On asymptotic properties for some parameter dependent variational inequalities

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Abstract

This paper is concerned with asymptotic and monotonicity properties of some parameter dependent variational inequalities. The main part of the study deals with inequalities modelling friction problems as normal compliance and Tresca's model in which the parameter stands for the friction coefficient. The corresponding inequalities are (generalizations) of variational inequalities of the second kind. We then study an inequality of the first kind representing the elastoplastic torsion problem where the parameter represents the plasticity yield.

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Abbreviated title: Analysis of some parameter dependent variational inequalities.

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1 Introduction

Problems written with weak formulations involving variational inequalities represent various nonlinear phenomena which occur in mechanics and physics. The nonlinearity corresponding to the inequality in the problem is located either in the entire domain of study (e.g., obstacle problem, elastoplastic torsion problem, Bingham fluid...) or only on the boundary of the domain (e.g., Signorini or contact problems, friction models...). Generally we deal with first and second kind classes of variational inequalities. We recall that a variational inequality of the first (resp. second) kind is of the form:

$$u \in C \ : \qquad a(u, v-u) \geq L(v-u), \qquad \forall v \in C,$$

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respectively,

$$u \in X \ : \qquad a(u, v-u) + j(v) - j(u) \geq L(v-u), \qquad \forall v \in X,$$

where X is an Hilbert space, $C \subset X$ is a nonempty closed convex set, $a(\cdot,\cdot)$ is bilinear, X-elliptic and continuous on $X \times X$, $L(\cdot)$ is linear and continuous on X, $j(\cdot)$ is proper convex and lower semicontinuous on X (with values in $\mathbb{R} \cup \{+\infty\}$). More details concerning variational inequalities of the first and second kind can be found in e.g., [1, 6]. Our aim in this paper is to establish some properties for parameter dependent variational inequalities of the first, second kind and also more general inequalities when the parameter varies. To our knowledge the kind of estimates we prove have not been considered previously. An outline of the paper is as follows.

Section 2 is concerned with variational inequalities modelling friction phenomena which roughly speaking correspond to nonlinear boundary value problems. We first consider the well-known compliance model in elasticity introduced in [17, 15] (see also [11]). The corresponding inequality is neither of the first nor of the second kind since it corresponds to a more complex type. Denoting by $\mathbf{u}_\varepsilon$ the solution of the problem with a friction parameter ε, we show that $\varepsilon^{-1}(\mathbf{u}_\varepsilon - \mathbf{u}_0)$ converges as ε vanishes and we characterize the limit as a solution of a specific variational inequality. Then we consider an elementary friction model introduced in [6] where the unknown of the variational inequality (of the second kind) is a scalar valued function. We obtain similar results as in the previous case. Moreover we show that a blocking property occurs when the friction is large enough and we mention that the former studies can be applied to more realistic friction models involving the elasticity operator and vector valued functions (the so-called Tresca's friction model). Finally we show in the scalar case a monotonicity result which claims that the solution decreases when the parameter increases.

In Section 3 we consider a variational inequality of the first kind, the so-called elasto-plastic torsion problem. The nonnegative parameter denoted ε, stands for the plasticity yield and u_ε is the corresponding solution. We prove that $u_\varepsilon/\varepsilon$ converges as ε vanishes towards a function which we characterize and which can be explicitly given in physical meaningful cases. We end this section with a monotonicity result for $u_\varepsilon/\varepsilon$.

2 Inequalities modelling friction problems

In this section we consider some inequality problems modelling friction phenomena. For the derivation and the discussion of various friction models in a more general framework we refer the reader to, e.g., [19].

2.1 A friction problem with normal compliance

We consider an elastic body that occupies a region Ω in $\mathbb{R}^n$ $(n = 2, 3)$. The body is held fixed on a part Γ_D of the boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_C}$, can come into contact with a rigid foundation over the part Γ_C and is subject to volume forces $\mathbf{f} \in (L^2(\Omega))^n$. On the contact surface we use a normal compliance condition (see [13, 14] and the references therein). Let us denote by $\mathbf{u} = (u_i)_{1 \leq i \leq n}$ the displacement vector and by $\boldsymbol{\sigma} = (\sigma_{ij})_{1 \leq i,j \leq n}$ the stress tensor such that

$$\sigma_{ij}(\mathbf{u}) = a_{ijhk} \frac{\partial u_h}{\partial x_k}, \ \ i, j, h, k \in \{1, ..., n\},$$

where the summation convention of repeated indices is adopted. The functions $a_{ijhk} \in L^\infty(\Omega)$ are the coefficients of a fourth order tensor, representing the elastic properties of the material. As usual we assume that $a_{ijhk} = a_{jihk} = a_{hkij}$ and the ellipticity condition $a_{ijhk}\xi_{ij}\xi_{hk} \geq \alpha|\xi|^2$, $\forall\, \xi_{ij} = \xi_{ji}$, for some $\alpha > 0$.

The frictional contact problem with normal compliance in elastostatics is to find the displacement field $\mathbf{u}$ such that equations (1)–(3) hold:

$$\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{2}$$

where $\mathbf{div}$ denotes the divergence operator of tensor valued functions. For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following notation ($\mathbf{n}$ stands for the outward unit normal on $\partial\Omega$):

$$\mathbf{v} = v_N\mathbf{n} + \mathbf{v}_T \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_N(\mathbf{v})\mathbf{n} + \boldsymbol{\sigma}_T(\mathbf{v}).$$

Then the conditions of normal compliance with friction on Γ_C are:

$$\begin{cases} \sigma_N(\mathbf{u}) = -c_N(u_N)_+^{m_N}, \\ |\boldsymbol{\sigma}_T(\mathbf{u})| \leq c_T(u_N)_+^{m_T} \quad \text{if} \quad \mathbf{u}_T = \mathbf{0}, \\ \begin{cases} |\boldsymbol{\sigma}_T(\mathbf{u})| = c_T(u_N)_+^{m_T} \quad \text{if} \quad \mathbf{u}_T \neq \mathbf{0}, \\ \mathbf{u}_T \cdot \boldsymbol{\sigma}_T(\mathbf{u}) \leq 0, \end{cases} \end{cases} \tag{3}$$

where $(.)_+$ stands for the positive part so that $(u_N)_+$ represents the penetration of the body into the foundation. The constants $m_N \geq 1$, $m_T \geq 1$ as well as the nonnegative functions c_N and c_T in $L^\infty(\Gamma_C)$ stand for interface parameters characterizing the contact behavior between the body and the rigid foundation. The set of admissible displacements

$$\mathbf{V} = \left\{\mathbf{v} \in (H^1(\Omega))^n \;:\; \mathbf{v} = \mathbf{0} \textit{ on } \Gamma_D\right\}$$

is endowed with the norm of $(H^1(\Omega))^n$. We denote by $a(\cdot,\cdot)$ the standard bilinear form of linear elasticity

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x = \int_\Omega a_{ijhk}\frac{\partial u_i}{\partial x_j}\frac{\partial v_h}{\partial x_k}\,\mathrm{d}x,$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1\leq i,j\leq n}$ stands for the strain tensor where $\varepsilon_{ij}(\mathbf{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$. If Γ_D has positive superficial measure it is well known that the bilinear form $a(\cdot,\cdot)$ is $\mathbf{V}$-elliptic

$$a(\mathbf{v},\mathbf{v}) \geq \alpha \int_\Omega \sum_{i,j=1}^n \left|\frac{\partial v_i}{\partial x_j}\right|^2 \mathrm{d}x \geq \beta\|\mathbf{v}\|^2_{H^1(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{V}.$$

The previous boundary value problem leads to the following variational inequality (see, e.g., [13, 14])

$$\mathbf{u} \in \mathbf{V} \;:\; a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j_N(\mathbf{u},\mathbf{v}-\mathbf{u}) + j_T(\mathbf{u},\mathbf{v}) - j_T(\mathbf{u},\mathbf{u}) \geq L(\mathbf{v}-\mathbf{u}),\ \forall\, \mathbf{v} \in \mathbf{V} \tag{4}$$

where the linear form L is given by $L(\mathbf{v}) = \int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x)\, \mathrm{d}x$ and

$$j_N(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_C} c_N (u_N)_+^{m_N} v_N \, \mathrm{d}\sigma(x), \;\; j_T(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_C} c_T (u_N)_+^{m_T} |\mathbf{v}_T| \, \mathrm{d}\sigma(x).$$

We assume that

$$1 \leq m_N, m_T \text{ if } n = 2, \;\; 1 \leq m_N, m_T < 3 \text{ if } n = 3. \tag{5}$$

