A NEW METRIC INVARIANT FOR BANACH SPACES

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Abstract. We show that if the Szlenk index of a Banach space X is larger than the first infinite ordinal ω or if the Szlenk index of its dual is larger than ω, then the tree of all finite sequences of integers equipped with the hyperbolic distance metrically embeds into X. We show that the converse is true when X is assumed to be reflexive. As an application, we exhibit new classes of Banach spaces that are stable under coarse-Lipschitz embeddings and therefore under uniform homeomorphisms.

1. Introduction

In 1976 Ribe proved in [22] that two uniformly homeomorphic Banach spaces are finitely representable in each other. This theorem gave birth to the "Ribe program" (see [4] or [17] for a detailed description). Local properties of Banach spaces are properties which only involve finitely many vectors. These are properties which are stable under finite representability. In view of Ribe's result the "Ribe program" aims at looking for metric invariants that characterize local properties of Banach spaces. The first occurence of the "Ribe program" is Bourgain's metric characterization of superreflexivity given in [4]. The metric invariant discovered by Bourgain is the collection of the hyperbolic dyadic trees of arbitrarily large height N. If we denote $\Omega_0 = \{\emptyset\}$, the root of the tree. Let $\Omega_i = \{-1,1\}^i$, $B_N = \bigcup_{i=0}^N \Omega_i$. Thus B_N endowed with its shortest path metric ρ is the hyperbolic dyadic tree of height N.

Let us recall some definitions. Let (M,d) and (N,δ) be two metric spaces and let $f : M \to N$ be an injective map. The *distortion* of f is

$$\text{dist}(f) := \|f\|_{Lip}\|f^{-1}\|_{Lip} = \sup_{x\neq y\in M} \frac{\delta(f(x),f(y))}{d(x,y)}.\sup_{x\neq y\in M} \frac{d(x,y)}{\delta(f(x),f(y))}.$$

If $\text{dist}(f)$ is finite, we say that f is a Lipschitz or metric embedding of M into N. If there exists an embedding f from M into N, with $\text{dist}(f) \le C$, we use the notation $M \overset{C}{\hookrightarrow} N$.

Bourgain's characterization is the following:

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Theorem 1.1. (Bourgain 1986 [4]**)** *Let X be a Banach space. Then X is not superreflexive if and only if there exists a constant $C \geq 1$ such that for all $N \in \mathbb{N}$, $(B_N, \rho) \overset{C}{\hookrightarrow} X$.*

It has been proved in [1] that this is also equivalent to the metric embedding of the infinite hyperbolic dyadic tree (B_∞, ρ) where $B_\infty = \bigcup_{N=0}^{\infty} B_N$. It should also be noted that in [4] and [1], the embedding constants are bounded above by a universal constant.

We also recall that it follows from the Enflo-Pisier renorming theorem ([6] and [21]) that superreflexivity is equivalent to the existence of an equivalent uniformly convex and (or) uniformly smooth norm.

In the series of papers [5], [16], [17] local properties such as linear type and linear cotype are deeply studied and other occurrences of "Ribe's program" are given.

In a similar vein our paper is an attempt to investigate which asymptotic properties admit a metrical characterization. Asymptotic properties have been intensively studied in [9], [7] and [20] and we refer to [11] for a precise definition of the asymptotic structure of a Banach space. The main result of this paper is an analogue of Bourgain's theorem in the asymptotic setting. Let us first introduce a few notation and definitions. For a positive integer N, We denote $T_N = \bigcup_{i=0}^{N} \mathbb{N}^i$, where $\mathbb{N}^0 := \{\emptyset\}$. Then $T_\infty = \bigcup_{N=1}^{\infty} T_N$ is the set of all finite sequences of positive integers. For $s \in T_\infty$, we denote by $|s|$ the length of s. There is a natural ordering on T_∞ defined by $s \leq t$ if t extends s. If $s \leq t$, we will say that s is an *ancestor* of t. If $s \leq t$ and $|t| = |s| + 1$, we will say that s is the *predecessor* of t and t is a *successor* of s and we will denote $s = t^-$. Then we equip T_∞, and by restriction every T_N, with the hyperbolic distance ρ, which is defined as follows. Let s and s' be two elements of T_∞ and let $u \in T_\infty$ be their greatest common ancestor. We set

$$\rho(s, s') = |s| + |s'| - 2|u| = \rho(s, u) + \rho(s', u).$$

We now define the asymptotic version of uniform convexity and uniform smoothness that we will consider. Let $(X, \|\ \|)$ be a Banach space and $\tau > 0$. We denote by B_X its closed unit ball and by S_X its unit sphere. For $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(\tau, x, Y) = \sup_{y \in S_Y} \|x + \tau y\| - 1 \quad \text{and} \quad \overline{\delta}(\tau, x, Y) = \inf_{y \in S_Y} \|x + \tau y\| - 1.$$

Then

$$\overline{\rho}(\tau) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \overline{\rho}(\tau, x, Y) \quad \text{and} \quad \overline{\delta}(\tau) = \inf_{x \in S_X} \sup_{\dim(X/Y) < \infty} \overline{\delta}(\tau, x, Y).$$

The norm $\| \ \|$ is said to be *asymptotically uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\overline{\rho}(\tau)}{\tau} = 0.$$

It is said to be *asymptotically uniformly convex* if

$$\forall \tau > 0 \quad \overline{\delta}(\tau) > 0.$$

These moduli have been first introduced by Milman in [18].

We can now state the main result of our paper in a way that is clearly an asymptotic analogue of Bourgain's theorem.

Theorem 1.2. *Let X be a reflexive Banach space. The following assertions are equivalent.*

(i) There exists $C \geq 1$ such that $T_\infty \overset{C}{\hookrightarrow} X$.

(ii) There exists $C \geq 1$ such that for any N in $\mathbb{N}$, $T_N \overset{C}{\hookrightarrow} X$.

(iii) X does not admit any equivalent asymptotically uniformly smooth norm <u>or</u> X does not admit any equivalent asymptotically uniformly convex norm.

The main tool for our proof will be the so-called *Szlenk index*. We now recall the definition of the Szlenk derivation and the Szlenk index that have been first introduced in [24] and used there to show that there is no universal space for the class of separable reflexive Banach spaces. So consider a real separable Banach space X and K a weak*-compact subset of X^*. For $\varepsilon > 0$ we let $\mathcal{V}$ be the set of all relatively weak*-open subsets V of K such that the norm diameter of V is less than ε and $s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$. We define inductively $s_\varepsilon^\alpha K$ for any ordinal α, by $s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K)$ and $s_\varepsilon^\alpha K = \cap_{\beta<\alpha} s_\varepsilon^\beta K$ if α is a limit ordinal. Then we define $Sz(X,\varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $Sz(X,\varepsilon) = \infty$. The *Szlenk index* of X is finally defined by $Sz(X) = \sup_{\varepsilon>0} Sz(X,\varepsilon)$.

We denote ω the first infinite ordinal and ω_1 the first uncountable ordinal. Note that the dual of a separable Banach space X is separable if and only if $Sz(X) < \omega_1$ (this is a consequence of Baire's theorem on the pointwise limit of sequences of continuous functions). We will essentially deal with the condition $Sz(X) \leq \omega$. The weak*-compactness of B_{X^*} implies that this is equivalent to the condition: $Sz(X,\varepsilon) < \omega$, for all $\varepsilon > 0$. Also, it follows from a theorem of Knaust, Odell and Schlumprecht ([11]) that a separable Banach space admits an equivalent asymptotically uniformly smooth norm if and only if $Sz(X) \leq \omega$. Then it is easy to see that for a reflexive Banach space the condition $Sz(X^*) \leq \omega$ is equivalent to the existence of an equivalent asymptotically uniformly convex norm on X. Therefore condition (iii) in Theorem (1.2) is equivalent to

$$(iv) \quad Sz(X) > \omega \text{ or } Sz(X^*) > \omega.$$

With this information at hand, we can almost forget the formulations in terms of renormings and work essentially with the notion of the Szlenk index of a Banach space.

