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Square functions and H^∞ calculus on subspaces of L^p and on Hardy spaces

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Abstract. Let X be a (closed) subspace of L^p with $1 \leq p < \infty$, and let A be any sectorial operator on X. We consider associated square functions on X, of the form $\|x\|_F = \big\|\big(\int_0^\infty \big|F(tA)x\big|^2 \frac{dt}{t}\big)^{\frac{1}{2}}\big\|_{L^p}$, and we show that if A admits a bounded H^∞ functional calculus on X, then these square functions are equivalent to the original norm of X. Then we deduce a similar result when $X = H^1(\mathbb{R}^N)$ is the usual Hardy space, for an appropriate choice of $\|\ \|_F$. For example if $N = 1$, the right choice is the sum $\|h\|_F = \big\|\big(\int_0^\infty \big|F(tA)h\big|^2 \frac{dt}{t}\big)^{\frac{1}{2}}\big\|_{L^1} + \big\|\big(\int_0^\infty \big|H\big(F(tA)h\big)\big|^2 \frac{dt}{t}\big)^{\frac{1}{2}}\big\|_{L^1}$ for $h \in H^1(\mathbb{R})$, where H denotes the Hilbert transform.

1. Introduction

Let X be a Banach space, and let A be a sectorial operator on X. In this paper we investigate relationships between H^∞ functional calculus and square functions associated with A when X is a subspace of some L^p-space, for $1 \leq p < \infty$. This includes the case when $X = H^1(\mathbb{R}^N)$ is the usual Hardy space on $\mathbb{R}^N$. Following usual convention, we let Σ_θ denote the open sector of all $z \in \mathbb{C} \setminus \{0\}$ such that $|\mathrm{Arg}(z)| < \theta$, for any angle $\theta \in (0, \pi)$. Then we let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded holomorphic functions $f\colon \Sigma_\theta \to \mathbb{C}$, and we let $H_0^\infty(\Sigma_\theta)$ denote the subalgebra of all $f \in H^\infty(\Sigma_\theta)$ for which there exists a positive number $s > 0$ such that $|f(z)| = O(|z|^{-s})$ at ∞, and $|f(z)| = O(|z|^s)$ at 0.

Let $1 \leq p < \infty$, and assume that $X = L^p(\Omega)$ for some measure space Ω. If A is sectorial of type $\omega \in (0, \pi)$ on X, and if F is a non zero function belonging to $H_0^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi)$, the associated square function is defined by

$$\|x\|_F = \left\| \left(\int_0^\infty \big|F(tA)x\big|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\Omega)} \tag{1.1}$$

for any $x \in L^p(\Omega)$. These quantities were introduced on Hilbert spaces (i.e. $p = 2$) in the early days of H^∞ functional calculus by McIntosh [15] (see also [16]), and on any L^p-space by Cowling, Doust, McIntosh, and Yagi [4]. In a recent paper [11], the second named author showed that if A is actually R-sectorial of type ω, then all these square functions are pairwise equivalent. That is, for any F and G as above, there is a positive constant $K > 0$ such that $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for any $x \in L^p(\Omega)$. Furthermore it follows from [4] and [11] that if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus and $F \in H_0^\infty(\Sigma_{\theta'}) \setminus \{0\}$ for some $\theta' > \theta$, then $\| \ \|_F$ is equivalent to the original norm on $X = L^p(\Omega)$. In Section 2 below, we will extend these equivalence results to the case when X is a (closed) subspace of $L^p(\Omega)$. In this context, the square functions will be also defined by (1.1). To study a sectorial operator A on L^p, it is often convenient to use the adjoint operator A^* and its associated square functions. Indeed in that case, A^* is a sectorial operator acting on $L^{p'}$ (if $p \neq 1$). The new difficulty appearing in the case when A acts on $X \subset L^p$ is that the dual space of X is no longer a subspace of some $L^{p'}$. Thus we do not have any convenient square functions for A^* at our disposal.

In Section 3, we will turn to Hardy spaces and will consider a sectorial operator A acting on $X = H^1(\mathbb{R}^N)$. Using a natural isometric embedding of $H^1(\mathbb{R}^N)$ into some L^1-space, we will derive equivalence results which also extend those on L^p. However the definition of square functions has to be adapted. For example if $N = 1$, they will be defined for any $h \in H^1(\mathbb{R})$ by

$$\|h\|_F = \left\| \left(\int_0^\infty \left|F(tA)h\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^1(\mathbb{R})} + \left\| \left(\int_0^\infty \left|H\big(F(tA)h\big)\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^1(\mathbb{R})},$$

where H denotes the Hilbert transform on $L^1(\mathbb{R})$. Thus we will obtain that the above $\| \ \|_F$ is an equivalent norm of $H^1(\mathbb{R})$ provided that A has a bounded H^∞ functional calculus on $H^1(\mathbb{R})$.

2. H^∞ calculus on subspaces of L^p

We shall briefly recall standard definitions and basic results on sectorial operators and their H^∞ functional calculus. For details and complements, the reader is referred to the classical papers [15,16,4,9], as well as to [17, Section 8.1] or [12].

Let X be a Banach space, and let $B(X)$ be the space of all bounded linear operators on X. Let A be a closed and densely defined linear operator on X. The domain and the spectrum of A will be denoted by $D(A)$ and $\sigma(A)$ respectively. For any $z \notin \sigma(A)$, we let $R(z, A) = (z - A)^{-1} \in B(X)$ denote the associated resolvent operator. We say that A is a sectorial operator of type $\omega \in (0, \pi)$ if A has dense range, $\sigma(A) \subset \overline{\Sigma_\theta}$, and for any $\theta \in (\omega, \pi)$, there is a constant $C_\theta \geq 0$ such that

$$\|zR(z, A)\| \leq C_\theta, \qquad z \notin \overline{\Sigma_\theta}.$$

Such an operator A is automatically one-one (see e.g. [4, Theorem 3.8]). In some circumstances, the dense range assumption is omitted in the definition of sectoriality, however it is necessary for our purposes.

