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Prescribed Szlenk index of separable Banach spaces

by

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Abstract. In a previous work, the first named author described the set $\mathcal{P}$ of all values of the Szlenk indices of separable Banach spaces. We complete this result by showing that for any integer n and any ordinal α in $\mathcal{P}$, there exists a separable Banach space X such that the Szlenk index of the dual of order k of X is equal to the first infinite ordinal ω for all k in $\{0,\dots,n-1\}$ and equal to α for $k=n$. One of the ingredients is to show that the Lindenstrauss space and its dual both have Szlenk index equal to ω. We also show that any element of $\mathcal{P}$ can be realized as the Szlenk index of a reflexive Banach space with an unconditional basis.

1. Introduction and notation. In this paper we exhibit some new properties of the Szlenk index, an ordinal index associated with a Banach space. More precisely we study the values that can be achieved as the Szlenk index of a Banach space and of its iterated duals. Let us first recall the definition of the Szlenk index.

Let X be a Banach space, K a weak*-compact subset of its dual X^* and $\varepsilon>0$. Then we define

$$s_\varepsilon^1(K)=\{x^*\in K : \text{for any weak}^*\text{-neighborhood } U \text{ of } x^*,\ \operatorname{diam}(K\cap U)\geq\varepsilon\}$$

and inductively the sets $s_\varepsilon^\alpha(K)$ for α ordinal as follows: $s_\varepsilon^{\alpha+1}(K)=s_\varepsilon^1(s_\varepsilon^\alpha(K))$ and $s_\varepsilon^\alpha(K)=\bigcap_{\beta<\alpha}s_\varepsilon^\beta(K)$ if α is a limit ordinal.

Then we let $\mathrm{Sz}(K,\varepsilon)=\inf\{\alpha : s_\varepsilon^\alpha(K)=\emptyset\}$ if it exists, and $\mathrm{Sz}(K,\varepsilon)=\infty$ otherwise. Next we define $\mathrm{Sz}(K)=\sup_{\varepsilon>0}\mathrm{Sz}(K,\varepsilon)$. The closed unit ball of X^* is denoted B_{X^*}, and the *Szlenk index* of X is $\mathrm{Sz}(X)=\mathrm{Sz}(B_{X^*})$.

The Szlenk index was first introduced by W. Szlenk [21], in a slightly different form, in order to prove that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces. The key ingredients in [21] are that the Szlenk index of a separable reflexive space is always countable and that for any countable ordinal α, there exists

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a separable reflexive Banach space with Szlenk index larger than α. It has been remarked in [15] that, when it is different from ∞, the Szlenk index of a Banach space is always of the form ω^α for some ordinal α. Here, ω denotes the first infinite ordinal. On the other hand, it follows from the work of Bessaga and Pełczyński [4] and Samuel [20] that if K is an infinite, countable, compact topological space, then the Szlenk index of the space of continuous functions on K is $\omega^{\alpha+1}$, where α is the unique countable ordinal such that $\omega^\alpha \leq \mathrm{CB}(K) < \omega^{\alpha+1}$ and $\mathrm{CB}(K)$ is the Cantor–Bendixson index of K. Finally, the set of all possible values for the Szlenk index of a Banach space was completely described in [7, Theorem 1.5]. One consequence of this general result is that for any countable ordinal α, there exists an infinite-dimensional separable Banach space X with $\mathrm{Sz}(X) = \alpha$ if and only if $\alpha \in \Gamma \setminus \Lambda$, where

$$\Gamma = \{\omega^\xi : \xi \in [1, \omega_1)\} \quad \text{and} \quad \Lambda = \{\omega^{\omega^\xi} : \xi \in [1, \omega_1) \text{ and } \xi \text{ is a limit ordinal}\}.$$

Our first result shows that there is quite some freedom in prescribing the Szlenk indices of the iterated duals of a separable Banach space. We shall use the notation $Z^{(n)}$ for the nth dual of a Banach space Z. Then our statement is the following.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $\alpha \in \Gamma \setminus \Lambda$. Then there exists a separable Banach space Z_n such that for all $k \in \{0, \ldots, n-1\}$,*

$$\mathrm{Sz}(Z_n^{(k)}) = \omega \quad \textit{and} \quad \mathrm{Sz}(Z_n^{(n)}) = \alpha.$$

The above result relies on a statement of independent interest. Let us first recall that in [16], J. Lindenstrauss constructed, for any separable Banach space X, a Banach space Z such that Z^{**}/Z is isomorphic to X. We prove the following.

Theorem 1.2. *For any separable Banach space X, the associated Lindenstrauss space Z satisfies*

$$\mathrm{Sz}(Z) = \mathrm{Sz}(Z^*) = \omega.$$

Theorem 1.2 and then Theorem 1.1 are proved in Section 2. In Section 3, we show the following refinement of [7, Theorem 1.5].

Theorem 1.3. *For any $\alpha \in \Gamma \setminus \Lambda$ there exists a separable reflexive Banach space G_α with an unconditional basis such that*

$$\mathrm{Sz}(G_\alpha) = \alpha \quad \textit{and} \quad \mathrm{Sz}(G_\alpha^*) = \omega.$$

We conclude this introduction by recalling the definitions of some uniform asymptotic properties of norms that we will use. For a Banach space $(X, \|\ \|)$ we denote by B_X the closed unit ball of X and by S_X its unit sphere. The following definitions are due to V. Milman [18] and we follow the notation from [13]. For $t \in [0, \infty)$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}_X(t,x,Y) = \sup_{y\in S_Y} (\|x+ty\|-1) \quad \text{and} \quad \overline{\delta}_X(t,x,Y) = \inf_{y\in S_Y} (\|x+ty\|-1).$$

Then

$$\overline{\rho}_X(t,x) = \inf_{\dim(X/Y)<\infty} \overline{\rho}_X(t,x,Y) \quad \text{and} \quad \overline{\delta}_X(t,x) = \sup_{\dim(X/Y)<\infty} \overline{\delta}_X(t,x,Y).$$

Finally,

$$\overline{\rho}_X(t) = \sup_{x\in S_X} \overline{\rho}_X(t,x) \quad \text{and} \quad \overline{\delta}_X(t) = \inf_{x\in S_X} \overline{\delta}_X(t,x).$$

The norm $\|\ \|$ is said to be *asymptotically uniformly smooth* (*AUS* for short) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

It is *asymptotically uniformly convex* (*AUC*) if

$$\forall t>0, \quad \overline{\delta}_X(t)>0.$$

Let $p\in(1,\infty)$ and $q\in[1,\infty)$. We say that the norm of X is

- *p-AUS* if there exists $c>0$ such that $\overline{\rho}_X(t)\le ct^p$ for all $t\in[0,\infty)$;
- *q-AUC* if there exists $c>0$ such that $\overline{\delta}_X(t)\ge ct^q$ for all $t\in[0,1]$.

Similarly, there is on X^* a modulus of weak* asymptotic uniform convexity defined by

$$\overline{\delta}^*_X(t) = \inf_{x^*\in S_{X^*}} \sup_E \inf_{y^*\in S_E} (\|x^*+ty^*\|-1),$$

where E runs through all weak*-closed subspaces of X^* of finite codimension. The norm of X^* is said to be *weak* asymptotically uniformly convex* (for short *weak*-AUC*) if $\overline{\delta}^*_X(t)>0$ for all t in $(0,\infty)$. If there exist $c>0$ and $q\in[1,\infty)$ such that $\overline{\delta}^*_X(t)\ge ct^q$ for all $t\in[0,1]$, we say that the norm of X^* is *q-weak*-AUC*.

We will need the following classical duality result concerning these moduli (see for instance [10, Corollary 2.3] for a precise statement).

Proposition 1.4. *Let X be a Banach space. Then $\|\ \|_X$ is AUS if and only if $\|\ \|_{X^*}$ is weak*-AUC.*

If $p,q\in(1,\infty)$ are conjugate exponents, then $\|\ \|_X$ is p-AUS if and only if $\|\ \|_{X^*}$ is q-weak*-AUC.

Finally, let us recall the following fundamental result, due to Knaust, Odell and Schlumprecht [14], which relates the existence of equivalent asymptotically uniformly smooth norms and the Szlenk index.