In order to have a complete view of the analogy between our result and Bourgain's theorem, it is worth noting at this point that the superreflexivity can be similarly characterized by the behavior of an ordinal index. For a given weak*-compact convex subset C of X^* and a given $\varepsilon > 0$, let us denote $\mathcal{S}$ be the set of all relatively weak*-open slices S of C such that the norm diameter of S is less than ε and $d_\varepsilon C = C \setminus \cup\{S : S \in \mathcal{S}\}$. We then define inductively $d_\varepsilon^\alpha(C)$ for α ordinal as before and $Dz(X,\varepsilon)$ to be the least ordinal α so that $d_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $Dz(X,\varepsilon) = \infty$. Finally, the *weak*-dentability index* of X is $Dz(X) = \sup_{\varepsilon>0} Dz(X,\varepsilon)$. Then it follows from [12] (see also the survey [13]) that the following conditions are equivalent:

(i) X is super-reflexive.

(ii) $Dz(X) \leq \omega$.

(iii) $Dz(X^*) \leq \omega$.

Let us now describe the organization of this article. In Section 2 we give the construction of several embeddings and finally prove that T_∞ Lipschitz-embeds into X, whenever $Sz(X) > \omega$ or $Sz(X^*) > \omega$. In Section 3 we show the converse statement in the reflexive case. This will conclude the proof of Theorem 1.2. In the last section we describe a few applications of our result to the stability of certain classes of Banach spaces under coarse-Lipschitz embeddings or uniform homeomorphisms. The main consequence of our work is that the class of all separable reflexive spaces X so that $Sz(X) \leq \omega$ and $Sz(X^*) \leq \omega$ is stable under coarse-Lipschitz embeddings. It seems also interesting to us that a metric invariant (the embeddability of T_∞ in this case) is used to prove stability results, whereas the metric invariant is more often found after the class is already known to be stable.

2. Construction of the embeddings

Before starting, we need to introduce more notation concerning our trees. For $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_m)$ in T_∞, we denote

$$s \frown t = (s_1, \dots, s_n, t_1, \dots, t_m) \text{ and also } \emptyset \frown t = t \frown \emptyset = t.$$

For $t \in T_\infty$ and $k \le |t|$, we denote $t_{|_k}$ the ancestor of t of length k.
For $s \le t$ in T_∞, we denote $[s,t] = \{u \in T_\infty,\ s \le u \le t\}$.
For N in $\mathbb{N}$ and $T \subset T_N$, we say that a map $\Phi : T_N \to T$ is a *tree isomorphism* if $\Phi(T_N) = T$, $\Phi(\emptyset) = \emptyset$ and for all $s \in T_{N-1}$ and $n \in \mathbb{N}$ $\Phi(s \frown n) = \Phi(s) \frown k_{s,n}$ with $k_{s,n} \in \mathbb{N}$ and $k_{s,n} < k_{s,m}$ whenever $n < m$. A subset T of T_N is called a *full subtree* of T_N if there exists a tree isomorphism from T_N onto T or equivalently if $\emptyset \in T$ and for all $s \in T \cap T_{N-1}$, the set of successors of s that also belong to T is infinite.

We now begin with a very simple lemma.

Lemma 2.1. *Let $(x_n^*)_{n=0}^\infty$ be a weak*-null sequence in X^* such that $\|x_n^*\| \ge 1$ for all n in $\mathbb{N}$ and let F be a finite dimensional subspace of X^*. Then there exists a sequence $(x_n)_n$ in B_X such that for all $y^* \in F$, $y^*(x_n) = 0$ and* $\liminf\ x_n^*(x_n) \ge \frac{1}{2}$

Proof. It is a classical consequence of Mazur's technique for constructing basic sequences (see for instance [14]), that $\liminf d(x_n^*, F) \ge \frac{1}{2}$. Denote $E := \{x \in X \quad \forall x^* \in F \quad x^*(x) = 0\}$ be the pre-orthogonal of F. Since F is finite dimensional, we have that $F = E^\perp$. Therefore, for any $x^* \in X^*$, $d(x^*, F) = \|x^*_{|_E}\|_{E^*}$. This finishes the proof. $\square$

Let now X be a separable Banach space. It follows from the metrizability of the weak* topology on B_{X^*} that if $\mathrm{Sz}(X, \varepsilon) > \omega$ then, for all $N \in \mathbb{N}$ there exists $(y_s^*)_{s \in T_N}$ in B_X^* such that for all $s \in T_{N-1}$ and all $n \in \mathbb{N}$, $\|y^*_{s\frown n} - y^*_s\| \ge \varepsilon/2 := \varepsilon'$ and $y^*_{s \frown n} \overset{w*}{\to} y^*_s$.
It is an easy and well known fact that the map $\varepsilon \mapsto \mathrm{Sz}(X, \varepsilon)$ is submultiplicative (see for instance [13]). So, if $\mathrm{Sz}(X) > \omega$, then $\mathrm{Sz}(X, \varepsilon) > \omega$ for any $\varepsilon \in (0,1)$. Therefore, in the above choice of $(y_s^*)_{s \in T_N}$ we can take $\varepsilon' = \frac{1}{3}$.
By considering $z_s^* = y_s^* - y^*_{s^-}$ for $s \ne \emptyset$, $z^*_\emptyset = y^*_\emptyset$ and re-scaling, this is clearly equivalent to the existence, for all $N \in \mathbb{N}$ of $(z_s^*)_{s \in T_N}$ in X^* so that

- $\forall s \in T_N \setminus \{\emptyset\}$, $\|z_s^*\| \ge 1$,
- $\forall s \in T_{N-1}$, $z^*_{s \frown n} \overset{w*}{\to} 0$,
- $\forall s \in T_N$, $\|\sum_{t \le s} z_t^*\| \le 3$.

In our next proposition, we improve the above statement by constructing an almost biorthogonal system associated with $(z_s^*)_{s \in T_N}$.

Proposition 2.2. *Let X be a separable Banach space. If $Sz(X) > \omega$, then for all $N \in \mathbb{N}$ and $\delta > 0$ there exist $(x_s^*)_{s\in T_N}$ in X^* and $(x_s)_{s\in T_N}$ in B_X such that*

- $\forall s \in T_{N-1},\ \ x^*_{s\frown n} \overset{w*}{\to} 0$,
- $\forall s \in T_N \setminus \{\emptyset\},\ \ \|x_s^*\| \geq 1$ *and* $\forall s \in T_N,\ \ \|\sum_{t\leq s} x_t^*\| \leq 3$,
- $\forall s \in T_N,\ \ x_s^*(x_s) \geq \frac{1}{3}\|x_s^*\|$,
- $\forall s \neq t,\ \ |x_s^*(x_t)| < \delta$.

Proof. Let $f : \mathbb{N} \to T_N$ be a bijection such that

$$\forall s < t \in T_N \quad f^{-1}(s) < f^{-1}(t)$$

and

$$\forall s \in T_{N-1} \ \ \forall n < m \in \mathbb{N} \ \ f^{-1}(s \frown n) < f^{-1}(s \frown m).$$

Denote $s_i = f(i)$. In particular, $\emptyset = s_1$.
We now build inductively a tree isomorphism $\Phi : T_N \to \Phi(T_N) \subset T_N$ and a family $(z_{\Phi(s)})_{s\in T_N}$ in B_X such that

$$z^*_{\Phi(s)}(z_{\Phi(s)}) \geq \frac{1}{3},\ \ s \in T_N \text{ and } |z^*_{\Phi(s)}(z_{\Phi(t)})| < \delta,\ \ s \neq t \in T_N. \tag{2.1}$$

So set $\Phi(\emptyset) = \emptyset$, pick $z_{\Phi(\emptyset)}$ in B_X so that $z^*_{\Phi(\emptyset)}(z_{\Phi(\emptyset)}) \geq \frac{1}{3}\|z^*_{\Phi(\emptyset)}\|$ and assume that $\Phi(s_1), \dots, \Phi(s_k)$ and $z_{\Phi(s_1)}, \dots, z_{\Phi(s_k)}$ have been constructed accordingly to (2.1). Then, there exists $i \in \{1, \dots, k\}$ and $p \in \mathbb{N}$ such that $s_{k+1} = s_i \frown p$. Since $(z^*_{\Phi(s_i)\frown n})_{n\geq 1}$ is a weak*-null sequence, Lemma 2.1 insures that we can pick $n \in \mathbb{N}$ and $z_{\Phi(s_i)\frown n}$ in B_X such that $|z^*_{\Phi(s_i)\frown n}(z_{\Phi(s_j)})| < \delta$ for all $j \leq k$, $z^*_{\Phi(s_j)}(z_{\Phi(s_i)\frown n}) = 0$ for all $j \leq k$ and $z^*_{\Phi(s_i)\frown n}(z_{\Phi(s_i)\frown n}) \geq \frac{1}{3}$. We now set $\Phi(s_{k+1}) = \Phi(s_i) \frown n$. If n is chosen large enough all the required properties, including those needed for making Φ a tree isomorphism, are satisfied.
We conclude the proof by setting $x_s^* = z^*_{\Phi(s)}$ and $x_s = z_{\Phi(s)}$, for s in T_N.