The hypotheses (5) guarantee the imbedding

$$H^1(\Omega) \hookrightarrow L^q(\Gamma_C)$$

for any $q \in [1, +\infty[$ if $n = 2$ and for any $q \in [1, 4]$ if $n = 3$. If $n = 2$ we have for any $\mathbf{u}, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in (H^1(\Omega))^2$

$$|j_N(\mathbf{u}, \mathbf{v})| \leq \|c_N\|_{L^\infty(\Gamma_C)} \|\mathbf{u}\|^{m_N}_{L^{2m_N}(\Gamma_C)} \|\mathbf{v}\|_{L^2(\Gamma_C)} \leq C \|\mathbf{u}\|^{m_N}_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \tag{6}$$

and

$$\begin{aligned} |j_T(\mathbf{u}, \mathbf{v}_1) - j_T(\mathbf{u}, \mathbf{v}_2)| &\leq \int_{\Gamma_C} c_T (u_N)_+^{m_T} |(\mathbf{v}_1 - \mathbf{v}_2)_T| \, \mathrm{d}\sigma(x) \\ &\leq \|c_T\|_{L^\infty(\Gamma_C)} \|\mathbf{u}\|^{m_T}_{L^{2m_T}(\Gamma_C)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_C)} \\ &\leq C \|\mathbf{u}\|^{m_T}_{H^1(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(\Omega)}. \end{aligned} \tag{7}$$

If $n = 3$ we obtain similar bounds. Indeed, consider $p \in [1, 4]$ such that $1/p + m_N/4 = 1$ (which is possible since $1 \leq m_N < 3$) and by Hölder inequality one gets

$$|j_N(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|^{m_N}_{L^4(\Gamma_C)} \|\mathbf{v}\|_{L^p(\Gamma_C)} \leq C \|\mathbf{u}\|^{m_N}_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \tag{8}$$

and similarly

$$|j_T(\mathbf{u}, \mathbf{v}_1) - j_T(\mathbf{u}, \mathbf{v}_2)| \leq C \|\mathbf{u}\|^{m_T}_{L^4(\Gamma_C)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^p(\Gamma_C)} \leq C \|\mathbf{u}\|^{m_T}_{H^1(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(\Omega)}. \tag{9}$$

Let us analyze now the continuity of the applications j_N, j_T with respect to the first argument. If $m_N, m_T = 1$ it is easily seen that

$$\begin{aligned} \max\{|j_N(\mathbf{u}_1, \mathbf{v}) - j_N(\mathbf{u}_2, \mathbf{v})|, |j_T(\mathbf{u}_1, \mathbf{v}) - j_T(\mathbf{u}_2, \mathbf{v})|\} &\leq C \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(\Gamma_C)} \|\mathbf{v}\|_{L^2(\Gamma_C)} \\ &\leq C \|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}. \end{aligned}$$

When $m_N, m_T > 1$ we use the inequality

$$\begin{aligned} |(a)_+^m - (b)_+^m| &\leq m |(a)_+ - (b)_+| \left((a)_+^{m-1} + (b)_+^{m-1} \right) \\ &\leq m |a - b| \left(|a|^{m-1} + |b|^{m-1} \right) \qquad a, b \in \mathbb{R}, m \geq 1. \end{aligned}$$

If $n = 2$ we have for any $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in (H^1(\Omega))^2$

$$\begin{aligned} & |j_N(\mathbf{u}_1, \mathbf{v}) - j_N(\mathbf{u}_2, \mathbf{v})| \\ \leq\ & C \int_{\Gamma_C} |\mathbf{u}_1 - \mathbf{u}_2| \left(|\mathbf{u}_1|^{m_N - 1} + |\mathbf{u}_2|^{m_N - 1} \right) |\mathbf{v}| \, \mathrm{d}\sigma(x) \\ \leq\ & C \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(\Gamma_C)} \left\{ \|\mathbf{u}_1\|^{m_N - 1}_{L^{4(m_N - 1)}(\Gamma_C)} + \|\mathbf{u}_2\|^{m_N - 1}_{L^{4(m_N - 1)}(\Gamma_C)} \right\} \|\mathbf{v}\|_{L^4(\Gamma_C)} \\ \leq\ & C \|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(\Omega)} \left\{ \|\mathbf{u}_1\|^{m_N - 1}_{H^1(\Omega)} + \|\mathbf{u}_2\|^{m_N - 1}_{H^1(\Omega)} \right\} \|\mathbf{v}\|_{H^1(\Omega)}. \end{aligned} \tag{10}$$

If $n = 3$ we consider as before $p \in [1, 4]$ such that $1/4 + (m_N - 1)/4 + 1/p = 1$ and therefore one gets

$$\begin{aligned} |j_N(\mathbf{u}_1, \mathbf{v}) - j_N(\mathbf{u}_2, \mathbf{v})| &\leq C\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^4(\Gamma_C)} \left\{ \|\mathbf{u}_1\|^{m_N-1}_{L^4(\Gamma_C)} + \|\mathbf{u}_2\|^{m_N-1}_{L^4(\Gamma_C)} \right\} \|\mathbf{v}\|_{L^p(\Gamma_C)} \\ &\leq C\|\mathbf{u}_1 - \mathbf{u}_2\|_{H^1(\Omega)} \left\{ \|\mathbf{u}_1\|^{m_N-1}_{H^1(\Omega)} + \|\mathbf{u}_2\|^{m_N-1}_{H^1(\Omega)} \right\} \|\mathbf{v}\|_{H^1(\Omega)}. \quad (11) \end{aligned}$$

Similar inequalities hold for the application j_T.

Remark 2.1 *The above computations show that for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ the functions $j_N(\mathbf{u}, \cdot)$, $j_T(\mathbf{u}, \cdot)$ and $j_N(\cdot, \mathbf{v})$, $j_T(\cdot, \mathbf{v})$ are continuous with respect to the strong topology of $\mathbf{V}$.*

Remark 2.2 *It is easily seen that the functions j_N, j_T are strongly continuous on $\mathbf{V} \times \mathbf{V}$*

$$\lim_{k \to +\infty} (j_N, j_T)(\mathbf{u}_k, \mathbf{v}_k) = (j_N, j_T)(\mathbf{u}, \mathbf{v})$$

for any sequences $(\mathbf{u}_k, \mathbf{v}_k)_k$ converging strongly in $\mathbf{V} \times \mathbf{V}$ towards $(\mathbf{u}, \mathbf{v})$.

Under the above conditions it has been proved in [13] that the problem (4) has solutions. Moreover, uniqueness results are available for sufficiently small coefficients c_N, c_T. Therefore, for $\varepsilon > 0$ small enough, the problems associated to the coefficients εc_N, εc_T are well posed:

$$\mathbf{u}_\varepsilon \in \mathbf{V} \ : \ a(\mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{v} - \mathbf{u}_\varepsilon) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{v}) - \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \geq L(\mathbf{v} - \mathbf{u}_\varepsilon), \quad (12)$$

for all $\mathbf{v} \in \mathbf{V}$. It is easily seen by taking $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = \mathbf{u}_\varepsilon$ in (12) that the previous problem reduces to

$$a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) = L(\mathbf{u}_\varepsilon) \quad (13)$$

$$a(\mathbf{u}_\varepsilon, \mathbf{v}) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{v}) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{v}) \geq L(\mathbf{v}), \ \forall \, \mathbf{v} \in \mathbf{V}. \quad (14)$$

Observing that $j_N(\mathbf{v}, \mathbf{v}) \geq 0$ for any $\mathbf{v} \in \mathbf{V}$ and $j_T \geq 0$ we deduce that $\sup_{\varepsilon > 0} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)} \leq \beta^{-1} \|\mathbf{f}\|_{L^2(\Omega)}$ since for any $\varepsilon > 0$ we have

$$\beta \, \|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)} \leq a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \leq L(\mathbf{u}_\varepsilon) \leq \|\mathbf{f}\|_{L^2(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}.$$

Let us consider $\mathbf{u}_0$ the solution of the variational equality

$$\mathbf{u}_0 \in \mathbf{V} \ : \ a(\mathbf{u}_0, \mathbf{v}) = L(\mathbf{v}), \ \forall \, \mathbf{v} \in \mathbf{V}. \quad (15)$$

Combining (12) written with $\mathbf{v} = \mathbf{u}_0$ and (15) written with $\mathbf{v} = \mathbf{u}_\varepsilon - \mathbf{u}_0$ implies

$$\beta \, \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|^2_{H^1(\Omega)} \leq \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{u}_0 - \mathbf{u}_\varepsilon) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_0) - \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon). \quad (16)$$

By the inequalities (6), (8) we have

$$|j_N(\mathbf{u}_\varepsilon, \mathbf{u}_0 - \mathbf{u}_\varepsilon)| \leq C \|\mathbf{u}_\varepsilon\|^{m_N}_{H^1(\Omega)} \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega)} \leq C \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega)}. \quad (17)$$