Theorem 1.5 (Knaust–Odell–Schlumprecht). *Let X be a separable infinite-dimensional Banach space. Then X admits an equivalent norm which is asymptotically uniformly smooth if and only if* $\mathrm{Sz}(X)=\omega$.

2. Prescribed Szlenk index of iterated duals

2.1. Renormings of the Lindenstraus space and of its dual. We recall the construction given by J. Lindenstrauss [16] (see also [17, Theorem 1.d.3]) and introduce notation that will be used throughout this section. We refer the reader to the textbooks [17] and [1] for a presentation of the standard notions of a Schauder, shrinking, boundedly complete or unconditional basis of a Banach space.

Let $(X, \| \ \|_X)$ be a separable Banach space. Assume $X \neq \{0\}$ and fix a dense sequence $(x_i)_{i=1}^\infty$ in the unit sphere S_X of X. Let

$$E = \Big\{ a = (a_i)_{i=1}^\infty \in \mathbb{R}^{\mathbb{N}} : \\ \|a\|_E = \sup_{0=p_0<p_1<\cdots<p_k} \Big(\sum_{j=1}^{k} \Big\| \sum_{i=p_{j-1}+1}^{p_j} a_i x_i \Big\|_X^2 \Big)^{1/2} < \infty \Big\}.$$

Then $(E, \| \ \|_E)$ is a Banach space. Let $(e_i)_{i=1}^\infty$ be the canonical algebraic basis of c_{00}, the space of finitely supported real-valued sequences. It is clear that $(e_i)_{i=1}^\infty$ is a boundedly complete basis of E. It follows that E is isometric to the dual Y^* of a Banach space Y with a shrinking basis. If $(e_i^*)_{i=1}^\infty$ is the sequence of coordinate functionals associated with the basis $(e_i)_{i=1}^\infty$ of E, then the canonical image of Y in its bidual Y^{**} is the closed linear span of $\{e_i^* : i \geq 1\}$ and $(e_i^*)_{i=1}^\infty$ can be seen as a shrinking basis of Y. Note now that if $a = (a_i)_{i=1}^\infty \in E$, then the series $\sum_{i=1}^\infty a_i x_i$ is converging in X. It is important to note that the density of $(x_i)_{i=1}^\infty$ in S_X implies that the map $Q : E \to X$ defined by $Q(a) = \sum_{i=1}^\infty a_i x_i$ is linear, onto, $\|Q\| = 1$, and the open mapping constant of Q is 1. Consequently, Q^* is an isometry from X^* into Y^{**}. The main result of [16] is that

$$Y^{**} = \widehat{Y} \oplus Q^*(X^*),$$

where $\widehat{Y}$ is the canonical image of Y in Y^{**}, and the projection from Y^{**} onto $Q^*(X^*)$ with kernel $\widehat{Y}$ has norm 1. In particular, Y is isomorphic to the quotient space $Y^{**}/Q^*(X^*)$.

Now let Z denote the kernel of Q. Then Z is a subspace of $E = Y^*$ and its orthogonal $Z^\perp$ is clearly equal to $Q^*(X^*)$. It follows from the classical duality theory that Z^* is isometric to $Y^{**}/Q^*(X^*)$ and therefore isomorphic to Y. If I is the inclusion map from Z into Y^* and J_Y is the canonical injection from Y into Y^{**}, then an isomorphism from Y onto Z^* is given by $T = I^* J_Y$. Finally, if J_Z is the canonical injection from Z into Z^{**}, it is easy to check that $T^* J_Z = \mathrm{Id}_Z$. It follows immediately that $Z^{**}/J_Z(Z)$ (or simply Z^{**}/Z) is isomorphic to Y^*/Z and therefore to X.

The purpose of this subsection is to prove Theorem 1.2. In fact, our result is stronger.

THEOREM 2.1. *For any separable Banach space X, the associated Lindenstrauss space Z satisfies the following properties:*

(i) *The space Z^* admits an equivalent norm which is 2-AUS.*
(ii) *The space Z admits an equivalent norm which is 2-AUS.*

We start with the proof of the easy part (i) which can be precisely stated as follows.

PROPOSITION 2.2. *The norm $\|\ \|_E$ is 2-weak*-AUC on $Y^* = E$ and therefore $\|\ \|_Y$ is 2-AUS. In particular, Z^* admits an equivalent norm which is 2-AUS, there exists $C > 0$ such that $\mathrm{Sz}(Z^*, \varepsilon) \le C\varepsilon^{-2}$ for all $\varepsilon > 0$, and $\mathrm{Sz}(Y) = \mathrm{Sz}(Z^*) = \omega$.*

This result is an immediate consequence of the following elementary lemma.

LEMMA 2.3. *Let $a, b \in E$ and assume that there exists $k \in \mathbb{N}$ such that the sequence a is supported in $[1, k]$ while b is supported in $[k + 3, \infty)$. Then*

$$\|a + b\|_E^2 \ge \|a\|_E^2 + \|b\|_E^2.$$

Proof. Since a is supported in $[1, k]$, we can find a sequence $0 = p_0 < p_1 < \cdots < p_m = k + 1$ such that

$$\|a\|_E^2 = \sum_{j=1}^{m} \Big\| \sum_{i=p_{j-1}+1}^{p_j} a_i x_i \Big\|_X^2.$$

Fix $\eta > 0$. Since b is supported in $[k+3, \infty)$, we can find a sequence $k+1 = q_0 < q_1 < \cdots < q_r$ such that

$$\|b\|_E^2 \ge \sum_{j=1}^{r} \Big\| \sum_{i=q_{j-1}+1}^{q_j} b_i x_i \Big\|_X^2 - \eta.$$

Let $n_j = p_j$ for $j \in \{0, \ldots, m\}$ and $n_j = q_{j-m}$ for $m \le j \le m + r$. Then

$$\|a + b\|_E^2 \ge \sum_{j=1}^{m+r} \Big\| \sum_{i=n_{j-1}+1}^{n_j} (a + b)_i x_i \Big\|_X^2 \ge \|a\|_E^2 + \|b\|_E^2 - \eta. \ \blacksquare$$

We now turn to the proof of Theorem 2.1(ii), which will rely on the following technical lemma.

LEMMA 2.4. *Assume that $a^1, \ldots, a^N$ are skipped blocks with respect to the basis $(e_i)_{i=1}^\infty$ of E, meaning that there exist $0 = r_0 < r_1 < \cdots < r_N$ such that*

$$\forall k \in \{1, \ldots, N\}, \quad \mathrm{supp}(a^k) \subset (r_{k-1}, r_k),$$

and denote $\varepsilon_k = \|\sum_{i=1}^\infty a_i^k x_i\|_X$. *Then*

$$\Big\|\sum_{k=1}^N a^k\Big\|_E \le \sum_{k=1}^N \varepsilon_k + 2\Big(\sum_{k=1}^N \|a^k\|_E^2\Big)^{1/2}.$$

Proof. Fix $0 = p_0 < p_1 < \cdots < p_m$ and assume without loss of generality that $p_m \ge r_N$. Then for $j \in \{1,\dots,m\}$ we denote

$$A_j = \{k \le N : (r_{k-1}, r_k) \subset (p_{j-1}, p_j]\}, \quad A = \bigcup_{j=1}^m A_j, \quad B = \{1,\dots,m\}\backslash A.$$

We first estimate

$$\Big(\sum_{j=1}^m \Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in A} a_i^k\Big)x_i\Big\|_X^2\Big)^{1/2} \le \sum_{j=1}^m \Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in A} a_i^k\Big)x_i\Big\|_X$$
$$= \sum_{j=1}^m \Big\|\sum_{k\in A_j}\sum_{i=p_{j-1}+1}^{p_j} a_i^k x_i\Big\|_X \le \sum_{j=1}^m \sum_{k\in A_j}\Big\|\sum_{i=p_{j-1}+1}^{p_j} a_i^k x_i\Big\|_X$$

and obtain

$$\Big(\sum_{j=1}^m \Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in A} a_i^k\Big)x_i\Big\|_X^2\Big)^{1/2} \le \sum_{j=1}^m \sum_{k\in A_j}\varepsilon_k \le \sum_{k=1}^N \varepsilon_k. \tag{2.1}$$

