Stationary solutions for the one dimensional Nordström-Vlasov system

Mihai Bostan *

(19th March 2006)

Abstract

The Nordström-Vlasov system describes the evolution of self-gravitating collisionless particles. We prove the existence of stationary solutions in one dimension. We show also the propagation of impulsion moments and perform an asymptotic analysis.

Keywords: Nordström equation, Vlasov equation, mild solution.

AMS classification: 35B35, 35A05, 35B45, 82D10.

1 Introduction

We consider a population of particles interacting by fields created collectively and we neglect the collisions. Examples of such collisionless gases occur in plasma physics and in astrophysics. If the particles interact by electro-magnetic forces we deal with

*Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex France. E-mail : `mbostan@math.univ-fcomte.fr`

the Vlasov-Maxwell equations, see [4], [8], [10], [11], [12], [13], [14], [15], [17]. If the particles interact by gravitational forces the evolution of the system is given by the Einstein-Vlasov equations, see [1], [20], [21], [22]. Recently Calogero and Rein studied in [6], [7] a different relativistic model obtained by coupling the Vlasov equation to the Nordström scalar gravitation theory [18].

Let $F = F(t, x, p) \geq 0$ denote the density of particles in phase-space. Here $t \in \mathbb{R}$ represents time, $x \in \mathbb{R}^N$ position and $p \in \mathbb{R}^N$ impulsion, with $N \in \{1, 2, 3\}$. This density satisfies the Vlasov equation

$$\partial_t F + v(p) \cdot \nabla_x F - \Big((\partial_t \phi + v(p) \cdot \nabla_x \phi) p + (1 + |p|^2)^{-\frac{1}{2}} \nabla_x \phi \Big) \cdot \nabla_p F = 0,$$

coupled to the wave equation

$$\partial_t^2 \phi - \Delta_x \phi = -e^{(N+1)\phi(t,x)} \int_{\mathbb{R}^N} \frac{F(t, x, p)}{(1 + |p|^2)^{\frac{1}{2}}} \, dp.$$

We assume that the physical constants (the mass of particles, the gravitational constant and the speed of the light) are all equal to unity and $v(p) = \frac{p}{(1+|p|^2)^{1/2}}$ denotes the relativistic velocity of a particle with impulsion p (see [5] for more details on this model). After introducing the new unknown $f(t, x, p) = e^{(N+1)\phi(t,x)} F(t, x, p)$ the system becomes

$$\partial_t f + v(p) \cdot \nabla_x f - \Big((S\phi) p + (1 + |p|^2)^{-\frac{1}{2}} \nabla_x \phi \Big) \cdot \nabla_p f = (N + 1) f (S\phi), \tag{1}$$

$$\partial_t^2 \phi - \Delta_x \phi = -\mu(t, x), \tag{2}$$

$$\mu(t, x) = \int_{\mathbb{R}^N} \frac{f(t, x, p)}{(1 + |p|^2)^{\frac{1}{2}}} \, dp, \tag{3}$$

where $S = \partial_t + v(p) \cdot \nabla_x$ is the free-transport operator. We add the initial conditions

$$f(0, \cdot, \cdot) = f_0, \;\; \phi(0, \cdot) = \varphi_0, \;\; \partial_t \phi(0, \cdot) = \varphi_1. \tag{4}$$

The Nordström-Vlasov system (1), (2), (3), (4) was studied by Calogero and Rein. They proved in [6] the existence of local in time classical solutions in the three dimensional case. The existence of global weak solutions has been analyzed in [7].

The aim of this paper is to prove the existence of stationary solutions for the Nordström-Vlasov equations in one dimension

$$v(p)\partial_x f - (1+p^2)^{\frac{1}{2}}\phi'(x)\partial_p f = 2f(x,p)v(p)\phi'(x), \quad (x,p)\in]0,1[\times\mathbb{R}, \tag{5}$$

$$\phi''(x) = \int_{\mathbb{R}} \frac{f(x,p)}{(1+p^2)^{\frac{1}{2}}}\, dp, \quad x\in]0,1[, \tag{6}$$

with the boundary conditions

$$f(x=0,p>0) = g_0(p), \quad f(x=1,p<0) = g_1(p), \tag{7}$$

$$\phi(0) = \varphi_0, \quad \phi(1) = \varphi_1. \tag{8}$$

The proof follows by using the Schauder fixed point theorem. One of the key points is to establish a priori estimates. For results on permanent regimes for the Vlasov-Poisson system and Vlasov-Maxwell system the reader can refer to [2], [3], [9], [16], [19].

Our main result is the following

Theorem 1.1 *Assume that* $\varphi_0, \varphi_1 \in \mathbb{R}$, $g_0, g_1 \geq 0$ *such that*

$$G_0 = e^{-\varphi_0}\int_{p>0} v(p)g_0(p)\, dp - e^{-\varphi_1}\int_{p<0} v(p)g_1(p)\, dp < +\infty,$$

$$G_\infty = \max\{\|g_0\|_{L^\infty(]0,+\infty[)}, \|g_1\|_{L^\infty(]-\infty,0[)}\} < +\infty.$$

Then there is a mild solution $(f\geq 0, \phi) \in L^\infty(]0,1[;L^1(]0,1[))\cap L^\infty(]0,1[\times\mathbb{R})\times W^{2,\infty}(]0,1[)$ *for the stationary one dimensional Nordström-Vlasov problem* (5), (6), (7), (8). *Moreover we have*

$$e^{-\phi(x)}\int_{\mathbb{R}} v(p)f(x,p)\, dp = e^{-\varphi_0}\int_{p>p_0} v(p)g_0(p)\, dp + e^{-\varphi_1}\int_{p<p_1} v(p)g_1(p)\, dp, \quad x\in[0,1],$$

where

$$p_0 = \left(e^{2(\max_{x\in[0,1]}\phi(x)-\phi(0))}-1\right)^{\frac{1}{2}}, \quad p_1 = -\left(e^{2(\max_{x\in[0,1]}\phi(x)-\phi(1))}-1\right)^{\frac{1}{2}}.$$

We perform also an asymptotic analysis of the stationary one dimensional Nordström-Vlasov problem for singular boundary conditions $g_\varepsilon(p) = \frac{1}{\varepsilon^2} g(\frac{p}{\varepsilon}),\ \forall\ p > 0$ where $\varepsilon > 0$ is a small parameter. For any $\varepsilon > 0$ we consider a solution $(f_\varepsilon, \phi_\varepsilon)$ for (5), (6), (8) and

$$f_\varepsilon(x = 0, p > 0) = g_\varepsilon(p), \quad f_\varepsilon(x = 1, p < 0) = 0.$$

We denote by $\mathcal{M}_b([0,1] \times \mathbb{R})$ the space of bounded measures on $[0,1] \times \mathbb{R}$ and by δ the Dirac measure. We prove the following result

Theorem 1.2 *Assume that* $\varphi_0 > \varphi_1$, $0 \le g \in L^\infty(]0,+\infty[)$ *such that* $\gamma = \int_{p>0} p g(p)\, dp < +\infty$. *Then we have the convergences*

$$\lim_{\varepsilon \searrow 0} \phi_\varepsilon = \phi, \ \text{ in } C^1([0,1]),$$

$$\lim_{\varepsilon \searrow 0} f_\varepsilon = f, \ \text{ weakly } \star \text{ in } \mathcal{M}_b([0,1] \times \mathbb{R}),$$

where $\phi \in C^1([0,1]) \cap C^2(]0,1])$ *is the unique solution of*

$$\phi''(x) = \frac{\gamma}{e^{\varphi_0 - \phi(x)} \sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}}, \quad x \in]0,1[,$$

$$\phi(0) = \varphi_0, \quad \phi(1) = \varphi_1,$$

and

$$f(x,p) = \frac{\gamma}{\sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}}\, \delta\left(p - \sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}\right).$$

Our paper is organized as follows. In Section 2 we introduce the notions of weak and mild solutions for the stationary one dimensional Vlasov problem. We study the properties of the characteristics. Some technical proofs are postponed to the Appendix. We give also estimates for the impulsion moments. In Section 3 we show the existence of mild solution for the stationary one dimensional Nordström-Vlasov problem. We use the fixed point method. In the last section we perform the asymptotic analysis for singular boundary conditions.

2 The stationary Vlasov problem

In this section we assume that $\phi = \phi(x)$ is a given smooth function and we introduce the notions of weak solution and mild solution for the stationary one dimensional Vlasov problem (5), (7). Notice that (5) can be written

$$\partial_x \left(v(p) f(x,p)\, e^{-\phi(x)} \right) - \partial_p \left((1+p^2)^{\frac{1}{2}} \phi'(x) f(x,p)\, e^{-\phi(x)} \right) = 0,$$

and therefore we have the usual definition

Definition 2.1 *Assume that* $\phi \in W^{1,\infty}(]0,1[)$, $g_0 \in L^1_{\text{loc}}([0,+\infty[)$, $g_1 \in L^1_{\text{loc}}(]-\infty,0])$. *We say that* $f \in L^1_{\text{loc}}([0,1]\times\mathbb{R})$ *is a weak solution for the stationary Vlasov problem* (5), (7) *iff*

$$\begin{aligned} -\int_0^1\int_{\mathbb{R}} f e^{-\phi(x)} \left(v(p)\partial_x\theta - (1+p^2)^{\frac{1}{2}}\phi'(x)\partial_p\theta \right)\, dp\, dx \;&=\; e^{-\phi(0)}\int_{p>0} v(p) g_0(p)\theta(0,p)\, dp \\ &-\; e^{-\phi(1)}\int_{p<0} v(p) g_1(p)\theta(1,p)\, dp, \end{aligned}$$

for any test function $\theta \in C^1([0,1]\times\mathbb{R})$ *satisfying* $\theta(0,p<0) = \theta(1,p>0) = 0$.

Assume now that $\phi \in W^{2,\infty}(]0,1[)$ and consider the system of characteristics associated to (5)

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = -(1+|P(s)|^2)^{\frac{1}{2}}\phi'(X(s)), \tag{9}$$

with the conditions

$$X(0) = x, \quad P(0) = p. \tag{10}$$

Observe that for any $(x,p) \in (]0,1[\times\mathbb{R}) \cup (\{0\}\times[0;+\infty[) \cup (\{1\}\times]-\infty,0])$ there is a unique solution for (9), (10) denoted

$$(X(s),P(s)) = (X(s;x,p),P(s;x,p)),\ \forall\ s \in [s_{in}(x,p), s_{out}(x,p)],$$

where the entry/exit times are given by

$$s_{\text{in}}(x,p) = \inf\{\tau \le 0\ :\ X(s;x,p) \in]0,1[,\ \forall\ s \in]\tau,0[\},$$

respectively

$$s_{\text{out}}(x,p) = \sup\{\tau \geq 0 \ : \ X(s;x,p) \in]0,1[, \ \forall \ s \in]0,\tau[\}.$$

After multiplication of (5) by $e^{-2\phi(x)}$ we obtain

$$v(p)\partial_x \left(f(x,p)e^{-2\phi(x)}\right) - (1+p^2)^{\frac{1}{2}}\phi'(x)\partial_p \left(f(x,p)e^{-2\phi(x)}\right) = 0, \quad (x,p) \in]0,1[\times \mathbb{R},$$

saying that $f(x,p)e^{-2\phi(x)}$ is constant along any characteristic. We have the definition

Definition 2.2 *Assume that $\phi \in W^{2,\infty}(]0,1[)$. The mild solution of the stationary Vlasov problem* (5), (7) *is given by*

$$f(x,p) = e^{2\phi(x)-2\phi(k)} g_k(P(s_{\text{in}}(x,p);x,p)), \text{ if } s_{\text{in}}(x,p) > -\infty, X(s_{\text{in}}(x,p);x,p) = k, k \in \{0,1\},$$

and

$$f(x,p) = 0 \text{ if } s_{\text{in}}(x,p) = -\infty.$$

By definition the mild solution is unique. Unfortunately in general there is no uniqueness for the weak solution because f can take arbitrary values on the characteristics such that $s_{\text{in}} = -\infty$.