□

We shall improve progressively our embedding results and start with the following.

Proposition 2.3. *There is a universal constant $C \geq 1$ such that, whenever X is a separable Banach space with $Sz(X) > \omega$, we have that*

$$\forall N \in \mathbb{N} \quad T_N \overset{C}{\hookrightarrow} X \ \ \textit{and} \ \ T_N \overset{C}{\hookrightarrow} X^*.$$

Proof. Let $(x_s^*, x_s)_{s\in T_N}$ be the system given by Proposition 2.2. Our choice of δ, will be specified later.

We shall first embed the T_N's into X. For that purpose, we mimic the natural embedding of T_N into $\ell_1(T_N)$ (with $(x_t)_{t\in T_N}$ playing the role of the canonical basis of $\ell_1(T_N)$) and define $F: T_N \to X$ by

$$\forall s \in T_N \quad F(s) = \sum_{t\leq s} x_t.$$

Since $(x_t)_{t\in T_N} \subset B_X$, we clearly have that F is 1-Lipschitz for the metric ρ on T_N.

Let now $s \neq s'$ in T_N and let u be their greatest common ancestor. Denote $d = \rho(u,s)$ and $d' = \rho(u,s')$. Recall that $\rho(s,s') = d + d'$ and assume for instance that $d \geq d'$. Then

$$\langle \sum_{t\leq s} x_t^*, F(s) - F(s')\rangle \geq \frac{1}{3}d - \delta|s|(d+d') \geq \frac{d}{3} - 2N^2\delta \geq \frac{1}{4}d \geq \frac{1}{8}\rho(s,s'),$$

if δ was chosen less than $\frac{1}{24N^2}$.

Since $\|\sum_{t\leq s} x_t^*\| \leq 3$, we obtain that for all s, s' in T_N:

$$\|F(s) - F(s')\| \geq \frac{1}{24}\rho(s,s').$$

This finishes the proof of our first embedding result.

We now turn to the question of embedding the T_N's into X^*.

Our construction will copy the natural embedding of T_N into $c_0(T_N)$, with $(x_t^*)_{t\in T_N}$ replacing the canonical basis of $c_0(T_N)$. For $s \in T_N$, we denote $y_s^* = \sum_{t\leq s} x_t^*$. Then we define $G: T_N \to X^*$ by

$$\forall s \in T_N \quad G(s) = \sum_{t\leq s} y_t^*.$$

Since $(y_t^*)_{t\in T_N}$ is a subset of $3B_{X^*}$, it is immediate that G is 3-Lipschitz.

Let now $s \neq s'$ in T_N and denote again by u their greatest common ancestor, $d = \rho(u,s)$ and $d' = \rho(u,s')$. Assume for instance that $d \geq d'$. Let us name v the unique successor of u such that $v \leq s$ and w the unique successor of u such that $w \leq s'$ if it exists. Then

$$G(s) - G(s') = \sum_{v\leq t\leq s} y_t^* - \sum_{w\leq t\leq s'} y_t^*.$$

If $s' \leq s$, $[w,s']$ is empty. Otherwise,

$$\forall t \in [w,s'] \quad |\langle x_v, y_t^*\rangle| \leq \delta|t| \leq \delta N.$$

On the other hand

$$\forall t \in [v,s] \quad |\langle x_v, y_t^*\rangle| \geq \frac{1}{3} - \delta(|t|-1) \geq \frac{1}{3} - \delta N.$$

The two previous inequalities yield

$$\|G(s) - G(s')\| \geq |\langle x_v, G(s) - G(s')\rangle| \geq \frac{d}{3} - 2\delta N^2 \geq \frac{d}{4} \geq \frac{1}{8}\rho(s,s'),$$

if δ was chosen in $(0, \frac{1}{24N^2})$. This concludes our argument for the second embedding.

□

Remark 1. Let us just finally notice that in both cases we proved the statement for $C = 24$, but our argument allows us to get the result for any constant $C > 8$.

Remark 2. The end of this section will be devoted to various improvements of Proposition 2.3, which are not fully needed in order to read the last two sections.

We now turn to the problem of embedding T_∞. We shall refine our arguments in order to improve Proposition 2.3 and obtain:

Theorem 2.4. *There is a constant $C \geq 1$ such that for any separable Banach space X satisfying $Sz(X) > \omega$, we have*

$$T_\infty \overset{C}{\hookrightarrow} X \quad \text{and} \quad T_\infty \overset{C}{\hookrightarrow} X^*.$$

Although this statement implies our previous results, we have chosen to separate its proof in the hope of making it easier to read.

Proof. So assume that $Sz(X) > \omega$ and fix a decreasing sequence $(\delta_i)_{i=0}^\infty$ in $(0,1)$. By combining the technique of Proposition 2.2 and a proper enumeration of $\bigcup_{i=0}^\infty \{i\} \times T_{2^i}$, one can actually build for every $i \geq 0$: $(x^*_{i,s})_{s\in T_{2^i}}$ in X^* and $(x_{i,s})_{s\in T_{2^i}}$ in B_X such that

(i) $\forall i \geq 0,\ \forall s \in T_{2^i-1},\quad x^*_{i,s\frown n} \overset{w*}{\to} 0$,
(ii) $\forall i \geq 0,\ \forall s \in T_{2^i} \setminus \{\emptyset\},\quad \|x^*_{i,s}\| \geq 1$ and $\forall s \in T_{2^i},\ \|\sum_{t\leq s} x^*_{i,t}\| \leq 3$,
(iii) $\forall i \geq 0,\ \forall s \in T_{2^i},\quad x^*_{i,s}(x_{i,s}) \geq \frac{1}{3}\|x^*_{i,s}\|$,
(iv) $\forall (i,s) \neq (j,t),\quad |x^*_{i,s}(x_{j,t})| < \delta_i$.

Let us just emphasize the fact that the whole system $(x_{i,s}, x^*_{i,s})_{(i,s)}$ is almost biorthogonal. We wish also to note that the estimate given in (iv) depends only on i. This last fact relies on a careful application of Lemma 2.1.
For $i \geq 0$, we denote by F_i a translate of the map defined on $T_{2^{i+1}}$ in the proof of Proposition 2.3. So let

$$F_i(\emptyset) = 0 \ \text{ and } \ F_i(s) = \sum_{\emptyset < t \leq s} x_{i+1,t} \quad s \in T_{2^{i+1}} \setminus \{\emptyset\}.$$

Now we adopt the gluing technique introduced in [1] and also used in [2] and build our embedding as follows. For $s \in T_\infty \setminus \{\emptyset\}$ there exists $k \geq 0$ such that $2^k \leq |s| < 2^{k+1}$. We define

$$F(s) = \lambda_s F_k(s) + (1-\lambda_s) F_{k+1}(s), \quad \text{where} \quad \lambda_s = \frac{2^{k+1} - |s|}{2^k}.$$

Of course, we set $F(\emptyset)=0$. We clearly have that for all $s\in T_\infty$, $\|F(s)\|\le |s|$ and following the proof of Theorem 2.1 in [2] that F is 9-Lipschitz. Consider now $s\neq s'\in T_\infty\setminus\{\emptyset\}$ and assume for instance that $1\le |s'|\le |s|$. Let $2^k\le |s'|\le 2^{k+1}$ and $2^l\le |s|\le 2^{l+1}$, with $k\le l$. Then,

$$\begin{aligned} F(s)-F(s') =& \lambda_s\sum_{t\le s}x_{l+1,t}+(1-\lambda_s)\sum_{t\le s}x_{l+2,t}\\ &-\left(\lambda_{s'}\sum_{t\le s'}x_{k+1,t}+(1-\lambda_{s'})\sum_{t\le s'}x_{k+2,t}\right).\end{aligned}$$

Let u be the greatest common ancestor of s and s' and let $d=\rho(u,s)$ as before.
If we denote $(*)=\langle \sum_{u<t\le s}(x^*_{l+1,t}+x^*_{l+2,t}),F(s)-F(s')\rangle$, we get

$$\begin{aligned}(*)\ge& \lambda_s\frac{d}{3}+(1-\lambda_s)\frac{d}{3}\\ &-\delta_{l+1}(\lambda_s d(|s|-1)+(1-\lambda_s)d|s|+\lambda_{s'}d|s'|+(1-\lambda_{s'})d|s'|)\\ &-\delta_{l+2}((1-\lambda_s)d(|s|-1)+\lambda_s d|s|+\lambda_{s'}d|s'|+(1-\lambda_{s'})d|s'|)\\ \ge& \frac{d}{3}-2d|s|(\delta_{l+1}+\delta_{l+2})\\ \ge& \frac{d}{3}-2\cdot 2^{2l+2}(\delta_{l+1}+\delta_{l+2})\ge\frac{d}{4}\ge\frac{\rho(s,s')}{8},\end{aligned}$$

if the δ_i's were chosen small enough.
Since $\|\sum_{t\le s}x^*_{i,t}\|\le 3$ for all $i\ge 0$, we obtain the following lower bound

$$\|F(s)-F(s')\|\ge\frac{\rho(s,s')}{96}.$$

If $s'=\emptyset\neq s'$, the argument is similar but simpler. This concludes our proof.