So we may assume that B is not empty and enumerate $B = \{a^{k(1)},\dots,a^{k(L)}\}$ with $k(1) < \cdots < k(L)$. Note that for $1 \le l \le L$, $\mathrm{supp}(a_{k(l)}) \subset (r_{k(l)-1}, r_{k(l)}) \subset (r_{k(l-1)}, r_{k(l)})$, and $(r_{k(l-1)}, r_{k(l)})$ is not included in any of the sets $(p_{j-1}, p_j]$ for $1 \le j \le m$. Then we define $i_0 = 0$ and $i_l = \min\{i : p_i \ge r_{k(l)}\}$ for $1 \le l \le L$. From the definition of B, we see that $2 < i_1 < \cdots < i_L$ and $p_{i_l-1} < r_{k(l)} \le p_{i_l}$ for all $l \in \{1,\dots,L\}$. We can now write

$$\sum_{j=1}^m \Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in B} a_i^k\Big)x_i\Big\|_X^2 = \sum_{q=1}^L \sum_{j=i_{q-1}+1}^{i_q}\Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{l=1}^L a_i^{k(l)}\Big)x_i\Big\|_X^2.$$

Using the convention $a^{k(0)} = 0 = a^{k(L+1)}$ and the properties of our various sequences we get

$$\sum_{j=1}^m \Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in B} a_i^k\Big)x_i\Big\|_X^2 = \sum_{q=1}^L \sum_{j=i_{q-1}+1}^{i_q}\Big\|\sum_{i=p_{j-1}+1}^{p_j}(a_i^{k(q)} + a_i^{k(q+1)})x_i\Big\|_X^2$$
$$\le \sum_{q=1}^L \|a^{k(q)} + a^{k(q+1)}\|_E^2 \le 4\sum_{q=1}^L \|a^{k(q)}\|_E^2 \le 4\sum_{k=1}^N \|a^k\|_E^2,$$

which yields

$$(2.2)\qquad \Big(\sum_{j=1}^{m}\Big\|\sum_{i=p_{j-1}+1}^{p_j}\Big(\sum_{k\in B}a_i^k\Big)x_i\Big\|_X^2\Big)^{1/2}\le 2\Big(\sum_{k=1}^{N}\|a^k\|_E^2\Big)^{1/2}.$$

The conclusion now clearly follows from (2.1), (2.2) and the triangle inequality, by taking the supremum over all finite sequences $(p_j)_j$. ∎

Before we proceed with the proof of Theorem 2.1, we need to introduce some notation. For an infinite subset $\mathbb{M}$ of $\mathbb{N}$, we denote by $[\mathbb{M}]^{<\omega}$ the set of void or finite increasing sequences in $\mathbb{M}$. The void sequence is denoted $\emptyset$. For $E\in[\mathbb{N}]^{<\omega}$, we denote by $|E|$ the *length* of E, defined by $|E|=0$ if $E=\emptyset$ and $|E|=k$ if $E=(n_1,\dots,n_k)$. For $F=(n_1,\dots,n_l)$ in $[\mathbb{N}]^{<\omega}$, we write $E\prec F$ if $E=\emptyset$ or $E=(n_1,\dots,n_k)$ for some $k<l$, and we then say that E is a *proper initial segment* of F. We write $E\preceq F$ if $E<F$ or $E=F$ and we then say that E is an *initial segment* of F. For $E=(n_1,\dots,n_k)\in[\mathbb{N}]^{<\omega}$ and $n\in\mathbb{N}$ such that $n>n_k$, (E,n) denotes the sequence $(n_1,\dots,n_k,n)$, while $(\emptyset,n)$ is (n). For a Banach space X, we will call a family $(x_E)_{E\in[\mathbb{N}]^{<\omega}}$ in X a *tree in* X. Then a family $(x_E)_{E\in[\mathbb{N}]^{<\omega}}$ in a Banach space X is said to be a *weakly null tree* if for any E in $[\mathbb{N}]^{<\omega}$ the sequence $(x_{(E,n)})_n^\infty$ is weakly null. If $(x_E)_{E\in[\mathbb{N}]^{<\omega}}$ is a tree in the Banach space X and $\mathbb{M}$ is an infinite subset of $\mathbb{N}$, we call $(x_E)_{E\in[\mathbb{M}]^{<\omega}}$ a *refinement* or a *full subtree* of $(x_E)_{E\in[\mathbb{N}]^{<\omega}}$.

Proof of Theorem 2.1(ii). Fix a sequence $(\varepsilon_n)_{n=0}^\infty$ in $(0,\infty)$ such that $\sum_{n=0}^\infty\varepsilon_n^2\le 1/4$. Let $(z_F)_{F\in[\mathbb{N}]^{<\omega}}$ be a weakly null tree in the unit ball B_Z. By extracting a full subtree, we may assume that there exist $0=r_0<r_1<\cdots$ and for any $F\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\}$ there exist $a^F\in B_E$ such that

$$\forall F=(n_1,\dots,n_k)\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\},\qquad \operatorname{supp}(a^F)\subset(r_{n_k-1},r_{n_k})\ \text{and}\ \|a^F-z_F\|_E\le\varepsilon_k.$$

Since $(z_F)_{F\in[\mathbb{N}]^{<\omega}}$ is in the kernel of Q, the last condition implies

$$\forall F\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\},\qquad \Big\|\sum_{i=1}^{\infty}a_i^Fx_i\Big\|_X\le\varepsilon_k.$$

We can therefore apply Lemma 2.4 and the triangle inequality to deduce that for all $(\lambda_F)_{F\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\}}$ in $\mathbb{R}$ and all $F\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\}$,

$$\Big\|\sum_{\emptyset<G\le F}\lambda_Gz_G\Big\|_E\le 2\sum_{\emptyset<G\le F}|\lambda_G|\varepsilon_{|G|}+2\Big(\sum_{\emptyset<G\le F}\lambda_G^2\Big)^{1/2}.$$

It then follows from our initial choice of $(\varepsilon_n)_{n=0}^\infty$ and from the Cauchy–Schwarz inequality that

$$\forall F\in[\mathbb{N}]^{<\omega}\setminus\{\emptyset\},\qquad \Big\|\sum_{\emptyset<G\le F}\lambda_Gz_G\Big\|_E\le 3\Big(\sum_{\emptyset<G\le F}\lambda_G^2\Big)^{1/2}.$$

In the terminology introduced in [9] this means that Z satisfies ℓ_2 upper tree estimates. It then follows from [9, Theorem 1.1] that Z admits an equivalent norm which is 2-AUS. ∎

Remark 2.5. Statement (i) in Theorem 2.1 can be rephrased as follows: The space Z^* admits an equivalent norm whose dual norm is 2-weak*-AUC. It is important to note that this norm cannot be the dual norm of an equivalent norm on Z. Indeed, a bidual norm cannot be weak*-AUC unless the space is reflexive (see the proposition below). In particular, in Lindenstrauss' construction, the space Y is isomorphic but never isometric to Z^*.

For the convenience of the reader, we state and prove an elementary fact from which the previous remark follows.

Proposition 2.6. *Let Z be a non-reflexive Banach space. Then the norm of Z^{**} is not weak*-AUC.*

Proof. Assume that Z is not reflexive, so there exists $z^{**} \in S_{Z^{**}} \setminus Z$. Pick $\varepsilon > 0$ such that $\varepsilon < d(z^{**}, Z)$. Fix $\delta > 0$ so that $\varepsilon + \delta < d(z^{**}, Z)$ and a weak*-closed finite-codimensional subspace E of Z^{**}. We can write $E = \bigcap_{i=1}^n \operatorname{Ker} z_i^*$ with $z_i^* \in Z^*$. Fix now $\eta > 0$. Then Goldstine's theorem ensures that there exists $z \in B_Z$ such that $|(z^{**} - z)(z_i^*)| < \eta$ for all $i \le n$. If we denote by F the linear span of $z_1^*, \ldots, z_n^*$, it follows from elementary duality theory that

$$d(z^{**} - z, E) = \|z^{**} - z\|_{Z^{**}/F^\perp} = \|z^{**} - z\|_{F^*}.$$

So, if η was chosen small enough, we get $d(z^{**} - z, E) < \delta$. Thus we can pick $e^{**} \in E$ such that $\|z - z^{**} - e^{**}\| < \delta$. Note that this implies that $\|e^{**}\| > \varepsilon$.