2.1 Properties of characteristics

In this paragraph we establish several important properties of the characteristics. Some of them hold also true in the time dependent case. Therefore, for the sake of generality we work with smooth functions $\phi = \phi(t,x)$. In this case, in one dimension, the equation (1) can be written

$$\partial_t(fe^{-2\phi(t,x)}) + v(p)\partial_x(fe^{-2\phi(t,x)}) - \left((S\phi)p + (1+p^2)^{-\frac{1}{2}}\partial_x\phi\right)\partial_p(fe^{-2\phi(t,x)}) = 0,$$

and thus the characteristics to be considered are given by

$$\frac{d\tilde{X}}{ds} = v(\tilde{P}(s)), \quad \frac{d\tilde{P}}{ds} = -\left(\partial_t\phi + v(\tilde{P}(s))\partial_x\phi\right)(s,\tilde{X}(s))\tilde{P}(s) - \frac{\partial_x\phi(s,\tilde{X}(s))}{\left(1+|\tilde{P}(s)|^2\right)^{\frac{1}{2}}}. \tag{11}$$

If $\phi \in C^1([-T,T]\times[0,1])\cap L^\infty(]-T,T[\times]0,1[)$, $\partial_t\phi, \partial_x\phi \in L^\infty(]-T,T[;W^{1,\infty}(]0,1[))$, $\forall\, T>0$, then for any $(t,x,p) \in (\mathbb{R}\times]0,1[\times\mathbb{R})\cup(\mathbb{R}\times\{0\}\times[0;+\infty[)\cup(\mathbb{R}\times\{1\}\times]-\infty,0])$ there is a unique solution $(\tilde{X}(s),\tilde{P}(s))$ for (11) satisfying

$$\tilde{X}(s=t)=x, \quad \tilde{P}(s=t)=p. \tag{12}$$

For any $s\in[\tilde{s}_{\text{in}}(t,x,p),\tilde{s}_{\text{out}}(t,x,p)]$ we use the notation

$$(\tilde{X}(s),\tilde{P}(s)) = (\tilde{X}(s;t,x,p),\tilde{P}(s;t,x,p)),$$

where the entry/exit times are given by

$$\tilde{s}_{\text{in}}(t,x,p) = \inf\{\tau\le t \ :\ \tilde{X}(s;t,x,p)\in]0,1[,\ \forall\, s\in]\tau,t[\},$$

respectively

$$\tilde{s}_{\text{out}}(t,x,p) = \sup\{\tau\ge t \ :\ \tilde{X}(s;t,x,p)\in]0,1[,\ \forall\, s\in]t,\tau[\}.$$

Note that we have

$$\operatorname{div}_{(x,p)}(v(p), -(S\phi)p-(1+p^2)^{-\frac{1}{2}}\partial_x\phi) = -S\phi, \tag{13}$$

and since $\frac{d}{ds}\phi(s,\tilde{X}(s)) = \partial_t\phi(s,\tilde{X}(s)) + v(\tilde{P}(s))\partial_x\phi(s,\tilde{X}(s))$ we deduce that the jacobian matrix $\tilde{J}(s;t,x,p) = \frac{\partial(\tilde{X}(s;t,x,p),\tilde{P}(s;t,x,p))}{\partial(x,p)}$ has the determinant given by

$$\det \tilde{J}(s;t,x,p) = e^{-\phi(s,\tilde{X}(s;t,x,p))+\phi(t,x)} \ne 0. \tag{14}$$

Consider now the change of variables $\mathbb{R}\times]0,+\infty[\ni(t,p)\to(\tilde{X}(s;t,0,p),\tilde{P}(s;t,0,p))$. By computations we check that the jacobian matrix $\tilde{J}_0(s;t,p) = \frac{\partial(\tilde{X}(s;t,0,p),\tilde{P}(s;t,0,p))}{\partial(t,p)}$ has the determinant given by

$$\det \tilde{J}_0(s;t,p) = -e^{-\phi(s,\tilde{X}(s;t,0,p))+\phi(t,0)}v(p). \tag{15}$$

A similar result holds for the change $\mathbb{R}\times]-\infty,0[\ni(t,p)\to(\tilde{X}(s;t,1,p),\tilde{P}(s;t,1,p))$. We obtain that the jacobian matrix $\tilde{J}_1(s;t,p) = \frac{\partial(\tilde{X}(s;t,1,p),\tilde{P}(s;t,1,p))}{\partial(t,p)}$ has the determinant given by

$$\det \tilde{J}_1(s;t,p) = -e^{-\phi(s,\tilde{X}(s;t,1,p))+\phi(t,1)}v(p). \tag{16}$$

Let us interpret the equalities (15), (16) in the stationary case. Observe that in this case we have

$$\begin{aligned}(\tilde{X}(s;t,x,p),\tilde{P}(s;t,x,p)) &= (\tilde{X}(s-t;0,x,p),\tilde{P}(s-t;0,x,p)) \\ &= (X(s-t;x,p),P(s-t;x,p)).\end{aligned}$$

Consider the change of variables $\Omega_0 \ni (s,p) \to (X(s;0,p),P(s;0,p))$ where

$$\Omega_0 = \cup_{p>0}\left(]0,s_{\rm out}(0,p)[\times\{p\}\right),$$

and denote by $J_0(s,p)$ the jacobian matrix $J_0(s,p) = \frac{\partial(X(s;0,p),P(s;0,p))}{\partial(s,p)}$. We have

$$(X(s;0,p),P(s;0,p)) = (\tilde{X}(0;-s,0,p),\tilde{P}(0;-s,0,p)),$$

and therefore

$$\begin{aligned}\det J_0(s,p) &= -\det \tilde{J}_0(0;-s,p) \\ &= e^{-\phi(\tilde{X}(0;-s,0,p))+\phi(0)}v(p) \\ &= e^{-\phi(X(s;0,p))+\phi(0)}v(p). \end{aligned} \tag{17}$$

Similarly for the change $\Omega_1 \ni (s,p) \to (X(s;1,p),P(s;1,p))$ where

$$\Omega_1 = \cup_{p<0}\left(]0,s_{\rm out}(1,p)[\times\{p\}\right),$$

we obtain

$$\begin{aligned}\det J_1(s,p) &= \det\left(\frac{\partial(X(s;1,p),P(s;1,p))}{\partial(s,p)}\right) \\ &= e^{-\phi(X(s;1,p))+\phi(1)}v(p). \end{aligned} \tag{18}$$

Another important property in the stationary case is the conservation of $W(x,p) = (1+p^2)e^{2\phi(x)}$ along characteristics.

Proposition 2.1 *Assume that $\phi \in W^{2,\infty}(]0,1[)$. Then for any solution of* (9) *we have*

$$\frac{d}{ds}\{W(X(s),P(s))\} = 0,\ \forall\ s_{\rm in} < s < s_{\rm out}.$$

Proof. Compute the derivative of $W(X(s),P(s))$ with respect to s and use (9).

□

The previous conservation allows us to study the geometry of characteristics (for a similar result on the one dimensional Vlasov-Poisson equations see [16]). We introduce the critical impulsions

$$p_0 = \left(e^{2(\max_{x\in[0,1]}\phi(x)-\phi(0))} - 1\right)^{\frac{1}{2}}, \tag{19}$$

$$p_1 = -\left(e^{2(\max_{x\in[0,1]}\phi(x)-\phi(1))} - 1\right)^{\frac{1}{2}}. \tag{20}$$

The proof of the following proposition can be found in Appendix.

Proposition 2.2 *Assume that* $\phi \in W^{2,\infty}(]0,1[)$.

1) For any $0 < p < p_0$ *there are* $x_0 \in]0,1[$, $0 < s_0 \le s_{\text{out}}(0,p) \le +\infty$ *such that*

$$0 < X(s;0,p) < x_0, \quad P(s;0,p) > 0, \ \forall\ 0 < s < s_0,$$

$$\lim_{s \nearrow s_0} X(s;0,p) = x_0, \quad \lim_{s\nearrow s_0} P(s;0,p) = 0.$$

Moreover, if $\phi'(x_0) \neq 0$ *then* $s_{out}(0,p) = 2s_0 < +\infty$ *and*

$$X(s;0,p) = X(2s_0 - s;0,p), \ P(s;0,p) = -P(2s_0 - s;0,p), \forall s \in [0,2s_0].$$

In particular $X(s_{out}(0,p);0,p) = 0$.

2) For any $p > p_0$ *we have* $s_{out}(0,p) < +\infty$, $P(s;0,p) > 0$, $\forall\ 0 \le s \le s_{out}(0,p)$ *and* $X(s_{out}(0,p);0,p) = 1$.

3) For any $p_1 < p < 0$ *there are* $x_1 \in]0,1[$, $0 < s_1 \le s_{out}(1,p) \le +\infty$ *such that*

$$1 > X(s;1,p) > x_1, \quad P(s;1,p) < 0, \ \forall\ 0 < s < s_1,$$

$$\lim_{s\nearrow s_1} X(s;1,p) = x_1, \quad \lim_{s\nearrow s_1} P(s;1,p) = 0.$$

Moreover, if $\phi'(x_1) \neq 0$ *then* $s_{out}(1,p) = 2s_1 < +\infty$ *and*

$$X(s;1,p) = X(2s_1 - s;1,p), \ P(s;1,p) = -P(2s_1 - s;1,p), \forall s \in [0,2s_1].$$

In particular $X(s_{out}(1,p);1,p) = 1$.

4) For any $p < p_1$ *we have* $s_{out}(1,p) < +\infty$, $P(s;1,p) < 0$, $\forall\ 0 \le s \le s_{out}(1,p)$ *and* $X(s_{out}(1,p);1,p) = 0$.

When ϕ is convex we can show that the exit time is finite for all characteristics (see the Appendix for the proof).

Corollary 2.1 *Assume that ϕ is convex and belongs to $W^{2,\infty}(]0,1[)$. With the previous notations we have*

$$s_{\text{out}}(0,p) < +\infty, \ \forall\, p > 0, \ \text{ and } \ s_{\text{out}}(1,p) < +\infty \ \forall\, p < 0.$$

2.2 Properties of the mild solution

In this paragraph we give several properties of the mild solution of (5), (7).

Proposition 2.3 *Assume that ϕ belongs to $W^{2,\infty}(]0,1[)$, $g_0 \in L^\infty_{\text{loc}}([0,+\infty[), g_1 \in L^\infty_{\text{loc}}(]-\infty,0])$. Denote by f the mild solution of* (5), (7)*. Then*
1) if g_0, g_1 are nonnegative, f is nonnegative;
2) f belongs to $L^\infty_{\text{loc}}([0,1]\times\mathbb{R})$; moreover, if $g_0 \in L^\infty(]0,+\infty[)$, $g_1 \in L^\infty(]-\infty,0[)$, then $f \in L^\infty(]0,1[\times\mathbb{R})$ and

$$\|f\|_\infty \le e^{2\max_{x\in[0,1]}\phi(x) - 2\min\{\phi(0),\phi(1)\}} \max\{\|g_0\|_\infty, \|g_1\|_\infty\};$$

3) for any test function $\psi \in C^0_c([0,1]\times\mathbb{R})$ we have

$$\begin{aligned}\int_0^1\int_{\mathbb{R}} f\psi \; dp\, dx &= e^{-\phi(0)}\int_{p>0} v(p)g_0(p)\int_0^{s_{\text{out}}(0,p)} \psi(X(s;0,p),P(s;0,p))e^{\phi(X(s;0,p))}\, ds\, dp \\ &- e^{-\phi(1)}\int_{p<0} v(p)g_1(p)\int_0^{s_{\text{out}}(1,p)} \psi(X(s;1,p),P(s;1,p))e^{\phi(X(s;1,p))}\, ds\, dp \ ;\end{aligned} \tag{21}$$

4) f is a weak solution for (5), (7)*.*

Proof. The first statement and the last part of the second one follow immediately by the definition of the mild solution. Take now $R > 0$ and consider $C > 0$ such that

$$|g_0(p)| \le C, \ \ |g_1(-p)| \le C, \ \forall\, 0 < p \le (1+R^2)^{\frac{1}{2}} e^{2\|\phi\|_\infty}.$$