For any $\gamma \in (0, \pi)$, we let Γ_γ be the boundary of Σ_γ, oriented counterclockwise. Let A be a sectorial operator of type ω, and let $\theta \in (\omega, \pi)$. For any $f \in H_0^\infty(\Sigma_\theta)$, we set

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz, \tag{2.1}$$

where $\Gamma = \Gamma_\gamma$ for some $\gamma \in (\omega, \theta)$. Then $f(A)$ is a well defined bounded operator on X, whose definition does not depend on the choice of γ. Moreover the mapping $f \mapsto f(A)$ is a homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. Let us equip $H^\infty(\Sigma_\theta)$ with the supremum norm,

$$\|f\|_{\infty,\theta} = \sup\{|f(z)| \,:\, z \in \Sigma_\theta\}, \qquad f \in H^\infty(\Sigma_\theta).$$

We say that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if there is a constant $C > 0$ such that $\|f(A)\| \leq C\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$. In that case, there is a unique way to define a bounded operator $f(A)$ for any $f \in H^\infty(\Sigma_\theta)$, such that the resulting mapping $f \mapsto f(A)$ is a bounded homomorphism, and we have

$$\|f(A)\| \leq C\|f\|_{\infty,\theta}, \qquad f \in H^\infty(\Sigma_\theta). \tag{2.2}$$

Let us recall here the definitions of R-boundedness [3] and R-sectoriality [20, 9]. Consider a Rademacher sequence $(\varepsilon_k)_{k\geq 1}$ on a probability space $(\Omega_0, \mathbb{P})$. That is, the ε_k's are pairwise independent random variables on Ω_0 such that $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$ for any $k \geq 1$. For any finite family $x_1, \ldots, x_n$ in X, we define

$$\Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(X)} = \int_{\Omega_0} \Big\|\sum_{k=1}^n \varepsilon_k(w) x_k\Big\|_X \, d\mathbb{P}(w)\,.$$

A set $\mathcal{T} \subset B(X)$ is R-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathcal{T}$, and $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_{k=1}^n \varepsilon_k\, T_k(x_k)\Big\|_{\mathrm{Rad}(X)} \leq C\Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(X)}.$$

Now if A is a sectorial operator on X, we say that A is R-sectorial of R-type $\omega \in (0, \pi)$ if for any $\theta \in (\omega, \pi)$, the set $\{zR(z, A) \,:\, z \notin \overline{\Sigma_\theta}\} \subset B(X)$ is R-bounded.

Throughout this section, we let Ω be a measure space, we let $1 \leq p < \infty$, and we assume that X is a (closed) subspace of $L^p(\Omega)$. It is well-known that there is a constant $C_0 > 0$ (only depending on p) such that

$$C_0^{-1}\Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(X)} \leq \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq C_0\Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(X)} \tag{2.3}$$

for any finite family $x_1, \ldots, x_n$ in X. (See e.g. [13, 1.d.6].)

Given a sectorial operator A of type ω on X, an angle $\theta > \omega$, and $F \in H_0^\infty(\Sigma_\theta)\setminus\{0\}$, we let $\|x\|_F$ be defined by (1.1). More precisely for any $x \in X$, we temporarily set $x_F(t) = F(tA)x$ for any $t > 0$. It is easy to check that x_F is a continuous function from $(0, \infty)$ into $X \subset L^p(\Omega)$. Then we let $\|x\|_F$ be the norm of x_F in $L^p\big(\Omega; L^2(\mathbb{R}_+^*; \frac{dt}{t})\big)$ if x_F belongs to that space, and we let $\|x\|_F = \infty$ otherwise.

The following equivalence result was established in [11] in the case when $X = L^p(\Omega)$. Its proof extends almost verbatim to the case when X is merely a subspace of L^p, hence we omit it.

Theorem 2.1. *Let X be a subspace of $L^p(\Omega)$, with $1 \leq p < \infty$, and let A be an R-sectorial operator of R-type $\omega \in (0, \pi)$ on X. Let $\theta \in (\omega, \pi)$ and let F, G be two non zero functions belonging to $H_0^\infty(\Sigma_\theta)$. There exists a constant $K > 0$ such that we have*

$$K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G, \qquad x \in X.$$

We need two lemmas which will be used in Theorem 2.4 below. Lemma 2.2 is implicit in the proof of [4, Lemma 6.5]. Further details can be found in [8]. In that statement, $\langle \cdot\, , \cdot \rangle$ denotes the usual inner product on the Hilbert space $L^2(\mathbb{R}_+^*, \frac{dt}{t})$.

Lemma 2.2. *There exists a sequence $(b_j)_{j\geq 1}$ in $L^2(\mathbb{R}_+^*, \frac{dt}{t})$ satisfying the following two properties.*

(1) *For any $a \in L^2(\mathbb{R}_+^*, \frac{dt}{t})$, $\|a\|^2 = \sum_{j\geq 1} |\langle a, b_j\rangle|^2$.*
(2) *For any $0 < \theta < \delta < \pi$ and any $G \in H_0^\infty(\Sigma_\delta)$, let $G_z \in L^2(\mathbb{R}_+^*, \frac{dt}{t})$ be defined by $G_z(t) = G(tz)$ for $t > 0$. Then*

$$\sup_{z\in\Sigma_\theta} \sum_{j\geq 1} |\langle G_z, b_j\rangle| < \infty.$$

We need some notation which will be used throughout the rest of this section. Let $L^2(\mathbb{R}_+^*, \frac{dt}{t}; X)$ be the usual Banach space of strongly measurable functions $\phi\colon (0, \infty) \to X$ such that $t \mapsto \|\phi(t)\|_X$ belongs to $L^2(\mathbb{R}_+^*, \frac{dt}{t})$ (see e.g. [5, p.49-50]). We will usually write $L^2(X)$ for that space. Likewise, we will write L^p, L^2, and $L^p(L^2)$ for $L^p(\Omega)$, $L^2(\mathbb{R}_+^*, \frac{dt}{t})$ and $L^p\big(\Omega; L^2(\mathbb{R}_+^*, \frac{dt}{t})\big)$ respectively. The fact that p may be equal to 2 should not cause any confusion! For any $a \in L^2$ and $x \in X$, the elementary tensor $a \otimes x$ may be identified with the function $\phi(t) = a(t)x$. This yields a canonical embedding $L^2 \otimes X \subset L^2(X)$. It is well-known that $L^2 \otimes X$ is actually a dense subspace of $L^2(X)$. Since $L^2 \otimes X \subset L^2 \otimes L^p \simeq L^p \otimes L^2$, we have a similar canonical embedding $L^2 \otimes X \subset L^p(L^2)$.