Now, writing $z = z^{**} + e^{**} + z - z^{**} - e^{**}$ and using the fact that $z \in B_Z$, we deduce that $\|z^{**} + e^{**}\| \le 1 + \delta$. Finally, by convexity, there exists $\lambda \in (0,1)$ such that $\|\lambda e^{**}\| = \varepsilon$ and $\|z^{**} + \lambda e^{**}\| \le 1 + \delta$. Since δ could be chosen arbitrarily small, we deduce that for any weak*-closed finite-codimensional subspace E of Z^{**},

$$\inf_{y^{**} \in S_{E^{**}}} \|z^{**} + \varepsilon y^{**}\| \le 1,$$

which implies that $\overline{\delta}^*_{Z^*}(\varepsilon) = 0$ and finishes our proof. ∎

2.2. Proof of Theorem 1.1. We fix $\alpha \in \Gamma \setminus \Lambda$ and use induction on $n \in \mathbb{N}$.

For $n = 2$, let X_α (given by [7, Theorem 1.5]) be a separable Banach space such that $\mathrm{Sz}(X_\alpha) = \alpha$. Then denote by Z_2 the Lindenstrauss space such that Z_2^{**}/Z_2 is isomorphic to X_α. By Theorem 1.2 we have $\mathrm{Sz}(Z_2) = \mathrm{Sz}(Z_2^*) = \omega$. Next, by [6, Proposition 2.1], there exists $C > 0$ such that

$$\forall \varepsilon > 0, \quad \mathrm{Sz}(Z_2^{**}, \varepsilon) \le \mathrm{Sz}(Z_2^{**}/Z_2, \varepsilon/C)\, \mathrm{Sz}(Z_2, \varepsilon/C) < \alpha.$$

The last inequality follows from $\mathrm{Sz}(Z_2^{**}/Z_2, \varepsilon/C) < \alpha$, $\mathrm{Sz}(Z_2, \varepsilon) < \omega$ and elementary properties of multiplication of ordinal numbers. We deduce that $\mathrm{Sz}(Z_2^{**})$ is at most α and therefore $\mathrm{Sz}(Z_2^{**}) = \alpha$, since $\mathrm{Sz}(Z_2^{**}) \geq \mathrm{Sz}(Z_2^{**}/Z_2) = \mathrm{Sz}(X_\alpha) = \alpha$. Thus we can choose $Z_1 = Z_2^*$.

Assume now that $n \geq 3$ and that spaces $Z_1, \dots, Z_{n-1}$ have been constructed with the required indices of the duals. Then denote by Z_n the Lindenstrauss space such that Z_n^{**}/Z_n is isomorphic to Z_{n-2}. We already know that $\mathrm{Sz}(Z_n) = \mathrm{Sz}(Z_n^*) = \omega$. Since $\mathrm{Sz}(Z_{n-2}) = \omega$, we can use the fact that having Szlenk index ω is a three-space property (see [6]) to deduce that $\mathrm{Sz}(Z_n^{**}) = \omega$. Then using elementary facts about duality, we find that for all $k \geq 3$ the space $Z_n^{(k)}$ is isomorphic to $Z_n^{(k-2)} \oplus Z_{n-2}^{(k-2)}$, which implies that $\mathrm{Sz}(Z^{(k)}) = \max\{\mathrm{Sz}(Z_n^{(k-2)}), \mathrm{Sz}(Z_{n-2}^{(k-2)})\}$ (see [8]). It now clearly follows that $\mathrm{Sz}(Z_n^{(k)}) = \omega$ for all $k \in \{0, \dots, n-1\}$ and $\mathrm{Sz}(Z_n^{(n)}) = \alpha$. ∎

3. Prescribing Szlenk indices of reflexive Banach spaces. We now turn to the proof of Theorem 1.3, which will take a few steps.

First we describe a general construction of a Banach space associated with a given Banach space with a Schauder basis, which will be essential further on. As will be clear, this resembles Lindenstrauss' construction. The crucial difference is that the dense sequence $(x_i)_{i=1}^\infty$ in X will be replaced by a normalized Schauder basis of X.

So assume that $(x_i)_{i=1}^\infty$ is a normalized Schauder basis of the Banach space X and denote again by $(e_i)_{i=1}^\infty$ the canonical algebraic basis of c_{00}. We define X^{ℓ_2} as the completion of c_{00} with respect to the norm

$$\Big\|\sum_{i=1}^\infty a_i e_i\Big\|_{X^{\ell_2}} = \sup\Big\{\Big(\sum_{i=1}^\infty \Big\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\Big\|_X^2\Big)^{1/2} : 0 \leq k_0 < k_1 < \cdots\Big\}.$$

This construction is presented in [19, Section 3] in a more general setting. With the notation from [19], the space X^{ℓ_2} is $Z^V(E)$ with $Z = X$, $V = \ell_2$ and E the finite-dimensional decomposition of X into the one-dimensional spaces spanned by the basis vectors $(x_i)_{i=1}^\infty$ of X. Clearly, the definition of X^{ℓ_2} depends on our choice of $(x_i)_{i=1}^\infty$. However, we shall omit reference to this dependence in notation.

Note first that $(e_i)_{i=1}^\infty$ is a basis for X^{ℓ_2} which is an unconditional basis for X^{ℓ_2} if $(x_i)_{i=1}^\infty$ is unconditional in X. Furthermore, the map $e_i \mapsto x_i$ extends to a well defined linear operator $I : X^{\ell_2} \to X$ of norm 1. Note also that $(e_i)_{i=1}^\infty$ is a bimonotone basis for X^{ℓ_2}, even if $(x_i)_{i=1}^\infty$ is not bimonotone in X.

PROPOSITION 3.1. *Assume that $(x_i)_{i=1}^\infty$ is a shrinking basis of X. Then:*

(i) *The space X^{ℓ_2} is reflexive. In particular, $(e_i)_{i=1}^\infty$ is a shrinking and boundedly complete basis of X^{ℓ_2}.*

(ii) *The space $(X^{\ell_2})^*$ is 2-AUS. In particular,* $\mathrm{Sz}((X^{\ell_2})^*) = \omega$.

Proof. Statement (i) is a particular case of [19, Corollary 3.4].

(ii) Since $(e_i)_{i=1}^\infty$ is shrinking, $(X^{\ell_2})^*$ can be seen as the closed linear span of $\{e_i^* : i \in \mathbb{N}\}$. Now it is clear that if $x^*, y^* \in (X^{\ell_2})^*$ with $\max \operatorname{supp}(x^*) < \min \operatorname{supp}(y^*)$, then $\|x^* + y^*\|^2 \le \|x^*\|^2 + \|y^*\|^2$. Here, the support is meant with respect to the basis $(e_i^*)_{i=1}^\infty$ of $(X^{\ell_2})^*$. Hence $(X^{\ell_2})^*$ is 2-AUS and has Szlenk index ω.

Note that this also implies that the bidual norm on $(X^{\ell_2})^{**}$ is weak*-AUC and, by Proposition 2.6, re-proves the fact that X^{ℓ_2} is reflexive, since we know that $(e_i)_{i=1}^\infty$ is shrinking. ∎

Our next proposition provides a crucial estimate for $\operatorname{Sz}(X^{\ell_2})$.

PROPOSITION 3.2. *Assume that $(x_i)_{i=1}^\infty$ is a shrinking basis of X. Then $\operatorname{Sz}(X^{\ell_2}) \le \operatorname{Sz}(X)$.*

Our strategy will be to show that $\operatorname{Sz}(X^{\ell_2}) \le \operatorname{Sz}(\ell_2(X))$, where $\ell_2(X)$ is the space of sequences $(x_n)_{n=1}^\infty$ in X such that $\sum_{n=1}^\infty \|x_n\|_X^2$ is finite, equipped with its natural norm,

$$\|(x_n)_{n=1}^\infty\|_{\ell_2(X)} = \Big(\sum_{n=1}^\infty \|x_n\|_X^2\Big)^{1/2}.$$

Then the conclusion will follow from the well known fact that $\operatorname{Sz}(\ell_2(X)) = \operatorname{Sz}(X)$ when X is infinite-dimensional (see [5] for a general study of the behavior of the Szlenk index under direct sums).