For any $(x,p) \in [0,1] \times [-R,R]$ consider $(X(s),P(s)) = (X(s;x,p),P(s;x,p))$, $\forall\, s_{\text{in}}(x,p) \le s \le s_{\text{out}}(x,p)$. By Proposition 2.1 we have

$$(1+|P(s)|^2)e^{2\phi(X(s))} = (1+p^2)e^{2\phi(x)},$$

and therefore we obtain for any $s \in [s_{\text{in}}(x,p), s_{\text{out}}(x,p)]$

$$|P(s)| \le (1+R^2)^{\frac{1}{2}} e^{2\|\phi\|_\infty}.$$

By the definition of the mild solution one gets

$$|f(x,p)| \le e^{2\max\phi - 2\min\{\phi(0),\phi(1)\}} C,\ \forall\ (x,p) \in [0,1]\times[-R,R],$$

saying that $f \in L^\infty_{\text{loc}}([0,1]\times\mathbb{R})$. Let us prove 3). From the previous point we know that f is locally bounded and therefore is locally integrable. Observe that $|f|$ is the mild solution of (5) with the boundary conditions $|g_0|, |g_1|$ and therefore it is sufficient to prove (21) when $g_0, g_1 \ge 0$ and $\psi \in C^0_c([0,1]\times\mathbb{R})$, $\psi \ge 0$. Combining the definition of the mild solution and (17), (18) yields

$$\begin{aligned}
\int_0^1\!\!\int_{\mathbb{R}} f\psi\, dp\, dx &= \int_{\Omega_0} f(X(s;0,p),P(s;0,p))\psi(X(s;0,p),P(s;0,p))|\det J_0(s,p)|\, ds\, dp \\
&+ \int_{\Omega_1} f(X(s;1,p),P(s;1,p))\psi(X(s;1,p),P(s;1,p))|\det J_1(s,p)|\, ds\, dp \\
&= e^{-\phi(0)} \int_{p>0} v(p) g_0(p) \int_0^{s_{\text{out}}(0,p)} \psi(X(s;0,p),P(s;0,p)) e^{\phi(X(s;0,p))}\, ds\, dp \\
&- e^{-\phi(1)} \int_{p<0} v(p) g_1(p) \int_0^{s_{\text{out}}(1,p)} \psi(X(s;1,p),P(s;1,p)) e^{\phi(X(s;1,p))}\, ds\, dp. \quad (22)
\end{aligned}$$

In order to prove 4) use the mild formulation (21) with the test function

$$\psi(x,p) = -e^{-\phi(x)}(v(p)\partial_x\theta - (1+p^2)^{\frac{1}{2}}\phi'(x)\partial_p\theta),\ \forall\ (x,p)\in[0,1]\times\mathbb{R},$$

where $\theta \in C^1_c([0,1]\times\mathbb{R})$ satisfying $\theta(0,p<0) = \theta(1,p>0) = 0$.

$\square$

We intend now to estimate the moments $\rho(x) = e^{-\phi(x)} \int_{\mathbb{R}} f(x,p)\, dp$ and $j(x) = e^{-\phi(x)} \int_{\mathbb{R}} v(p) f(x,p)\, dp$, $x \in [0,1]$. We introduce also $j^\pm$ given by

$$j^\pm(x) = e^{-\phi(x)} \int_{\mathbb{R}} v^\pm(p) f(x,p)\, dp,\quad x \in [0,1],$$

where $v^\pm(p) = \max\{0, \pm v(p)\}$, $\forall p \in \mathbb{R}$.

Proposition 2.4 *Assume that $\phi \in W^{2,\infty}(]0,1[)$ (not necessarily convex), $g_0, g_1 \geq 0$ such that*

$$G_0 := e^{-\phi(0)} \int_{p>0} v(p) g_0(p)\, dp - e^{-\phi(1)} \int_{p<0} v(p) g_1(p)\, dp < +\infty.$$

Denote by f the mild solution of (5), (7). *Then we have*

$$\|j^+ + j^-\|_{L^\infty(]0,1[)} = \left\| e^{-\phi(\cdot)} \int_{\mathbb{R}} |v(p)| f(\cdot, p)\, dp \right\|_{L^\infty(]0,1[)} \leq 2G_0,$$

and for any $R \geq e^{2\|\phi\|_{L^\infty}}$

$$\left\| e^{-\phi(\cdot)} \int_{|p|>R} |v(p)| f(\cdot, p)\, dp \right\|_{L^\infty(]0,1[)} \leq 2G_0^r,$$

where

$$G_0^r := e^{-\phi(0)} \int_{p>r} v(p) g_0(p)\, dp - e^{-\phi(1)} \int_{p<-r} v(p) g_1(p)\, dp,$$

and $r = e^{-2\|\phi\|_{L^\infty}} \left(1 + R^2 - e^{4\|\phi\|_{L^\infty}}\right)^{\frac{1}{2}}$.

Proof. We use the mild formulation (21) with the test function $\psi(x,p) = e^{-\phi(x)} v^+(p) \theta(x)$ where $\theta \in L^1(]0,1[)$, $\theta \geq 0$ (actually we need to apply first (21) with the test function $\psi_R(x,p) = \psi(x,p)\chi_R(p)$ where $\chi \in C_c^1(\mathbb{R})$, $\chi(u) = 1$ if $|u| \leq 1$, $\chi(u) = 0$ if $|u| \geq 2$, $\chi \geq 0$, $\chi_R(\cdot) = \chi(\frac{\cdot}{R})$, and then let $R \to +\infty$; we skip these standard arguments). We obtain

$$\begin{aligned} \int_0^1 j^+(x)\theta(x)\, dx &= \int_0^1 \int_{\mathbb{R}} f(x,p)\psi(x,p)\, dp\, dx \\ &= e^{-\phi(0)} \int_{p>0} v(p) g_0(p) \int_0^{s_{\mathrm{out}}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\, ds\, dp \\ &- e^{-\phi(1)} \int_{p<0} v(p) g_1(p) \int_0^{s_{\mathrm{out}}(1,p)} v^+(P(s;1,p))\theta(X(s;1,p))\, ds\, dp \\ &= I_0 + I_1. \end{aligned} \tag{23}$$

We analyze the term I_0. Consider the critical impulsion p_0 given by (19). For any $p > p_0$ we know by Proposition 2.2 that $s_{\mathrm{out}}(0,p) < +\infty, P(s;0,p) > 0$ for any

$s \in [0, s_{\rm out}(0,p)]$, $X(s_{\rm out}(0,p); 0, p) = 1$. We have

$$\begin{aligned}
\int_0^{s_{\rm out}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\ ds &= \int_0^{s_{\rm out}(0,p)} v(P(s;0,p))\theta(X(s;0,p))\ ds \\
&= \int_0^{s_{\rm out}(0,p)} \frac{dX}{ds}\theta(X(s;0,p))\ ds \\
&= \|\theta\|_{L^1(]0,1[)}. \qquad (24)
\end{aligned}$$

Take now $0 < p < p_0$. If $s_{\rm out}(0,p) = +\infty$ we know by Proposition 2.2 that there is $x_0 \in]0,1[$ such that $\lim_{s\to+\infty} X(s;0,p) = x_0$, $P(s;0,p) > 0$, $\forall s > 0$. We find

$$\begin{aligned}
\int_0^{s_{\rm out}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\ ds &= \int_0^{s_{\rm out}(0,p)} \frac{dX}{ds}\theta(X(s;0,p))\ ds \\
&= \int_0^{x_0} \theta(u)\ du \\
&\leq \|\theta\|_{L^1(]0,1[)}. \qquad (25)
\end{aligned}$$

Consider now $0 < p < p_0$ with $s_{\rm out}(0,p) < +\infty$. In this case there is $x_0 \in]0,1[$ such that $X(s_{\rm out}(0,p)/2; 0,p) = x_0$, $P(s;0,p) > 0\ \forall\ s \in [0, s_{\rm out}(0,p)/2[$, $P(s_{\rm out}(0,p)/2;0,p) = 0$ and $P(s;0,p) < 0\ \forall\ s \in]s_{\rm out}(0,p)/2, s_{\rm out}(0,p)]$. We obtain

$$\begin{aligned}
\int_0^{s_{\rm out}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\ ds &= \int_0^{\frac{s_{\rm out}(0,p)}{2}} \frac{dX}{ds}\theta(X(s;0,p))\ ds \\
&= \int_0^{x_0} \theta(u)\ du \\
&\leq \|\theta\|_{L^1(]0,1[)}. \qquad (26)
\end{aligned}$$

Combining (24), (25), (26) yields

$$I_0 \leq e^{-\phi(0)} \int_{p>0} v(p) g_0(p)\ dp\ \|\theta\|_{L^1(]0,1[)}.$$

Similarly we deduce that

$$I_1 \leq -e^{-\phi(1)} \int_{p<0} v(p) g_1(p)\ dp\ \|\theta\|_{L^1(]0,1[)},$$

and therefore (23) implies

$$\int_0^1 j^+(x)\theta(x)\ dx \leq G_0 \|\theta\|_{L^1(]0,1[)},$$

for any nonnegative function $\theta \in L^1(]0,1[)$, saying that $\|j^+\|_{L^\infty(]0,1[)} \leq G_0$. By similar computations we obtain $\|j^-\|_{L^\infty(]0,1[)} \leq G_0$ and thus the first statement of our proposition follows. Take now $R \geq e^{2\|\phi\|_{L^\infty}}$ and let us use (21) with the test function $\psi(x,p) = e^{-\phi(x)} v^+(p)\theta(x)\mathbf{1}_{\{|p|>R\}}$, where $\theta \in L^1(]0,1[)$, $\theta \geq 0$. We obtain

$$\begin{aligned}
\int_0^1 \theta(x) e^{-\phi(x)} \int_{|p|>R} v^+(p) f(x,p)\, dp\, dx \;&=\; e^{-\phi(0)} \int_{p>0} v(p) g_0(p) \int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p)) \\
&\times\; \theta(X(s;0,p))\mathbf{1}_{\{|P(s;0,p)|>R\}}\, ds\, dp \\
&-\; e^{-\phi(1)} \int_{p<0} v(p) g_1(p) \int_0^{s_{\text{out}}(1,p)} v^+(P(s;1,p)) \\
&\times\; \theta(X(s;1,p))\mathbf{1}_{\{|P(s;1,p)|>R\}}\, ds\, dp \\
&=\; I_0^R + I_1^R. \qquad (27)
\end{aligned}$$

Observe that for any $0 < p \leq e^{-2\|\phi\|_{L^\infty}} \left(1 + R^2 - e^{4\|\phi\|_{L^\infty}}\right)^{\frac{1}{2}} =: r$ we have

$$\begin{aligned}
1 + |P(s;0,p)|^2 \;&=\; (1+p^2) e^{2\phi(0) - 2\phi(X(s;0,p))} \\
&\leq\; 1 + R^2,\ \forall\, 0 < s < s_{\text{out}}(0,p), \qquad (28)
\end{aligned}$$

and thus $|P(s;0,p)| \leq R$, $\forall\, 0 < s < s_{\text{out}}(0,p)$. We deduce that

$$\int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\mathbf{1}_{\{|P(s;0,p)|>R\}}\, ds = 0, \qquad (29)$$

for any $0 < p \leq r$. By the previous computations we know also that

$$\begin{aligned}
\int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\mathbf{1}_{\{|P(s;0,p)|>R\}}\, ds \;&\leq\; \int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\, ds \\
&\leq\; \|\theta\|_{L^1(]0,1[)},\ \forall\, p > 0, p \neq p_0. \qquad (30)
\end{aligned}$$

Combining (29), (30) we obtain for any $p > 0, p \neq p_0$

$$\int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p))\theta(X(s;0,p))\mathbf{1}_{\{|P(s;0,p)|>R\}}\, ds \leq \mathbf{1}_{\{p>r\}} \|\theta\|_{L^1(]0,1[)}, \qquad (31)$$

and therefore one gets for any nonnegative function $\theta \in L^1(]0,1[)$

$$I_0^R \leq e^{-\phi(0)} \int_{p>r} v(p) g_0(p)\, dp\, \|\theta\|_{L^1(]0,1[)}. \qquad (32)$$