Lemma 2.3. *Let ϕ be in $L^p\big(\Omega; L^2(\mathbb{R}_+^*, \frac{dt}{t})\big) \cap L^2(\mathbb{R}_+^*, \frac{dt}{t}; X)$. There exists a net $(\phi_\alpha)_\alpha$ in $L^2 \otimes X$ such that $\phi_\alpha \to \phi$ in $L^2(X)$, and $\|\phi_\alpha\|_{L^p(L^2)} \leq \|\phi\|_{L^p(L^2)}$ for any α.*

Proof. Let I_X denote the identity operator on X. According to [5, Lemma III.2.1], there is a net of finite rank contractive mappings $E_\alpha\colon L^2 \to L^2$ such that $E_\alpha \otimes I_X\colon L^2 \otimes X \to L^2 \otimes X$ extends to a contraction $\widehat{E_\alpha}\colon L^2(X) \to L^2(X)$, and $\|\widehat{E_\alpha}(\phi) - \phi\|_{L^2(X)} \to 0$ for any $\phi \in L^2(X)$. Assume that ϕ belongs to $L^p(L^2) \cap L^2(X)$, and let $\phi_\alpha = \widehat{E_\alpha}(\phi)$. Since E_α is finite rank, ϕ_α belongs to $L^2 \otimes X$. Indeed, $\widehat{E_\alpha}$ is valued in the vector space $\mathrm{Ran}(E_\alpha) \otimes X$. On the other hand, $I_{L^p} \otimes E_\alpha\colon L^p \otimes$

$L^2 \to L^p \otimes L^2$ extends to a bounded operator $\widetilde{E_\alpha}\colon L^p(L^2) \to L^p(L^2)$ with $\|\widetilde{E_\alpha}\| = \|E_\alpha\|$. Since ϕ_α is clearly equal to $\widetilde{E_\alpha}(\phi)$, we deduce that

$$\|\phi_\alpha\|_{L^p(L^2)} \leq \|E_\alpha\|\,\|\phi\|_{L^p(L^2)} \leq \|\phi\|_{L^p(L^2)}.$$

□

Theorem 2.4. *Let X be a subspace of $L^p(\Omega)$, with $1 \leq p < \infty$, and let A be a sectorial operator on X. Assume that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$. Then for any non zero function F belonging to $H_0^\infty(\Sigma_{\theta'})$, with $\theta' > \theta$, there exists a constant $K > 0$ such that we have*

$$K^{-1}\|x\|_F \leq \|x\| \leq K\|x\|_F, \qquad x \in X. \tag{2.4}$$

Proof. The left hand side inequality $\|x\|_F \leq K\|x\|$ was proved in [4, Theorem 6.6] in the case when $X = L^p(\Omega)$. The arguments in that proof turn out to extend to the case when X is merely a subspace of $L^p(\Omega)$. We will therefore omit the details. Instead we will outline a variant of this proof in Remark 2.5 below.

We will now concentrate on the right hand side inequality. Before going into the proof, we outline the main idea. For a certain function F in $H_0^\infty(\Sigma_{\theta'})$, and for any x in X, we will approximate x by sums of the form $\sum_j g_j(A)f_j(A)x$, where $(f_j)_{j\geq 1}$ and $(g_j)_{j\geq 1}$ are sequences of bounded holomorphic functions, $(g_j)_{j\geq 1}$ satisfies the estimate (2.13) below, and $(f_j)_{j\geq 1}$ satisfies an estimate $\left\|\sum_j \varepsilon_j\, f_j(A)x\right\|_{\mathrm{Rad}(X)} \leq C''\|x\|_F$. Then we write

$$x \sim \sum_j g_j(A)f_j(A)x = \int_{\Omega_0}\Big(\sum_j \varepsilon_j(w)g_j(A)\Big)\Big(\sum_j \varepsilon_j(w)f_j(A)x\Big)\,d\mathbb{P}(w)\,,$$

where $(\varepsilon_j)_j$ is a Rademacher sequence, and we can conclude that $\|x\|\leq CC'C''\|x\|_F$.

We now turn to the proof, including the technical details. According to [9, Theorem 5.3], the fact that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on X implies that A is R-sectorial of type θ. Indeed subspaces of L^p (with $1 \leq p < \infty$) have the property (Δ) discussed in the latter paper. Thus it is enough by Theorem 2.1 to prove the right hand side inequality for a special function F. We now explain how to choose it. Let $\theta < \delta < \nu < \pi$. There exist two functions F and G in $H_0^\infty(\Sigma_\delta)$ and a constant $M > 0$ such that for all $f \in H_0^\infty(\Sigma_\nu)$, there exists $b \in L^1 \cap L^\infty(\mathbb{R}_+^*, \frac{dt}{t})$ satisfying the following two properties:

$$\forall z \in \Sigma_\delta, \quad f(z) = \int_0^\infty b(t)F(tz)G(tz)\,\frac{dt}{t}\,; \tag{2.5}$$

and

$$\|b\|_\infty \leq M\|f\|_{\infty,\nu}. \tag{2.6}$$

The existence of such functions follows from [4], namely by combining part of the proof of Theorem 4.4 and Example 4.7 from that paper. From now on F and G will be those two functions in $H_0^\infty(\Sigma_\delta)$ and we will prove the right hand side inequality for F.