In order to embed T_∞ into X^*, we use exactly the same technique. For $i\ge 0$ and $s\in T_{2^i}$ denote $y^*_{i,s}=\sum_{t\le s}x^*_{i,t}$ and

$$G_i(\emptyset)=0\ \text{ and }\ G_i(s)=\sum_{\emptyset<t\le s}y^*_{i+1,t},\quad s\in T_{2^{i+1}}\setminus\{\emptyset\}.$$

Then again, we set $G(\emptyset)=0$ and for $s\in T_\infty\setminus\{\emptyset\}$:

$$G(s)=\lambda_s G_k(s)+(1-\lambda_s)G_{k+1}(s).$$

Following again the proof in [2], we obtain first that G is 27-Lipschitz.
Consider now $s\neq s'\in T_\infty$ such that for instance $0\le |s'|\le |s|$, $2^l\le |s|\le 2^{l+1}$ and $2^k\le |s'|\le 2^{k+1}$ with $k\le l$ or $s'=\emptyset$. Let u be the greatest common ancestor of s and s' and v be the successor of u such that $v\le s$. In a very

similar way, by evaluating $\langle x_{l+1,v}+x_{l+2,v}, G(s)-G(s')\rangle$, we can show that a proper choice for the δ_i's implies that

$$\|G(s)-G(s')\| \geq \frac{\rho(s,s')}{16}.$$

This concludes the proof of this proposition.

□

We will now study the condition "$\mathrm{Sz}(X^*)>\omega$". We already know that if $\mathrm{Sz}(X^*)>\omega$, then T_∞ Lipschitz embeds into X^{**} and therefore, when X is reflexive, T_∞ Lipschitz embeds into X. We will show how to drop the reflexivity assumption in this statement. As before, we start with finite trees.

Proposition 2.5. *There is a universal constant $C \geq 1$ such that, whenever X is a separable Banach space with $Sz(X^*)>\omega$, we have that*

$$\forall N \in \mathbb{N}, \quad T_N \overset{C}{\hookrightarrow} X.$$

Proof. If X^* is non separable, then $\mathrm{Sz}(X)>\omega$ and our problem is settled by Proposition 2.3. Thus we assume that X^* is separable. Then, for a given positive integer N and a given $\delta>0$, Proposition 2.2 provides us with $(x_s^*)_{s\in T_N}$ in B_{X^*} and $(x_s^{**})_{s\in T_N}$ in X^{**} such that

- $\forall s \in T_{N-1},\ \ x^{**}_{s\frown n} \overset{w^*}{\to} 0,$
- $\forall s \in T_N\setminus\{\emptyset\},\ \ \|x_s^{**}\|\geq 1$ and $\ \forall s\in T_N,\ \|\sum_{t\leq s} x_t^{**}\|\leq 3,$
- $\forall s\in T_N,\ \ x_s^{**}(x_s^*)\geq \frac{1}{3}\|x_s^{**}\|,$
- $\forall s\neq t,\ \ |x_s^{**}(x_t^*)|<\delta.$

Let $\{s_i,\ i\in\mathbb{N}\}$ be an enumeration of $\{s\in T_N,\ |s|=N\}$ and let $\mathcal{B}_i=\{t\in T_N,\ t\leq s_i\}$ be the corresponding branches of T_N.

For $s\in T_N$ denote $y_s^{**}=\sum_{t\leq s}x_t^{**}$.

Let us now fix $\eta>0$. Agreeing that $\mathcal{B}_0$ is the empty set, for a given $s\in T_N$, there is a unique $i=i_s\in\mathbb{N}$ such that $s\in\mathcal{B}_{i_s}\setminus\mathcal{B}_{i_s-1}$. Then, we can pick y_s in X so that

$$\|y_s\|\leq 3 \quad \text{and} \quad \forall t\in\bigcup_{j=1}^{i_s}\mathcal{B}_j\ \ |\langle x_t^*, y_s^{**}-y_s\rangle|<\eta. \tag{2.2}$$

In particular

$$\forall t\leq s \quad |\langle x_t^*, y_s^{**}-y_s\rangle|<\eta. \tag{2.3}$$

We now define $G: T_N\to X$ by

$$\forall s\in T_N \ \ G(s)=\sum_{t\leq s} y_t.$$

Since $(y_t)_{t\in T_N}$ is a subset of $3B_X$, it is immediate that G is 3-Lipschitz.
Let now $s \neq s'$ in T_N and denote again by u their greatest common ancestor, $d = \rho(u,s)$ and $d' = \rho(u,s')$, v the successor of u so that $v \leq s$ and w the successor of u so that $w \leq s'$, if they exist.
Assume first that s and s' are comparable and for instance that $s' \leq s$. Then $u = s'$, v exists, w does not and by (2.3)

$$\langle x_v^*, G(s) - G(s')\rangle \geq \langle x_v^*, \sum_{v\leq t\leq s} y_t^{**}\rangle - \eta d \geq \frac{1}{4}d, \tag{2.4}$$

for δ and η chosen small enough.
Suppose now that s and s' are not comparable. Then v and w are defined and not comparable. Therefore $i_v \neq i_w$. For instance $i_v < i_w$. We will then consider two cases.
(a) If $d' \geq 24d$. Then $\|G(s) - G(s')\| \geq \|\sum_{u<t\leq s'} y_t\| - 3d$. From (2.4) it follows that

$$\|G(s) - G(s')\| \geq \frac{1}{4}d' - 3d \geq \frac{1}{8}d' \geq \frac{1}{16}\rho(s,s').$$

(b) Assume now that $d' < 24d$.
We clearly have that for all t in $[v,s]\cup[w,s']$, $i_t \geq i_v$ and therefore it follows from (2.2)

$$\forall t \in [v,s]\cup[w,s'],\quad |\langle x_v^*, y_t^{**} - y_t\rangle| < \eta.$$

It follows that

$$\|G(s) - G(s')\| \geq \langle x_v^*, \sum_{v\leq t\leq s} y_t^{**} - \sum_{w\leq t\leq s'} y_t^{**}\rangle - (d+d')\eta \geq \frac{1}{4}d \geq \frac{1}{100}\rho(s,s'),$$

if δ and η were beforehand carefully chosen small enough.

□

We now state the last result of this section.

Theorem 2.6. *There is a universal constant $C \geq 1$ such that, whenever X is a separable Banach space with $Sz(X^*) > \omega$, we have that $T_\infty \overset{C}{\hookrightarrow} X$.*

Proof. Again, we may directly assume that X^* is separable. The gluing argument that we used before to embed T_∞ does not seem to be efficient in this case. We shall develop another technique. Fix first an integer $K \geq 2$. Then choose a decreasing sequence $(\delta_i)_i$ in $(0,1)$. Assuming that $\mathrm{Sz}(X^*) > \omega$ we can build

$(x_{i,s}^{**})_{s\in T_{K^i+1}}$ in X^{**} and $(x_{i,s}^*)_{s\in T_{K^i+1}}$ in B_{X^*} such that

- $\forall i \geq 0,\ \forall s \in T_{K^i},\quad x_{i,s\frown n}^{**} \overset{w^*}{\to} 0,$
- $\forall i \geq 0,\ \forall s \in T_{K^i+1}\setminus\{\emptyset\},\ \|x_{i,s}^{**}\| \geq 1$ and $\forall s \in T_{K^i+1},\ \|\sum_{t\leq s} x_{i,t}^{**}\| \leq 3,$

- $\forall i \geq 0,\ \forall s \in T_{K^i+1},\quad x^{**}_{i,s}(x^*_{i,s}) \geq \frac{1}{3}\|x^{**}_{i,s}\|$,
- $\forall (i,s) \neq (j,t),\quad |x^{**}_{i,s}(x^*_{j,t})| < \delta_i$.