Similarly we obtain

$$I_1^R \le -e^{-\phi(1)} \int_{p<-r} v(p) g_1(p)\ dp\ \|\theta\|_{L^1(]0,1[)}. \tag{33}$$

From (27), (32), (33) one gets for any nonnegative function $\theta \in L^1(]0,1[)$

$$\begin{aligned}\int_0^1 \theta(x) e^{-\phi(x)} \int_{|p|>R} v^+(p) f(x,p)\ dp\ dx \ &\le\ e^{-\phi(0)} \int_{p>r} v(p) g_0(p)\ dp\ \|\theta\|_{L^1(]0,1[)} \\ &-\ e^{-\phi(1)} \int_{p<-r} v(p) g_1(p)\ dp\ \|\theta\|_{L^1(]0,1[)}, \quad (34)\end{aligned}$$

saying that

$$\left\| e^{-\phi(\cdot)} \int_{|p|>R} v^+(p) f(\cdot,p)\ dp \right\|_{L^\infty} \le e^{-\phi(0)} \int_{p>r} v(p) g_0(p)\ dp - e^{-\phi(1)} \int_{p<-r} v(p) g_1(p)\ dp. \tag{35}$$

In the same manner we obtain

$$\left\| e^{-\phi(\cdot)} \int_{|p|>R} v^-(p) f(\cdot,p)\ dp \right\|_{L^\infty} \le e^{-\phi(0)} \int_{p>r} v(p) g_0(p)\ dp - e^{-\phi(1)} \int_{p<-r} v(p) g_1(p)\ dp, \tag{36}$$

and the second statement of our proposition follows from (35), (36). □

Remark 2.1 *In the previous proof we obtain the inequality*

$$\int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p)) \theta(X(s;0,p))\ ds \le \|\theta\|_{L^1(]0,1[)},$$

for any nonnegative function $\theta \in L^1(]0,1[)$ *and* $p > 0, p \ne p_0$. *If* ϕ *is convex we check easily (see Corollary* 2.1*) that for* $p = p_0 > 0$ *we have the equality*

$$\int_0^{s_{\text{out}}(0,p)} v^+(P(s;0,p)) \theta(X(s;0,p))\ ds = \|\theta\|_{L^1(]0,1[)}.$$

Corollary 2.2 *Assume that* $\phi \in W^{2,\infty}(]0,1[)$, $g_0, g_1 \ge 0$, $g_0 \in L^\infty(]0,+\infty[), g_1 \in L^\infty(]-\infty,0[)$ *such*

$$G_0 = e^{-\phi(0)} \int_{p>0} v(p) g_0(p)\ dp - e^{-\phi(1)} \int_{p<0} v(p) g_1(p)\ dp < +\infty.$$

Denote by f *the mild solution of* (5), (7). *Then* $\rho(\cdot) = e^{-\phi(\cdot)} \int_{\mathbb{R}} f(\cdot,p)\ dp$, $j(\cdot) = e^{-\phi(\cdot)} \int_{\mathbb{R}} v(p) f(\cdot,p)\ dp$, $\mu(\cdot) = \int_{\mathbb{R}} \frac{f(\cdot,p)}{(1+p^2)^{\frac{1}{2}}}\ dp$ *belong to* $L^\infty(]0,1[)$ *and we have*

$$\max\{\|j\|_{L^\infty}, \|\rho\|_{L^\infty}\} \le 2e^{5\|\phi\|_{L^\infty}} G_\infty + 2\sqrt{2} G_0,$$

$$\|\mu\|_{L^\infty} \le 2e^{2\max_{x\in[0,1]}\phi(x) - 2\min\{\phi(0),\phi(1)\}} G_\infty + 2\sqrt{2} e^{\max_{x\in[0,1]}\phi(x)} G_0,$$

where $G_\infty = \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$.

Proof. We already know by Proposition 2.3 that $f \in L^\infty(]0,1[\times\mathbb{R})$ and $\|f\|_{L^\infty} \leq e^{2\max_{x\in[0,1]}\phi(x) - 2\min\{\phi(0),\phi(1)\}} G_\infty$. We can write

$$
\begin{aligned}
\max\{|j(x)|, \rho(x)\} &= \rho(x) = e^{-\phi(x)} \int_{|p|\leq 1} f(x,p)\,dp + e^{-\phi(x)} \int_{|p|>1} f(x,p)\,dp \\
&\leq 2e^{\|\phi\|_\infty} \|f\|_\infty + e^{-\phi(x)} \int_{|p|>1} \frac{|v(p)|}{v(1)} f(x,p)\,dp \\
&\leq 2e^{5\|\phi\|_\infty} G_\infty + \sqrt{2} e^{-\phi(x)} \int_{\mathbb{R}} |v(p)| f(x,p)\,dp \\
&\leq 2e^{5\|\phi\|_\infty} G_\infty + 2\sqrt{2} G_0, \quad x \in [0,1].
\end{aligned}
$$

For the density μ we have

$$
\begin{aligned}
\mu(x) &\leq \int_{|p|\leq 1} f(x,p)\,dp + \int_{|p|>1} f(x,p)\,dp \\
&\leq 2\|f\|_\infty + \int_{|p|>1} \frac{|v(p)|}{v(1)} f(x,p)\,dp \\
&\leq 2e^{2\max\phi - 2\min\{\phi(0),\phi(1)\}} G_\infty + \sqrt{2} e^{\phi(x)} (j^+(x) + j^-(x)) \\
&\leq 2e^{2\max\phi - 2\min\{\phi(0),\phi(1)\}} G_\infty + 2\sqrt{2} e^{\max\phi} G_0, \quad x \in [0,1].
\end{aligned}
$$

$\square$

Observe also that the current j is constant with respect to x. Indeed, if ϕ is convex, by Corollary 2.1 we know that $s_{\text{out}}(0,p) < +\infty$, $\forall\, p > 0$, $s_{\text{out}}(1,p) < +\infty$, $\forall\, p < 0$ and thus for any function $\theta \in C_c^1(]0,1[)$ we have

$$
\begin{aligned}
\int_0^1 j(x)\theta'(x)\,dx &= \int_0^1 \int_{\mathbb{R}} f(x,p) e^{-\phi(x)} v(p) \theta'(x)\,dp\,dx \\
&= e^{-\phi(0)} \int_{p>0} v(p) g_0(p) \int_0^{s_{\text{out}}(0,p)} \frac{dX}{ds} \theta'(X(s;0,p))\,ds\,dp \\
&- e^{-\phi(1)} \int_{p<0} v(p) g_1(p) \int_0^{s_{\text{out}}(1,p)} \frac{dX}{ds} \theta'(X(s;1,p))\,ds\,dp \\
&= 0.
\end{aligned}
$$

When ϕ is convex the current j depends only on $\phi(0), \phi(1), g_0, g_1$.

Proposition 2.5 *Assume that ϕ is convex and belongs to $W^{2,\infty}(]0,1[)$, $g_0, g_1 \geq 0$ such that*

$$
G_0 = e^{-\phi(0)} \int_{p>0} v(p) g_0(p)\,dp - e^{-\phi(1)} \int_{p<0} v(p) g_1(p)\,dp < +\infty.
$$

Denote by f the mild solution of (5), (7). *Then we have*

$$e^{-\phi(x)} \int_{\mathbb{R}} v(p) f(x,p)\ dp = e^{-\phi(0)} \int_{p>p_0} v(p) g_0(p)\ dp + e^{-\phi(1)} \int_{p<p_1} v(p) g_1(p)\ dp,\ \forall\ x \in [0,1],$$

where p_0, p_1 are given by (19), (20).

Proof. Since the current is constant, it is sufficient to check the above equality for $x = 0$. In order to fix the ideas assume that $\phi(0) \le \phi(1)$ and therefore

$$p_0 = \left(e^{2(\phi(1)-\phi(0))} - 1\right)^{\frac{1}{2}} \ge 0,\ \ p_1 = 0.$$

We can write

$$\begin{aligned} j(0) &= e^{-\phi(0)} \int_{p<-p_0} v(p) f(0,p)\ dp + e^{-\phi(0)} \int_{-p_0}^{p_0} v(p) f(0,p)\ dp + e^{-\phi(0)} \int_{p>p_0} v(p) g_0(p)\ dp \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Observe that $I_2 = 0$. Indeed, this is obvious when $p_0 = 0$ and if $p_0 > 0$ it follows immediately since by Proposition 2.2 and the definition of the mild solution we have

$$f(0,p) = g_0(-p),\ \forall\ -p_0 < p < 0.$$

In order to prove our statement it remains to verify that

$$I_1 = e^{-\phi(0)} \int_{p<-p_0} v(p) f(0,p)\ dp = e^{-\phi(1)} \int_{p<0} v(p) g_1(p)\ dp.$$

Note that for any $p < -p_0$ we have

$$f(0,p) e^{-2\phi(0)} = g_1(q) e^{-2\phi(1)},$$

where $p(q) = -\sqrt{(1+q^2)e^{2\phi(1)-2\phi(0)} - 1}$. By change of variable we have

$$\begin{aligned} I_1 &= e^{\phi(0)} \int_{p<-p_0} v(p) g_1(q(p)) e^{-2\phi(1)}\ dp \\ &= e^{-\phi(1)} \int_{q<0} v(q) g_1(q)\ dq. \end{aligned}$$

□

Consider also the moment $j_1(\cdot) = \int_{\mathbb{R}} pf(\cdot, p)\, dp$. We can prove

Proposition 2.6 *Assume that $\phi \in W^{2,\infty}(]0,1[)$ (not necessarily convex), $g_0, g_1 \geq 0$ such that*

$$G_1 := \int_{p>0} pg_0(p)\, dp - \int_{p<0} pg_1(p)\, dp < +\infty.$$

Denote by f the mild solution of (5), (7)*. Then we have*

$$\left\| \int_{\mathbb{R}} |p| f(\cdot, p)\, dp \right\|_{L^\infty(]0,1[)} \leq 2G_1.$$

If ϕ is convex we have $\frac{d}{dx} j_1 = 0,\ x \in]0,1[$ and

$$\int_{\mathbb{R}} pf(x,p)\, dp = \int_{p>p_0} pg_0(p)\, dp + \int_{p<p_1} pg_1(p)\, dp,\ \forall\, x \in [0,1].$$

Proof. It is very similar to those of Propositions 2.4, 2.5. For any nonnegative function $\theta \in L^1(]0,1[)$ we have

$$\begin{aligned}
\int_0^1 \int_{\mathbb{R}} \theta(x) |p| f(x,p)\, dp\, dx &= e^{-\phi(0)} \int_{p>0} v(p) g_0(p) \int_0^{s_{\text{out}}(0,p)} e^{\phi(X(s;0,p))} |P(s;0,p)| \\
&\times\ \theta(X(s;0,p))|\, ds\, dp \\
&-\ e^{-\phi(1)} \int_{p<0} v(p) g_1(p) \int_0^{s_{\text{out}}(1,p)} e^{\phi(X(s;1,p))} |P(s;1,p)| \\
&\times\ \theta(X(s;1,p))|\, ds\, dp. \qquad (37)
\end{aligned}$$

By the computations in the proof of Proposition 2.4 we know that for $k \in \{0,1\}$ we have

$$\int_0^{s_{\text{out}}(k,p)} |v(P(s;k,p))|\, \theta(X(s;k,p))\, ds \leq 2\|\theta\|_{L^1(]0,1[)},\ \forall\, (-1)^k p > 0, p \neq p_k. \quad (38)$$