Throughout the rest of the proof x will be an element in X and η an element in X^*. We take two auxilliary functions f in $H_0^\infty(\Sigma_\nu)$ and g in $H_0^\infty(\Sigma_\delta)$. In the last step of the proof f and g will tend to 1. Let $b \in L^1 \cap L^\infty(\mathbb{R}_+^*, \frac{dt}{t})$ be satisfying (2.5) and (2.6). By Fubini's theorem we have

$$f(A) = \int_0^\infty b(t)F(tA)G(tA)\,\frac{dt}{t}\,.$$

We define $\phi\colon (0,\infty) \to X$ and $\psi\colon (0,\infty) \to X^*$ by letting

$$\phi(t) = b(t)F(tA)x \quad \text{and} \quad \psi(t) = g(A)^* G(tA)^* \eta,$$

for $t > 0$, so that we have

$$\langle g(A)f(A)x, \eta\rangle = \int_0^\infty \langle \phi(t), \psi(t)\rangle\,\frac{dt}{t}\,. \tag{2.7}$$

It follows from well-known computations (see e.g. [1, Section (E)]) that

$$\sup_{t>0} \|F(tA)\| < \infty \qquad \text{and} \qquad \int_0^\infty \|g(A)G(tA)\|\,\frac{dt}{t} < \infty.$$

Since $b \in L^1 \cap L^\infty(\mathbb{R}_+^*, \frac{dt}{t})$, we deduce that ϕ is in $L^2(X)$ and that ψ is in $L^2(X^*)$. These properties will be used later on in the proof.

Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on X, the left hand side inequality in Theorem 2.4 implies that the function $t \mapsto F(tA)x$ belongs to $L^p(L^2)$. Thus ϕ is in $L^p(L^2)$, with

$$\begin{aligned}\|\phi\|_{L^p(L^2)} &= \left\|\left(\int_0^\infty \left|b(t)F(tA)x\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p} \leq \|b\|_\infty \left\|\left(\int_0^\infty \left|F(tA)x\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p} \\ &\leqslant \|b\|_\infty \|x\|_F.\end{aligned}$$

Hence using (2.6) we obtain the estimate

$$\|\phi\|_{L^p(L^2)} \leq M\,\|f\|_{\infty,\nu}\|x\|_F. \tag{2.8}$$

We now consider the sequence $(b_j)_j$ given by Lemma 2.2. For a and a' scalar functions in $L^2(\mathbb{R}_+^*, \frac{dt}{t})$ we have:

$$\int_0^\infty a(t)a'(t)\frac{dt}{t} = \sum_{j\geq 1} \int_0^\infty a(t)\overline{b_j(t)}\,\frac{dt}{t} \int_0^\infty a'(t)b_j(t)\,\frac{dt}{t}\,.$$

Thus for $\varphi = \sum_{k=1}^K a_k \otimes x_k \in L^2 \otimes X$ we have:

$$\begin{aligned}\int_0^\infty \langle \varphi(t), \psi(t)\rangle\,\frac{dt}{t} &= \sum_{k=1}^K \int_0^\infty a_k(t)\langle x_k, \psi(t)\rangle\,\frac{dt}{t} \\ &= \sum_{k=1}^K \sum_{j\geq 1} \int_0^\infty a_k(t)\overline{b_j(t)}\,\frac{dt}{t} \int_0^\infty \langle x_k, \psi(t)\rangle b_j(t)\,\frac{dt}{t} \\ &= \sum_{j\geq 1} \Big\langle \int_0^\infty \sum_{k=1}^K a_k(t)x_k\,\overline{b_j(t)}\,\frac{dt}{t}\,, \int_0^\infty \psi(t)\,b_j(t)\,\frac{dt}{t}\Big\rangle.\end{aligned}$$

So we have for $\varphi \in L^2 \otimes X$:

$$\int_0^\infty \langle \varphi(t), \psi(t)\rangle \,\frac{dt}{t} \;=\; \sum_{j\geq 1} \Big\langle \int_0^\infty \varphi(t)\overline{b_j(t)}\,\frac{dt}{t}\,,\, \int_0^\infty \psi(t) b_j(t)\,\frac{dt}{t}\Big\rangle. \qquad (2.9)$$

We noticed that the vector valued function ϕ both belongs to $L^p(L^2)$ and $L^2(X)$. Hence using Lemma 2.3 we obtain a net $(\phi_\alpha)_\alpha$ in $L^2 \otimes X$ such that $\phi_\alpha \to \phi$ in $L^2(X)$, with

$$\|\phi_\alpha\|_{L^p(L^2)} \leq \|\phi\|_{L^p(L^2)}. \qquad (2.10)$$

Since $\psi \in L^2(X^*)$, the above convergence property yields

$$\int_0^\infty \langle \phi(t), \psi(t)\rangle \,\frac{dt}{t} \;=\; \lim_\alpha \int_0^\infty \langle \phi_\alpha(t), \psi(t)\rangle \,\frac{dt}{t}\,. \qquad (2.11)$$

For each α, the function ϕ_α belongs to $L^2 \otimes X$, hence we obtain by applying (2.9) with $\varphi = \phi_\alpha$ that

$$\int_0^\infty \langle \phi_\alpha(t), \psi(t)\rangle \,\frac{dt}{t} \;=\; \sum_{j\geq 1} \langle x_j^\alpha, \eta_j\rangle, \qquad (2.12)$$

where $x_j^\alpha \in X$ and $\eta_j \in X^*$ are defined by

$$x_j^\alpha = \int_0^{+\infty} \phi_\alpha(t)\overline{b_j(t)}\,\frac{dt}{t} \qquad \text{and} \qquad \eta_j = \int_0^\infty \psi(t) b_j(t)\,\frac{dt}{t}\,.$$

We define $g_j(z) = \int_0^\infty G(tz)\overline{b_j(t)}\,\frac{dt}{t}$ for $z \in \Sigma_\theta$. Since g belongs to $H_0^\infty(\Sigma_\theta)$, we have by Fubini's theorem that

$$g(A)g_j(A) = \int_0^\infty g(A)G(tA)\overline{b_j(t)}\,\frac{dt}{t}\,,$$

so that we have $\eta_j = g_j(A)^* g(A)^* \eta$.