Let M_1 be the set of all sequences $(y_i^*)_{i=1}^\infty$ in $B_{\ell_2(X^*)}$ such that there exist $n \in \mathbb{N}$ and $0 = k_0 < \cdots < k_{n-1}$ with the following properties: for every $1 \le i < n$, y_i^* belongs to the linear span of $\{x_j^* : k_{i-1} < j \le k_i\}$, y_n^* belongs to the closed linear span of $\{x_j^* : j > k_{n-1}\}$ and $y_i^* = 0$ for all $i > n$. Then we denote by M_2 the set of all sequences $(y_i^*)_{i=1}^\infty$ in $B_{\ell_2(X^*)}$ such that there exists an infinite sequence $0 = k_0 < k_1 < \cdots$ such that for all $i \in \mathbb{N}$, y_i^* belongs to the linear span of $\{x_j^* : k_{i-1} < j \le k_i\}$. Finally, we set $M = M_1 \cup M_2$.

It is easy to check that M is weak*-compact in $\ell_2(X^*) = \ell_2(X)^*$.

Recall that $I : X^{\ell_2} \to X$ denotes the continuous linear map such that $I(e_i) = x_i$ and $\|I\| = 1$, and define $j : M \to (X^{\ell_2})^*$ by

$$\forall y^* = (y_i^*)_{i=1}^\infty \in M, \quad j(y^*) = \sum_{i=1}^\infty I^* y_i^*.$$

An elementary application of the Cauchy–Schwarz inequality shows that j is well defined and

$$\forall y^* \in M, \quad \|j(y^*)\|_{(X^{\ell_2})^*} \le \|y^*\|_{\ell_2(X^*)}.$$

It is also easy to verify that j is weak*-to-weak* continuous.

Note that the set $j(M)$ can be less formally described as the set of all $\sum_{j=1}^{\infty} b_j e_j^*$ such that there exists an increasing finite or infinite sequence $(F_k)_{k\in A}$ of blocks of $\mathbb{N}$ such that

$$\sum_{k\in A}\Big\|\sum_{j\in F_k} b_j x_j^*\Big\|_{X^*}^2 \le 1.$$

So we now consider the weak*-compact subset $K = j(M)$ of $B_{(X^{\ell_2})^*}$. First we will need to show that K is norming for X^{ℓ_2}. More precisely, we have:

Claim 3.3. *There exists a constant $c > 0$ such that*

$$\forall x \in X^{\ell_2}, \qquad \|x\|_{X^{\ell_2}} \ge c \sup_{x^*\in K} x^*(x).$$

Proof. Let $C \ge 1$ be the bimonotonicity constant of the Schauder basis $(x_i)_{i=1}^{\infty}$ of X, let $x = \sum_{i=1}^{\infty} a_i e_i \in X^{\ell_2}$ and $\varepsilon > 0$. Pick $0 \le k_0 < \cdots < k_n$ such that

$$\Big(\sum_{i=1}^{n}\Big\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\Big\|_X^2\Big)^{1/2} \ge \|x\|_{X^{\ell_2}} - \varepsilon.$$

It follows from the Hahn–Banach theorem that for all $1 \le i \le n$, there exists $u_i^* \in X^*$ with $\mathrm{supp}(u_i^*) \subset (k_{i-1}, k_i]$ and such that

$$u_i^*\Big(\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\Big) = \Big\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\Big\|_X \quad \text{and} \quad \|u_i^*\|_{X^*} \le C.$$

We now set

$$y_i^* = \frac{\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\|_X u_i^*}{C(\sum_{i=1}^{n}\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\|_X^2)^{1/2}} \text{ for } 1 \le i \le n, \qquad y_i^* = 0 \text{ for } i > n.$$

It is then clear that $y^* = (y_i^*)_{i=1}^{\infty} \in M$ and

$$j(y^*)(x) = \frac{1}{C}\Big(\Big\|\sum_{j=k_{i-1}+1}^{k_i} a_j x_j\Big\|_X^2\Big)^{1/2} \ge \frac{\|x\|_{X^{\ell_2}} - \varepsilon}{C}. \ \blacksquare$$

Claim 3.4. *The function $j : M \to K$ is $2C$-Lipschitz, where C is the bimonotonicity constant of the basis $(x_i)_{i=1}^{\infty}$ in X.*

Proof. Fix $y^* = (y_i^*)_{i=1}^{\infty}, z^* = (z_i^*)_{i=1}^{\infty} \in M$. Then there exist $S, T \subset \mathbb{N}$ and sequences $(I_s)_{s\in S}, (J_t)_{t\in T}$ of successive intervals, where S, T are (possibly infinite) initial segments of $\mathbb{N}$, $\{i : y_i^* \ne 0\} \subset S$, $\{i : z_i^* \ne 0\} \subset T$, and for each $s \in S$ and $t \in T$, $\mathrm{supp}(y_s^*) \subset I_s$ and $\mathrm{supp}(z_t^*) \subset J_t$ (here the supports of y_s^* and z_t^* are meant with respect to the basis $(x_j^*)_{j=1}^{\infty}$ of X^*). By allowing either $I_s = \emptyset$ or $J_t = \emptyset$ for $s > \max S$ or $t > \max T$, we may assume $S = T = \mathbb{N}$. For each $i \in \mathbb{N}$, consider three cases:

(a) $J_i \subset I_i$,
(b) $I_i \subset J_i$,
(c) neither (a) nor (b) holds.

If (a) holds, let

$$u_i^* = y_i^* - z_i^* \in \operatorname{span}\{x_j^* : j \in I_i\} \quad \text{and} \quad v_i^* = 0 \in \operatorname{span}\{x_j^* : j \in J_i\}.$$

If (b) holds, let

$$u_i^* = 0 \in \operatorname{span}\{x_j^* : j \in I_i\} \quad \text{and} \quad v_i^* = y_i^* - z_i^* \in \operatorname{span}\{x_j^* : j \in J_i\}.$$

If (c) holds, let

$$u_i^* = P_{I_i \setminus J_i}^*(y_i^* - z_i^*) \in \operatorname{span}\{x_j^* : j \in I_i\},$$
$$v_i^* = P_{J_i}^*(y_i^* - z_i^*) \in \operatorname{span}\{x_j^* : j \in J_i\}.$$

Here, for an interval I, $P_I : X \to \operatorname{span}\{x_j : j \in I\}$ denotes the basis projection. Note that in case (c), $I_i \setminus J_i$ is an interval. Then, since each vector u_i^*, v_i^* is either zero or an interval projection of $y_i^* - z_i^*$, we see that for each i, $\|u_i^*\|_{X^*} \le C\|y_i^* - z_i^*\|_{X^*}$ and $\|v_i^*\|_{X^*} \le C\|y_i^* - z_i^*\|_{X^*}$. It follows that $u^* = (u_i^*)_{i=1}^\infty$, $v^* = (v_i^*)_{i=1}^\infty$ lie in $\ell_2(X)^*$ and $\|u^*\|_{\ell_2(X)^*}, \|v^*\|_{\ell_2(X)^*} \le C\|y^* - z^*\|_{\ell_2(X)^*}$. Because the $(u_i^*)_{i=1}^\infty$ are successively supported, another application of the Cauchy–Schwarz inequality implies that $\sum_{i=1}^\infty u_i^*$ is norm convergent in $(X^{\ell_2})^*$ with $\|\sum_{i=1}^\infty u_i^*\|_{(X^{\ell_2})^*} \le C\|y^* - z^*\|_{\ell_2(X)^*}$. Similarly, $\|\sum_{i=1}^\infty v_i^*\|_{\ell_2(X)^*} \le C\|y^* - z^*\|_{\ell_2(X)^*}$. Since $j(y^*) - j(z^*) = \sum_{i=1}^\infty y_i^* - z_i^* = \sum_{i=1}^\infty u_i^* + v_i^*$, we conclude that

$$\|j(y^*) - j(z^*)\|_{(X^{\ell_2})^*} \le 2C\|y^* - z^*\|_{\ell_2(X)^*}. \ \blacksquare$$