Using also Proposition 2.1 one gets for $k \in \{0,1\},\ (-1)^k p > 0, p \neq p_k$

$$\begin{aligned}
\int_0^{s_{\text{out}}(k,p)} e^{\phi(X(s;k,p))} |P(s;k,p)|\, \theta(X(s;k,p))\, ds &= \int_0^{s_{\text{out}}(k,p)} e^{\phi(k)} (1+p^2)^{\frac{1}{2}} |v(P(s;k,p))| \\
&\times\ \theta(X(s;k,p))\, ds \\
&\leq\ 2e^{\phi(k)} (1+p^2)^{\frac{1}{2}} \|\theta\|_{L^1(]0,1[)}. \qquad (39)
\end{aligned}$$

Combining (37), (38), (39) yields

$$\int_0^1 \theta(x) \int_{\mathbb{R}} |p| f(x,p)\, dp\, dx \leq 2\|\theta\|_{L^1(]0,1[)} G_1,$$

saying that

$$\left\| \int_{\mathbb{R}} |p| f(\cdot, p)\ dp \right\|_{L^\infty(]0,1[)} \le 2G_1.$$

Suppose now that ϕ is convex. Take $\theta \in C_c^1(]0,1[)$ and let us compute

$$\begin{aligned}
\int_0^1 j_1(x)\theta'(x)\ dx &= \int_0^1 \int_{\mathbb{R}} f(x,p) p\ \theta'(x)\ dp\ dx \\
&= \sum_{k=0}^{1} (-1)^k e^{-\phi(k)} \int_{(-1)^k p>0} v(p) g_k(p) \int_0^{s_{\mathrm{out}}(k,p)} e^{\phi(X(s;k,p))} P(s;k,p) \\
&\times \ \theta'(X(s;k,p))\ ds\ dp \\
&= \sum_{k=0}^{1} (-1)^k \int_{(-1)^k p>0} p g_k(p) \int_0^{s_{\mathrm{out}}(k,p)} \frac{dX(s;k,p)}{ds} \theta'(X(s;k,p))\ ds\ dp \\
&= 0.
\end{aligned}$$

Therefore $\frac{dj_1}{dx} = 0,\ \ x \in]0,1[$. Assume for example that $\phi(0) \le \phi(1)$. Thus

$$p_0 = \left(e^{2(\phi(1)-\phi(0))} - 1\right)^{\frac{1}{2}} \ge 0,\ \ p_1 = 0.$$

For any $x \in [0,1]$ we can write

$$\begin{aligned}
j_1(x) = j_1(0) &= \int_{p<-p_0} p f(0,p)\ dp + \int_{-p_0}^{p_0} p f(0,p)\ dp + \int_{p>p_0} p g_0(p)\ dp \\
&= \int_{p<-p_0} p f(0,p)\ dp + \int_{p>p_0} p g_0(p)\ dp.
\end{aligned}$$

By change of variable we check as before that

$$\int_{p<-p_0} p f(0,p)\ dp = \int_{p<0} p g_1(p)\ dp.$$

□

The reader can check easily the following generalizations

Proposition 2.7 *Assume that $\phi \in W^{2,\infty}(]0,1[)$ (not necessarily convex), $g_0, g_1 \ge 0$ such that for some $m \in \mathbb{R}$*

$$G_m := e^{(m-1)\phi(0)} \int_{p>0} v(p)(1+p^2)^{\frac{m}{2}} g_0(p)\ dp - e^{(m-1)\phi(1)} \int_{p<0} v(p)(1+p^2)^{\frac{m}{2}} g_1(p)\ dp < +\infty.$$

Denote by f the mild solution of (5), (7). *Then we have*

$$\left\| e^{(m-1)\phi(\cdot)} \int_{\mathbb{R}} |v(p)|(1+p^2)^{\frac{m}{2}} f(\cdot,p)\ dp \right\|_{L^\infty(]0,1[)} \leq 2G_m.$$

If ϕ is convex then $\frac{d}{dx} j_m = 0,\ x \in]0,1[$ where $j_m(\cdot) = e^{(m-1)\phi(\cdot)} \int_{\mathbb{R}} v(p)(1+p^2)^{\frac{m}{2}} f(\cdot,p)\ dp$, and we have for any $x \in [0,1]$

$$j_m(x) = e^{(m-1)\phi(0)} \int_{p>p_0} v(p)(1+p^2)^{\frac{m}{2}} g_0(p)\ dp + e^{(m-1)\phi(1)} \int_{p<p_1} v(p)(1+p^2)^{\frac{m}{2}} g_1(p)\ dp.$$

3 Existence of stationary solution for the one dimensional Nordström-Vlasov equations

Suppose that $\varphi_0, \varphi_1 \in \mathbb{R}$, g_0, g_1 are nonnegative functions such that

$$G_0 = e^{-\varphi_0} \int_{p>0} v(p) g_0(p)\ dp - e^{-\varphi_1} \int_{p<0} v(p) g_1(p)\ dp < +\infty,$$

$$G_\infty = \max\{\|g_0\|_{L^\infty(]0,+\infty[)}, \|g_1\|_{L^\infty(]-\infty,0[)}\} < +\infty.$$

In this section we prove the existence of solution (f,ϕ) for the stationary Nordström-Vlasov system

$$v(p)\partial_x f - (1+p^2)^{\frac{1}{2}} \phi'(x) \partial_p f = 2f(x,p)v(p)\phi'(x),\ \ (x,p) \in]0,1[\times\mathbb{R}, \tag{40}$$

$$\phi''(x) = \int_{\mathbb{R}} \frac{f(x,p)}{(1+p^2)^{\frac{1}{2}}}\ dp,\ \ x \in]0,1[, \tag{41}$$

with the boundary conditions

$$f(x=0, p>0) = g_0(p),\ \ f(x=1, p<0) = g_1(p), \tag{42}$$

$$\phi(0) = \varphi_0,\ \ \phi(1) = \varphi_1, \tag{43}$$

where ϕ is smooth and f is solution by characteristics of (40), (42). We call such solution (f,ϕ) a mild solution for the Nordström-Vlasov system. We use the Schauder fixed point theorem. We consider the set

$$\begin{aligned} \mathcal{D} &= \{\phi \in W^{2,\infty}(]0,1[)\ :\ \phi'' \geq 0, \phi(0) = \varphi_0, \phi(1) = \varphi_1, \\ &\quad \|\phi\|_{L^\infty} \leq \max\{|\varphi_0|,|\varphi_1|\} + M,\ \|\phi'\|_{L^\infty} \leq |\varphi_1 - \varphi_0| + M,\ \|\phi''\|_{L^\infty} \leq M\}, \end{aligned}$$

where $M = 2e^{2|\varphi_1 - \varphi_0|}G_\infty + 2\sqrt{2}e^{\max\{\varphi_0,\varphi_1\}}G_0$. Note that $\mathcal{D}$ is convex and compact in $C^1([0,1])$. For any $\phi \in \mathcal{D}$ we define $\mathcal{F}\phi = \tilde{\phi}$ where $\tilde{\phi}$ solves (41), (43) with $\mu(\cdot) = \int_{\mathbb{R}} f(\cdot,p)(1+p^2)^{-\frac{1}{2}}\, dp$ and f is the mild solution of (40), (42) corresponding to ϕ.

Proposition 3.1 *Assume that $\varphi_0, \varphi_1 \in \mathbb{R}$, $g_0, g_1 \geq 0$ such that $G_0 < +\infty$, $G_\infty < +\infty$. Then the set $\mathcal{D}$ is left invariant by the application $\mathcal{F}$.*

Proof. Take $\phi \in \mathcal{D}$ and denote by f the mild solution of (40), (42) corresponding to ϕ and let $\mu = \int_{\mathbb{R}} (1+p^2)^{-\frac{1}{2}} f\, dp$. By Corollary 2.2 we have

$$\|\mu\|_{L^\infty} \leq 2e^{2\max_{x\in[0,1]}\phi(x) - 2\min\{\varphi_0,\varphi_1\}}G_\infty + 2\sqrt{2}e^{\max_{x\in[0,1]}\phi(x)}G_0.$$

Since ϕ is convex, we have $\max_{x\in[0,1]}\phi(x) = \max\{\varphi_0,\varphi_1\}$ and thus the previous inequality implies

$$\|\mu\|_{L^\infty} \leq M.$$

Observe that $\tilde{\phi} = \mathcal{F}\phi$ can be written $\tilde{\phi} = (1-x)\varphi_0 + x\varphi_1 + \tilde{\tilde{\phi}}$ where

$$\tilde{\tilde{\phi}}''(x) = \mu(x),\ x \in]0,1[,\ \ \tilde{\tilde{\phi}}(0) = \tilde{\tilde{\phi}}(1) = 0.$$

Obviously $\tilde{\phi}$ is convex and verifies $\|\tilde{\phi}''\|_{L^\infty} \leq M$. Since $\tilde{\tilde{\phi}}(0) = \tilde{\tilde{\phi}}(1) = 0$, there is $c \in]0,1[$ such that $\tilde{\tilde{\phi}}'(c) = 0$ and therefore $|\tilde{\tilde{\phi}}\,'(x)| = \left|\int_c^x \tilde{\tilde{\phi}}''(y)\, dy\right| \leq M,\ \forall\, x \in [0,1]$. We deduce that

$$|\tilde{\phi}\,'(x)| \leq |\varphi_1 - \varphi_0| + M,\ \forall\, x \in [0,1].$$

Observe also that $|\tilde{\tilde{\phi}}(x)| = \left|\int_0^x \tilde{\tilde{\phi}}\,'(y)\, dy\right| \leq M,\ \forall\, x \in [0,1]$ and thus

$$|\tilde{\phi}(x)| \leq \max\{|\varphi_0|, |\varphi_1|\} + M,\ \forall\, x \in [0,1].$$

$\square$

Proposition 3.2 *Assume that $\varphi_0, \varphi_1 \in \mathbb{R}$, $g_0, g_1 \geq 0$ such $G_0 < +\infty$, $G_\infty < +\infty$. Then the application $\mathcal{F}$ is continuous on $\mathcal{D}$ with respect to the topology of $C^1([0,1])$.*

Proof. The arguments are standard. Take $(\phi_n)_n \subset \mathcal{D}$ such that $\lim_{n\to+\infty}\phi_n = \phi$ in $C^1([0,1])$. Denote by f_n the mild solution of (40), (42) corresponding to ϕ_n and let $\mu_n = \int_{\mathbb{R}}(1+p^2)^{-\frac{1}{2}} f_n \; dp$, $\forall n$. By Proposition 2.3 and Corollary 2.2 we have

$$\|f_n\|_{L^\infty(]0,1[\times\mathbb{R})} \leq e^{2|\varphi_1-\varphi_0|}G_\infty, \;\; \|\mu_n\|_{L^\infty(]0,1[)} \leq M, \; \forall n,$$

and we can extract a subsequence $(f_{n_k})_k$ converging weakly $\star$ in $L^\infty(]0,1[\times\mathbb{R})$ towards some function f satisfying $0 \leq f \leq e^{2|\varphi_1-\varphi_0|}G_\infty$. By elementary but tedious arguments (which we omit) we can check that f is the mild solution of (40), (42) corresponding to ϕ. We use here the geometry of characteristics (see Section 4 for this kind of computations). In fact all the sequence $(f_n)_n$ converges weakly $\star$ in $L^\infty(]0,1[\times\mathbb{R})$ towards f. By using the uniform bound $\sup_n \|\phi_n\|_{L^\infty} \leq \max\{|\varphi_0|,|\varphi_1|\} + M$ and the second statement of Proposition 2.4 we can show that $(\mu_n)_n$ converges weakly $\star$ in $L^\infty(]0,1[)$ towards $\mu = \int_{\mathbb{R}}(1+p^2)^{-\frac{1}{2}} f \; dp$. Observe also that we can extract a subsequence $(\phi_{n_k})_k$ such that $(\tilde{\phi}_{n_k})_k = (\mathcal{F}\phi_{n_k})_k$ converges in $C^1([0,1])$ towards some function $\omega \in \mathcal{D}$. By passing to the limit in distributions sense in the equation $\frac{d^2}{dx^2}\tilde{\phi}_{n_k} = \mu_{n_k}(x), \; x \in]0,1[$, one gets that $\omega = \mathcal{F}\phi$. We deduce that $\lim_{n\to+\infty}\mathcal{F}\phi_n = \mathcal{F}\phi$ in $C^1([0,1])$. $\square$

By applying the Schauder fixed point theorem we obtain our existence result stated in Theorem 1.1.

