainly difficult to hope for emergence of an approach that would be applicable to the whole variety of non-local fluxes, but some general ideas do work ; starting on particular examples provided by different authors and difficulties related to them, we secretly hope to discover some ideas that go beyond one example.

One much used idea is to exploit finest structural stability results (continuous dependence on coefficients and nonlinearities) for a scalar conservation law, see e.g. [BCM, ASh, Mer]. This is, in particular, the case when non-locality appears through an advection coefficient given by convolution of the unknown (or a function of it) with a regularizing kernel. In this case, results of the kind Bouchut, Perthame [BP], Karlsen, Risebro [KaRi] and recent refinements by Colombo et al. [CMR] permit to get reduced to the Gronwall inequality and establish uniqueness. It should be stressed that these techniques are $BV$-based. When kernels become singular (which is not less interesting from the modelling point of view), this idea may break down. In this case, either subtler tools should be designed, or non-uniqueness conjecture should be tested (this is quite difficult due to the lack of explicit solutions for the non-local case ; cf. [AlAn] where an indirect method is used to construct multiple solutions for fractal Burgers equation).

Non-local fluxes may arise, for instance, in systems of equations such as the Keller-Segel model (with the steady elliptic equation for the chemoattractant and a possibly degenrate parabolic equation for the cell population, see e.g. Dalibard and Perthame [DP], Burger, Dolak, Schmeiser [BDS]), from the simple interpretation of the second equation of the system as providing a non-local expression to be inserted in the first one. They may also appear directly in modelling, for the cases where long-range interaction is built into the model (radiation dynamics, Rohde et Yong [RoY] ; models in population dynamics, e.g. Burger et al. [BCM] ; one-dimensional nonlocal avalanche models, Amadori and Shen [ASh]),

$$p_t + \Big(\frac{p-1}{p}\exp\Big(\int_x^0 \frac{p(t,y)-1}{p(t,y)}\,dy\Big)\Big)_x = 0 \tag{5}$$

(with a non-straightforward interpretation of $t$, $p$). For a rather celebrated example, let us mention the 2D surface quasi-geostorphic equations (derived from 3D athmosphere model in certain regimes), the 2D Euler equations in the vorticity form and the "Burgers-QG" equation

$$\theta_t = u[\theta]\cdot\nabla\theta + \mathcal{L}[\theta] \tag{6}$$

where $\theta$ is a 2D vector or (in the last case) a scalar, and $u[\theta]$ is a nonlocal (usually pseudodifferential) operator involving e.g. a Riesz or a Hilbert transform (if $u$ is a local function of $\theta$, we are in case (2) of Task 2). These different cases are studied e.g. by Córdoba et al. [CCCF], by Kiselev et al.[KNV, Kis], by Cafarelli and Vasseur [CV]. The $2D$ examples are very difficult and related to major open issues in fluid mechanics, so we are firstly interested in a simplified 1D model of "active scalars" studied in particular by Kiselev et al. in a series of papers [KNV, Kis]. While these very difficult results are rather concerned with regularity of solutions, our interest comes (by analogy with Task 2) to solutions of lowest regularity ; we are indeed interested in establishing a well-posedness theory (notion of "entropy" solution, existence and uniqueness) for merely $L^\infty$ data. This is also the case for equation (5) for which a $BV$ theory was already established in the work [ASh]. On the contrary, we think that in the work [BCM], where uniqueness of entropy solutions was established for the equation

$$u_t + \mathrm{div}\,(u\nabla(K * u)) - \Delta a(u) = 0, \tag{7}$$

the regularity assumptions on the kernel $K$ make it possible to stay within the class of weak solutions. However, for a singular kernel, well-posedness is unclear even for entropy solutions.

Thus, it could be said that a common difficulty about generalizing to singular kernels the study of several models with convolution-kind (Green function, etc.) nonlocality lies in the fact that $BV$ tools become difficult to use. We will search for formulations based upon less regularity.

We also mention here the simplified 1D (once more !) models like Rohde's ([Ro]) model

$$u_t + f(u)_x = \mathcal{L}u + \Big(K * u - u\Big)_x, \tag{8}$$

where we have a combination of nonlinear local and linear nonlocal fluxes ; the latter term can be interpreted as a dispersion one. A problem similar in form is the one by DiFrancesco et al. [DFFL] coming from radiation models, cf. also [RoY]. Classical solutions exist when $\mathcal{L}$ is the laplacian and $K$ is not singular ; it may be expected that entropy solutions are needed in more general context, and their precise definition and uniqueness are to be understood.