Let $(\varepsilon_j)_j$ be any sequence taking values in $\{-1, 1\}$. Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on X, we have an estimate

$$\Big\|\sum_{j=1}^N \varepsilon_j\, g_j(A)\Big\| \leq C \sup_{z\in\Sigma_\theta} \Big|\sum_{j=1}^N \varepsilon_j\, g_j(z)\Big|,$$

by (2.2). Hence

$$\Big\|\sum_{j=1}^N \varepsilon_j\, g_j(A)\Big\| \leq C \sup_{z\in\Sigma_\theta} \sum_{j=1}^N |g_j(z)|.$$

Since $g_j(z) = \langle G_z, b_j\rangle$, it follows from Lemma 2.2 that the right hand side in the last inequality is bounded by a constant C' independent of N and ε_j. Therefore we obtain that

$$\forall N \geq 1,\ \forall \varepsilon_j = \pm 1, \quad \Big\|\sum_{j=1}^N \varepsilon_j\, g_j(A)\Big\| \leq C\, C'. \qquad (2.13)$$

For any α and $N \geq 1$, we have

$$\sum_{j=1}^{N} \langle x_j^\alpha, \eta_j \rangle = \sum_{j=1}^{N} \big\langle g(A) g_j(A) x_j^\alpha, \eta \big\rangle.$$

Moreover if $(\varepsilon_j)_j$ is now a Rademacher sequence, we have

$$\sum_{j=1}^{N} g(A) g_j(A) x_j^\alpha = \int_{\Omega_0} \Big(\sum_{j=1}^{N} \varepsilon_j(w) g_j(A) g(A)\Big) \Big(\sum_{j=1}^{N} \varepsilon_j(w) x_j^\alpha\Big)\, d\mathbb{P}(w)\,.$$

Thus

$$\sum_{j=1}^{N} \langle x_j^\alpha, \eta_j \rangle = \Big\langle \int_{\Omega_0} \Big(\sum_{j=1}^{N} \varepsilon_j(w) g_j(A)\Big)\, g(A) \Big(\sum_{j=1}^{N} \varepsilon_j(w) x_j^\alpha\Big)\, d\mathbb{P}(w)\,,\ \eta \Big\rangle.$$

Applying the estimate (2.13), we obtain that

$$\begin{aligned} \Big|\sum_{j=1}^{N} \langle x_j^\alpha, \eta_j \rangle\Big| &\leq C\, C' \, \|g(A)\| \left[\int_{\Omega_0} \Big\| \sum_{j=1}^{N} \varepsilon_j(w) x_j^\alpha \Big\|\, d\mathbb{P}(w) \right] \|\eta\| \\ &\leq C^2\, C' \|g\|_{\infty,\theta} \Big\| \sum_{j=1}^{N} \varepsilon_j\, x_j^\alpha \Big\|_{\mathrm{Rad}(X)} \|\eta\|. \end{aligned}$$

Then we consider the operator V_N from $L^2(\mathbb{R}_+^*, \frac{dt}{t})$ to ℓ_2^N defined by $V_N(a) = (\langle a, b_j \rangle)_{j=1}^N$. By Lemma 2.2, this operator has norm at most 1. Hence its tensor extension $I_{L^p} \otimes V_N$ from $L^p(L^2)$ to $L^p(\ell_2^N)$ is a contraction. Since $(x_j^\alpha)_{j=1}^N = (Id_{L^p} \otimes V_N)(\phi_\alpha)$, this implies that

$$\Big\| \Big(\sum_{j=1}^{N} |x_j^\alpha|^2 \Big)^{1/2} \Big\|_{L^p} \leq \|\phi_\alpha\|_{L^p(L^2)}.$$

Since X is a subspace of L^p, this yields

$$\Big\| \sum_{j=1}^{N} \varepsilon_j\, x_j^\alpha \Big\|_{\mathrm{Rad}(X)} \leq C_0 \|\phi_\alpha\|_{L^p(L^2)}$$

by (2.3), and hence

$$\Big|\sum_{j=1}^{N} \langle x_j^\alpha, \eta_j \rangle\Big| \leq C^2\, C'\, C_0\, \|g\|_{\infty,\theta}\, \|\phi_\alpha\|_{L^p(L^2)}\, \|\eta\|.$$

Using (2.10) and (2.8), we obtain that

$$\Big|\sum_{j=1}^{N} \langle x_j^\alpha, \eta_j \rangle\Big| \leq C^2\, C'\, C_0\, M\, \|g\|_{\infty,\theta}\, \|f\|_{\infty,\nu}\, \|x\|_F\, \|\eta\|.$$

On the other hand, combining (2.7), (2.11) and (2.12) we have

$$\langle g(A)f(A)x,\eta\rangle = \lim_\alpha \sum_{j\geq 1}\langle x_j^\alpha,\eta_j\rangle.$$

Hence we finally obtain that

$$|\langle g(A)f(A)x,\eta\rangle| \leq C^2\,C'\,C_0\,M\,\|g\|_{\infty,\theta}\,\|f\|_{\infty,\nu}\,\|x\|_F\,\|\eta\|.$$

To conclude the proof, we apply this last inequality with f_n and g_n in place of f and g, where $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are bounded sequences respectively in $H_0^\infty(\Sigma_\nu)$ and $H_0^\infty(\Sigma_\theta)$, such that $f_n(A)$ and $g_n(A)$ converge pointwise to I_X. That such functions exist is well-known, using the fact that A has a dense range (take e.g. $f_n(z) = g_n(z) = n^2 z(n+z)^{-1}(1+nz)^{-1}$). This yields an inequality $|\langle x,\eta\rangle| \leq K\|x\|_F\|\eta\|$. Taking the supremum over η in the unit ball of X^*, we obtain the desired inequality $\|x\| \leq K\|x\|_F$. □