Proof of Proposition 3.2. It is easily seen that if E and F are Banach spaces, $B \subset E^*$ and $C \subset F^*$ are weak*-compact and $f : B \to C$ is a weak*-to-weak* continuous Lipschitz surjection from B to C, then $\mathrm{Sz}(C) \le \mathrm{Sz}(B)$ (see [7, Lemma 2.5(i)]). It follows from this fact and Claim 3.4 that $\mathrm{Sz}(K) \le \mathrm{Sz}(M)$. On the other hand, since $M \subset B_{\ell_2(X)^*}$, we deduce from [5] that $\mathrm{Sz}(M) \le \mathrm{Sz}(\ell_2(X)) = \mathrm{Sz}(X)$. Combining these yields $\mathrm{Sz}(K) \le \mathrm{Sz}(X)$. Denote by L the weak*-closed convex hull of K. It follows from Claim 3.3 and the geometric Hahn–Banach theorem that $cB_{(X^{\ell_2})^*} \subset L \subset B_{(X^{\ell_2})^*}$. Finally, we can apply [7, Theorem 1.1] to deduce $\mathrm{Sz}(L) \le \mathrm{Sz}(X)$ from $\mathrm{Sz}(K) \le \mathrm{Sz}(X)$. $\blacksquare$

The construction of our family $(G_\alpha)_{\alpha \in \Gamma \setminus \Lambda}$ of spaces will also rely on the use of the Schreier families. These were introduced in [2]. Let us now recall the definition of the Schreier family $\mathcal{S}_\alpha$ for α a countable ordinal. Recall that $[\mathbb{N}]^{<\omega}$ denotes the set of finite subsets of $\mathbb{N}$, which we identify with the set of void or finite, strictly increasing sequences in $\mathbb{N}$. We complete the notation introduced in Section 2 by writing $E < F$ to mean $\max E < \min F$, and

$n \le E$ to mean $n \le \min E$. For each countable ordinal α, $\mathcal{S}_\alpha$ will be a subset of $[\mathbb{N}]^{<\omega}$. We let

$$\mathcal{S}_0 = \{\emptyset\} \cup \{(n) : n \in \mathbb{N}\},$$
$$\mathcal{S}_{\alpha+1} = \{\emptyset\} \cup \Big\{\bigcup_{i=1}^n E_i : n \in \mathbb{N},\, \emptyset \ne E_i \in \mathcal{S}_\alpha,\, E_1 < \cdots < E_n,\, n \le E_1\Big\},$$

and if $\alpha < \omega_1$ is a limit ordinal, we fix an increasing sequence $(\alpha_n)_{n=1}^\infty$ tending to α and let

$$\mathcal{S}_\alpha = \{E \in [\mathbb{N}]^{<\omega} : \exists n \le E \in \mathcal{S}_{\alpha_n}\}.$$

In what follows, $[\mathbb{N}]^{<\omega}$ will be topologized by the identification $[\mathbb{N}]^{<\omega} \ni E \leftrightarrow 1_E \in \{0,1\}^{\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}}$ is equipped with the Cantor topology.

Given $(m_i)_{i=1}^k, (n_i)_{i=1}^k$ in $[\mathbb{N}]^{<\omega}$, we say $(n_i)_{i=1}^k$ is a *spread* of $(m_i)_{i=1}^k$ if $m_i \le n_i$ for each $1 \le i \le k$.

We say that a subset $\mathcal{F}$ of $[\mathbb{N}]^{<\omega}$ is

(i) *spreading* if it contains all spreads of its members,
(ii) *hereditary* if it contains all subsets of its members,
(iii) *regular* if it is spreading, hereditary, and compact.

Given $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\mathbb{N}}$, we let

$$\mathcal{F}[\mathcal{G}] = \{\emptyset\} \cup \Big\{\bigcup_{i=1}^n E_i : n \in \mathbb{N},\, \emptyset \ne E_i \in \mathcal{G},\, E_1 < \cdots < E_n,\, (\min E_i)_{i=1}^n \in \mathcal{F}\Big\}.$$

We refer to [8] for a detailed presentation of these notions and their fundamentals properties.

For a topological space $\mathcal{F}$, we denote $\mathcal{F}^1$ its Cantor–Bendixson derived set (the set of its accumulation points), for an ordinal α we let $\mathcal{F}^\alpha$ be its Cantor–Bendixson derived set of order α, and finally $\mathrm{CB}(\mathcal{F})$ is its Cantor–Bendixson index.

We note that if $\mathcal{F}$ and $\mathcal{G}$ are regular subsets of $[\mathbb{N}]^{<\omega}$, then $\mathcal{F}[\mathcal{G}]$ is regular, and if the Cantor–Bendixson indices of $\mathcal{F}$ and $\mathcal{G}$ are $\alpha+1$ and $\beta+1$, respectively, then the Cantor–Bendixson index of $\mathcal{F}[\mathcal{G}]$ is $\beta\alpha+1$ (see [8, Proposition 3.1]).

For each $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \{E \in [\mathbb{N}]^{<\omega} : |E| \le n\}.$$

It is well known that for each $\alpha < \omega_1$, $\mathcal{S}_\alpha$ is regular with Cantor–Bendixson index $\omega^\alpha+1$. Moreover, for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is regular with Cantor–Bendixson index $n+1$. These facts together with those cited from [8] yield the following.

Lemma 3.5. *Fix an ordinal $\alpha < \omega_1$ and $n \in \mathbb{N}$.*

(i) *$\mathcal{A}_n[\mathcal{S}_\alpha]$ is regular with Cantor–Bendixson index $\omega^\alpha n+1$.*
(ii) *For any $\beta < \omega_1$, $\mathcal{S}_\beta[\mathcal{S}_\alpha]$ is regular with Cantor–Bendixson index $\omega^{\alpha+\beta}+1$.*

Lemma 3.6. *If $\mathcal{F}$ and $\mathcal{G}$ are regular families, $E < F \neq \emptyset$, and $E, E \cup F \in \mathcal{F}[\mathcal{G}]$, then either $E \in \mathcal{F}^1[\mathcal{G}]$ or $F \in \mathcal{G}$.*

Proof. Write $E \cup F = \bigcup_{i=1}^n E_i$, where $\emptyset \neq E_i \in \mathcal{G}$, $E_1 < \cdots < E_n$, and $(\min E_i)_{i=1}^n \in \mathcal{F}$.

If $E \cap E_n = \emptyset$, then there exists $1 \leq m \leq n$ such that $E \cap E_i \neq \emptyset$ for each $i < m$ and $E \cap E_i = \emptyset$ for each $m \leq i \leq n$.

If $m = 1$, then $E = \emptyset \in \mathcal{F}^1$, since $\emptyset \prec (\min E_i)_{i=1}^n \in \mathcal{F}$.

If $m > 1$, then the representation

$$E = \bigcup_{i=1}^{m-1} (E \cap E_i)$$

witnesses that $E \in \mathcal{F}^1[\mathcal{G}]$, since $(\min E_i)_{i=1}^{m-1} \in \mathcal{F}^1$.

Now if $E \cap E_n \neq \emptyset$, then $F = E_n \setminus E \subset E_n$, and $F \in \mathcal{G}$. ∎

We are now ready to prove Theorem 1.3, that is, to construct for each $\alpha \in \Gamma \setminus \Lambda$ a reflexive Banach space G_α with an unconditional basis and such that $\mathrm{Sz}(G_\alpha) = \alpha$ and $\mathrm{Sz}(G_\alpha^*) = \omega$.

So, let $\alpha \in \Gamma \setminus \Lambda$. We write $\alpha = \omega^\delta$ with $\delta \in (0, \omega_1)$. Then by standard facts about ordinals, either $\delta = \omega^\xi$ for some ordinal $\xi \in [0, \omega_1)$, or $\delta = \beta + \gamma$ for some $\beta, \gamma < \delta$. We shall separate our construction into these two main cases.

3.1. First case: $\delta = \omega^\xi$ with $\xi \in [0, \omega_1)$. In this situation, ξ must be either 0 or a successor ordinal, otherwise $\alpha \in \Lambda$.

If $\xi = 0$, let $\mathcal{F}_n = \mathcal{S}_0$ for all $n \in \mathbb{N} \cup \{0\}$.