For s in T_{K^i+1}, we define $y^{**}_{i,s} = \sum_{t\leq s} x^{**}_{i,s}$.
Let $N_i = \sum_{k=0}^{i} K^k$, choose an enumeration $\{s^i_r,\ r \in \mathbb{N}\}$ of $\{s \in T_{N_i},\ |s| = N_i\}$ and denote $\mathcal{B}^i_r = \{t \in T_{N_i}, t \leq s^i_r\}$ the branch of T_{N_i} whose endpoint is s^i_r. We will also use an enumeration $\{t^i_r,\ r \in \mathbb{N}\}$ of the terminal nodes of T_{K^i+1} and the corresponding branches $\mathcal{C}^i_r = \{t \in T_{K^i+1},\ t \leq t^i_r\}$.

Let us first describe the general idea. We set $G(\emptyset) = 0$. Consider now $s \in T_\infty \setminus \{\emptyset\}$. Then, there exists $n \in \mathbb{N}$ and $s_0, \ldots, s_n$ in T_∞ such that $|s_j| = K^j$ for $j \leq n-1$, $1 \leq |s_n| \leq K^n$ and $s = s_0 \frown \ldots \frown s_n$. For $j \leq n-1$, $s_0 \frown \ldots \frown s_j$ is a terminal node of T_{N_j} that we denote $s^j_{r_j}$. We shall now define

$$G(s) = \sum_{\emptyset < t \leq s_0} y_{t,0} + \ldots + \sum_{r_{j-1} \leq t \leq r_{j-1} \frown s_j} y_{t,j} + \ldots + \sum_{r_{n-1} \leq t \leq r_{n-1} \frown s_n} y_{t,n}$$

where $y_{t,j}$ is a proper weak*-approximation of $y^{**}_{t,j}$.

We now detail the rather technical construction of the $y_{t,j}$'s.
So let $s = (s(1), \ldots, s(k)) \in T_{K^i+1} \setminus \{\emptyset\}$. We recall that $s^{i-1}_{s(1)}$ is the $s(1)^{\text{th}}$ terminal node of $T_{N_{i-1}}$. So it can be written $s^{i-1}_{s(1)} = s_0 \frown \ldots \frown s_{i-1}$, with $|s_j| = K^j$ for $j \leq i-1$. Then, for any $j \leq i-1$, $s_0 \frown \ldots \frown s_j$ is a terminal node of T_{N_j} that we denote $s^j_{r_j}$. Besides, $r_{j-1} \frown s_j$ is a terminal node of T_{K^j+1} that we denote $t^j_{k_j}$. Let also $k_i \in \mathbb{N}$ be such that $s \in \mathcal{C}^i_{k_i} \setminus \bigcup_{k=1}^{k_i-1} \mathcal{C}^i_k$. Then we pick $y_{s,i}$ in $3B_X$ satisfying the following conditions:

$$\forall j \leq i \ \ \forall t \in \bigcup_{k=1}^{k_j} \mathcal{C}^j_k \quad |\langle y^{**}_{s,i} - y_{s,i}, x^*_{t,j}\rangle| \leq \delta_i \tag{2.5}$$

Since any $y_{s,i}$ belongs to $3B_X$, it is clear that G is 3-Lipschitz.

We now start a discussion to prove that G^{-1} is Lipschitz. So let $s \neq s'$ in $T_\infty \setminus \{\emptyset\}$ and n, m non negative integers so that $N_{n-1} < |s| \leq N_n$ and $N_{m-1} < |s'| \leq N_m$ (with the convention $N_{-1} := 0$). As usual, u is the greatest common ancestor of s and s' and we denote p the integer such that $N_{p-1} < |u| \leq N_p$, $d = \rho(u,s)$ and $d' = \rho(u,s')$. So we can write $s = s_0 \frown \ldots \frown s_n$, $s' = s'_0 \frown \ldots \frown s'_m$ and $u = u_0 \frown \ldots \frown u_p$, with $|s_j| = K^j$ for $j \leq n-1$, $0 < |s_n| \leq K^n$, $|s'_j| = K^j$ for $j \leq m-1$, $0 < |s'_m| \leq K^m$, $|u_j| = K^j$ for $j \leq p-1$ and $0 < |u_p| \leq K^p$. Then, we have that $u_j = s_j = s'_j$ for $j \leq p-1$ and that u_p is the greatest common ancestor of s_p and s'_p in T_{K^p}. Finally, if we denote $s_0 \frown \ldots \frown s_j = s^j_{r_j}$ for $j \leq n-1$

and $s'_0 \frown \ldots \frown s'_j = s^j_{r'_j}$ for $j \leq m-1$, we can write

$$G(s) - G(s') = \sum_{r_{p-1} \frown u_p < t \leq r_{p-1} \frown s_p} y_{t,p} + \ldots + \sum_{r_{n-1} \leq t \leq r_{n-1} \frown s_n} y_{t,n}$$
$$- \sum_{r_{p-1} \frown u_p < t \leq r_{p-1} \frown s'_p} y_{t,p} - \ldots - \sum_{r'_{m-1} \leq t \leq r'_{m-1} \frown s'_m} y_{t,m}.$$

a) Assume first that $n \geq m+2$.
Denote $x^* = x^*_{r_{n-2},n-1}$. Then

$$\begin{aligned}
&\|G(s) - G(s')\| \\
&\geq \langle x^*, \sum_{r_{n-2} \leq t \leq r_{n-2} \frown s_{n-1}} y_{t,n-1} + \sum_{r_{n-1} \leq t \leq r_{n-1} \frown s_n} y_{t,n} \rangle - 6N_{n-2} \\
&\geq \langle x^*, \sum_{r_{n-2} \leq t \leq r_{n-2} \frown s_{n-1}} y^{**}_{t,n-1} + \sum_{r_{n-1} \leq t \leq r_{n-1} \frown s_n} y^{**}_{t,n} \rangle - \delta_{n-1}(K^{n-1} + K^n) - 6N_{n-2} \\
&\geq \frac{1}{3}K^{n-1} - \delta_{n-1}\big(K^{n-1} + K^n + K^{n-1}K^{n-1} + K^n(K^n+1)\big) - 6N_{n-2} \geq \frac{1}{4}K^{n-1},
\end{aligned}$$

if K was chosen big enough and the δ_n's small enough.
In that case $\rho(s,s') \leq 2N_n$. So

$$\|G(s) - G(s')\| \geq \frac{\rho(s,s')}{L},$$

where L is a constant depending only on K.

b) Assume that $n = m+1$ and $m = p$.
Denote $x^* = x^*_{r_{n-1},n}$, $a = |s_n|$ and $b = K^{n-1} - |u_{n-1}|$. Notice that $a + b = d$ and $d' \leq b$. Then

$$\langle x^*, G(s) - G(s') \rangle \geq \frac{a}{4} - 3b - 3d' \geq \frac{a}{4} - 6b, \tag{2.6}$$

if the δ_n's were chosen small enough.
Let v_{n-1} be the successor of u_{n-1} so that $v_{n-1} \leq s_{n-1}$ and w_{n-1} be the successor of u_{n-1} so that $w_{n-1} \leq s'_{n-1}$. Denote now $y^* = x^*_{r_{n-2} \frown v_{n-1}}$ and $z^* = x^*_{r_{n-2} \frown w_{n-1}}$.
Assume first that there exists an integer k such that $r_{n-2} \frown v_{n-1} \in \mathcal{C}^{n-1}_k$ and $r_{n-2} \frown w_{n-1} \notin \bigcup_{l=1}^k \mathcal{C}^{n-1}_l$. Then, for small enough δ_n's

$$\langle y^*, G(s) - G(s') \rangle \geq \frac{b}{4}. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\langle x^* + 25y^*, G(s) - G(s') \rangle \geq \frac{a+b}{4} \geq \frac{d+d'}{8}.$$