Remark 2.5. Using some of the arguments in the above proof, we can now give a functional analytic proof of the left hand side of Theorem 2.4. Since this is a simple adaptation of a similar result proved in [8] for sectorial operators on non commutative L^p-spaces, we will only give a sketch and refer to the latter paper for missing technical details. Assume that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $X \subset L^p(\Omega)$, and let $G \in H_0^\infty(\Sigma_\delta)$, for some $\delta > \theta$. We will show that $\|x\|_G \leq K\|x\|$ for some constant $K > 0$ not depending on $x \in X$. We let $(b_j)_j$ be given by Lemma 2.2, and we define g_j as in the proof of Theorem 2.4. Using (2.13) and (2.3), we find that

$$\forall\, N\geq 1,\quad \Big\|\Big(\sum_{j=1}^N \big|g_j(A)x\big|^2\Big)^{1/2}\Big\|_{L^p} \leq K\,\|x\|,\qquad x\in X, \tag{2.14}$$

for some $K > 0$ not depending either on N or on x. Let $g \in H_0^\infty(\Sigma_\delta)$ be an arbitrary function. According to Lemma 2.2 (1), we let $V\colon L^2(\mathbb{R}_+^*, \frac{dt}{t}) \to \ell^2$ be the isometry defined by $V(a) = (\langle a, b_j\rangle)_{j\geq 1}$. Then one can show (see [8]) that for any $x \in X$ and any $\eta \in X^*$, the function $t \mapsto \langle G(tA)g(A)x,\eta\rangle$ belongs to $L^2(\mathbb{R}_+^*, \frac{dt}{t})$, and that

$$V\big(\langle G(\cdot\,A)g(A)x,\eta\rangle\big) = \big(\langle g_j(A)g(A)x,\eta\rangle\big)_{j\geq 1}.$$

Using a tensor extension of V^*, it is not hard to deduce that

$$\|g(A)x\|_G = \big\|G(\cdot\,A)g(A)x\big\|_{L^p(L^2)} \leq \sup_{N\geq 1}\Big\|\Big(\sum_{j=1}^N \big|g_j(A)g(A)x\big|^2\Big)^{1/2}\Big\|_{L^p}. \tag{2.15}$$

Combining (2.14) and (2.15), we deduce that $\|g(A)x\|_G \leq K\|g(A)x\|$. Then it suffices to apply that estimate with g replaced by a bounded sequence $(g_n)_n$ such that $g_n(A)x \to x$ to get the desired inequality.

Remark 2.6. Let X be a subspace of $L^p(\Omega)$, with $1 \leq p < \infty$, and let A be a sectorial operator of type $\omega \in (0, \pi)$ on X. Let $\theta \in (\omega, \pi)$, and let F be a non zero function in $H_0^\infty(\Sigma_\theta)$. If A is R-sectorial of R-type ω, then there is a constant $K > 0$ such that

$$\|f(A)x\|_F \leq K\|x\|_F \qquad \text{for any } \ f \in H_0^\infty(\Sigma_\theta) \ \text{ and any } \ x \in X.$$

Indeed this is proved in [11] when $X = L^p(\Omega)$ and the proof works as well if X is a subspace. This yields the following converse to Theorem 2.4: if A is R-sectorial of R-type ω, and if (2.4) holds true for a non zero $F \in H_0^\infty(\Sigma_\theta)$, with $\theta > \omega$, then A has a bounded $H_0^\infty(\Sigma_\theta)$ functional calculus. We do not know if (2.4) implies a bounded functional calculus for A without any R-sectoriality assumption.

Remark 2.7. Let Λ be a Banach lattice with finite cotype (see e.g. [13]). Let $X \subset \Lambda$ be a subspace and assume that A is a sectorial operator of type $\omega \in (0, \pi)$ on X. For any $\theta > \omega$ and any $F \in H_0^\infty(\Sigma_\theta)$, one may define a square function by letting

$$\|x\|_F = \left\| \left(\int_0^\infty \left|F(tA)x\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_\Lambda, \qquad x \in X.$$

Then it is not hard to see that Theorems 2.1 and 2.4 hold true in that setting.

3. Square functions on Hardy spaces

Let $N \geq 1$ be an integer. In this section we will be interested in H^∞ functional calculus and square functions for sectorial operators on the Hardy space $H^1(\mathbb{R}^N)$. We refer the reader to e.g. [18], [7], or [14] for general information and background on Hardy spaces. We let $R_1, \ldots, R_N$ denote the Riesz transforms, so that

$$H^1(\mathbb{R}^N) = \big\{ h \in L^1(\mathbb{R}^N) \ : \ R_j(h) \in L^1(\mathbb{R}^N) \ \text{ for any } \ j = 1, \ldots, N \big\}.$$

This space admits several equivalent norms for which it is a Banach space. Here we choose to work with

$$\|h\|_{H^1} = \|h\|_1 + \sum_{j=1}^N \|R_j(h)\|_1, \qquad h \in H^1(\mathbb{R}^N), \tag{3.1}$$

where $\|\cdot\|_1$ denotes the usual norm on $L^1(\mathbb{R}^N)$.

We observe that $H^1(\mathbb{R}^N)$ equipped with $\|\cdot\|_{H^1}$ is isometrically isomorphic to a subspace of L^1. Indeed let $J\colon H^1(\mathbb{R}^N) \to \ell^1_{N+1}\big(L^1(\mathbb{R}^N)\big)$ be defined by letting

$$J(h) = \big(h, R_1(h), \ldots, R_N(h)\big)$$

for any $h \in H^1(\mathbb{R}^N)$, and let $X = \mathrm{Ran}(J)$. Then J is a linear isometry. Moreover we may clearly identify $\ell^1_{N+1}\big(L^1(\mathbb{R}^N)\big)$ with $L^1(\Omega_N)$, where Ω_N is equal to the disjoint union of $(N+1)$ copies of $\mathbb{R}^N$. Thus $H^1(\mathbb{R}^N)$ is isometrically isomorphic to $X \subset L^1(\Omega_N)$.