Similarly (7), (9) imply

$$|j_T(\mathbf{u}_\varepsilon, \mathbf{u}_0) - j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)| \leq C \|\mathbf{u}_\varepsilon\|^{m_T}_{H^1(\Omega)} \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega)} \leq C \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega)}. \quad (18)$$

Therefore we deduce by (16), (17), (18) that

$$\sup_{\varepsilon>0} \frac{\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega)}}{\varepsilon} < +\infty. \tag{19}$$

In particular one gets the strong convergence $\lim_{\varepsilon\searrow 0} \mathbf{u}_\varepsilon = \mathbf{u}_0$ in $\mathbf{V}$, saying that $\mathbf{u}_\varepsilon = \mathbf{u}_0 + \mathcal{O}(\varepsilon)$. But in some applications the parameter $\varepsilon > 0$ is not too small and in such cases the above approximation is not satisfactory: we need to determine the first order correction $\mathbf{r}$ such that $\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon\mathbf{r} + \mathcal{O}(\varepsilon^2)$. In the sequel we establish the strong convergence in $\mathbf{V}$ of the fluctuations $\varepsilon^{-1}(\mathbf{u}_\varepsilon - \mathbf{u}_0)$ towards $\mathbf{r}$ and we characterize the first order correction $\mathbf{r}$ as the unique solution of a variational inequality on some convex set. We denote by $\mathcal{K}$ the set

$$\mathcal{K} = \{\mathbf{q} \in \mathbf{V} \;:\; a(\mathbf{q}, \mathbf{v} - \mathbf{u}_0) + j_N(\mathbf{u}_0, \mathbf{v} - \mathbf{u}_0) + j_T(\mathbf{u}_0, \mathbf{v}) - j_T(\mathbf{u}_0, \mathbf{u}_0) \geq 0,\ \forall\, \mathbf{v} \in \mathbf{V}\}.$$

Observe that $j_N(\mathbf{u}_0, \cdot), j_T(\mathbf{u}_0, \cdot)$ are convex and therefore the set $-\mathcal{K}$ coincides with the subgradient of $j_N(\mathbf{u}_0, \cdot) + j_T(\mathbf{u}_0, \cdot)$ at $\mathbf{u}_0$, with respect to the inner product $a(\cdot, \cdot)$. Therefore $\mathcal{K}$ is a non empty closed convex set in $\mathbf{V}$.

Theorem 2.1 *For any $\varepsilon > 0$ let $\mathbf{r}_\varepsilon = \varepsilon^{-1}(\mathbf{u}_\varepsilon - \mathbf{u}_0)$ where $\mathbf{u}_\varepsilon$ solves (12). Then $\lim_{\varepsilon\searrow 0} \mathbf{r}_\varepsilon = \mathbf{r}$ strongly in $\mathbf{V}$ where $\mathbf{r}$ is the unique solution of the variational inequality*

$$\mathbf{r} \in \mathcal{K} \;:\; a(\mathbf{r}, \mathbf{q} - \mathbf{r}) \geq 0,\ \forall\, \mathbf{q} \in \mathcal{K}. \tag{20}$$

Proof. By (19) we know that $(\mathbf{r}_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{V}$ and thus there exists $\mathbf{r} \in \mathbf{V}$ and a sequence $(\varepsilon_k)_k$ converging towards zero such that $(\mathbf{r}_k)_k := (\mathbf{r}_{\varepsilon_k})_k$ converges weakly towards $\mathbf{r}$ in $\mathbf{V}$. Combining (14) and (15) one gets for any $\mathbf{v} \in \mathbf{V}$

$$a(\mathbf{u}_\varepsilon, \mathbf{v}) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{v}) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{v}) \geq L(\mathbf{v}) = a(\mathbf{u}_0, \mathbf{v})$$

and therefore we obtain for any k, with the notation $\mathbf{u}_k = \mathbf{u}_{\varepsilon_k}$

$$a(\mathbf{r}_k, \mathbf{v}) + j_N(\mathbf{u}_k, \mathbf{v}) + j_T(\mathbf{u}_k, \mathbf{v}) \geq 0.$$

Passing to the limit for $k \to +\infty$ we deduce by Remark 2.1 and by using the weak convergence of $(\mathbf{r}_k)_k$ that

$$a(\mathbf{r}, \mathbf{v}) + j_N(\mathbf{u}_0, \mathbf{v}) + j_T(\mathbf{u}_0, \mathbf{v}) \geq 0,\ \ \mathbf{v} \in \mathbf{V}. \tag{21}$$

Combining now (13) and (15) we obtain

$$a(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) = L(\mathbf{u}_\varepsilon) = a(\mathbf{u}_0, \mathbf{u}_\varepsilon)$$

and therefore one gets

$$a(\mathbf{r}_k, \mathbf{u}_k) + j_N(\mathbf{u}_k, \mathbf{u}_k) + j_T(\mathbf{u}_k, \mathbf{u}_k) = 0,\ \forall\, k. \tag{22}$$

Since $(\mathbf{r}_k)_k$ converges weakly in $\mathbf{V}$ and $(\mathbf{u}_k)_k$ converges strongly in $\mathbf{V}$ we deduce that $\lim_{k\to+\infty} a(\mathbf{r}_k, \mathbf{u}_k) = a(\mathbf{r}, \mathbf{u}_0)$. Notice also that by Remark 2.2 we have

$$\lim_{k\to+\infty} (j_N, j_T)(\mathbf{u}_k, \mathbf{u}_k) = (j_N, j_T)(\mathbf{u}_0, \mathbf{u}_0).$$

Passing to the limit for $k \to +\infty$ in (22) yields

$$a(\mathbf{r}, \mathbf{u}_0) + j_N(\mathbf{u}_0, \mathbf{u}_0) + j_T(\mathbf{u}_0, \mathbf{u}_0) = 0. \tag{23}$$

We deduce by (21), (23) that

$$a(\mathbf{r}, \mathbf{v} - \mathbf{u}_0) + j_N(\mathbf{u}_0, \mathbf{v} - \mathbf{u}_0) + j_T(\mathbf{u}_0, \mathbf{v}) - j_T(\mathbf{u}_0, \mathbf{u}_0) \geq 0, \ \forall \mathbf{v} \in \mathbf{V} \tag{24}$$

saying that $\mathbf{r}$ belongs to the set $\mathcal{K}$. Using now (12) with $\mathbf{v} = \mathbf{u}_0$ and (15) implies

$$a(\mathbf{u}_\varepsilon, \mathbf{u}_0 - \mathbf{u}_\varepsilon) + \varepsilon j_N(\mathbf{u}_\varepsilon, \mathbf{u}_0 - \mathbf{u}_\varepsilon) + \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_0) - \varepsilon j_T(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \geq L(\mathbf{u}_0 - \mathbf{u}_\varepsilon) = a(\mathbf{u}_0, \mathbf{u}_0 - \mathbf{u}_\varepsilon)$$

and therefore, replacing ε by ε_k one gets after division by ε_k^2

$$a(\mathbf{r}_k, \mathbf{r}_k) \leq -j_N(\mathbf{u}_k, \mathbf{r}_k) + \frac{1}{\varepsilon_k} \left(j_T(\mathbf{u}_k, \mathbf{u}_0) - j_T(\mathbf{u}_k, \mathbf{u}_k) \right), \ \forall \, k. \tag{25}$$

Taking now $\mathbf{v} = \mathbf{u}_k$ in (24) yields

$$a(\mathbf{r}, \mathbf{r}_k) \geq -j_N(\mathbf{u}_0, \mathbf{r}_k) + \frac{1}{\varepsilon_k} \left(j_T(\mathbf{u}_0, \mathbf{u}_0) - j_T(\mathbf{u}_0, \mathbf{u}_k) \right), \ \forall \, k. \tag{26}$$

Since the positive part function is non decreasing we have for any k

$$\begin{aligned} J_N^k : &= j_N(\mathbf{u}_k, \mathbf{r}_k) - j_N(\mathbf{u}_0, \mathbf{r}_k) \\ &= \frac{1}{\varepsilon_k} \int_{\Gamma_C} c_N \left((u_{kN})_+^{m_N} - (u_{0N})_+^{m_N} \right) (u_{kN} - u_{0N}) \, \mathrm{d}\sigma(x) \geq 0. \end{aligned} \tag{27}$$

We compute now the limit $l = \lim_{k \to +\infty} J_T^k$ where

$$J_T^k := \frac{1}{\varepsilon_k} \left(j_T(\mathbf{u}_k, \mathbf{u}_k) - j_T(\mathbf{u}_k, \mathbf{u}_0) + j_T(\mathbf{u}_0, \mathbf{u}_0) - j_T(\mathbf{u}_0, \mathbf{u}_k) \right).$$