Thus

$$\|G(s)-G(s')\| \geq \frac{\rho(s,s')}{208}.$$

Assume now that there exists an integer k such that $r_{n-2} \frown w_{n-1} \in \mathcal{C}_k^{n-1}$ and $r_{n-2} \frown v_{n-1} \notin \bigcup_{l=1}^k \mathcal{C}_l^{n-1}$. Then, still for small δ_n's,

$$\langle y^*, G(s)-G(s')\rangle \geq \frac{b}{4} - 3d' \quad \text{and} \quad \langle z^*, G(s)-G(s')\rangle \geq \frac{d'}{4}.$$

It follows from the above and (2.6) that

$$\langle x^* + 25y^* + 301z^*, G(s)-G(s')\rangle \geq \frac{d+d'}{4} \quad \text{and} \quad \|G(s)-G(s')\| \geq \frac{\rho(s,s')}{1308}.$$

c) Assume that $n = m+1$ and $p \leq m-1$.
Denote $x^* = x^*_{r_{n-1},n}$, $y^* = x^*_{r_{n-2},n-1}$ and $z^* = x^*_{r'_{n-2},n-1}$. Note that $y^* \neq z^*$. We also denote $a = |s_n|$, $b = |s_{n-1}| = K^{n-1}$, $b' = |s'_{n-1}|$ and
$c = |s_0 \frown \ldots \frown s_{n-2}| - |u| = |s'_0 \frown \ldots \frown s'_{n-2}| - |u|$.
First, we have that for small enough δ_n's

$$\langle x^*, G(s)-G(s')\rangle \geq \frac{a}{4} - 3b - 3b' - 6c \geq \frac{a}{4} - 6b - 6c. \tag{2.8}$$

Assume first that there exists an integer k such that $r_{n-2} \in \mathcal{C}_k^{n-1}$ and $r'_{n-2} \notin \bigcup_{l=1}^k \mathcal{C}_l^{n-1}$. Then, for small enough δ_n's

$$\langle y^*, G(s)-G(s')\rangle \geq \frac{b}{4} - 6c.$$

This, together with (2.8) yields

$$\langle x^* + 25y^*, G(s)-G(s')\rangle \geq \frac{a+b}{4} - 156c.$$

A previous choice of a big enough K insures in this situation that

$$\frac{a+b}{4} - 156c \geq \frac{\rho(s,s')}{10} - 156c \geq \frac{\rho(s,s')}{20}.$$

Therefore

$$\|G(s)-G(s')\| \geq \frac{\rho(s,s')}{520}.$$

Otherwise, there exists an integer k such that $r'_{n-2} \in \mathcal{C}_k^{n-1}$ and $r_{n-2} \notin \bigcup_{l=1}^k \mathcal{C}_l^{n-1}$. Then a proper choice for the δ_n's yields

$$\langle y^*, G(s)-G(s')\rangle \geq \frac{b}{4} - 3b' - 6c \quad \text{and} \quad \langle z^*, G(s)-G(s')\rangle \geq \frac{b'}{4} - 6c \tag{2.9}$$

¿From (2.8) and (2.9) we deduce

$$\langle x^* + 25y^* + 300z^*, G(s)-G(s')\rangle \geq \frac{a+b}{4} - 1956c.$$

Again, our starting choice of a very large K will insure the existence of a universal constant L so that in this situation

$$\|G(s) - G(s')\| \geq \frac{\rho(s,s')}{L}.$$

d) Assume that $n = m = p$. We just have to follow the proof of Proposition 2.5

e) Assume that $n = m$ and $p \leq n-2$.
Denote $y^* = x^*_{r_{n-2},n-1}$, $z^* = x^*_{r'_{n-2},n-1}$, $a = |s_n|$, $a' = |s'_n|$, $b = |s_{n-1}| = |s'_{n-1}| = K^{n-1}$ and $c = |s_0 \frown \ldots \frown s_{n-2}| - |u| = |s'_0 \frown \ldots \frown s'_{n-2}| - |u|$.
It follows from the condition (2.5) and a proper choice of the δ_n's that

$$\text{either} \quad \langle y^*, G(s) - G(s')\rangle \geq \frac{b}{4} - 6c \quad \text{or} \quad \langle z^*, G(s) - G(s')\rangle \geq \frac{b}{4} - 6c.$$

If K was chosen big enough we then obtain that

$$\|G(s) - G(s')\| \geq \frac{K^{n-1}}{8} \geq \frac{\rho(s,s')}{L},$$

for some universal constant L.

f) Finally assume that $n = m$ and $p = n-1$.
Let v_{n-1} be the successor of u_{n-1} so that $v_{n-1} \leq s_{n-1}$ and w_{n-1} be the successor of u_{n-1} so that $w_{n-1} \leq s'_{n-1}$. Denote now $x^* = x^*_{r_{n-1},n}$, $y^* = x^*_{r_{n-2}\frown v_{n-1}}$ and $z^* = x^*_{r_{n-2}\frown w_{n-1}}$. We also denote $|a| = |s_n|$ and $b = |s_0 \frown \ldots \frown s_{n-1}| - |u| = |s'_0 \frown \ldots \frown s'_{n-1}| - |u|$.
First, we have

$$\|G(s) - G(s')\| \geq \|G(s') - G(u)\| - 3d \geq \alpha d' - 3d,$$

where $\alpha \in (0,1)$ is a universal constant given by case (b). If $d' \geq Md$, with $M = \dfrac{6}{\alpha}$, we obtain that

$$\|G(s) - G(s')\| \geq \frac{\alpha}{2} d' \geq \frac{\alpha \rho(s,s')}{4}.$$

So, we may as well assume that $d' < Md$. Now, with our usual careful choice of small δ_n's we get

$$\langle x^*, G(s) - G(s')\rangle \geq \frac{a}{4} - 6b \quad \text{and}$$

$$\text{either} \quad \langle y^*, G(s) - G(s')\rangle \geq \frac{b}{4} \quad \text{or} \quad \langle z^*, G(s) - G(s')\rangle \geq \frac{b}{4}.$$

Then, using $x^* + 25y^*$ or $x^* + 25z^*$, we obtain that

$$\|G(s) - G(s')\| \geq \frac{d}{104} = \frac{(M+1)d}{104(M+1)} \geq \frac{d + d'}{104(M+1)} = \frac{\rho(s,s')}{104(M+1)}.$$

All possible cases have been considered and our discussion is finished.

□

3. On the non-embeddability of the hyperbolic trees

Our aim is now to prove in the reflexive case the converse of the results given in the previous section. More precisely, the main result of this section is the following.

Theorem 3.1. *Assume that X is a separable reflexive Banach space and that there exists $C \geq 1$ such that $T_N \overset{C}{\hookrightarrow} X$ for all N in $\mathbb{N}$.*
Then either $Sz(X) > \omega$ or $Sz(X^*) > \omega$.

Before proceeding with the proof of this theorem, we need to recall two very convenient renorming theorems essentially due to Odell and Schlumprecht. We refer to [19] and [20] for a complete exposition of the links between the Szlenk index of a Banach space and its embeddability into a Banach space with a finite dimensional decomposition with upper and lower estimates.

Theorem 3.2. *Let X be a separable reflexive Banach space. Then, the following properties are equivalent.*

(i) $Sz(X) \leq \omega$.

(ii) There exist $1 < p < \infty$ and an equivalent norm $\|\cdot\|$ on X such that if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, $x \in X$ and $(x_n)_{n=1}^\infty$ is any bounded sequence with $\lim_{n\in\mathcal{U}} x_n = 0$ weakly

$$\lim_{n\in\mathcal{U}} \|x + x_n\| \leq \lim_{n\in\mathcal{U}} (\|x\|^p + \|x_n\|^p)^{1/p}. \tag{3.1}$$

This is contained in the proof of Theorem 3 of [20].

Theorem 3.3. *Let X be a separable reflexive Banach space. Then, the following properties are equivalent.*

(i) $Sz(X) \leq \omega$ and $Sz(X^*) \leq \omega$.