If $\xi = \zeta + 1$, let $\mathcal{F}_0 = \mathcal{S}_0$ and $\mathcal{F}_{n+1} = \mathcal{S}_{\omega^\zeta}[\mathcal{F}_n]$ for $n \in \mathbb{N}$.

In both cases, denote

$$M_n = \Big\{ 2^{-n} \sum_{i \in E} e_i^* : E \in \mathcal{F}_n \Big\} \quad \text{for } n \in \{0\} \cup \mathbb{N} \quad \text{and} \quad M = \bigcup_{n=0}^{\infty} M_n,$$

where $(e_i^*)_{i=1}^\infty$ is the sequence of coordinate functionals defined on c_{00}.

Then we define $\mathfrak{G}_\alpha$ to be the completion of c_{00} with respect to the norm

$$\|x\|_{\mathfrak{G}_\alpha} = \sup_{x^* \in M} |x^*(x)|.$$

Note that the canonical basis of c_{00} is a 1-suppression unconditional basis of $\mathfrak{G}_\alpha$. To keep our notation consistent, we shall denote by $(x_i)_{i=1}^\infty$ this basis of $\mathfrak{G}_\alpha$. The reason is that we need next to set $G_\alpha = \mathfrak{G}_\alpha^{\ell_2}$, where this construction is meant with respect to the basis $(x_i)_{i=1}^\infty$, which we shall later call the canonical basis of $\mathfrak{G}_\alpha$. On the other hand $(e_i)_{i=1}^\infty$ will still denote the canonical basis of c_{00} considered as a basis of G_α. Finally, we define the following subsets of $\mathfrak{G}_\alpha^*$:

$$K_n = \Big\{2^{-n} \sum_{i \in E} x_i^* : E \in \mathcal{F}_n\Big\} \quad \text{for } n \in \{0\} \cup \mathbb{N} \quad \text{and} \quad K = \bigcup_{n=0}^{\infty} K_n.$$

Later, the sets M_n and M will be considered as subsets of G_α^*.

It is easily checked that $\mathfrak{G}_\omega = c_0$ and $G_\omega = \ell_2$. Clearly G_ω is reflexive with an unconditional basis and $\mathrm{Sz}(G_\omega) = \mathrm{Sz}(G_\omega^*) = \omega$. So we shall now assume that ξ is different from 0 and is therefore a countable successor ordinal.

Proposition 3.7. *Assume $\alpha = \omega^{\omega^\xi}$, where ξ is a countable successor ordinal. Then $\mathrm{Sz}(\mathfrak{G}_\alpha) \le \alpha$.*

Proof. By [7, Theorem 1.1], it is sufficient to prove that $\mathrm{Sz}(K) \le \alpha$, since $B_{\mathfrak{G}_\alpha^*}$ is the weak*-closed, absolutely convex hull of K.

First, it is easy to see that for any $\varepsilon > 0$ and any ordinal η,

$$s_\varepsilon^\eta(K) \subset \{0\} \cup \bigcup_{n=0}^{\infty} s_\varepsilon^\eta(K_n),$$

whence

$$\mathrm{Sz}(K, \varepsilon) \le \Big(\sup_{n \in \mathbb{N} \cup \{0\}} \mathrm{Sz}(K_n, \varepsilon)\Big) + 1.$$

Thus it suffices to show that $\sup_{n \in \mathbb{N} \cup \{0\}} \mathrm{Sz}(K_n, \varepsilon) < \alpha$ for each $\varepsilon > 0$.

For a given $\varepsilon > 0$, we will provide an upper estimate for $\mathrm{Sz}(K_n, 2\varepsilon)$ in one of two ways, depending on whether n is large or small relative to ε. The Cantor–Bendixson index of K_n is an easy upper bound for $\mathrm{Sz}(K_n, 2\varepsilon)$, which is a good upper bound for small n. We note that the map $\phi_n : \mathcal{F}_n \to K_n$ given by $\phi_n(E) = \sum_{i \in E} x_i^*$ is a homeomorphism from $\mathcal{F}_n$ to K_n, where K_n is endowed with its weak* topology. It follows that for any $n \in \mathbb{N} \cup \{0\}$ and any $\varepsilon > 0$,

$$\mathrm{Sz}(K_n, \varepsilon) \le \mathrm{CB}(K_n) = \mathrm{CB}(\mathcal{F}_n).$$

We now turn to bounding $\mathrm{Sz}(K_n, 2\varepsilon)$ for large n. Recall that $\xi = \zeta + 1$ with $\zeta \in [0, \omega_1)$. We now prove that if $2^{-m} < \varepsilon$, then for any $n > m$ and any ordinal η,

$$s_{2\varepsilon}^\eta(K_n) \subset \Big\{2^{-n} \sum_{i \in E} x_i^* : E \in \mathcal{F}_m^\eta[\mathcal{F}_{n-m}]\Big\}.$$

The proof is by induction on η, with the base case following from the fact that $\mathcal{F}_a[\mathcal{F}_b] = \mathcal{F}_{a+b}$ for any $a, b \in \mathbb{N}$. The limit ordinal case follows by taking intersections. Finally, assume we have the result for some η and

$$2^{-n} \sum_{i \in E} x_i^* \in s_{2\varepsilon}^{\eta+1}(K_n),$$

so that the inductive hypothesis guarantees that $E \in \mathcal{F}_m^\eta[\mathcal{F}_{n-m}]$. Then there exists a sequence

$$\Big(2^{-n}\sum_{i\in E_j} x_i^*\Big)_{j=1}^\infty \subset s_{2\varepsilon}^\eta(K_n,\varepsilon) \subset \Big\{2^{-n}\sum_{i\in E} x_i^* : E \in \mathcal{F}_m^\eta[\mathcal{F}_{n-m}]\Big\}$$

converging weak* to $2^{-n}\sum_{j\in E} x_i^*$ and such that

$$\liminf_{j\to\infty} \Big\|2^{-n}\sum_{i\in E} x_i^* - 2^{-n}\sum_{i\in E_j} x_i^*\Big\|_{\mathfrak{G}_\alpha^*} \geq \varepsilon.$$

Of course, this means that $E_j \to E$ in $\mathcal{F}_n$, so that, after passing to another subsequence, we may assume $E_j = E \cup F_j$ for some $F_j \neq \emptyset$ with $E < F_j$. Now since $E, E_j \in \mathcal{F}_m^\eta[\mathcal{F}_{n-m}]$ for each j, by Lemma 3.6 either $F_j \in \mathcal{F}_{n-m}$ or $E \in \mathcal{F}_m^{\eta+1}[\mathcal{F}_{n-m}]$. However, if $F_j \in \mathcal{F}_{n-m}$, then $2^{m-n}\sum_{i\in F_j} x_i^* \in B_{\mathfrak{G}_\alpha^*}$ and

$$\forall j \in \mathbb{N}, \quad \Big\|2^{-n}\sum_{i\in E} x_i^* - 2^{-n}\sum_{i\in E_j} x_i^*\Big\|_{\mathfrak{G}_\alpha^*} = 2^{-m}\Big\|2^{m-n}\sum_{i\in F_j} x_i^*\Big\|_{\mathfrak{G}_\alpha^*} \leq 2^{-m} < \varepsilon,$$

a contradiction. This concludes the successor case.

We now deduce from the inclusion just proved that

$$s_{2\varepsilon}^{\omega^{\omega^\zeta m}+1}(K_n) \subset \Big\{2^{-n}\sum_{i\in E} x_i^* : E \in \mathcal{F}_m^{\omega^{\omega^\zeta m}+1}[\mathcal{F}_{n-m}]\Big\} = \emptyset.$$

So, we can estimate

$$\mathrm{Sz}(K_n, 2\varepsilon) \leq \begin{cases} \omega^{\omega^\zeta n} + 1, & n \leq \log_2(1/\varepsilon), \\ \omega^{\omega^\zeta \lceil \log_2(1/\varepsilon)\rceil} + 1, & n > \log_2(1/\varepsilon), \end{cases}$$

which finishes the proof of Proposition 3.7. ∎

Proof of Theorem 1.3 in the first case. Let $\alpha = \omega^{\omega^\xi}$, where ξ is a countable successor ordinal and $\mathfrak{G}_\alpha$, G_α are constructed as above.