We have for any k

$$\begin{aligned} J_T^k &= \int_{\Gamma_C} c_T (u_{kN})_+^{m_T} \frac{|\mathbf{u}_{kT}| - |\mathbf{u}_{0T}|}{\varepsilon_k} \, \mathrm{d}\sigma(x) \\ &- \int_{\Gamma_C} c_T (u_{0N})_+^{m_T} \frac{|\mathbf{u}_{kT}| - |\mathbf{u}_{0T}|}{\varepsilon_k} \, \mathrm{d}\sigma(x) \\ &= \int_{\Gamma_C} c_T \left((u_{kN})_+^{m_T} - (u_{0N})_+^{m_T} \right) \frac{|\mathbf{u}_{kT}| - |\mathbf{u}_{0T}|}{\varepsilon_k} \, \mathrm{d}\sigma(x) \end{aligned}$$

and therefore we obtain as before (see for example (10), (11)) that

$$\begin{aligned} |J_T^k| &\leq C \int_{\Gamma_C} \left| (u_{kN})_+^{m_T} - (u_{0N})_+^{m_T} \right| |\mathbf{r}_k| \, \mathrm{d}\sigma(x) \\ &\leq C \|\mathbf{u}_k - \mathbf{u}_0\|_{H^1(\Omega)} \|\mathbf{r}_k\|_{H^1(\Omega)} \\ &= C \varepsilon_k \|\mathbf{r}_k\|_{H^1(\Omega)}^2. \end{aligned}$$

Since $(\mathbf{r}_k)_k$ is bounded in $\mathbf{V}$ we deduce that $\lim_{k \to +\infty} J_T^k = 0$. Combining (25), (26), (27) yields

$$a(\mathbf{r}, \mathbf{r}_k) - a(\mathbf{r}_k, \mathbf{r}_k) \geq J_N^k + J_T^k \geq J_T^k$$

and after passing to the limit for $k \to +\infty$ one gets

$$a(\mathbf{r},\mathbf{r}) \geq \limsup_{k\to+\infty}(a(\mathbf{r}_k,\mathbf{r}_k) + J_T^k) = \limsup_{k\to+\infty} a(\mathbf{r}_k,\mathbf{r}_k).$$

Since $(\mathbf{r}_k)_k$ converges weakly towards $\mathbf{r}$ in $\mathbf{V}$ we have also

$$a(\mathbf{r},\mathbf{r}) \leq \liminf_{k\to+\infty} a(\mathbf{r}_k,\mathbf{r}_k)$$

implying that $\lim_{k\to+\infty} a(\mathbf{r}_k,\mathbf{r}_k) = a(\mathbf{r},\mathbf{r})$, which says that $(\mathbf{r}_k)_k$ converges strongly towards $\mathbf{r}$ in $\mathbf{V}$. Take now any element $\mathbf{q} \in \mathcal{K}$. By the definition of $\mathcal{K}$ (with $\mathbf{v} = \mathbf{u}_k$) we have

$$a(\mathbf{q},\mathbf{r}_k) + j_N(\mathbf{u}_0,\mathbf{r}_k) + \frac{1}{\varepsilon_k} j_T(\mathbf{u}_0,\mathbf{u}_k) - \frac{1}{\varepsilon_k} j_T(\mathbf{u}_0,\mathbf{u}_0) \geq 0.$$

Using one more time (25) we deduce that

$$\begin{aligned} a(\mathbf{q},\mathbf{r}_k) - a(\mathbf{r}_k,\mathbf{r}_k) &\geq j_N(\mathbf{u}_k,\mathbf{r}_k) - j_N(\mathbf{u}_0,\mathbf{r}_k) \\ &+ \frac{1}{\varepsilon_k}(j_T(\mathbf{u}_k,\mathbf{u}_k) - j_T(\mathbf{u}_k,\mathbf{u}_0) - j_T(\mathbf{u}_0,\mathbf{u}_k) + j_T(\mathbf{u}_0,\mathbf{u}_0)) \\ &= J_N^k + J_T^k \geq J_T^k, \ \forall\, k. \end{aligned}$$

Passing to the limit for $k \to +\infty$ yields

$$a(\mathbf{q}-\mathbf{r},\mathbf{r}) \geq 0, \ \ \mathbf{q} \in \mathcal{K}.$$

Therefore the sequence $(\mathbf{r}_k)_k$ converges strongly in $\mathbf{V}$ towards the solution of the variational inequality (20). Since this solution is unique, all the family $(\varepsilon^{-1}(\mathbf{u}_\varepsilon - \mathbf{u}_0))_{\varepsilon>0}$ converges strongly in $\mathbf{V}$ towards this solution, as ε goes to zero.

□

2.2 A simplified friction problem with applications to Tresca's friction model

Now we consider a simplified friction model introduced in [6] (see also [3, 7]) involving a variational inequality of the second kind. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with a smooth boundary denoted Γ. Let $V = H^1(\Omega)$ and set:

$$u_\varepsilon \in V \ : \qquad a(u_\varepsilon, v-u_\varepsilon) + \varepsilon j(v) - \varepsilon j(u_\varepsilon) \geq L(v-u_\varepsilon), \qquad \forall\, v \in V, \qquad (28)$$

where ε is a nonnegative constant representing in some sense the friction coefficient (see Remark 2.3 hereafter) and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx, \ \forall\, u,v \in V,$$

$$L \in V',$$

$$j(v) = \int_\Gamma |v(x)| \, d\sigma(x), \ \forall\, v \in V.$$

It is easy to check (when $L(v) = \int_\Omega fv$) that the strong formulation corresponding to (28) is $-\Delta u_\varepsilon + u_\varepsilon = f$ in Ω, $|\partial u_\varepsilon/\partial \mathbf{n}| \le \varepsilon$ if $u_\varepsilon = 0$ on $\partial\Omega$ and $\partial u_\varepsilon/\partial \mathbf{n} = -\varepsilon u_\varepsilon/|u_\varepsilon|$ if $u_\varepsilon \ne 0$ on $\partial\Omega$. The inequality (28) is equivalent to the two following conditions:

$$u_\varepsilon \in V \ : \qquad a(u_\varepsilon, v) + \varepsilon j(v) \ \ge \ L(v), \qquad \forall\, v \in V, \tag{29}$$

$$a(u_\varepsilon, u_\varepsilon) + \varepsilon j(u_\varepsilon) \ = \ L(u_\varepsilon). \tag{30}$$

It is easily seen that j is convex and continuous on V (thanks to the trace theorem). Thus, Lions-Stampacchia's theorem ensures the existence of a unique solution to (28), see, e.g., [6, 12].

The main aim of the present section is to study the asymptotic behavior with respect to ε for the solutions of (28). Let us first mention a straightforward property dealing with the behavior of the solution when the loads increase:

Proposition 2.1 *Let $L_1,\ L_2 \in V'$ such that:*

$$L_1(v) \ge L_2(v),\ \forall\, v \in V,\ \ v \ge 0.$$

Let $u_{1,\varepsilon}$ be the solution to (28) with $L = L_1$ and let $u_{2,\varepsilon}$ be the solution to (28) with $L = L_2$. Then

$$u_{1,\varepsilon} \ge u_{2,\varepsilon}.$$

Proof. As here, ε is fixed, for simplicity we denote the solutions of (28) corresponding to L_1, L_2 by u_1, u_2 respectively. Taking $L = L_1$ and $v = u_1 + (u_1 - u_2)_-$ in (28), we get

$$a(u_1, (u_1 - u_2)_-) + \varepsilon j(u_1 + (u_1 - u_2)_-) - \varepsilon j(u_1) \ge L_1((u_1 - u_2)_-). \tag{31}$$

Similarly, with $L = L_2$ and $v = u_2 - (u_1 - u_2)_-$ in (28) we obtain,

$$a(u_2, -(u_1 - u_2)_-) + \varepsilon j(u_2 - (u_1 - u_2)_-) - \varepsilon j(u_2) \ge L_2(-(u_1 - u_2)_-). \tag{32}$$

Since

$$j(u_1 + (u_1 - u_2)_-) + j(u_2 - (u_1 - u_2)_-) - j(u_1) - j(u_2) = 0,$$

we get by adding (31) and (32):

$$a(u_1 - u_2, (u_1 - u_2)_-) \ge (L_1 - L_2)((u_1 - u_2)_-) \ge 0,$$

and thus

$$-a((u_1 - u_2)_-, (u_1 - u_2)_-) \ge 0,$$

which proves the announced result. □

2.2.1 Asymptotic behavior for small friction coefficient

When $\varepsilon = 0$, the variational inequality (28) simply reduces to the equality

$$u_0 \in V \ : \qquad a(u_0, v) = L(v), \qquad \forall\, v \in V. \tag{33}$$

Choosing $v = u_0$ in (28) and $v = u_\varepsilon - u_0$ in (33) we get

$$a(u_\varepsilon, u_0 - u_\varepsilon) + \varepsilon j(u_0) - \varepsilon j(u_\varepsilon) \geq L(u_0 - u_\varepsilon),$$

and

$$a(u_0, u_\varepsilon - u_0) = L(u_\varepsilon - u_0).$$

By addition, we obtain

$$\begin{aligned} \|u_\varepsilon - u_0\|^2_{H^1(\Omega)} \leq \varepsilon j(u_0) - \varepsilon j(u_\varepsilon) \leq \varepsilon j(u_\varepsilon - u_0) &\leq \varepsilon |\Gamma|^{1/2} \|u_\varepsilon - u_0\|_{L^2(\Gamma)} \\ &\leq C(\Omega)\varepsilon \|u_\varepsilon - u_0\|_{H^1(\Omega)}, \end{aligned} \quad (34)$$

where Cauchy-Schwarz inequality and the trace theorem have been used, and where $|\Gamma|$ denotes the measure of Γ. Hence u_ε converges towards u_0 in V as ε vanishes. Now we prove the following theorem.