(ii) There exist $1 < p < q < \infty$ and an equivalent norm $\|\cdot\|$ on X such that if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, $x \in X$ and $(x_n)_{n=1}^\infty$ is any bounded sequence with $\lim_{n\in\mathcal{U}} x_n = 0$ weakly

$$\lim_{n\in\mathcal{U}} (\|x\|^q + \|x_n\|^q)^{1/q} \leq \lim_{n\in\mathcal{U}} \|x + x_n\| \leq \lim_{n\in\mathcal{U}} (\|x\|^p + \|x_n\|^p)^{1/p}. \tag{3.2}$$

Let us remark that (ii) is equivalent to the statements that $\overline{\delta}(\tau) \geq (1 + \tau^q)^{1/q} - 1$ and $\overline{\rho}(\tau) \leq (1 + \tau^p)^{1/p} - 1$. This result follows directly from Theorem 7 of [20].

Proof of Theorem 3.1. Let X be a reflexive Banach space such that $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$. We will assume that the norm satisfies (3.2) and we may assume for convenience that p and q are conjugate i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Let us suppose that there is a constant $C \geq 1$ so that for every $N \in \mathbb{N}$, we have $T_N \overset{C}{\hookrightarrow} X$. We will show that for large enough N this produces a contradiction. Let us pick $a \in \mathbb{N}$ such that $a > (2C)^q$. We then pick $m \in \mathbb{N}$ with $m > (2C)^q$ and $N = a^{m+1}$.

Suppose now that $u : T_N \to X$ is a map such that $u(\emptyset) = 0$ and:

$$\forall s, s' \in T_N \quad \rho(s, s') \leq \|u(s) - u(s')\| \leq C\rho(s, s'). \tag{3.3}$$

We now consider an ultraproduct $\mathcal{X}$ of X modeled on the set $\mathbb{N}^N$; this idea is inspired by similar considerations in [15]. Let $\mathcal{U}$ be a fixed non-principal ultrafilter on $\mathbb{N}$ and define the seminorm on $Z = \ell_\infty(\mathbb{N}^N, X)$ by

$$\|x\|_{\mathcal{X}} = \lim_{n_1 \in \mathcal{U}} \cdots \lim_{n_N \in \mathcal{U}} \|x(n_1, \ldots, n_N)\|.$$

If we factor out the set $\{x : \|x\|_{\mathcal{X}} = 0\}$ this induces an ultraproduct $\mathcal{X}$.

For $x \in Z$ and $0 \leq k \leq N$ we define

$$\mathcal{E}_k(x)(n_1, \ldots, n_N) = \lim_{m_{k+1} \in \mathcal{U}} \cdots \lim_{m_N \in \mathcal{U}} x(n_1, \ldots, n_k, m_{k+1}, \ldots m_N)$$

where each limit is with respect to the weak topology on X (recall that X is reflexive). For $k < 0$ it is convenient to write $\mathcal{E}_k x = 0$. It will be useful to introduce $\mathcal{F}_k = I - \mathcal{E}_k$ for the complementary projections.

We now use (3.2) to deduce that if $\mathcal{F}_k x = 0$ and $\mathcal{E}_k y = 0$ then

$$(\|x\|_{\mathcal{X}}^q + \|y\|_{\mathcal{X}}^q)^{1/q} \leq \|x + y\|_{\mathcal{X}} \leq (\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p)^{1/p}.$$

From this it follows that the projections $\mathcal{F}_k$ are contractive. Also if $0 = k_0 < k_1 < \cdots < k_r$ and $x_j \in Z$ with $\mathcal{F}_{k_j} x_j = 0$ and $\mathcal{E}_{k_{j-1}} x_j = 0$ for $1 \leq j \leq r$ then

$$\left(\sum_{j=1}^r \|x_j\|_{\mathcal{X}}^q \right)^{1/q} \leq \| \sum_{j=1}^r x_j \|_{\mathcal{X}} \leq \left(\sum_{j=1}^r \|x_j\|_{\mathcal{X}}^p \right)^{1/p}. \tag{3.4}$$

Let us now define $z_j \in Z$ for $1 \leq j \leq N$ by

$$z_j(n_1, \ldots, n_N) = u(n_1, \ldots, n_j) - u(n_1, \ldots, n_{j-1}).$$

Here we understand that $z_1(n_1, \ldots, n_N) = u(n_1)$.

We then define $w_{j0} = z_j - \mathcal{E}_{j-1} z_j$ and then

$$w_{jk} = \mathcal{E}_{j-a^{k-1}} z_j - \mathcal{E}_{j-a^k} z_j, \qquad 1 \leq k < \infty.$$

Then

$$z_j = \sum_{k=0}^\infty w_{jk}$$

and by (3.4)

$$\begin{aligned}\sum_{k=1}^{m}\|w_{jk}\|_{\mathcal{X}} &\le m^{1/p}(\sum_{k=0}^{\infty}\|w_{jk}\|_{\mathcal{X}}^q)^{1/q}\\ &\le m^{1/p}\|z_j\|_{\mathcal{X}}\le Cm^{1/p}.\end{aligned}$$

This implies that

$$\sum_{j=1}^{N}\sum_{k=1}^{m}\|w_{jk}\|_{\mathcal{X}}\le Cm^{1/p}N. \tag{3.5}$$

On the other hand if $0\le r\le r+s\le N$ we note that by (3.3),

$$\lim_{n'_{r+1}\in\mathcal{U}}\lim_{n'_{r+2}\in\mathcal{U}}\cdots\lim_{n'_{r+s}\in\mathcal{U}}\|u(n_1,\ldots,n_r,n'_{r+1},\ldots n'_{r+s})-u(n_1,\ldots,n_{r+s})\|\ge 2s.$$

Hence if $v\in\ell_\infty(\mathbb{N}^r,X)$ we have

$$\lim_{n_{r+1}\in\mathcal{U}}\cdots\lim_{n_{r+s}\in\mathcal{U}}\|u(n_1,\ldots,n_{r+s})-v(n_1,\ldots,n_r)\|\ge s.$$

In particular if we let

$$v(n_1,\ldots,n_r)=\lim_{n_{r+1}\in\mathcal{U}}\cdots\lim_{n_{r+s}\in\mathcal{U}}u(n_1,\ldots,n_{r+s})$$

(with limits in the weak topology) we obtain

$$\|\mathcal{F}_r(\sum_{j=r+1}^{r+s}z_j)\|_{\mathcal{X}}\ge s.$$

Now suppose $s=a^k$ where $k\ge 1$. If $r\le N-a^k$ we have

$$\begin{aligned}a^k &\le \|\mathcal{F}_r(\sum_{j=r+1}^{r+a^k}z_j)\|_{\mathcal{X}}\\ &\le \|\sum_{j=r+1}^{r+a^k}\mathcal{F}_{j-a^k}z_j\|_{\mathcal{X}}.\end{aligned}$$

The last inequality follows from the fact that $\mathcal{F}_k\mathcal{F}_l=\mathcal{F}_l\mathcal{F}_k=\mathcal{F}_l$, whenever $k\le l$ and from the contractivity of $\mathcal{F}_r$.

On the other hand

$$\begin{aligned}\|\sum_{j=r+1}^{r+a^k}\mathcal{F}_{j-a^{k-1}}z_j\|_{\mathcal{X}} &= \|\sum_{j=r+1}^{r+a^{k-1}}\sum_{i=0}^{a-1}\mathcal{F}_{j+(i-1)a^{k-1}}z_{j+ia^{k-1}}\|_{\mathcal{X}}\\ &\le \sum_{j=r+1}^{r+a^{k-1}}\left(\sum_{i=0}^{a-1}\|\mathcal{F}_{j+(i-1)a^{k-1}}z_{j+ia^{k-1}}\|_{\mathcal{X}}^p\right)^{1/p}\\ &\le Ca^{k-1}a^{1/p}\\ &\le a^k/2.\end{aligned}$$

Combining these statements we have that if $r = \lambda a^k$ with $1 \le k \le m$ and $0 \le \lambda \le a^{m+1-k} - 1$ (in particular $r \le N - a^k = a^{m+1} - a^k$)

$$\sum_{j=r+1}^{r+a^k} \|w_{jk}\|_{\mathcal{X}} \ge a^k/2$$

and hence

$$\sum_{j=1}^{N} \|w_{jk}\|_{\mathcal{X}} = \sum_{\lambda=0}^{a^{m+1-k}-1} \sum_{j=\lambda a^k+1}^{(\lambda+1)a^k} \|w_{jk}\|_{\mathcal{X}} \ge \frac{N}{2}.$$

This implies

$$\sum_{j=1}^{N}\sum_{k=1}^{m} \|w_{jk}\|_{\mathcal{X}} \ge \frac{mN}{2}. \tag{3.6}$$

Now (3.5) and (3.6) give a contradiction since $m > (2C)^q$. $\square$

As an immediate consequence of Theorem 3.1 and section 2 we obtain the following characterization, which yields Theorem 1.2 announced in our introduction.