Consider also the moment $k(\cdot) = \int_{\mathbb{R}} v(p) p f(\cdot, p) \; dp$. We have the following conservation for the total energy

Proposition 3.3 *Under the hypotheses of Theorem* 1.1 *let* (f,ϕ) *be the mild solution constructed above. Assume also that*

$$K := \int_{p>0} v(p) p g_0(p) \; dp + \int_{p<0} v(p) p g_1(p) \; dp < +\infty.$$

Then $\int_{\mathbb{R}} v(p) p f(\cdot,p) \; dp \in L^\infty(]0,1[)$ *and*

$$\frac{d}{dx}\left\{\int_{\mathbb{R}} v(p) p f(x,p) \; dp + \frac{1}{2}|\phi\,'(x)|^2\right\} = 0, \;\; x \in]0,1[.$$

Proof. Since g_0, g_1 are bounded, the hypothesis $K < +\infty$ implies that

$$G_1 := \int_{p>0} p g_0(p)\, dp - \int_{p<0} p g_1(p)\, dp < +\infty,$$

and by Proposition 2.6 we know that $\int_{\mathbb{R}} |p| f(\cdot, p)\, dp \in L^\infty(]0,1[)$. Since f is bounded we deduce that $k(\cdot) = \int_{\mathbb{R}} v(p) p f(\cdot, p)\, dp \in L^\infty(]0,1[)$. For any $\theta \in C_c^1(]0,1[)$ we can write

$$\int_0^1 \theta\,'(x) k(x)\, dx = \int_0^1 \int_{\mathbb{R}} f(x,p) \theta\,'(x) v(p) p\, dp\, dx \tag{44}$$

$$= \sum_{k=0}^{1} (-1)^k e^{-\phi(k)} \int_{(-1)^k p>0} v(p) g_k(p) \int_0^{s_{\mathrm{out}}(k,p)} e^{\phi(X(s;k,p))} \frac{d}{ds}\{\theta(X(s;k,p))\} P(s;k,p)\, ds\, dp.$$

After integration by parts we have for $k \in \{0,1\}$

$$\int_0^{s_{\mathrm{out}}(k,p)} e^{\phi(X(s;k,p))} \frac{d}{ds}\{\theta(X(s;k,p))\} P(s;k,p)\, ds = \int_0^{s_{\mathrm{out}}(k,p)} \theta(X(s;k,p)) \times e^{\phi(X(s;k,p))} \frac{\phi\,'(X(s;k,p))}{(1+|P(s;k,p)|^2)^{\frac{1}{2}}}\, ds. \tag{45}$$

Combining (44), (45) and by using one more time the mild formulation one gets

$$\int_0^1 \theta\,'(x) k(x)\, dx = \int_0^1 \int_{\mathbb{R}} f(x,p) \frac{\phi\,'(x)}{(1+p^2)^{\frac{1}{2}}} \theta(x)\, dp\, dx$$

$$= \int_0^1 \theta(x) \phi\,'(x) \mu(x)\, dx, \tag{46}$$

where $\mu = \int_{\mathbb{R}} (1+p^2)^{-\frac{1}{2}} f\, dp$. We deduce that $\frac{d}{dx} k = -\phi\,'(x)\mu(x),\ x \in]0,1[$. From (41) we have

$$\frac{1}{2} \frac{d}{dx} |\phi\,'(x)|^2 = \phi\,'(x) \mu(x), \quad x \in]0,1[, \tag{47}$$

and therefore $\frac{d}{dx}\{k(x) + \frac{1}{2}|\phi\,'(x)|^2\} = 0, \quad x \in]0,1[$. □

4 An asymptotic analysis of the one dimensional Nordström-Vlasov system

We end this paper with an asymptotic analysis for the stationary Nordström-Vlasov equations similar to those performed in [9] for the Vlasov-Poisson equations, when studying the Child-Langmuir law. Assume that $\varphi_0, \varphi_1 \in \mathbb{R}$, $\varphi_0 > \varphi_1$, $g \geq 0$, $g \in L^\infty(]0, +\infty[)$ such that $\int_{\mathbb{R}} p g(p)\ dp < +\infty$. For any $\varepsilon > 0$ we consider the Nordström-Vlasov system

$$v(p)\partial_x f_\varepsilon - (1+p^2)^{\frac{1}{2}} \phi_\varepsilon{}'(x)\partial_p f_\varepsilon = 2 f_\varepsilon(x,p) v(p) \phi_\varepsilon{}'(x), \ \ (x,p) \in]0,1[\times\mathbb{R}, \qquad (48)$$

$$\phi_\varepsilon''(x) = \int_{\mathbb{R}} \frac{f_\varepsilon(x,p)}{(1+p^2)^{\frac{1}{2}}}\ dp, \ \ x \in]0,1[, \qquad (49)$$

with the boundary conditions

$$f_\varepsilon(x=0, p>0) = g_\varepsilon(p), \ \ f_\varepsilon(x=1, p<0) = 0, \qquad (50)$$

$$\phi_\varepsilon(0) = \varphi_0, \ \ \phi_\varepsilon(1) = \varphi_1, \qquad (51)$$

where $g_\varepsilon(p) = \frac{1}{\varepsilon^2} g\left(\frac{p}{\varepsilon}\right)$, $\forall\, p > 0$. The choice of the particular scaling for the boundary condition g_ε is motivated by the following computation involving the incoming current density

$$e^{-\varphi_0} \int_{p>0} v(p) g_\varepsilon(p)\ dp = e^{-\varphi_0} \int_{u>0} \frac{u g(u)}{(1+\varepsilon^2 u^2)^{\frac{1}{2}}}\ du \leq e^{-\varphi_0}\gamma, \ \forall\, \varepsilon > 0, \qquad (52)$$

with $\gamma = \int_{u>0} u g(u)\ du < +\infty$. By Theorem 1.1 we know that for any $\varepsilon > 0$ there is (at least) one mild solution $(f_\varepsilon \geq 0, \phi_\varepsilon)$ for (48), (49), (50), (51). Notice that under the assumption $\varphi_0 > \varphi_1$, the critical impulsions are given by

$$p_{0,\varepsilon} = 0, \ \ p_{1,\varepsilon} = -\left(e^{2(\varphi_0 - \varphi_1)} - 1\right)^{\frac{1}{2}} =: p_1, \ \forall\, \varepsilon > 0.$$

Actually the condition $\varphi_0 > \varphi_1$ ensures that all particles injected by the left boundary $x = 0$ leave the interval $]0,1[$ by the right boundary $x = 1$. We intend to investigate the behavior of solutions $(f_\varepsilon, \phi_\varepsilon)$ when ε goes to 0. We establish uniform estimates with respect to $\varepsilon > 0$.

Proposition 4.1 *Assume that $0 \leq g \in L^\infty(]0, +\infty[)$ such that $\int_{\mathbb{R}} p g(p)\ dp < +\infty$. Then there is a constant C such that*

$$\left\| \int_{\mathbb{R}} p v(p) f_\varepsilon(\cdot, p)\ dp \right\|_{L^\infty(]0,1[)} + \|\phi_\varepsilon\|_{L^\infty(]0,1[)} + \|\phi_\varepsilon{}'\|_{L^\infty(]0,1[)} \leq C, \ \forall\, \varepsilon > 0.$$

Proof. By Proposition 3.3 we know that $k_\varepsilon(\cdot) = \int_{\mathbb{R}} pv(p) f_\varepsilon(\cdot, p)\, dp \in L^\infty(]0,1[)$ for any $\varepsilon > 0$. Actually the L^∞ bound is uniform with respect to $\varepsilon > 0$. Indeed, for any nonnegative function $\theta \in L^1(]0,1[)$ we have

$$\begin{aligned}
\int_0^1 k_\varepsilon(x)\theta(x)\, dx &= \int_0^1 \int_{\mathbb{R}} pv(p) f_\varepsilon(x,p)\theta(x)\, dp\, dx \\
&= e^{-\varphi_0} \int_{p>0} v(p) g_\varepsilon(p) \int_0^{s_{\text{out}}(0,p)} e^{\phi_\varepsilon(X_\varepsilon(s))} \theta(X_\varepsilon(s)) P_\varepsilon(s) v(P_\varepsilon(s))\, ds\, dp \\
&= \int_{p>0} p g_\varepsilon(p) \int_0^{s_{\text{out}}(0,p)} \theta(X_\varepsilon(s)) |v(P_\varepsilon(s))|^2\, ds\, dp \\
&\le \int_{p>0} p g_\varepsilon(p) \int_0^{s_{\text{out}}(0,p)} \theta(X_\varepsilon(s)) |v(P_\varepsilon(s))|\, ds\, dp. \qquad (53)
\end{aligned}$$

In the above computation we have used the conservation

$$e^{\phi_\varepsilon(X_\varepsilon(s))} \left(1 + |P_\varepsilon(s)|^2\right)^{\frac{1}{2}} = e^{\varphi_0}(1+p^2)^{\frac{1}{2}}, \ \forall\, s \in [0, s_{\text{out}}(0,p)],$$

where $(X_\varepsilon, P_\varepsilon)$ are the characteristics associated to ϕ_ε. Now by using the inequality

$$\int_0^{s_{\text{out}}(0,p)} \theta(X_\varepsilon(s)) |v(P_\varepsilon(s))|\, ds \le 2\|\theta\|_{L^1(]0,1[)}, \ \forall\, p > 0,$$

one gets

$$\int_0^1 k_\varepsilon(x)\theta(x)\, dx \le 2\|\theta\|_{L^1} \int_{u>0} ug(u)\, du,$$

and therefore

$$\|k_\varepsilon\|_{L^\infty(]0,1[)} \le 2\int_{u>0} ug(u)\, du = 2\gamma, \ \forall\, \varepsilon > 0.$$

By Proposition 3.3 we know also that $\frac{d}{dx}\{k_\varepsilon(x) + \frac{1}{2}|\phi_\varepsilon{}'(x)|^2\} = 0,\ x \in]0,1[$. Take $x_\varepsilon \in]0,1[$ such that $\phi_\varepsilon{}'(x_\varepsilon) = \varphi_1 - \varphi_0$. Therefore we obtain for any $x \in]0,1[$ and $\varepsilon > 0$

$$\begin{aligned}
\frac{1}{2}|\phi_\varepsilon{}'(x)|^2 &\le k_\varepsilon(x) + \frac{1}{2}|\phi_\varepsilon{}'(x)|^2 = k_\varepsilon(x_\varepsilon) + \frac{1}{2}|\phi_\varepsilon{}'(x_\varepsilon)|^2 \\
&\le 2\gamma + \frac{1}{2}(\varphi_1 - \varphi_0)^2,
\end{aligned}$$

which implies that $\|\phi_\varepsilon{}'\|_{L^\infty} \le 2\gamma^{\frac{1}{2}} + \varphi_0 - \varphi_1$. By taking into account that $|\phi_\varepsilon(x) - \varphi_0| \le \|\phi_\varepsilon{}'\|_\infty \le 2\gamma^{\frac{1}{2}} + \varphi_0 - \varphi_1$ for any $x \in [0,1]$ and $\varepsilon > 0$, our conclusion follows with $C = 2\gamma + |\varphi_0| + 2(2\gamma^{\frac{1}{2}} + \varphi_0 - \varphi_1)$. □

Proposition 4.2 *Assume that* $\varphi_0 > \varphi_1$, $0 \le g \in L^\infty(]0,+\infty[)$ *such that* $\int_{u>0} ug(u)\,du < +\infty$. *Then we have*

$$0 \le \phi_\varepsilon''(x) \le \frac{\int_{u>0} ug(u)\,du}{\sqrt{2x(\varphi_0-\varphi_1)}},\ \forall\, x \in]0,1[,\ \forall\, \varepsilon > 0.$$