Theorem 2.2 *Let*

$$D = \{ z \in V \ : \quad a(z, v - u_0) + j(v) - j(u_0) \geq 0, \quad \forall\, v \in V \},$$

($-D$ is the nonempty closed convex set of subgradients of j at u_0 with respect to the inner product of V). Then, when ε goes to zero, $(u_\varepsilon - u_0)/\varepsilon$ strongly converges in V to the unique solution of

$$y \in D \ : \qquad a(y, z - y) \geq 0, \qquad \forall\, z \in D. \quad (35)$$

Proof. Let $y_\varepsilon = (u_\varepsilon - u_0)/\varepsilon$. From (34) we deduce that $\sup_{\varepsilon>0} \|y_\varepsilon\|_{H^1(\Omega)} < +\infty$. Hence there exist $y \in V$ and a sequence $(\varepsilon_k)_k \searrow 0$ such that $(y_k)_k := (y_{\varepsilon_k})_k$ weakly converges in V to y. From (29) and (33) we get that

$$a(y_\varepsilon, v) + j(v) \geq 0, \qquad \forall\, v \in V.$$

Taking $\varepsilon = \varepsilon_k$ and passing to the limit, we obtain:

$$a(y, v) + j(v) \geq 0, \qquad \forall\, v \in V. \quad (36)$$

Besides, the substraction of (33) (where $v = u_\varepsilon$) with (30) yields

$$a(y_\varepsilon, u_\varepsilon) + j(u_\varepsilon) = 0.$$

We take $\varepsilon = \varepsilon_k$. From the strong convergence of $(u_{\varepsilon_k})_k$ and the weak convergence of $(y_{\varepsilon_k})_k$ we get:

$$a(y, u_0) + j(u_0) = 0. \quad (37)$$

Recombining (36) and (37) we deduce that y satisfies the following inequality:

$$a(y, v - u_0) + j(v) - j(u_0) \geq 0, \qquad \forall\, v \in V, \quad (38)$$

saying that $y \in D$. Now, from (33) and (28) we have

$$a(u_0, u_\varepsilon - u_0) = L(u_\varepsilon - u_0),$$

$$a(u_\varepsilon, u_0 - u_\varepsilon) + \varepsilon j(u_0) - \varepsilon j(u_\varepsilon) \geq L(u_0 - u_\varepsilon).$$

By addition and division by ε^2 we obtain

$$a(y_\varepsilon, y_\varepsilon) \leq \varepsilon^{-1} \left(j(u_0) - j(u_\varepsilon) \right). \tag{39}$$

Taking now $v = u_\varepsilon$ in the limit problem (38) and dividing by ε, one gets

$$a(y, y_\varepsilon) \geq \varepsilon^{-1} \left(j(u_0) - j(u_\varepsilon) \right). \tag{40}$$

Putting together (39) and (40) written with $\varepsilon = \varepsilon_k$ we deduce

$$\limsup_{k\to+\infty} a(y_{\varepsilon_k}, y_{\varepsilon_k}) \leq \limsup_{k\to+\infty} a(y, y_{\varepsilon_k}) = a(y, y) \leq \liminf_{k\to+\infty} a(y_{\varepsilon_k}, y_{\varepsilon_k}).$$

Thus $(a(y_{\varepsilon_k}, y_{\varepsilon_k}))_k$ converges to $a(y, y)$ and therefore $\lim\limits_{k\to+\infty} \|y_{\varepsilon_k} - y\|_{H^1(\Omega)} = 0$. And finally let $z \in D$. Then, by the definition of D (with $v = u_\varepsilon$) we have

$$a(z, y_\varepsilon) \geq \varepsilon^{-1} \left(j(u_0) - j(u_\varepsilon) \right).$$

Together with (39) the last inequality implies

$$a(z, y_\varepsilon) \geq a(y_\varepsilon, y_\varepsilon).$$

Taking $\varepsilon = \varepsilon_k$ and passing to the limit for $k \to +\infty$ we obtain that y solves (35). And since (35) admits a unique solution, the proof is complete. □

Remark 2.3 *In this remark we show that the result obtained in Theorem 2.2 can be extended to a more physically relevant case dealing with the elasticity operator and vector valued functions. More precisely, let us consider an elastic body Ω in $\mathbb{R}^n$ with $n = 2$ or $n = 3$ where plane small strain assumptions are made. The regular boundary Γ of Ω consists in two nonoverlapping parts Γ_D and Γ_C of positive measures. The normal unitary outward vector on Γ is denoted by $\mathbf{n}$. In its initial stage, the body is in contact on Γ_C with a rigid foundation (the extension to two elastic bodies in contact can be easily made, at least for small strain models) and we suppose that the body slips on Γ_C with friction. The body is clamped on Γ_D and it is subjected to volume forces $\mathbf{f} \in (L^2(\Omega))^n$. We assume that the normal constraint $S = S(x) \geq 0$ is given on Γ_C and $\mu \geq 0$ denotes the friction coefficient between the body and the rigid foundation (this corresponds to a so-called bilateral contact model with Tresca friction).*

The frictional contact problem in linear elasticity consists in finding the displacement field $\mathbf{u} := \mathbf{u}_\mu : \Omega \to \mathbb{R}^n$ verifying the equations and conditions (41),(42):

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_D. \end{cases} \tag{41}$$

The conditions describing bilateral contact with Tresca friction (using the same notation as in section 2.1) on Γ_C are:

$$\begin{cases} u_N = 0, \\ |\boldsymbol{\sigma}_T(\mathbf{u})| \leq \mu S, \quad \text{if} \quad \mathbf{u}_T = \mathbf{0}, \\ \boldsymbol{\sigma}_T(\mathbf{u}) = -\mu S \dfrac{\mathbf{u}_T}{|\mathbf{u}_T|}, \quad \text{if} \quad \mathbf{u}_T \neq \mathbf{0}. \end{cases} \tag{42}$$

The variational inequality issued from (41),(42) is: find $\mathbf{u}_\mu \in \mathbf{V}$ *such that*

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}_\mu) : \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}_\mu)dx+\mu\int_{\Gamma_C} S|\mathbf{v}_T|d\sigma(x)-\mu\int_{\Gamma_C} S|(\mathbf{u}_\mu)_T|d\sigma(x) \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_\mu)dx, \quad (43)$$

for all $\mathbf{v} \in \mathbf{V}$ *where*

$$\mathbf{V} = \Big\{\mathbf{v} \in (H^1(\Omega))^n \ : \ \mathbf{v} = \mathbf{0} \ \text{on } \Gamma_D, \ v_N = 0 \ \text{on } \Gamma_C\Big\}.$$

For any nonnegative μ *there is a unique solution* $\mathbf{u}_\mu$ *of (43) and the results in Theorem 2.2 hold straightforwardly in this case.*

2.2.2 Asymptotic behavior for large friction coefficients

We consider now the behavior for the solutions of (28) when the friction coefficient goes to infinity. Let $f \in L^2(\Omega)$ and denote by u_∞ the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_\infty(x) + u_\infty(x) = f(x), & x \in \Omega, \\ u_\infty(x) = 0, & x \in \Gamma. \end{cases}$$

Proposition 2.2 *Assume that* f *and* Ω *are such that* $\frac{\partial u_\infty}{\partial n} \in L^\infty(\Gamma)$ *where* $\boldsymbol{n}$ *represents the unitary outward normal to* Ω*. Consider the linear application* $L \in V'$ *given by* $L(v) = \int_\Omega f(x)v(x)\,dx$ *for any* $v \in V$*. Then for any* $\varepsilon \geq \left\|\frac{\partial u_\infty}{\partial n}\right\|_{L^\infty(\Gamma)}$ *the solution of (28) coincides with* u_∞*.*