Corollary 3.4. *Let X be a separable reflexive Banach space. The following assertions are equivalent*

(i) $Sz(X) > \omega$ or $Sz(X^*) > \omega$.

(ii) There exists $C \ge 1$ such that $T_\infty \overset{C}{\hookrightarrow} X$.

(iii) There exists $C \ge 1$ such that for any N in $\mathbb{N}$, $T_N \overset{C}{\hookrightarrow} X$.

Remark. Let us mention that we do not know if (iii) implies (i) for general Banach spaces.

4. Applications to coarse Lipschitz embeddings and uniform homeomorphisms between Banach spaces

We need to recall some definitions and notation. Let (M, d) and (N, δ) be two unbounded metric spaces. We define for $f : M \to N$:

$$\forall t > 0 \quad \omega_f(t) = \sup\{\delta(f(x), f(y)),\ x, y \in M,\ d(x, y) \le t\}.$$

We say that f is *uniformly continuous* if $\lim_{t\to 0} \omega_f(t) = 0$. The map f is said to be *coarsely continuous* if $\omega_f(t) < \infty$ for some $t > 0$.
Let us now introduce

$$L_\theta(f) = \sup_{t \ge \theta} \frac{\omega_f(t)}{t}, \quad \text{for } \theta > 0$$

and

$$L(f) = \sup_{\theta > 0} L_\theta(f), \qquad L_\infty(f) = \inf_{\theta > 0} L_\theta(f).$$

A map is Lipschitz if and only if $L(f) < \infty$. We will say that it is *coarse Lipschitz* if $L_\infty(f) < \infty$. Clearly, a coarse Lipschitz map is coarsely continuous. If f is bijective, we will say that f is a *uniform homeomorphism* (respectively, *coarse homeomorphism, Lipschitz homeomorphism, coarse Lipschitz homeomorphism*) if f and f^{-1} are uniformly continuous (respectively, coarsely continuous, Lipschitz, coarse Lipschitz). Finally we say that f is a *coarse Lipschitz embedding* if it is a coarse Lipschitz homeomorphism from X onto $f(X)$.

We conclude this brief introduction with the following easy and well known fact: if X and Y are Banach spaces, then for any map $f : X \to Y$, ω_f is a subadditive function. It follows that any coarsely continuous map $f : X \to Y$ is coarse Lipschitz. In particular, any uniform homeomorphism is a coarse Lipschitz homeomorphism.

Theorem 4.1. *Let X and Y be separable Banach spaces and suppose that there is a coarse Lipschitz embedding of X into Y. Suppose Y is reflexive and $Sz\,(Y) = \omega$. Then X is reflexive.*

Proof. We can assume by Theorem 3.2 that Y is normed to satisfy (3.1) for some $1 < p < \infty$.

Now let $f : X \to Y$ be a coarse Lipschitz embedding. We may assume that there exists $C \geq 1$ such that

$$\|x_1 - x_2\| - 1 \leq \|f(x_1) - f(x_2)\| \leq C\|x_1 - x_2\| + 1 \qquad x_1, x_2 \in X.$$

Suppose that X is a non reflexive Banach space and fix $\theta \in (0,1)$. Then, James' Theorem [8] insures the existence of a sequence $(x_n)_n$ in B_X such that $\|y - z\| \geq \theta$, for all $n \in \mathbb{N}$, all y in the convex hull of $\{x_i\}_{i=1}^n$ and all z in the convex hull of $\{x_i\}_{i\geq n+1}$. In particular

$$\|x_{n_1}+..+x_{n_k}-(x_{m_1}+..+x_{m_k})\| \geq \theta k, \quad n_1 < .. < n_k < m_1 < .. < m_k. \tag{4.1}$$

For $k \in \mathbb{N}$ let $\mathbb{N}^{[k]}$ denote the collection of all k-subsets of $\mathbb{N}$ (written in the form $(n_1, \dots, n_k)$ where $n_1 < n_2 < \cdots < n_k$. We define $h : \mathbb{N}^{[k]} \to X$ by

$$h(n_1, \dots, n_k) = x_{n_1} + \cdots + x_{n_k}.$$

On $\mathbb{N}^{[k]}$ we define the distance

$$d((n_1, \dots, n_k), (m_1, \dots, m_k)) = |\{j : \ n_j \neq m_j\}|.$$

Then h is Lipschitz with constant at most 2. Furthermore $f \circ h$ has Lipschitz constant at most $2C + 1$. By Theorem 4.2 of [10] there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that diam $f \circ h(\mathbb{M}^{[k]}) \leq 3(2C + 1)k^{1/p}$. If $n_1 < n_2 < \cdots < n_k <$

$m_1 < \cdots < m_k \in \mathbb{M}$ we thus have

$$\theta k - 1 \leq \|f(x_{n_1} + \cdots + x_{n_k}) - f(x_{m_1} + \cdots + x_{m_k})\| \leq 3(2C+1)k^{1/p}.$$

For large enough k this is a contradiction. □

It is proved in [7] (Theorem 5.5) that the condition "having a Szlenk index equal to ω" is stable under uniform homeomorphisms. So we immediately deduce.

Corollary 4.2. *The class of all reflexive Banach spaces with Szlenk index equal to ω is stable under uniform homeomorphisms.*

As a final application we now state the main result of this section.

Theorem 4.3. *Let Y be a reflexive Banach space such that $Sz(Y) \leq \omega$ and $Sz(Y^*) \leq \omega$ and assume that X is a Banach space which coarse Lipschitz embeds into Y. Then X is reflexive, $Sz(X) \leq \omega$ and $Sz(X^*) \leq \omega$.*

Proof. First, it follows from Theorem 4.1 that X is reflexive. Assume now that $\mathrm{Sz}(X)$ or $\mathrm{Sz}(X^*)$ is greater than ω. Then, we know from Theorem 2.4 that T_∞ Lipschitz embeds into X and therefore into Y. This is in contradiction with Theorem 3.1. □

Remark 3. Theorem 4.1, Corollary 4.2 and Theorem 4.3 should be compared to the fact that in general reflexivity is not preserved under coarse Lipschitz embeddings or even uniform homeomorphisms. Indeed, Ribe proved in [23] that $\ell_1 \oplus (\sum_n \oplus \ell_{p_n})_{\ell_2}$ is uniformly homeomorphic to $(\sum_n \oplus \ell_{p_n})_{\ell_2}$, if $(p_n)_n$ is strictly decreasing and tending to 1 (we also refer to Theorem 10.28 in [3] for a generalization of this result). The space $X = (\sum_n \oplus \ell_{p_n})_{\ell_2}$ is of course reflexive and standard computations yield that its Szlenk index is equal to ω^2. On the other hand, if the p_n's are chosen in $(1,2]$, it is also easy to show that the natural norm of X^* is asymptotically uniformly smooth with a modulus of asymptotic smoothness $\overline{\rho}(t) = t^2$. Thus, $\mathrm{Sz}(X^*) = \omega$.

So, in view of Corollary 4.2 and Theorem 4.3, Ribe's example is optimal.

Let us now recall that for a separable Banach space the condition "$\mathrm{Sz}(X) \leq \omega$" is equivalent to the existence of an equivalent asymptotically uniformly smooth norm on X and that for a reflexive separable Banach space the condition "$\mathrm{Sz}(X^*) \leq \omega$" is equivalent to the existence of an equivalent asymptotically uniformly convex norm on X (see [20] for a survey on these results and proper references). Let us now denote as in [20]:

$$\mathcal{C}_{auc} = \{Y :\ Y \textit{ is separable reflexive and has an equivalent a.u.c. norm}\}$$

and

$\mathcal{C}_{aus} = \{Y : Y\ is\ separable\ reflexive\ and\ has\ an\ equivalent\ a.u.s.\ norm\}.$

Then, we can restate Corollary 4.2 and Theorem 4.3 as follows

Theorem 4.4. *The class $\mathcal{C}_{aus}$ is stable under uniform homeomorphisms and the class $\mathcal{C}_{auc} \cap \mathcal{C}_{aus}$ is stable under coarse Lipschitz embeddings.*

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