In particular we have

$$|\phi_\varepsilon{}'(x) - \phi_\varepsilon{}'(y)| \le \frac{\sqrt{2}\int_{u>0} ug(u)\,du}{\sqrt{\varphi_0-\varphi_1}}|x-y|^{\frac{1}{2}},\ \forall\, x,y \in [0,1],\ \forall\, \varepsilon > 0.$$

Proof. The density $f_\varepsilon(x,p)$ is given by

$$f_\varepsilon(x,p) = e^{2\phi_\varepsilon(x)-2\varphi_0} g_\varepsilon\left(\sqrt{(1+p^2)e^{2\phi_\varepsilon(x)-2\varphi_0}-1}\right) \mathbf{1}_{\{p>\sqrt{e^{2\varphi_0-2\phi_\varepsilon(x)}-1}\}},$$

for any $(x,p) \in [0,1]\times\mathbb{R}$, $\varepsilon > 0$. In order to estimate $\phi_\varepsilon''(x)$ we perform for any fixed $x \in]0,1[$ the change of variable

$$\sqrt{(1+p^2)e^{2\phi_\varepsilon(x)-2\varphi_0}-1} = \varepsilon u,\ \ u > 0.$$

We obtain for any $x \in]0,1[$, $\varepsilon > 0$

$$\begin{aligned} 0 \le \phi_\varepsilon''(x) &= \mu_\varepsilon(x) \le \int_{\mathbb{R}} f_\varepsilon(x,p)\,dp \\ &= \int_{\mathbb{R}} e^{2\phi_\varepsilon(x)-2\varphi_0}\frac{1}{\varepsilon^2} g\left(\varepsilon^{-1}\sqrt{(1+p^2)e^{2\phi_\varepsilon(x)-2\varphi_0}-1}\right) \mathbf{1}_{\{p>\sqrt{e^{2\varphi_0-2\phi_\varepsilon(x)}-1}\}}\,dp \\ &= \int_{u>0} \frac{ug(u)}{\sqrt{(1+\varepsilon^2u^2)e^{2\varphi_0-2\phi_\varepsilon(x)}-1}}\,du. \end{aligned} \quad (54)$$

Since ϕ_ε is convex, we have also the inequality

$$\varphi_0 - \phi_\varepsilon(x) \ge x(\varphi_0-\varphi_1),\ \forall\, x \in [0,1],\ \varepsilon > 0, \quad (55)$$

and therefore

$$(1+\varepsilon^2u^2)e^{2\varphi_0-2\phi_\varepsilon(x)} - 1 \ge e^{2x(\varphi_0-\varphi_1)} - 1 \ge 2x(\varphi_0-\varphi_1),\ \forall\, x \in [0,1],\ \varepsilon > 0. \quad (56)$$

Combining (54), (56) yields

$$0 \le \phi_\varepsilon''(x) \le \frac{\int_{u>0} ug(u)\,du}{\sqrt{2x(\varphi_0-\varphi_1)}},\ \forall\, x \in]0,1],\ \varepsilon > 0. \quad (57)$$

For any $0 \leq x \leq y \leq 1$ we obtain

$$\begin{aligned} 0 &\leq \phi_\varepsilon'(y) - \phi_\varepsilon'(x) = \int_x^y \phi_\varepsilon''(z)\, dz \leq \int_x^y \frac{\gamma}{\sqrt{2z(\varphi_0 - \varphi_1)}}\, dz \\ &= \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}}(\sqrt{y} - \sqrt{x}) \leq \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}}\sqrt{y - x}. \end{aligned}$$

□

By using Arzela-Ascoli theorem we deduce from Propositions 4.1, 4.2 that there is a sequence $(\varepsilon_k)_k$ converging towards 0 such that $\lim_{k\to+\infty}\phi_{\varepsilon_k} = \phi$ in $C^1([0,1])$. Obviously we have

$$\phi(0) = \varphi_0, \;\; \phi(1) = \varphi_1,$$

$$\varphi_0 - \phi(x) \geq x(\varphi_0 - \varphi_1), \; \forall\, x \in [0,1],$$

$$\phi(x) \geq \varphi_1 - 2\gamma^{\frac{1}{2}}, \; \forall\, x \in [0,1],$$

and

$$0 \leq \phi'(y) - \phi'(x) \leq \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}}(\sqrt{y} - \sqrt{x}), \; \forall\, 0 \leq x \leq y \leq 1.$$

By the computations of (54), (56) we have

$$\int_0^1 \int_{\mathbb{R}} f_\varepsilon(x,p)\, dp\, dx \leq \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}}, \; \forall\, \varepsilon > 0.$$

After extraction of a subsequence (still denoted $(\varepsilon_k)_k$) we have

$$f_{\varepsilon_k} \rightharpoonup f, \text{ weakly } \star \text{ in } \mathcal{M}_b([0,1]\times\mathbb{R}),$$

where $\mathcal{M}_b([0,1]\times\mathbb{R})$ is the space of bounded measures on $[0,1]\times\mathbb{R}$.

Proposition 4.3 *Assume that $\varphi_0 > \varphi_1$, $0 \leq g \in L^\infty(]0,+\infty[)$ such that $\int_{u>0} ug(u)\, du < +\infty$. Then the limit $(f,\phi) = \lim_{k\to+\infty}(f_{\varepsilon_k}, \phi_{\varepsilon_k})$ satisfies*

$$f(x,p) = \frac{\int_{u>0} ug(u)\, du}{\sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}}\, \delta(p - \sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}),$$

and

$$\phi''(x) = \frac{\int_{u>0} ug(u)\, du}{e^{\varphi_0 - \phi(x)}\sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}}, \; \forall\, x \in]0,1[,$$

$$\phi(0) = \varphi_0, \;\; \phi(1) = \varphi_1,$$

where δ denotes the Dirac measure.

Proof. For any function $\psi \in C_c^0([0,1]\times\mathbb{R})$ we have

$$\begin{aligned}
\int_0^1\int_{\mathbb{R}} f_{\varepsilon_k}(x,p)\psi(x,p)\;dp\;dx &= \frac{1}{\varepsilon_k^2}\int_0^1\int_{\mathbb{R}} e^{2\phi_{\varepsilon_k}(x)-2\varphi_0} g\left(\varepsilon_k^{-1}\sqrt{(1+p^2)e^{2\phi_{\varepsilon_k}(x)-2\varphi_0}-1}\right)\\
&\times\; \mathbf{1}_{\{p>\sqrt{e^{2\varphi_0-2\phi_{\varepsilon_k}(x)}-1}\}}\psi(x,p)\;dp\;dx \qquad (58)\\
&= \int_0^1\int_{u>0} \frac{ug(u)\psi(x,\sqrt{(1+\varepsilon_k^2u^2)e^{2\varphi_0-2\phi_{\varepsilon_k}(x)}-1}}{\sqrt{(1+\varepsilon_k^2u^2)e^{2\varphi_0-2\phi_{\varepsilon_k}(x)}-1}}\;du\;dx.
\end{aligned}$$

By using inequality (56) we obtain for any $(x,u)\in]0,1[\times]0,+\infty[$ and $k\geq 1$

$$\left|\frac{ug(u)\psi(x,\sqrt{(1+\varepsilon_k^2u^2)e^{2\varphi_0-2\phi_{\varepsilon_k}(x)}-1}}{\sqrt{(1+\varepsilon_k^2u^2)e^{2\varphi_0-2\phi_{\varepsilon_k}(x)}-1}}\right| \;\leq\; \frac{ug(u)}{\sqrt{2x(\varphi_0-\varphi_1)}}\|\psi\|_{L^\infty}\in L^1(]0,1[\times]0,+\infty[).$$

By using the dominated convergence theorem one gets

$$\lim_{k\to+\infty}\int_0^1\int_{\mathbb{R}} f_{\varepsilon_k}(x,p)\psi(x,p)\;dp\;dx \;=\; \int_0^1\int_{u>0} ug(u)\frac{\psi\left(x,\sqrt{e^{2\varphi_0-2\phi(x)}-1}\right)}{\sqrt{e^{2\varphi_0-2\phi(x)}-1}}\;du\;dx,$$

and therefore

$$\lim_{k\to+\infty} f_{\varepsilon_k} = \frac{\gamma}{\sqrt{e^{2\varphi_0-2\phi(x)}-1}}\delta\left(p-\sqrt{e^{2\varphi_0-2\phi(x)}-1}\right),$$

weakly $\star$ in $\mathcal{M}_b([0,1]\times\mathbb{R})$. Take now $\theta\in C_c^1(]0,1[)$. We have

$$\begin{aligned}
-\int_0^1 \phi_{\varepsilon_k}'(x)\theta'(x)\;dx &= \int_0^1 \mu_{\varepsilon_k}(x)\theta(x)\;dx\\
&= \int_0^1\int_{\mathbb{R}} f_{\varepsilon_k}(x,p)\frac{\theta(x)}{(1+p^2)^{\frac{1}{2}}}\;dp\;dx. \qquad (59)
\end{aligned}$$

By using one more time the dominated convergence theorem we obtain as before that

$$\lim_{k\to+\infty}\int_0^1 \mu_{\varepsilon_k}(x)\theta(x)\;dx \;=\; \int_0^1\int_{u>0}\frac{ug(u)\theta(x)}{e^{\varphi_0-\phi(x)}\sqrt{e^{2\varphi_0-2\phi(x)}-1}}\;du\;dx. \qquad (60)$$

From (59), (60) one gets

$$-\int_0^1 \phi'(x)\theta'(x)\;dx \;=\; \gamma\int_0^1 \frac{\theta(x)}{e^{\varphi_0-\phi(x)}\sqrt{e^{2\varphi_0-2\phi(x)}-1}}\;dx,$$

saying that

$$\phi''(x) = \frac{\gamma}{e^{\varphi_0-\phi(x)}\sqrt{e^{2\varphi_0-2\phi(x)}-1}},\quad x\in]0,1[.$$

□

Remark 4.1 *Since ϕ is convex we have $\varphi_0 - \phi(x) \geq x(\varphi_0 - \varphi_1)$, $\forall\ x \in [0,1]$ and therefore*

$$0 \leq \phi''(x) \leq \frac{\gamma}{\sqrt{2x(\varphi_0 - \varphi_1)}} \in L^1(]0,1[).$$

In the following proposition we study the limit problem

$$\phi''(x) = \frac{\gamma}{e^{\varphi_0 - \phi(x)}\sqrt{e^{2\varphi_0 - 2\phi(x)} - 1}}, \quad x \in]0,1[, \tag{61}$$

$$\phi(0) = \varphi_0, \quad \phi(1) = \varphi_1, \tag{62}$$

where $\varphi_0 > \varphi_1$ and $\gamma \geq 0$.

Proposition 4.4 *Assume that $\varphi_0 > \varphi_1$, $\gamma \geq 0$. Then there is a unique solution $\phi \in C^1([0,1]) \cap C^2(]0,1])$ for (61), (62) satisfying $\varphi_0 - \phi(x) \geq x(\varphi_0 - \varphi_1)$, $\forall\ x \in [0,1]$.*

Proof. Denote by $h :]-\infty, \varphi_0[\to \mathbb{R}$ the function

$$h(\varphi) = \frac{\gamma}{e^{\varphi_0 - \varphi}\sqrt{e^{2\varphi_0 - 2\varphi} - 1}}, \ \forall \varphi < \varphi_0.$$

Take $\phi_1, \phi_2 \in C^1([0,1]) \cap C^2(]0,1])$ two solutions for (61), (62). We have for any $0 < \eta < 1$

$$\int_\eta^1 (h(\phi_1(x)) - h(\phi_2(x)))(\phi_1(x) - \phi_2(x))\ dx = \int_\eta^1 (\phi_1''(x) - \phi_2''(x))(\phi_1(x) - \phi_2(x))\ dx$$
$$= (\phi_1'(x) - \phi_2'(x))(\phi_1(x) - \phi_2(x))|_\eta^1 - \int_\eta^1 |\phi_1'(x) - \phi_2'(x)|^2\ dx. \tag{63}$$

Observe that h is strictly increasing and therefore we obtain

$$\int_\eta^1 |\phi_1'(x) - \phi_2'(x)|^2\ dx \leq -(\phi_1'(\eta) - \phi_2'(\eta))(\phi_1(\eta) - \phi_2(\eta)), \ \forall\ \eta > 0.$$

Letting $\eta \searrow 0$ yields $\phi_1' = \phi_2'$ and thus $h(\phi_1(x)) = h(\phi_2(x))$ for any $x \in]0,1]$, implying that $\phi_1 = \phi_2$.