In parallel, numerical studies will be performed and analyzed on some of the models (especially on simplified 1D scalar models) in order to test different approximation strategies and analysis tools. The most ambitious question we forsee here is : starting from the 1D "active scalars" equation (6) with $u[\theta] = H\theta$ the Hilbert transform, what ideas can be transposed for approximating the surface quasi-geostrophic equation ? Here, the experience of U.R. on analysis of solutions of fluid mechanics problems ([Raz1, Raz2]) may be as useful as his numerical skills.

Thus, Task 4 will contain the following questions :

- models in population dynamics with non-local aggregation (e.g. (7)) and singular kernels, B.A. ;
- singular kernels in different models, N.A.+B.A. ;
- well-posedness in $L^\infty$ of the nonlocal avalanche equation (5), B.A.+C.D.+N.A. ;
- study in $L^\infty$ of the 1D active scalar equation (6), N.A.+B.A.+M.B. ;
- numerical approaches to the 1D active scalar equation of [CCCF], U.R.+B.A.+N.A. ;
- numerical approaches to the 2D surface quasi-geostrophic equation, U.R.+B.A.+N.A..

It is understood that, this field being a particularly quickly growing one, the program of work should be modified according to the future evolution of the community understanding of problems with nonlocal fluxes.

An exterior collaborator for this Task is M. Bendahmane (U. Bordeaux) ; potential collaborators are E.Yu. Panov (Novgorod), Ch. Rohde (U. Stuttgart), D. Amadori (U. l'Aquila), G. Karch (U. Wrocław), P. Azerad (Montpellier), A. Benabdallah et F. Hubert (Marseilles), N. Seguin (Paris IV), F. Lagoutière (Orsay), G. Vallet (Pau), J. Rossi (Alicante).

The fundamental questions [A], [B], [E], [F] are addressed in this Task.

<u>Task 5</u> Solution profiles and applications (C.D.+N.A.+U.R.+B.A.).

We consider the general conservation law with a dissipative linear source of the form

$$u_t + \sum_{\alpha=1}^{n} \partial_\alpha f_\alpha(u) = Lu. \tag{9}$$

Of course the non local sources considered in Task 2 are allowed. Other fundamental examples are of course $L = 0$, the inviscid conservation law, $L = \varepsilon\Delta u$, the viscous conservation law and (nonlocal) the Rosenau model (elliptic regularization) $Lu = K * u - u$, where $K$ is a non-negative kernel of mass one. In [Se1] one can find an overview on the basic properties of these last three cases and on the (rather different) relaxation model.

Our goal is to understand better how the properties of the source impact the structure of the solutions and, in particular, its large time profile. We plan to approach this huge theme from three directions.

First of all, we intend to study in the details the formation and interaction of shock profiles for fractional conservation laws. It is known since the work of N. Alibaud et al. [AlDV] that the regularizing effect of the fractional laplacian with $\lambda < 1$ is not sufficient to rule out shock formation in finite time (as for the inviscid conservation law) in general. Nevertheless one can prove that these shock discontinuities are smoothed out in finite time, so that the asymptotic profile of the solutions is regular. One could be interested in comparing the "fractional laplacian" solution and the solution of the viscous equation (L=laplacian).

For the viscous approximation of conservation laws in one space dimension (case $L = \varepsilon\Delta$) we know that the results on the asymptotic expansions by Goodman and Xin [GX] can be extendend to the case in which the profiles actually interact (see Shen and Xu, [SX], and Bressan and Donadello [BD1], [BD2]). Is such an expansion valid for the solution of the fractional conservation law, at least far from the jumps (*i. e.* exixtence of an outer expansion) ? A positive answer to this question could motivate the study of numerical "front tracing" schemes for the fractional laplacian equation.

By formally writing the asymptotic expansion in terms of powers of $\varepsilon$ of the Rosenau solution and comparing it to the asymptotic expansion of the viscous solution we can see that they coincide up to the second order. This formal correspondence leads to the conjecture (object of an ongoing collaboration of C. D. and D. Serre) that the Rosenau approximation and the viscous approximation could be closer to each other than to the inviscid solution. Is it possible to check how close are the fractional laplacian solutions as the parameter $\lambda$ varies ? Is it possible to compare them to the (nonlocal) Rosenau approximation or to the viscous approximation ?

