Some remarks on time-dependent variational problems and their asymptotic behaviour

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Abstract

The subject matter of this paper deals with asymptotic behaviour for quasi-static variational inequalities, with respect to physical parameters like friction coefficient, compliance coefficient, etc. By convex duality the quasi-static problems can be recast into standard evolution problems, whose study rely on well-known methods. In this framework the stability with respect to small friction coefficients reduces to long time behaviour for evolution problems.

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1 Introduction

Many problems in mechanics are formulated in terms of variational inequalities. We deal with such models when studying obstacle problems, visco-plastic Bingham fluids, elasto-plastic torsion problems, Signorini or contact problems, Coulomb friction law, etc. The solution of these models depend on some physical parameters, like friction coefficient, torsion, compliance coefficient. A natural question concerns the stability of the solutions with respect to these coefficients. For example we want to identify the limit of solutions $u_\varepsilon$ when the physical parameter, let say $\varepsilon > 0$ becomes very small. In that case if $\varepsilon \sim 0$, then $u_\varepsilon$ behaves like the limit solution $u^0 = \lim_{\varepsilon \searrow 0} u_\varepsilon$. But in many situations the approximation $u_\varepsilon \sim u^0$ is not satisfactory; we also need to compute the first order corrections in the formal expansion

$$u_\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon).$$

In other words we have to compute the limit $\lim_{\varepsilon \searrow 0} (u_\varepsilon - u^0)/\varepsilon$.

A simplified friction model was introduced in [8] (see also [7], [9]). Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be any bounded open set with smooth boundary and $V = H^1(\Omega)$ endowed with the norm $\|v\| = (\int_{\Omega} |v(x)|^2 + |\nabla v|^2 \, dx)^{1/2}$ for any $v \in V$. For any $\varepsilon > 0$ consider the variational inequality

$$u_\varepsilon \in V : \int_{\Omega} \{u_\varepsilon (v - u_\varepsilon) + \nabla u_\varepsilon \cdot (\nabla v - \nabla u_\varepsilon)\} \, dx + \varepsilon \int_{\partial \Omega} \{|v| - |u_\varepsilon|\} \, dS(x) \geq \int_{\Omega} F(v - u_\varepsilon) \, dx, \forall v \in V$$

where $F \in L^2(\Omega)$. Surely the continuous linear form $v \in V \rightarrow \int_{\Omega} F v \, dx$ can be written as the scalar product $v \rightarrow \int_{\Omega} \{Fv + \nabla F \cdot \nabla v\} \, dx$ for some element $F \in V$. Here $\varepsilon$ is a small parameter corresponding to the friction coefficient. Clearly the above problem can be formulated in abstract form: if $(H, (\cdot, \cdot))$ is a Hilbert space find $u_\varepsilon \in H$ such that

$$a(u_\varepsilon, v - u_\varepsilon) + \varepsilon j(v) - \varepsilon j(u_\varepsilon) \geq (F, v - u_\varepsilon), \forall v \in H.$$

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where $a : H \times H \to \mathbb{R}$ is a bilinear coercive form, $j : H \to ]-\infty, +\infty]$ is a proper, convex, l.s.c. function on $H$ and $F \in H$. Therefore for any $\varepsilon > 0$ Lions-Stampacchia’s theorem ensures the well-posedness of (2) cf. [8], [12]. We inquire about the asymptotic behaviour of the family of solutions $(u_\varepsilon)_{\varepsilon > 0}$ for small $\varepsilon$. For example we are looking for expansion like

$$u_\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...$$  \hspace{1cm} (3)

Plugging this ansatz in (2) yields

$$a(u^0 + \varepsilon u^1 + ..., v - u^0 - \varepsilon u^1 - ...) + \varepsilon j(v) - \varepsilon j(u^0 + \varepsilon u^1 + ...) \geq (F, v - u^0 - \varepsilon u^1 - ...)$$ \hspace{1cm} (4)

and passing, at least formally, to the limit when $\varepsilon \searrow 0$ leads to

$$u^0 \in H : a(u^0, v - u^0) \geq (F, v - u^0), \forall v \in H$$

which is equivalent to

$$u^0 \in H : a(u_0, v) = (F, v), \forall v \in H.$$  \hspace{1cm} (5)