In order to prove the existence consider the fixed point application $\mathcal{F}\phi = \tilde{\phi}$ where $\tilde{\phi}$ solves

$$\tilde{\phi}''(x) = h(\phi(x)), \quad x \in]0,1[,$$

$$\tilde{\phi}(0) = \varphi_0, \quad \tilde{\phi}(1) = \varphi_1,$$

for any $\phi \in \mathcal{D}$ where $\mathcal{D}$ is the set of functions $\phi \in C^1([0,1])$ satisfying

$$\phi(0) = \varphi_0,\ \phi(1) = \varphi_1,\ \varphi_0 - \phi(x) \geq x(\varphi_0 - \varphi_1),\ \phi(x) - \varphi_1 \geq -2\gamma^{\frac{1}{2}},\ \forall\, x \in [0,1],$$

$$|\phi\,'(x)| \leq \varphi_0 - \varphi_1 + \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}},\ \forall\, x \in [0,1],$$

and

$$0 \leq \phi\,'(y) - \phi\,'(x) \leq \frac{\gamma\sqrt{2}}{\sqrt{\varphi_0 - \varphi_1}}(\sqrt{y} - \sqrt{x}),\ \forall\, 0 \leq x \leq y \leq 1.$$

We check easily that $\mathcal{D}$ is convex and compact in $C^1([0,1])$. Moreover $\mathcal{D}$ is left invariant by $\mathcal{F}$ and $\mathcal{F}$ is continuous on $\mathcal{D}$ with respect to the $C^1([0,1])$ topology. By using the Schauder fixed point theorem we deduce that there is $\phi \in \mathcal{D}$ such that

$$\phi''(x) = h(\phi(x)), \quad x \in]0,1[$$

$$\phi(0) = \varphi_0, \quad \phi(1) = \varphi_1.$$

□

Combining Propositions 4.3, 4.4 one gets Theorem 1.2.

5 Appendix

Proof (of Proposition 2.2) We justify only the first two statements. Similar arguments apply for the other ones.

1) Assume that $p_0 > 0$ and consider $p \in]0, p_0[$. Therefore we can take $\tilde{x} \in]0,1[$ such that $W(0,p) < W(\tilde{x}, 0)$. We claim that $X(s;0,p) \in [0, \tilde{x}[$ for any $s \in [0, s_{\text{out}}(0,p)[$. Indeed, if there is $\tilde{\tilde{s}}$ such that $X(\tilde{\tilde{s}};0,p) \geq \tilde{x} > 0 = X(0;0,p)$, then there is $\tilde{s} \in]0, \tilde{\tilde{s}}]$ such that $X(\tilde{s};0,p) = \tilde{x}$. By Proposition 2.1 we can write

$$W(\tilde{x}, P(\tilde{s})) = W(X(\tilde{s}), P(\tilde{s})) = W(0,p) < W(\tilde{x}, 0) \leq W(\tilde{x}, P(\tilde{s})),$$

and thus one gets a contradiction. Let us consider

$$s_0 = \sup\{0 < \tau \leq s_{\text{out}}(0,p)\ :\ P(s;0,p) > 0,\ \forall\, s \in [0, \tau[\},$$

and consider $x_0 = \lim_{s \nearrow s_0} X(s;0,p)$. Since for any $s \in]0, s_0[$ we have $0 < X(s;0,p) < \tilde{x}$ and $X(\cdot;0,p)$ is strictly increasing on $[0, s_0[$ we deduce that $0 < x_0 \leq \tilde{x} < 1$ and $X(s;0,p) < x_0 \; \forall \; s \in [0, s_0[$. By construction we have $P(s;0,p) > 0 \; \forall \; s \in [0, s_0[$ and by Proposition 2.1 we have

$$(1 + |P(s)|^2) e^{2\phi(X(s))} = (1 + p^2) e^{2\phi(0)}, \; \forall \; s \in [0, s_0[,$$

and therefore $P(s;0,p)$ has a finite limit as s goes to s_0

$$\lim_{s \nearrow s_0} P(s;0,p) = \eta \geq 0.$$

We claim that $\eta = 0$. Indeed, in the case $s_0 = +\infty$, if $\eta > 0$, then for s large enough we have $P(s;0,p) > \frac{\eta}{2}$, $\forall s > s_1$. Taking into account that $v(P(s;0,p)) > v\left(\frac{\eta}{2}\right) = \frac{\eta}{2}\left(1 + \left(\frac{\eta}{2}\right)^2\right)^{-1/2}$, $\forall s > s_1$, we obtain a contradiction since

$$1 > X(s) - X(s_1) \geq v\left(\frac{\eta}{2}\right)(s - s_1), \; \forall \; s > s_1.$$

In the case $s_0 < +\infty$ if $\eta > 0$ then $P(s_0) = \lim_{s \nearrow s_0} P(s;0,p) = \eta > 0$ and there is $\tilde{s}_0 > s_0$ such that $P(\tau) > 0$ for any $\tau \in [s_0, \tilde{s}_0]$ which is in contradiction with the definition of s_0. Therefore in both cases $\lim_{s \nearrow s_0} P(s;0,p) = 0$. Notice also that in the case $s_0 < +\infty$, $(X^{\pm}(s), P^{\pm}(s)) := (X(s_0 \pm s), \pm P(s_0 \pm s))$ verify (9) and the condition $(X^{\pm}(0), P^{\pm}(0)) = (x_0, 0)$. By the uniqueness of the characteristics we deduce that $(X^+, P^+) = (X^-, P^-)$ saying that $s_{out} = 2s_0$ and $X(s;0,p) = X(2s_0 - s;0,p)$, $P(s;0,p) = -P(2s_0 - s;0,p)$ for any $s \in [0, 2s_0]$. Therefore it remains to prove that the condition $\phi'(x_0) \neq 0$ ensures that $s_0 < +\infty$. Suppose that $s_0 = +\infty$. By Proposition 2.1 we have for any $s \in [0, s_0[$

$$e^{2\phi(X(s))} \leq (1 + |P(s)|^2) e^{2\phi(X(s))} = W(X(s), P(s)) = W(x_0, 0) = e^{2\phi(x_0)},$$

and thus

$$\phi(X(s)) \leq \phi(x_0), \; \forall \; s \in [0, s_0[. \tag{64}$$

Since $X(s) < x_0 \; \forall \; s \in [0, s_0[$ we deduce that $\phi'(x_0) > 0$. We have

$$P(s) = \left(e^{2\phi(x_0) - 2\phi(X(s))} - 1\right)^{\frac{1}{2}}, \; \forall \; s \in [0, s_0[. \tag{65}$$

We obtain for any $s \in [0, s_0[$

$$\frac{dX}{ds} = v(P(s)) = \frac{\left(e^{2\phi(x_0)-2\phi(X(s))} - 1\right)^{\frac{1}{2}}}{e^{\phi(x_0)-\phi(X(s))}},$$

which implies

$$\frac{X'(s)}{(e^{2\phi(x_0)-2\phi(X(s))} - 1)^{\frac{1}{2}}} = e^{\phi(X(s))-\phi(x_0)} \geq e^{-2\|\phi\|_\infty}, \ \forall\, s \in [0, s_0[.$$

After integration with respect to s one gets for any $0 \leq s_2 < s_3 < s_0$

$$\int_{X(s_2)}^{X(s_3)} \frac{dx}{(e^{2\phi(x_0)-2\phi(x)} - 1)^{\frac{1}{2}}} \geq e^{-2\|\phi\|_\infty}(s_3 - s_2). \tag{66}$$

Observe that

$$\lim_{x \nearrow x_0} \frac{\left(e^{2\phi(x_0)-2\phi(x)} - 1\right)^{\frac{1}{2}}}{(x_0 - x)^{\frac{1}{2}}} = \sqrt{2}(\phi'(x_0))^{\frac{1}{2}} > (\phi'(x_0))^{\frac{1}{2}}.$$

Since $\lim_{s \nearrow s_0} X(s) = x_0$ we deduce that for $s_2 < s_3$ large enough we have

$$\frac{\left(e^{2\phi(x_0)-2\phi(x)} - 1\right)^{\frac{1}{2}}}{(x_0 - x)^{\frac{1}{2}}} > (\phi'(x_0))^{\frac{1}{2}}, \ \forall\, x \in [X(s_2), X(s_3)]. \tag{67}$$

Combining (66), (67) yields for s_2, s_3 large enough

$$e^{-2\|\phi\|_\infty}(s_3 - s_2) \leq \frac{1}{(\phi'(x_0))^{\frac{1}{2}}} \int_{X(s_2)}^{X(s_3)} \frac{dx}{(x_0 - x)^{\frac{1}{2}}} \leq 2\frac{x_0^{\frac{1}{2}}}{(\phi'(x_0))^{\frac{1}{2}}} < +\infty, \tag{68}$$

and therefore we obtain a contradiction. Thus we have $s_0 < +\infty$.

2) Take now $p > p_0$ and denote $\delta = (p^2 - p_0^2)e^{2\phi(0)-2\max_{x\in[0,1]}\phi(x)} > 0$. We have

$$\begin{aligned}
(1 + |P(s)|^2)e^{2\phi(X(s))} &= W(X(s), P(s)) = W(0, p) = (1 + p^2)e^{2\phi(0)} \\
&= (1 + p_0^2)e^{2\phi(0)} + (p^2 - p_0^2)e^{2\phi(0)} \\
&\geq e^{2\phi(X(s))} + \delta e^{2\phi(X(s))}, \ \forall\, s \in [0, s_{\text{out}}(0, p)[,
\end{aligned}$$

and therefore $|P(s)|^2 \geq \delta > 0$ for any $s \in [0, s_{\text{out}}(0, p)[$. In particular we deduce that $s_{\text{out}}(0, p) < +\infty$, $P(s) > 0$ for any $s \in [0, s_{\text{out}}(0, p)]$ and $X(s_{\text{out}}(0, p); 0, p) = 1$.

Proof (of Corollary 2.1) Let us prove for the moment that $s_{\text{out}}(0,p) < +\infty$ for any $p \in]0,+\infty[-\{p_0\}$. By Proposition 2.2 it is sufficient to consider $0 < p < p_0$ and to check that $\phi'(x_0) \neq 0$, with x_0 given in statement 1) of Proposition 2.2. By Proposition 2.1 we have $e^{2\phi(0)} < (1+p^2)e^{2\phi(0)} = e^{2\phi(x_0)}$ implying that $\phi(x_0) > \phi(0)$. Therefore x_0 is not a minimum point of ϕ on $[0,1]$ and since ϕ is convex, thus $\phi'(x_0) \neq 0$. It remains to analyze the case $p = p_0 > 0$. Since ϕ is convex, we deduce by (19) that $\max_{x\in[0,1]} \phi(x) = \phi(1) > \phi(0)$. In fact we have $\phi(1) > \phi(x)$, $x \in [0,1[$. For any $s \in [0, s_{\text{out}}(0,p)[$ we obtain

$$(1+|P(s)|^2)e^{2\phi(1)} > (1+|P(s)|^2)e^{2\phi(X(s))} = (1+p_0^2)e^{2\phi(0)} = e^{2\phi(1)},$$

which implies $P(s) > 0$, $\forall\, s \in [0, s_{\text{out}}(0,p)[$. In this case we obtain

$$\lim_{s \nearrow s_{\text{out}}(0,p)} X(s;0,p) = 1, \quad \lim_{s \nearrow s_{\text{out}}(0,p)} P(s;0,p) = 0.$$

Moreover, since ϕ is convex, $\phi(0) < \phi(1)$, we have $\phi'(1) > 0$ and as in the proof of Proposition 2.2 we deduce that $s_{\text{out}}(0,p_0) < +\infty$.

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