Finally, one more direction is to be foreseen, even if it depends on the success of an ongoing collaboration. C. D. and A. Bressan are actively working on the generalization of the results in [BaB] to general strictly hyperbolic systems (no more conditions on the genuine nonlinearity or the linear degeneracy of the characteristic families). In practice, this project consist in introducing in the space of solutions a Riemann-type metric, uniformly equivalent to the $L^1$ distance and with respect to which the flux of the system is contractive (the flux generated by a system is in general non contractive in $L^1$). This goal should be achieved relying on the results by S. Bianchini in [Bi], where a Glimm type functional was introduced for general hyperbolic systems, and by A. Bressan and S. Bianchini in [BiB]. Results concerning the continuous dependence of solutions of the system with dissipative source

$$u_t + A(u)u_x = Lu, \tag{10}$$

on the initial data are currently available, see [AG], [Ch], and were obtained by directly studying the solutions. Once the Riemann-type metric is available, it could offer an alternative approach to such problems. In particular, we are interested in providing a characterization of the sources (and here we could even consider some non dissipative sources) for which continuous dependence of initial data can be achieved in terms of the properties of the semigroups $\mathcal{S}_t$ and $\mathcal{S}_t^*$ associated with the flux-splitting

$$u_t + A(u)u_x = 0 \qquad \text{and} \qquad u_t = Lu,$$

so that the selection of the source term could be done *a priori*, without first solving the whole system.

Thus, Task 5 will contain the following stages :

- Formation and interaction of shock profiles for fractal conservation laws, C.D. + N.A. ;
- Comparison of semigroup solutions associated to different dissipative sources, C.D.+B.A. ;
- Some numerical experiences by U.R. are planned on this part ;
- Continuous dependence on the initial data for strictly hyperbolic systems with source, C.D..

Exterior collaborators for this task are D. Serre (ENS Lyon), A. Bressan (Penn State), potential collaborator G. Karch (U. Wrocław).

The fundamental questions [B], [C], [E], [F] are addressed in this Task.

Task 6 Simple 1D and 2D fluid-particle interaction models (B.A.+C.D.+U.R.+N.A.).

In the work [LST] of Lagoutière, Takahashi and Seguin, the following 1D model of interaction of a point particle with a Burgers fluid via a drag force was introduced :

$$\begin{cases} u_t + (u^2/2)_x = -\lambda\ (u - h'(t))\ \delta_0(x - h(t)), \\ \\ mh''(t) = \lambda\ (u(t, h(t)) - h'(t)), \end{cases} \tag{11}$$

$\delta_0$ being the Dirac distribution, $m$ the particle mass, and $\lambda$, a drag force coefficient. Thus, a singular source term features in the right-hand side of the Burgers equation, leading to a non-conservative coupling of the “two Burgers equations” on the two sides from the particle path $x = h(t)$. This model is interesting because there is good hope for a rather complete analysis of the problem, and it provides the playground for testing ideas on efficient numerical simulation. It has more realistic features compared e.g. to the model in [VZ] that does not allow particles to collide.

The non-conservative product in the right-hand side of (11) is defined in [LST] via passage to the limit from regularized problems (the limit is essentially independent of the regularization). Further, it was realized that within this model, the particle is “moved by the conservation of the quantity of movement”, and, naturally the energy of admissible solutions should be dissipated. Analysis in terms of the Riemann problem was carried out in [LST].

Toghether with N. Seguin, B.A. applied the general approach of [AKR] to dissipative interface coupling of conservation laws in order to study the auxiliary problem with "frozen" particle located at $x = 0$ :

$$u_t + (u^2/2)_x = -\lambda\ u\ \delta_0(x). \tag{12}$$

It should be stressed that the passage from (12) to the case of a particle with general prescribed path $x = h(t)$ is not difficult (a change of variables plus piecewise affine approximation of $h(\cdot)$). The outcome is the well-posedness result for (12) for suitably defined entropy solutions, and a strikingly simple adaptation of monotone finite volume schemes in order to take into account the non-trivial interface coupling. These results on problem (12) were summarized in [ALST10], and are being prepared for publication. Presently B.A. continues the research with N. Seguin on problem (12) because there is good hope for obtaining also $BV$ well-posedness (looking at a version of the wave-front tracking algorithm in order to get precise interaction control on the interface). This work would open the way to well-posedness results in $BV$ for the full problem (11) (while the presently obtained $L^\infty$ theory would lead only to an existence result, via a fixed point argument sketched in [ALST10]). Further, we want to prove convergence of a practical numerical scheme. For the time being, we are intended to use a finite volume-based scheme with splitting and with Glimm-kind random choice method involved in approximation of the particle path ; we want to avoid explicit use of any Riemann solver. We will also look at the fractional Burgers equation coupled with a moving particle, this being a good example for studying the general issue of interface coupling for fractional convection-diffusion operators.

From the point of view of applications, are multi-dimensional generalization (thus involving a point particle coupled with the Euler system of conservation laws) is of particular interest. Clearly, within the framework of this project we cannot hope for a well-posedness theory such a system ; but we would like to understand the coupling conditions (thus putting forward a notion of admissible weak solution), to understand some of the solutions qualitative behavior, and to advance ideas for computing admissible solutions numerically.