Remark 2.4 *A similar blocking phenomenon (in the entire body and not only on the boundary) occurs in the case of a Bingham fluid when the external loads are small enough (which corresponds in a certain sense to a large yield limit, see [2]).*

Proof. (of Proposition 2.2) Let $v \in V$. We have $u_\infty \in V$ and

$$\begin{aligned} & a(u_\infty, v-u_\infty) + \varepsilon j(v) - \varepsilon j(u_\infty) \\ = \quad & \int_\Omega \nabla u_\infty \cdot \nabla(v-u_\infty)dx + \int_\Omega u_\infty(x)(v(x)-u_\infty(x))dx + \varepsilon j(v) \\ = \quad & \int_\Gamma \frac{\partial u_\infty}{\partial n}(v(x)-u_\infty(x))d\sigma(x) + L(v-u_\infty) + \varepsilon j(v) \\ = \quad & \int_\Gamma \left(\frac{\partial u_\infty}{\partial n}\ v(x) + \varepsilon|v(x)|\right) d\sigma(x) + L(v-u_\infty) \\ \geq \quad & L(v-u_\infty). \end{aligned}$$

Hence $u_\varepsilon = u_\infty$ for any $\varepsilon \geq \left\|\frac{\partial u_\infty}{\partial n}\right\|_{L^\infty(\Gamma)}$. □

Remark 2.5 *Once again the results of Proposition 2.2 can be applied to the inequality (43) dealing with the elasticity operator. More precisely, let* $\mathbf{f} \in (L^2(\Omega))^n$ *and let* $\mathbf{u}_\infty$ *denote the solution of the problem*

$$\begin{cases} -\mathbf{div}\ \boldsymbol{\sigma}(\mathbf{u}_\infty) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}_\infty = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Then, if $\mu S(x) \geq |\boldsymbol{\sigma}_T(\mathbf{u}_\infty(x))|$ *a.e.* $x \in \Gamma_C$ *then* $\mathbf{u}_\mu = \mathbf{u}_\infty$*.*

2.2.3 Monotonicity of the solution with respect to the friction

In this paragraph we prove that the solution of (28) increases when ε decreases, namely, that

$$0 < \varepsilon_1 \leq \varepsilon_2 \Rightarrow u_{\varepsilon_1} \geq u_{\varepsilon_2}. \tag{44}$$

We prove this result for nonnegative solutions. Observing that the solution of (28) with $L = 0$ is $u_\varepsilon = 0$ and applying Proposition 2.1 we deduce that nonnegative solutions are obtained for instance if L is such that $L(v) \geq 0$ for any $v \in V,\ v \geq 0$.

Let us first consider the regularized problem

$$u_\varepsilon^\delta \in V \ : \qquad a(u_\varepsilon^\delta, v - u_\varepsilon^\delta) + \varepsilon j_\delta(v) - \varepsilon j_\delta(u_\varepsilon^\delta) \geq L(v - u_\varepsilon^\delta), \qquad \forall\ v \in V, \tag{45}$$

where for any $v \in V$ and $\delta \geq 0$ the convex function j_δ is given by

$$j_\delta(v) = \int_\Gamma \sqrt{v(x)^2 + \delta^2}\, d\sigma(x).$$

The following proposition shows that it is sufficient to prove (44) for any regularized problem (45) with $\delta > 0$.

Proposition 2.3 *For any fixed $\varepsilon > 0$, when $\delta \searrow 0$, $(u_\varepsilon^\delta)_\delta$ strongly converges in V to u_ε, the unique solution to Problem (28).*

Proof. Since ε is fixed in this proposition, we use the simplified notation $u^\delta := u_\varepsilon^\delta$ and $u := u_\varepsilon$. From (45) and (28) we have for any $\delta > 0$:

$$a(u^\delta, u - u^\delta) + \varepsilon j_\delta(u) - \varepsilon j_\delta(u^\delta) \geq L(u - u^\delta),$$

and

$$a(u, u^\delta - u) + \varepsilon j(u^\delta) - \varepsilon j(u) \geq L(u^\delta - u).$$

By adding the two previous inequalities we get

$$\begin{aligned} a(u - u^\delta, u - u^\delta) &\leq \varepsilon j_\delta(u) - \varepsilon j_\delta(u^\delta) + \varepsilon j(u^\delta) - \varepsilon j(u) \\ &\leq \varepsilon j_\delta(u) - \varepsilon j(u) \\ &= \varepsilon \int_\Gamma \left(\frac{\delta^2}{\sqrt{u^2 + \delta^2} + |u|} \right) d\sigma(x) \\ &\leq \varepsilon \delta |\Gamma|, \end{aligned}$$

where $|\Gamma|$ denotes the measure of Γ. Hence the result. $\square$

We prove now the monotonicity property of u_ε^δ with respect to ε for any fixed $\delta > 0$.

Proposition 2.4 *Consider $\varepsilon_1, \varepsilon_2, \delta \in \mathbb{R}$ such that $0 < \varepsilon_1 \leq \varepsilon_2$, $\delta > 0$. Assume that the solutions $u_{\varepsilon_1}^\delta, u_{\varepsilon_2}^\delta$ are nonnegative. Then we have the inequality*

$$u_{\varepsilon_1}^\delta \geq u_{\varepsilon_2}^\delta.$$

Proof. We use the notations $u_1 := u^\delta_{\varepsilon_1}$, $u_2 := u^\delta_{\varepsilon_2}$. Choosing $v = u_i + hw$ in (28), with $h \in \mathbb{R}$ and $w \in V$ we get for $i \in \{1, 2\}$

$$a(u_i, hw) + \varepsilon_i j_\delta(u_i + hw) - \varepsilon_i j_\delta(u_i) \geq L(hw).$$

For $h > 0$ we obtain

$$a(u_i, w) + \frac{\varepsilon_i}{h} \int_\Gamma \left(\sqrt{(u_i + hw)^2 + \delta^2} - \sqrt{u_i^2 + \delta^2} \right) d\sigma(x) \geq L(w).$$

Letting $h \searrow 0$ we deduce using Lebesgue's theorem

$$a(u_i, w) + \varepsilon_i \int_\Gamma \frac{u_i w}{\sqrt{u_i^2 + \delta^2}} d\sigma(x) \geq L(w).$$

Similarly, considering $h < 0$, one gets

$$a(u_i, w) + \varepsilon_i \int_\Gamma \frac{u_i w}{\sqrt{u_i^2 + \delta^2}} d\sigma(x) \leq L(w).$$

Then we obtain for any $w \in V$ and $i \in \{1, 2\}$

$$a(u_i, w) + \varepsilon_i \int_\Gamma \frac{u_i w}{\sqrt{u_i^2 + \delta^2}} d\sigma(x) = L(w).$$

We introduce the increasing application $t \to \varphi(t) = \frac{t}{\sqrt{t^2+\delta^2}}$, $t \in \mathbb{R}$. The above formulation can be written

$$a(u_i, w) + \varepsilon_i \int_\Gamma \varphi(u_i) w \, d\sigma(x) = L(w), \quad w \in V, \ i \in \{1, 2\}.$$

Taking as test function $w = (u_1 - u_2)_-$ we obtain

$$a(u_1 - u_2, (u_1 - u_2)_-) + \int_\Gamma (\varepsilon_1 \varphi(u_1) - \varepsilon_2 \varphi(u_2))(u_1 - u_2)_- \, d\sigma(x) = 0,$$

implying that

$$\begin{aligned} a((u_1 - u_2)_-, (u_1 - u_2)_-) &= \int_\Gamma (\varepsilon_1 \varphi(u_1) - \varepsilon_2 \varphi(u_2))(u_1 - u_2)_- \, d\sigma(x) \\ &\leq \varepsilon_2 \int_\Gamma (\varphi(u_1) - \varphi(u_2))(u_1 - u_2)_- \, d\sigma(x). \end{aligned}$$

By the monotonicity of φ we deduce that $(\varphi(u_1) - \varphi(u_2))(u_1 - u_2)_- \leq 0$ and therefore

$$a((u_1 - u_2)_-, (u_1 - u_2)_-) \leq 0,$$

saying that $u_1 \geq u_2$. □

Combining now the Propositions 2.3, 2.4 we obtain the following result

Proposition 2.5 *Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that $0 < \varepsilon_1 \leq \varepsilon_2$ and assume that $L \in V'$ verifies $L(v) \geq 0$ for any $v \in V, v \geq 0$. Then we have the inequality $u_{\varepsilon_1} \geq u_{\varepsilon_2}$ where $(u_{\varepsilon_i})_{i \in \{1,2\}}$ are the solutions of (28) corresponding to $(\varepsilon_i)_{i \in \{1,2\}}$.*

Proof. Applying Proposition 2.4 we have $u^\delta_{\varepsilon_1} \geq u^\delta_{\varepsilon_2}$ for any $\delta > 0$. Passing to the limit when $\delta \searrow 0$ we obtain by Proposition 2.3 that $u_{\varepsilon_1} \geq u_{\varepsilon_2}$. □

2.3 Some open questions

An open question is concerned with the study of the widespread Coulomb friction model in linear elasticity. It is well known that this problem admits a solution if the friction coefficient is small enough (see e.g., [16, 10, 4, 5] and the references quoted therein). Note that uniqueness does not hold in the general case, at least for large friction coefficients, see [8, 9]. More recently a first uniqueness result has been obtained in [18] with the assumption that a "regular" solution exists (with some additional technical assumptions) and that the friction coefficient is sufficiently small. This uniqueness result could be the starting point of a asymptotic analysis similar to the one achieved in this work.