Not surprising, the dominant term in (3) solves the elliptic problem (5). The computation of the first order correction term $u^1$ follows by combining (4), (5). We obtain

$$\varepsilon a(u^1 + \varepsilon u^2 + ..., v - u^0 - \varepsilon u^1 - ...) + \varepsilon j(v) - \varepsilon j(u^0 + \varepsilon u^1 + ...) \geq 0.$$  \hspace{1cm} (6)

Simplifying by $\varepsilon$ and replacing $v$ by $u^0 + \varepsilon v$ yield

$$a(u^1 + \varepsilon u^2 + ..., \varepsilon(v - u^1) - \varepsilon^2 u^2 - ...) + j(u^0 + \varepsilon v) - j(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...) \geq 0$$

which is equivalent to

$$a(u^1 + \varepsilon u^2 + ..., v - u^1 - \varepsilon u^2 - ...) + \frac{j(u^0 + \varepsilon v) - j(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...)}{\varepsilon} \geq 0.$$  \hspace{1cm} (7)

Again, a formal passing to the limit when $\varepsilon \searrow 0$ leads to

$$a(u_1, v - u_1) + (\partial j(u^0), v - u_1) \geq 0, \forall v \in H.$$  \hspace{1cm} (8)
If we denote by $A : H \to H$ the linear operator associated to the bilinear form $a(\cdot, \cdot)$, the previous inequality says that $u^1$ belongs to the closed convex set $K = -A^{-1} \partial j(u^0)$. Actually it is not very hard to see that $u^1$ solves the variational problem

$$u^1 \in K : a(u^1, v - u^1) \geq 0, \forall v \in K. \quad (6)$$

Surely, once we have determined the terms $u^0, u^1, \ldots$ we need to check the validity of the asymptotic expansion (3), for example that $u_\varepsilon - u^0 = o(\varepsilon), u_\varepsilon - u^0 - \varepsilon u^1 = \varepsilon o(\varepsilon)$, etc. Such kind of results have been obtained in [4].

Here we intend to perform similar asymptotic analysis for quasistatic variational inequalities associated to (1): find $u_\varepsilon \in W^{1,p}_{\text{loc}}(\mathbb{R}_+; V)$ such that

$$u_\varepsilon(0) = u^0_\varepsilon,$$

$$u_\varepsilon(t) \in V : \int_\Omega \{ u_\varepsilon(t)(v - \dot{u}_\varepsilon(t)) + \nabla u_\varepsilon \cdot (\nabla v - \nabla \dot{u}_\varepsilon) \} \, dx + \varepsilon \int_{\partial \Omega} |v - |\dot{u}_\varepsilon| \} \, dS(x)$$

$$\geq \int_\Omega \{ F(t)(v - \dot{u}_\varepsilon(t)) + \nabla F(t) \cdot (\nabla v - \nabla \dot{u}_\varepsilon(t)) \} \, dx, \forall v \in V \quad (7)$$

where $F \in W^{1,p}_{\text{loc}}(\mathbb{R}_+; V)$ for some $p \in [1, +\infty]$. Notice that by the inclusion $W^{1,p}(0,T;V) \subset C(0,T;V)$ we have $F, u_\varepsilon \in C(0,T;V)$ for any $T > 0$ and thus the variational inequality in (7) is meaningful for any $t \in \mathbb{R}_+$. The well-posedness of quasistatic variational inequalities has been established in [10], [1], [2], [3], [11]. The existence can be obtained by using Euler backward finite difference approximation. Taking as usual $v = 0$ and $v = 2\dot{u}_\varepsilon(t)$ in (7) we obtain that (7) is equivalent to