Task 6 will contain the following stages :

- refining the theory (via a use of wave-front tracking) for the "frozen particle" case (12), B.A. ;
- well-posedness for $BV$ data of the coupled problem (11), B.A. ;
- analysis of numerical schemes for (11), B.A. ;
- mathematical formulation of the two-dimensional analogue of (11), B.A.+C.D. ;
- elements of well-posedness theory for the 2D problem (such as existence of solutions, uniqueness of very regular solutions, solutions with symmetry, typical profiles, asymptotic behavior in simple situations), B.A.+C.D. ;
- efficient numerical approximation of the two-dimensional analogue of (11), B.A.+U.R. ;
- fractional particle-in-Burgers model, N.A.+B.A..

There are is another group in France, around F. Lagoutière, working on the topic of Task 6 ; further questions may arise from our and their interactions.

Exterior collaborators for this task are N. Seguin (Paris IV), F. Lagoutière (Orsay) and T. Takahashi (U. Nancy).

The fundamental questions [A], [B], [D], [E], [F] are addressed in this Task.

Task 7 Relativistic heat equation (B.A.+N.A.+U.R.+C.D.).

There have been several efforts to correct, from the modelling point of view, the well-known infinite speed of propagation feature of the classical heat equation (indeed, although the heat kernel decreases very rapidly at infinity, there is a formal contradiction with the fundamental principles of the Einstein physics). One of the most interesting attempts, going back to Rosenau [R], is the following "Relativistic Heat Equation" (RHE) :

$$u_t - \operatorname{div} \nu \frac{|u|\nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}} = 0. \tag{13}$$

In a series of works by F. Andreu Vaillo et al. ([ACM1, ACM2, ACMM, ACM3],...) , a notion of entropy solution to (13) has been elaborated, well-posedness was shown, and it was justified that (13) admits entropy-admissible jumps that propagate at the "speed of light" $c$. The notion of solution is based on the $BV$ regularity, natural in view of the *a priori* estimates which are easily available. The notion is quite involved : it includes techniques of renormalization, composition of Radon measures by sublinear functions, Kruzhkov doubling of variables, and nonlinear semigroup techniques (the ground was prepared in the monograph [AVCM] by Andreu Vaillo, Caselles and Mazón). At the same time, the problem can be rewritten as a nonlinear equation combining advection and Hamilton-Jacobi terms ; this may provide a different point of view, e.g. for numerics.

Our main question is about approximation of solutions of (13) (and of related "limited flux" diffusion equations, because there are many : see e.g. [ACMM2]) by suitable numerical methods. While results on time-explicit finite difference approximation are already available, different kinds of approximation (in view of the discontinuity of typical solutions, finite volume or discontinuous Galerkin methods are of interest) will be studied. The only work in this direction we are aware of is the one by Marquina [Ma], using the finite difference time-explicit strategy. Our insight is provided by the DDFV methods, the co-volume methods developed by Handlovičová, Mikula et al. [HMS, HM], by the recent cell-centered Galerkin method of Di Pietro [DiP]. While implicit methods have good stability properties (related to the accretivity of the operator), the shock propagation is better captured by explicit methods ; the fact that the equation mixes hyperbolic and parabolic features makes quite delicate the choice of an optimal approximation strategy. We guess that, along with established methods in convection-diffusion context, for small values of $c$ (which may correspond rather to a unusual convection-diffusion model for population dynamics), methods inspired by Riemann solvers or wave-front tracking can be considered. Also, numerical strategies involving Hamilton-Jacobi techniques will be further investigated, cf. Marquina [Ma].

In addition, we want to further study the qualitative behavior of solutions (e.g. the $SBV$ regularity, with simplifications that this could bring to the uniqueness techniques ; solution profiles ; asymptotic behavior).

Further, we think that the limited flux diffusion equations could advantageously replace the classical diffusions in some problems of population dynamics. Indeed, porous medium kind equations were already used in this context, precisely in order to guarantee a finite speed of expansion for the population domain. Such models can only be of a phenomenological nature ; we will compare the qualitative properties of solutions (or numerical solutions) of limited flux equations (with low value of $c$) to descriptions of population movements in different applied contexts.

Task 7 will contain the following stages :

- finite volume methods for (13) and their convergence analysis, B.A. ;
- other numerical strategies for (13), B.A.+U.R. ;
- modelling and applications of limited flux diffusion equations to population dynamics, B.A.+U.R. ;
- Hamilton-Jacobi approach to (13), B.A.+N.A. ;
- "purely hyperbolic" numerical approaches in the regime of small $c$, B.A.+C.D. ;
- regularity and qualitative behavior of solutions, B.A.+C.D..