Our next goal is to explain how Theorems 2.1 and 2.4 for X 'transfer' to $H^1(\mathbb{R}^N)$. We record for further use that under the above identification, we have

$$L^1(\Omega_N;\mathcal{H}) \simeq \ell^1_{N+1}\big(L^1(\mathbb{R}^N;\mathcal{H})\big) \tag{3.2}$$

for any Hilbert space $\mathcal{H}$. Now we let

$$\mathcal{H} = L^2(\mathbb{R}_+^*;\, \tfrac{dt}{t}).$$

Let A be a sectorial operator of type $\omega \in (0,\pi)$ on the Banach space $H^1(\mathbb{R}^N)$. Let $\theta \in (\omega,\pi)$ and let $F \in H_0^\infty(\Sigma_\theta)$. For any $h \in H^1(\mathbb{R}^N)$, we let $[h]_F$ be the norm of the function $t \mapsto F(tA)h$ in $L^1(\mathbb{R}^N;\mathcal{H})$ (with the usual convention that $[h]_F = \infty$ if that function does not belong to $L^1(\mathbb{R}^N;\mathcal{H})$). Then if $T\colon H^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ is any bounded operator, we let $[h]_{TF}$ be the norm of $t \mapsto T\big(F(tA)h\big)$ in $L^1(\mathbb{R}^N;\mathcal{H})$, that is

$$[h]_{TF} = \left\| \left(\int_0^\infty \big|T\big(F(tA)h\big)\big|^2 \frac{dt}{t} \right)^{1/2} \right\|_1, \qquad h \in H^1(\mathbb{R}^N).$$

Note that $[h]_F = [h]_{TF}$ if T is equal to the canonical inclusion map $H^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$.

We now define square functions associated with A by letting

$$\|h\|_F = [h]_F + \sum_{j=1}^N [h]_{R_jF}, \qquad h \in H^1(\mathbb{R}^N), \tag{3.3}$$

for any $F \in H_0^\infty(\Sigma_\theta)\setminus\{0\}$ such that $\theta \in (\omega,\pi)$. Let $\widetilde{A} = JAJ^{-1}$ be the realization of A on $X \subset L^1(\Omega_N)$, let $h \in H^1(\mathbb{R}^N)$ and consider $\widetilde{h} = J(h) \in X$. Then we have

$$F(t\widetilde{A})\widetilde{h} = J\big(F(tA)h\big).$$

Hence applying (3.2) and (3.3), we have

$$\begin{aligned}
\big\|t \mapsto F(t\widetilde{A})\widetilde{h}\big\|_{L^1(\Omega_N;\mathcal{H})} &= \big\|t \mapsto J\big(F(tA)h\big)\big\|_{L^1(\Omega_N;\mathcal{H})} \\
&= \big\|t \mapsto F(tA)h\big\|_{L^1(\mathbb{R}^N;\mathcal{H})} \\
&\quad + \sum_{j=1}^N \big\|t \mapsto R_j\big(F(tA)h\big)\big\|_{L^1(\mathbb{R}^N;\mathcal{H})} \\
&= \|h\|_F.
\end{aligned}$$

This shows that the square function associated with A on $H^1(\mathbb{R}^N)$ and the corresponding square function associated with $\widetilde{A}$ on $X \subset L^1(\Omega_N)$ coincide. Therefore applying Theorem 2.1 and 2.4, we obtain the following results.

Corollary 3.1. *Let A be a sectorial operator on $H^1(\mathbb{R}^N)$.*

(1) *If A is R-sectorial or R-type $\omega \in (0,\pi)$, and if F, G are two non zero functions in $H_0^\infty(\Sigma_\theta)$ for some $\theta \in (\omega,\pi)$, then we have*

$$[h]_F + \sum_{j=1}^N [h]_{R_j F} \approx [h]_G + \sum_{j=1}^N [h]_{R_j G}, \qquad h \in H^1(\mathbb{R}^N).$$

(2) *If A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus, then for any $\theta' > \theta$ and any non zero function F in $H_0^\infty(\Sigma_{\theta'})$, we have*

$$\|h\|_{H^1} \approx [h]_F + \sum_{j=1}^N [h]_{R_j F}, \qquad h \in H^1(\mathbb{R}^N).$$

Of course in this statement, an equivalence $\mathcal{A}(h) \approx \mathcal{B}(h)$ means that there is a constant $K > 0$ not depending on h, such that $K^{-1}\mathcal{A}(h) \leq \mathcal{B}(h) \leq K\mathcal{A}(h)$.

Remark 3.2. If $N = 1$, then the Riesz transform R_1 is the Hilbert transform that we denote by H. Thus in that case square functions are given by

$$\|h\|_F = \left\| \left(\int_0^\infty \left| F(tA)h \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_1 + \left\| \left(\int_0^\infty \left| H\big(F(tA)h\big) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_1 \tag{3.4}$$

for any $h \in H^1(\mathbb{R})$.

Example 3.3. There are lots of examples of differential operators A on $L^2(\mathbb{R}^N)$ with the following properties: A has an $L^p(\mathbb{R}^N)$-realization A_p for any $1 \leq p < \infty$, the operator A_p has a bounded H^∞ functional calculus on $L^p(\mathbb{R}^N)$ if $p \neq 1$, but A_1 does not have a bounded H^∞ functional calculus on $L^1(\mathbb{R}^N)$. It turns out that sometimes, such an operator also has an $H^1(\mathbb{R}^N)$-realization, which has a bounded H^∞ functional calculus on $H^1(\mathbb{R}^N)$. The simplest such example (with $N = 1$) is the derivation operator $\frac{d}{dt}$, with domain equal to the Sobolev space $W^{1,p}(\mathbb{R})$ on $L^p(\mathbb{R})$. For any $1 \leq p < \infty$, this is a sectorial operator of type $\frac{\pi}{2}$. Furthermore for any $\theta \in (\frac{\pi}{2}, \pi)$, the operator $\frac{d}{dt}$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\mathbb{R})$ if and only if $1 < p < \infty$. It is easy to see that $A = \frac{d}{dt}$ acts as a sectorial operator on $H^1(\mathbb{R})$, and that it has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on that space. Indeed, for any $f \in H_0^\infty(\Sigma_\theta)$, the operator $f(\frac{d}{dt})$ is the Fourier mulitiplier operator associated to the function $t \mapsto f(it)$, and hence an estimate $\|f(A)\|_{H^1} \leq K\|f\|_{\infty,\theta}\|h\|_{H^1}$ follows by applying Mikhlin's Theorem on $H^1(\mathbb{R})$ (see e.g. [14, p. 99]).