Another line of research would consist of obtaining the higher order corrections and searching $\mathbf{r}_1, ..., \mathbf{r}_\alpha$ such that $\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon\mathbf{r}_1 + ... + \varepsilon^\alpha \mathbf{r}_\alpha + \mathcal{O}(\varepsilon^{\alpha+1})$ for the previous models.

Finally let us mention that we are not able to extend the results dealing with comparison of solutions such as positivity and monotonicity to the vector valued case involving the elasticity operator.

3 The elastoplastic torsion problem

Now we consider a simple variational inequality of the first kind modelling the elastoplastic torsion of a cylinder.

3.1 Problem set-up

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary. We consider the variational inequality modelling the torsion of an infinitely long elastoplastic cylinder of cross section Ω and plasticity yield $r > 0$: find the stress potential u such that

$$u \in K_r \ : \qquad a(u, v-u) \geq L(v-u), \qquad \forall\, v \in K_r, \tag{46}$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the bilinear form given by:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \ \forall\, u, v \in H_0^1(\Omega),$$

and $L \in H^{-1}(\Omega) = (H_0^1(\Omega))'$. The notation K_r represents the nonempty closed convex set of admissible stress potentials:

$$K_r = \left\{ v \in H_0^1(\Omega) \ : \ |\nabla v| \leq r, \text{ a.e. in } \Omega \right\}.$$

From Stampacchia's theorem we deduce that Problem (46) admits a unique solution (see also, e.g., [1, 3, 6, 7, 12]). More generally, for $\varepsilon > 0$ let u_ε be the unique solution of the variational inequality:

$$u_\varepsilon \in K_\varepsilon \ : \qquad a(u_\varepsilon, v-u_\varepsilon) \geq L(v-u_\varepsilon), \qquad \forall\, v \in K_\varepsilon, \tag{47}$$

where

$$K_\varepsilon = \left\{ v \in H_0^1(\Omega) \ : \ |\nabla v| \leq \varepsilon, \text{ a.e. in } \Omega \right\}.$$

When the solution of

$$u \in H_0^1(\Omega) \ : \qquad a(u,v) = L(v), \qquad \forall\, v \in H_0^1(\Omega),$$

lies in $W^{1,\infty}(\Omega)$ then it is easy to see that $u_\varepsilon = u$ for any $\varepsilon \geq \|\nabla u\|_{L^\infty(\Omega)}$. For $\varepsilon > 0$ we set

$$w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}.$$

Our aim is to study the behavior of w_ε as ε tends to zero. According to (47), we see that for any positive ε, w_ε is the unique solution of

$$w_\varepsilon \in K_1 \ : \qquad \varepsilon a(w_\varepsilon, v - w_\varepsilon) \geq L(v - w_\varepsilon), \qquad \forall\, v \in K_1. \tag{48}$$

3.2 Asymptotic behavior for small plasticity yield

In this paragraph we establish the convergence of the family $(w_\varepsilon)_{\varepsilon>0}$ as ε goes to zero. We introduce the problem

$$u \in K_1 \ : \qquad L(v - u) \leq 0, \qquad \forall\, v \in K_1, \tag{49}$$

and we denote by C the set of all the solutions of the above problem

$$C = \{u \in K_1 \ : \ L(v-u) \leq 0, \ \forall\, v \in K_1\}.$$

Proposition 3.1 *The set C is nonempty, closed in $H_0^1(\Omega)$ and convex.*

Proof. By definition K_1 is bounded in $H_0^1(\Omega)$ and therefore $\{L(v) \ : \ v \in K_1\}$ is bounded in $\mathbb{R}$. We denote by M the supremum

$$M = \sup\{L(v) \ : \ v \in K_1\},$$

and let us consider $(v_n)_n \subset K_1$ a maximizing sequence for M

$$M - \frac{1}{n} \leq L(v_n) \leq M, \ \forall\, n \geq 1. \tag{50}$$

After extraction of a subsequence we can assume that $v_{n_k} \rightharpoonup \tilde{v}$ weakly in $H_0^1(\Omega)$ and therefore $\lim_{k\to+\infty} L(v_{n_k}) = L(\tilde{v})$. Since K_1 is closed and convex, it is also weakly closed implying that $\tilde{v}$ belongs to K_1. Passing to the limit in (50) we obtain that $M = L(\tilde{v})$ and therefore $\tilde{v} \in C$, saying that C is nonempty. It is easily seen that C is closed and convex.
□

Theorem 3.1 *The family $(w_\varepsilon)_{\varepsilon>0}$ converges strongly in $H_0^1(\Omega)$ as ε goes to zero towards the element of minimal norm of the set C*

$$\lim_{\varepsilon \searrow 0} w_\varepsilon = w, \ \textit{strongly in } H_0^1(\Omega),$$

where

$$w \in C \ : \ \|\nabla w\|_{L^2(\Omega)} = \min_{v \in C} \|\nabla v\|_{L^2(\Omega)}. \tag{51}$$

Remark 3.1 *As L is linear and K_1 is not a strictly convex set (a convex set K is said to be strictly convex if for any u_1, u_2 in K, $u_1 \neq u_2$ and $\lambda \in]0,1[$, $\lambda u_1 + (1-\lambda)u_2$ lies in the interior of K) uniqueness to Problem (49) is not guaranteed in general. Let for instance $\Omega =]0,1[$ and $L(v) = \int_0^{1/4} v(x)dx$; then the set of solutions to (49) is precisely the set of $u \in K_1$ such that $u(x) = x$ in $[0, 1/4]$. Amongst all these solutions, a calculation shows that the one which solves (51) is given by $w(x) = (1-x)/3$ in $[1/4, 1]$.*

Proof. (of Theorem 3.1) Since $|\nabla w_\varepsilon| \leq 1$ a.e. in Ω, $(\nabla w_\varepsilon)_\varepsilon$ is bounded in $L^2(\Omega)$ implying that $(w_\varepsilon)_\varepsilon$ is bounded in $H_0^1(\Omega)$. Hence, there exists a sequence $(\varepsilon_k)_k$ converging to zero and $w \in H_0^1(\Omega)$, such that $(w_k)_k := (w_{\varepsilon_k})_k$ weakly converges in $H_0^1(\Omega)$ to w. As K_1 is closed and convex, $w \in K_1$. Taking $\varepsilon = \varepsilon_k$ in (48) and passing to the limit as $k \to +\infty$ we obtain that w solves (49). We claim that the sequence $(w_k)_k$ converges strongly in $H_0^1(\Omega)$. As $a(\cdot,\cdot)$ is symmetric, an equivalent formulation for (48) is

$$w_\varepsilon \in K_1, \qquad \frac{\varepsilon}{2} a(w_\varepsilon, w_\varepsilon) - L(w_\varepsilon) \leq \frac{\varepsilon}{2} a(v,v) - L(v), \qquad \forall\, v \in K_1. \tag{52}$$

Taking $\varepsilon = \varepsilon_k$ and $v = w$ in (52), we deduce from (49) that

$$\forall\, k \geq 0, \qquad \frac{\varepsilon_k}{2} a(w_k, w_k) - \frac{\varepsilon_k}{2} a(w,w) \leq L(w_k - w) \leq 0.$$

Hence, we obtain the inequality

$$\limsup_{k \to +\infty} a(w_k, w_k) \leq a(w,w),$$

while the weak convergence of $(w_k)_k$ and the lower semicontinuity of $a(\cdot,\cdot)$ with respect to the weak topology of $H_0^1(\Omega)$ imply

$$a(w,w) \leq \liminf_{k \to +\infty} a(w_k, w_k).$$

Therefore we have the convergence $a(w_k, w_k) \to a(w,w)$ as $k \to +\infty$. This in turn implies that $(a(w_k - w, w_k - w))_k$ converges to zero and thus the convergence of $(w_k)_k$ to w holds strongly in $H_0^1(\Omega)$.