A further direction is given by limited flux fractional diffusion equations, which can be viewed as non-local analogues to (13).

Exterior collaborators for this task are M. Ghilani and N. Marhraoui (Meknès) and G. Namah (ENS2M, Besançon), potential collaborators are N. Igbida (Limoges), J. Toledo and J. Mazon (Valencía), C. Imbert (Paris-Dauphine), G. Karch (Wrocław).

The fundamental questions [A], [C], [D], [E], [F] are addressed in this Task.

Task 8 Initiation to control of hyperbolic conservation laws (C.D.+B.A.+U.R.+M.B.+N.A.).

The fundamental question [G] will be addressed in this Task. It will involve participation to meetings on the subject, interaction (under the form of a working group) on selected directions with the members of the Besançon team on control (F. Ammar-Khodja, C. Dupaix, K. Mauffrey), invitation of exterior experts for short stays (the names we think of : J. LeRousseau, A. Benabdallah, A. Marson, O. Glass, T. Takahashi...) and one longer stay (e.g. F. Ancona, R. Colombo, G. Coclite). Particular orientations in relation with control of fractional conservation laws and the viscoelasticity system will be considered.

# 4 Consortium description

All of us are "Maître de Conférences" at the Besançon Math Laboratory CNRS UMR 6623. The project team consists of Nathaël Alibaud, Boris Andreianov (coordinator), Carlotta Donadello (80% of research time is dedicated to the project for each of them) and of Matthieu Brassart and Ulrich Razafison (20% each). Being conducted by members of CNRS UMR 6623 in Besançon, the project is widely open to national and international collaborations.

## 4.1 Partners description, relevance, complementarity

Relevance is inferred from the close correspondence of the different members'former research interests (as described below) to the starting points of our Tasks 1-7. Yet, the architecture of the project requires more and more collaboration and integration within the group. This will rely on sharing our (largely different) cultures and our common learning through problem-solving. Thus, complementarity of partners is a natural feature of our project and its essential engine.

N. Alibaud (N.A.) is a young researcher (PhD defended at the end of 2006) who actively developed an original direction of research centering on fractional (non-local) conservation laws and Hamilton-Jacobi equations. N.A. established a notion of well-posed solutions for both equations and studied their important qualitative properties. He possesses a command of many nonlinear analysis tools related to these equations, but also a good knowledge of linear tools and an interest in numerical analysis. N.A. is our guide to the world of non-local PDEs.

B. Andreianov (B.A.) has been active in the domain of nonlinear PDEs for more than ten years. He learnt from the Russian and French schools of nonlinear PDEs. B.A. worked on several aspects of scalar conservation laws, on related Leray-Lions kind convection-diffusion equations and on particular systems. His other focus was on discrete functional analysis tools for finite volume approximation of diffusion problems. During his PhD thesis B.A. was concerned with vanishing viscosity limits for systems of gas dynamics, and he followed with great interest the development of Bressan's theory. Conservation laws and systems constitute his favorite subject.

M. Brassart (M.B.) is an expert in linear aspects of hyperbolic equations, and in functional analysis applied to PDEs. His initial impulse was on equations in stochastic media and on the hierarchy of physical models (Schrödinger-Boltzmann). His more recent works concern the homogenization of second order hyperbolic equations. M.B. is presently working with N.A. on the functional-analytic approach to boundary-value problems for fractional conservation laws. M.B. has a deep commitment in mathematical reading : he is a precious source of information for the whole group.

C. Donadello (C.D.) obtained her PhD from SISSA (Trieste) in 2008 under the supervision of A. Bressan. Her contributions to the general theory of conservation laws concern the formation and interaction of viscous shock profiles and the stability of front tracking solutions. As a post-doc in the USA she worked under the guidance of G.Q. Chen on the hyperbolic-elliptic incompressible MHD system. C.D. is our guide to the study of hyperbolic systems.
With her recruitment in Besançon in 2010, the "hyperbolic group" has attained a critical mass and, even more importantly, a diversification that makes our project interesting.

U. Razafison (U.R.) possesses experience along the whole line PDE analysis – numerical analysis – scientific computing. U.R. worked on the numerical approximation of Saint-Venant equations and Hartree-Fock equations, and his thesis was dedicated to the analysis of a linearized Navier-Stokes model in anisotropically weighted Sobolev spaces. U.R. will coordinate our efforts in aspects of mathematical modelling and in solution approximation. He will carry out the computer experiments.

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