In the rest of this section, we describe a general framework where the ideas outlined in Example 3.3 apply. We fix an integer $N \geq 1$ and for simplicity, we write L^p and H^1 for $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ respectively. We suppose that for any $1 \leq p \leq 2$, A_p is a sectorial operator on L^p, with type ω not depending on p, and we assume that the family $\{A_p\}_p$ is consistent in the following sense: for any $1 \leq p, q \leq 2$, and for any $\lambda \notin \overline{\Sigma_\omega}$, the bounded operators $R(\lambda, A_p)$ and $R(\lambda, A_q)$

coincide on $L^p \cap L^q$. Clearly these assumptions imply that for any $\theta > \omega$, and any $f \in H_0^\infty(\Sigma_\theta)$, $f(A_p)$ and $f(A_q)$ also coincide on $L^p \cap L^q$.

We let $A = A_2$, and we assume further that A is a Fourier multiplier. By this we mean that there exists a measurable function $m\colon \mathbb{R}^N \to \mathbb{C}$ such that

$$\widehat{Ah} = m\,\widehat{h}, \qquad h \in D(A), \tag{3.5}$$

the domain of A being equal to the space of all $h \in L^2$ such that $m\,\widehat{h}$ belongs to L^2. In that case, m is essentially valued in $\overline{\Sigma_\omega}$. If (3.5) holds, we say that A is associated to m. Then for any $\lambda \notin \overline{\Sigma_\omega}$, the resolvent operator $R(\lambda, A)$ is equal to the Fourier multiplier associated to the bounded function $(\lambda - m(\cdot))^{-1}$. Likewise, for any $\theta \in (\omega, \pi)$ and $f \in H_0^\infty(\Sigma_\theta)$, the bounded operator $f(A)\colon L^2 \to L^2$ is the Fourier multiplier associated to $f \circ m$. This readily implies that $\|f(A)\| = \|f \circ m\|_\infty$. Consequently, we have $\|f(A)\| \leq \|f\|_{\infty,\theta}$, and hence A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on L^2. All these facts are well-known.

We now define a realization of A on H^1, denoted by A_H. Since A is a Fourier multiplier, then for any $\lambda \notin \overline{\Sigma_\omega}$, the operator $R(\lambda, A_1)$ commutes with the Riesz transforms. Thus $R(\lambda, A_1)$ maps H^1 into itself, and for any $j = 1, \ldots, N$, we have

$$R_j R(\lambda, A_1) = R(\lambda, A_1) R_j \quad \text{ on } \ H^1. \tag{3.6}$$

Then we define A_H by letting $A_H(h) = A_1(h)$ on the domain

$$D(A_H) = \bigl\{h \in H^1 \cap D(A_1) \ :\ A_1(h) \in H^1\bigr\},$$

Using (3.6), the following lemma is routine.

Lemma 3.4. *The operator A_H is sectorial of type ω on H^1. Moreover for any $\theta > \omega$ and any $f \in H_0^\infty(\Sigma_\theta)$, $f(A_1)$ maps H^1 into itself, and the corresponding restriction $f(A_1)_{|H^1 \to H^1}$ coincides with $f(A_H)$.*

For any $\theta > \omega$ and any $f \in H_0^\infty(\Sigma_\theta)$, $f(A) = K_f * \bullet$ is a convolution operator with respect to the tempered distribution $K_f \in \mathcal{S}'(\mathbb{R}^N)$ defined by $\widehat{K_f} = f \circ m$. We now make the *strong assumption* that any such operator $f(A)$ is a singular integral operator in the sense of [7, Section II.5]. That is, K_f coincides on $\mathbb{R}^N \setminus \{0\}$ with a locally integrable function, and there is a constant C_f such that for any $v \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{|u|>2|v|} \bigl|K_f(u-v) - K_f(u)\bigr|\, du \ \leq\ C_f. \tag{3.7}$$

Corollary 3.5. *Assume that for some $\theta > \omega$, there exists a constant $C > 0$ such that (3.7) holds true with $C_f \leq C\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$. Then A_H has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on H^1.*

Proof. By Lemma 3.4, $f(A_H)$ and $f(A)$ coincide on $L^2 \cap H^1$. Hence acccording to either [18, p. 114], or [7, p. 322], (3.7) ensures that $\|f(A_H)\| \leq B_0\, C_f$, where B_0 is an absolute constant. Thus we obtain that $\|f(A_H)\| \leq B_0\, C\, \|f\|_{\infty,\theta}$, and hence A_H has a bounded $H^\infty(\Sigma_\theta)$ functional calculus. □

Remark 3.6. We observe that for any $\theta > \omega$, any $F \in H_0^\infty(\Sigma_\theta)$, and any $t > 0$, we have $R_j F(tA_H) = F(tA_H)R_j$ on $H^1(\mathbb{R}^N)$. Hence $[h]_{R_j F} = [R_j h]_F$ for any $h \in H^1(\mathbb{R}^N)$. Thus the square functions associated with A_H can be expressed as

$$\|h\|_F = [h]_F + \sum_{j=1}^{N} [R_j h]_F, \qquad h \in H^1(\mathbb{R}^N).$$

Remark 3.7. The above discussion applies to $A = -\Delta$, where Δ is the Laplacian operator on $\mathbb{R}^N$. Indeed A satisfies (3.5) with $m(u) = |u|^2$, and it is well-known that the assumptions of Corollary 3.5 are verified for any $\theta > 0$. Thus A has an H^1-realization which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$. Let $k \geq 1$ be any positive integer, and consider the function F defined by $F(z) = z^k e^{-z}$. Clearly F belongs to $H_0^\infty(\Sigma_\theta)$ for any $\theta \in (0, \frac{\pi}{2})$. According to [2, Section 2.A] (see also [6,19]), a function $h \in L^1$ belongs to H^1 if and only if $[h]_F$ is finite. Moreover we have an equivalence

$$\|h\|_{H^1} \approx [h]_F, \qquad h \in H^1(\mathbb{R}^N). \tag{3.8}$$

Comparing with Corollary 3.1 (2) and Remark 3.6 (2), this is equivalent to saying that for any $j = 1, \ldots, N$, we have equivalences $[h]_F \approx [R_j(h)]_F$ on H^1. It would be interesting to have a 'H^∞ calculus proof' of these facts. It seems to be an open question whether (3.8) holds for any $F \in H_0^\infty(\Sigma_\theta)$.

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