Since generally (49) does not admit a unique solution (see Remark 3.1), we need to characterize w amongst the solutions of Problem (49). Let $z \in C$ be a solution of (49). Taking $\varepsilon = \varepsilon_k$ and $v = z$ in (52), we obtain as before

$$\forall\, k \geq 0, \qquad a(w_k, w_k) - a(z,z) \leq \frac{2}{\varepsilon_k} L(w_k - z) \leq 0.$$

Hence, passing to the limit as $k \to +\infty$ yields

$$\forall\, z \in C, \qquad a(w,w) \leq a(z,z).$$

In other words w is the unique solution of Problem (51). As w is unique, the whole family $(w_\varepsilon)_\varepsilon$ converges to w as ε goes to zero. The proof is complete. $\square$

Let $d_{\partial\Omega}$ denote the distance function with respect to $\partial\Omega$

$$d_{\partial\Omega}(x) = \inf_{y\in\partial\Omega} |x-y|, \; \forall\, x \in \overline{\Omega},$$

where $|\cdot|$ stands for the euclidian norm in $\mathbb{R}^n$. It is straightforward to check that $d_{\partial\Omega} \in K_1$. The following additional result holds:

Proposition 3.2 *Let w be the solution of (51) and assume that the linear application L satisfies*

$$L(v) \geq 0, \; \forall\, v \in H_0^1(\Omega), \;\; v \geq 0. \tag{53}$$

Then we have

$$L(w - d_{\partial\Omega}) = 0.$$

In particular, if $L(v) = \int_\Omega f(x)v(x)dx$ for any $v \in H_0^1(\Omega)$ with $f \in L^2(\Omega)$ and $f > 0$, then $\lim_{\varepsilon\searrow 0} w_\varepsilon = d_{\partial\Omega}$ strongly in $H_0^1(\Omega)$. If in addition there is some positive constant f_0 such that $f \geq f_0$, then

$$\lim_{\varepsilon\searrow 0} \varepsilon^{-1} \|d_{\partial\Omega} - w_\varepsilon\|_{L^1(\Omega)} = 0.$$

Proof. To prove the first part of the proposition, it suffices to take $v = d_{\partial\Omega}$ in (49) to obtain

$$L(d_{\partial\Omega} - w) \leq 0.$$

As $d_{\partial\Omega} - v \geq 0$ for any $v \in K_1$, we deduce by (53) that $L(d_{\partial\Omega} - w) \geq 0$ and thus $L(d_{\partial\Omega} - w) = 0$. In particular if $f > 0$ then $w = d_{\partial\Omega}$. Assume now that $f \geq f_0 > 0$ for some constant f_0. Choosing $v = w$ in (48) one gets

$$\frac{f_0}{\varepsilon}\|d_{\partial\Omega} - w_\varepsilon\|_{L^1(\Omega)} \leq \frac{1}{\varepsilon}L(d_{\partial\Omega} - w_\varepsilon) = \frac{1}{\varepsilon}L(w - w_\varepsilon) \leq a(w_\varepsilon, w - w_\varepsilon),$$

and therefore $\lim_{\varepsilon\searrow 0} \varepsilon^{-1}\|d_{\partial\Omega} - w_\varepsilon\|_{L^1(\Omega)} = 0.$ $\square$

Remark 3.2 *1. Let us mention that calculations similar to those in Proposition 3.2 were achieved in [6] to prove that the solution of (46) converges to $d_{\partial\Omega}$ when $f(=$ constant$)$ tends to infinity.*

2. The case f = constant corresponds to a physically meaningful model in which $w = d_{\partial\Omega}$ according to the previous proposition.

We end this section with a monotonicity property of $(w_\varepsilon)_{\varepsilon>0}$ with respect to ε.

Proposition 3.3 *Let $0 < \varepsilon_1 \leq \varepsilon_2$ and $L \in H^{-1}(\Omega)$ satisfy $L(v) \geq 0$ for any $v \in H_0^1(\Omega)$, $v \geq 0$. Then we have the inequality*

$$w_{\varepsilon_1} = \frac{u_{\varepsilon_1}}{\varepsilon_1} \geq \frac{u_{\varepsilon_2}}{\varepsilon_2} = w_{\varepsilon_2}.$$

Proof. We use the notations $w_1 := w_{\varepsilon_1}, w_2 := w_{\varepsilon_2}$. We intend to use the variational inequality (48) with $\varepsilon = \varepsilon_1$ and $v_1 = w_1 + (w_1 - w_2)_-$ (the notation $(\cdot)_-$ stands for the negative part). We need to check that v_1 belongs to K_1. For this observe that

$$v_1 = w_1 \mathbf{1}_{\{w_1 \geq w_2\}} + w_2 \mathbf{1}_{\{w_1 < w_2\}},$$

and therefore $v_1 \in H_0^1(\Omega)$, $|\nabla v_1| \leq \max\{|\nabla w_1|, |\nabla w_2|\} \leq 1$. We obtain

$$\varepsilon_1 a(w_1, (w_1 - w_2)_-) \geq L((w_1 - w_2)_-). \tag{54}$$

Taking now $\varepsilon = \varepsilon_2$ and $v = w_2 - (w_1 - w_2)_-$ one has

$$\varepsilon_2 a(w_2, -(w_1 - w_2)_-) \geq -L((w_1 - w_2)_-). \tag{55}$$

Combining (54), (55) yields

$$\varepsilon_1 \varepsilon_2 a(w_1 - w_2, (w_1 - w_2)_-) \geq (\varepsilon_2 - \varepsilon_1) L((w_1 - w_2)_-) \geq 0,$$

implying that $a((w_1 - w_2)_-, (w_1 - w_2)_-) \leq 0$. Therefore we deduce that $(w_1 - w_2)_- = 0$ saying that

$$\frac{u_{\varepsilon_1}}{\varepsilon_1} = w_1 \geq w_2 = \frac{u_{\varepsilon_2}}{\varepsilon_2}.$$

□

References

[1] K. Atkinson and W. Han, *Theoretical numerical analysis: a functional analysis framework*, Springer, New-York, Texts in Applied Mathematics, 39, 2001; second edition 2005.

[2] M. Bostan and P. Hild, *Starting flow analysis for Bingham Fluids*, Nonlinear Anal., 64 (2006), 1119–1139.

[3] G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Grundlehren der Mathematischen Wissenschaften, 219. Springer-Verlag, Berlin-New York, 1976.

[4] C. Eck and J. Jarušek, *Existence results for the static contact problem with Coulomb friction*, Math. Models Meth. Appl. Sci., 8 (1998) 445–468.

[5] C. Eck, J. Jarušek and M. Krbec, *Unilateral contact problems: variational methods and existence theorems*, Pure and Applied Mathematics Volume: 270, CRC Press, 2005.

[6] R. Glowinski, *Lectures on numerical methods for nonlinear variational problems.* Notes by M. G. Vijayasundaram and M. Adimurthi. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 65. Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1980.

[7] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical analysis of variational inequalities.* Studies in Mathematics and its Applications, 8. North-Holland Publishing Co., Amsterdam-New York, 1981.

[8] P. Hild, *Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity*, Quart. J. Mech. Appl. Math., 57 (2004) 225–235.

[9] P. Hild, *Multiple solutions of stick and separation type in the Signorini model with Coulomb friction*, Z. Angew. Math. Mech., 85 (2005) 673–680.

[10] J. Jarušek, *Contact problems with bounded friction. Coercive case*, Czechoslovak. Math. J., 33 (1983) 237–261.

[11] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Stud. Appl. Math. 8, SIAM, Philadelphia, 1988.

[12] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications.* Pure and Applied Mathematics, 88. Academic Press, New York-London, 1980.

[13] A. Klarbring, A. Mikelić and M. Shillor, *Frictional contact problems with normal compliance*, Internat. J. Engrg. Sci., 26 (1988) 811–832.

[14] A. Klarbring, A. Mikelić and M. Shillor, *On friction problems with normal compliance*, Nonlinear Anal., 13 (1989) 935–955.

[15] J.A.C. Martins and J.T. Oden, *Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws*, Nonlinear Anal., 11 (1987) 407–428.

[16] J. Nečas, J. Jarušek, and J. Haslinger, *On the solution of the variational inequality to the Signorini problem with small friction*, Boll. Unione Mat. Ital., 17 (1980) 796–811.

[17] J.T. Oden and J.A.C. Martins, *Models and computational methods for dynamic friction phenomena*, Comput. Methods. Appl. Mech. Engrg., 52 (1985) 527–634.

[18] Y. Renard, *A uniqueness criterion for the Signorini problem with Coulomb friction*, SIAM J. Math. Anal., 38 (2006) 458–467.

[19] M. Shillor, M. Sofonea and J.J. Telega, *Models and analysis of quasistatic contact. Varational methods.*, Springer, 2004.