Since the canonical basis $(x_i)_{i=1}^\infty$ of $\mathfrak{G}_\alpha$ is 1-suppression unconditional, it is clear that $(e_i)_{i=1}^\infty$ is a 1-suppression unconditional basis for G_α. Proposition 3.7 ensures that $\mathrm{Sz}(\mathfrak{G}_\alpha) \leq \alpha$ and therefore $\mathfrak{G}_\alpha$ does not contain ℓ_1. Then a classical result of R. C. James [12] shows that $(x_i)_{i=1}^\infty$ is a shrinking basis of $\mathfrak{G}_\alpha$. Thus we can apply Proposition 3.1 to deduce that G_α is reflexive and $\mathrm{Sz}(G_\alpha^*) = \omega$.

We also deduce from Proposition 3.2 that $\mathrm{Sz}(G_\alpha) \leq \mathrm{Sz}(\mathfrak{G}_\alpha) = \alpha$.

Now we have to prove that $\mathrm{Sz}(G_\alpha) \geq \alpha$. So write again $\alpha = \omega^{\omega^{\zeta+1}}$ with $\zeta \in [0, \omega_1)$. Suppose $n \in \mathbb{N}$ and $E < F$ are such that $F \in \mathcal{F}_n$. Fix $k \in F \setminus E$. Note that

$$2^{-n}\sum_{i\in F} e_i^* \in M_n$$

and

$$\Big\|2^{-n}\sum_{i\in E}e_i^* - 2^{-n}\sum_{i\in F}e_i^*\Big\|_{G_\alpha} \ge \Big|\Big(2^{-n}\sum_{i\in E}e_i^* - 2^{-n}\sum_{i\in F}e_i^*\Big)(e_k)\Big| = 2^{-n},$$

since $\|e_k\|_{G_\alpha} = 1$. From this and an easy induction argument, we see that $2^{-n}\sum_{i\in E}e_i^* \in s^{\mu}_{2^{-n-1}}(B_{G_\alpha^*})$ for any $n\in\mathbb{N}$, any $0\le\mu<\mathrm{CB}(\mathcal{F}_n)$ and any $E\in\mathcal{F}_n^{\mu}$. Since $\mathrm{CB}(\mathcal{F}_n) = (\omega^{\omega^\zeta})^n = \omega^{\omega^\zeta n}$, we deduce that

$$\mathrm{Sz}(G_\alpha) \ge \sup_{n\in\mathbb{N}}\omega^{\omega^\zeta n} = \omega^{\omega^{\zeta+1}} = \alpha.$$

This finishes the proof and our construction for $\alpha = \omega^{\omega^\xi}$ with ξ being a countable successor ordinal. ■

3.2. Second case: $\delta = \beta+\gamma$ for some $\beta,\gamma<\delta$. We will now slightly modify our construction in order to treat the case of $\alpha=\omega^{\beta+\gamma}$ with $\omega^\beta<\alpha$ and $\omega^\gamma<\alpha$. We have to consider two subcases.

First suppose γ is a limit ordinal. We fix $\gamma_0 = 0$ and an increasing sequence $(\gamma_n)_{n=1}^\infty$ such that $\sup_{n\in\mathbb{N}}\gamma_n=\gamma$. Then we set

$$\mathcal{F}_0 = \mathcal{S}_\beta \quad\text{and}\quad \mathcal{F}_n = \mathcal{S}_{\gamma_n}[\mathcal{S}_\beta] \quad \text{for } n\in\mathbb{N}.$$

If $\gamma=\zeta+1$ is a successor ordinal, we set

$$\mathcal{F}_0 = \mathcal{S}_{\beta+\zeta} \quad\text{and}\quad \mathcal{F}_n = \mathcal{A}_n[\mathcal{S}_{\beta+\zeta}] \quad \text{for } n\in\mathbb{N}.$$

In either case, let

$$M_n = \Big\{2^{-n}\sum_{i\in E}e_i^* : E\in\mathcal{F}_n\Big\} \quad \text{for } n\in\{0\}\cup\mathbb{N} \quad\text{and}\quad M=\bigcup_{n=0}^\infty M_n.$$

As in Subsection 3.1, we define $\mathfrak{G}_\alpha$ to be the completion of c_{00} with respect to the norm $\|x\|_{\mathfrak{G}_\alpha} = \sup_{x^*\in M}|x^*(x)|$ and let $G_\alpha = \mathfrak{G}_\alpha^{\ell_2}$, where this construction is meant with respect to the canonical basis $(x_i)_{i=1}^\infty$ of $\mathfrak{G}_\alpha$. As previously, we define

$$K_n = \Big\{2^{-n}\sum_{i\in E}x_i^* : E\in\mathcal{F}_n\Big\} \quad \text{for } n\in\{0\}\cup\mathbb{N} \quad\text{and}\quad K=\bigcup_{n=0}^\infty K_n.$$

PROPOSITION 3.8. *Assume that α is a countable ordinal that can be written as $\alpha=\omega^{\beta+\gamma}$ with $\omega^\beta<\alpha$ and $\omega^\gamma<\alpha$. Then $\mathrm{Sz}(\mathfrak{G}_\alpha)\le\alpha$.*

Proof. Again, it is sufficient to show that $\mathrm{Sz}(K)\le\alpha$. Arguing as in Proposition 3.7, we first note that for any $\varepsilon>0$ and $n\in\mathbb{N}$,

$$\mathrm{Sz}(K_n,\varepsilon) \le \mathrm{CB}(\mathcal{F}_n) = \begin{cases}\omega^{\beta+\gamma_n}+1, & \gamma \text{ a limit},\\ \omega^{\beta+\mu}n+1, & \gamma=\zeta+1.\end{cases}$$

Now for $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $2^{-n} < \varepsilon$, we claim that for any ordinal η,

$$s^\eta_{2\varepsilon}(K_n) \subset \begin{cases} \{2^{-n}\sum_{i\in E} x^*_i : E \in \mathcal{S}^\eta_{\gamma_n}[\mathcal{S}_\beta]\}, & \gamma \text{ a limit}, \\ \{2^{-n}\sum_{i\in E} x^*_i : E \in \mathcal{A}^\eta_n[\mathcal{S}_{\beta+\zeta}]\}, & \gamma = \zeta + 1. \end{cases}$$

The proof is even easier than the analogous proof in the first case, so we omit it. Note that in particular when γ is a limit ordinal and $2^{-n} < \varepsilon$, we have $\mathcal{S}^{\omega^\gamma}_{\gamma_n} = \emptyset$, whence the previous claim yields the estimate $\mathrm{Sz}(K_n, 2\varepsilon) \le \omega^\gamma < \omega^{\beta+\gamma}$ when $2^{-n} < \varepsilon$. Similarly, since $\mathcal{A}^\omega_n = \emptyset$, we see that $\mathrm{Sz}(K_n, 2\varepsilon) \le \omega < \omega^{\beta+\zeta+1}$ when $2^{-n} < \varepsilon$.

Therefore for $n \le \log_2(1/\varepsilon)$,

$$\mathrm{Sz}(K_n, 2\varepsilon) \le \mathrm{CB}(\mathcal{F}_n) = \begin{cases} \omega^{\beta+\gamma_n} + 1, & \gamma \text{ a limit}, \\ \omega^{\beta+\mu} n + 1, & \gamma = \zeta + 1, \end{cases}$$

and for $n > \log_2(1/\varepsilon)$,

$$\mathrm{Sz}(K_n, 2\varepsilon) \le \begin{cases} \omega^\gamma, & \gamma \text{ a limit}, \\ \omega, & \gamma = \zeta + 1. \end{cases}$$

Thus in either case, for every $\varepsilon > 0$, $\sup_{n\in\mathbb{N}\cup\{0\}} \mathrm{Sz}(K_n, \varepsilon) < \alpha$, yielding the result. ∎

Proof of Theorem 1.3 in the second case. The end of the proof is the same as for the first case, after noting that $\mathrm{CB}(\mathcal{F}_n) = \omega^{\beta+\gamma_n} + 1$ when γ is a limit ordinal, and $\mathrm{CB}(\mathcal{F}_n) = \omega^{\beta+\zeta} n + 1$ if $\gamma = \zeta + 1$. ∎

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