$$\int_\Omega \{ u_\varepsilon(t)\dot{u}_\varepsilon(t) + \nabla u_\varepsilon \cdot \nabla \dot{u}_\varepsilon \} \, dx + \varepsilon \int_{\partial \Omega} |\dot{u}_\varepsilon| \, dS(x) = \int_\Omega \{ F(t)\dot{u}_\varepsilon(t) + \nabla F(t) \cdot \nabla \dot{u}_\varepsilon(t) \} \, dx$$

$$\int_\Omega \{ u_\varepsilon(t)v + \nabla u_\varepsilon \cdot \nabla v \} \, dx + \varepsilon \int_{\partial \Omega} |v| \, dS(x) \geq \int_\Omega \{ F(t)v + \nabla F(t) \cdot \nabla v \} \, dx, \forall v \in V. \quad (7)$$

In particular the initial condition $u^0_\varepsilon$ should satisfy for any $v \in V$

$$\int_\Omega \{ u^0_\varepsilon(x)v(x) + \nabla u^0_\varepsilon \cdot \nabla v \} \, dx + \varepsilon \int_{\partial \Omega} |v(x)| \, dS(x) \geq \int_\Omega \{ F_0(x)v(x) + \nabla F_0 \cdot \nabla v \} \, dx. \quad (8)$$

It is possible to transform the quasistatic problem (7) into a standard evolution
problem. Indeed, the problem (7) is equivalent to
\[ A \left( \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \right) + \partial j(\dot{u}_\varepsilon(t)) \geq 0, \ t \in \mathbb{R}_+ \]
and thus to
\[ \left[ \dot{u}_\varepsilon(t), -A \left( \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \right) \right] \in \partial j, \ t \in \mathbb{R}_+. \quad (9) \]
For simplicity we assume that the function \( j \) is even \( i.e., \)
\[ D(j) = -D(j), \ j(v) = j(-v) \text{ for any } v \in D(j). \quad (10) \]
Observe that this is the case of the model (1). Under the assumption (10) it is easily seen that \( \partial j \) is odd \( i.e., \)
\[ [x, y] \in \partial j \iff [-x, -y] \in \partial j. \]
Therefore (9) becomes
\[ \left[ -\dot{u}_\varepsilon(t), A \left( \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \right) \right] \in \partial j. \]
Consider now the conjugate function \( j^* \) by convexity duality
\[ j^*(w) = \sup_{v \in H} \{ (w, v) - j(v) \}, \ w \in H. \]
It is well known [6] that \( j^* \) is proper, convex, l.s.c. and \( \partial j^* = (\partial j)^{-1} \). Therefore we obtain
\[ \left[ A \left( \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \right), -\dot{u}_\varepsilon(t) \right] \in (\partial j)^{-1} = \partial j^* \]
saying that \( u_\varepsilon \) solves the evolution problem
\[ \frac{du_\varepsilon}{dt} + \partial j^* A \left( \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \right) \geq 0, \ t \in \mathbb{R}_+. \]
Introducing the notation \( y_\varepsilon = \frac{u_\varepsilon(t) - u^0(t)}{\varepsilon} \) finally we deduce that the quasistatic problem (7) can be written
\[
\begin{cases}
    y_\varepsilon(0) = \frac{u^0_\varepsilon - u^0_0}{\varepsilon} =: y^0_\varepsilon \\
    \varepsilon \frac{dy_\varepsilon}{dt} + \partial j^* Ay_\varepsilon(t) \geq -\frac{du^0_\varepsilon}{dt}, \ t \in \mathbb{R}_+. \quad (11)
\end{cases}
\]
The well-posedness of (11) comes by standard results on evolution problems associated to maximal monotone operators [5]. We use the well-known result [5] pp. 72, which adapts easily to our case, due to the ellipticity of the operator $A$.

**Theorem 1.1** (Brezis) Let $\varphi : H \to [-\infty, +\infty]$ a proper, convex, l.s.c. function on a Hilbert space and $f \in L^2(0,T;H)$. Then for any $u^0 \in \partial\varphi = \partial(\partial\varphi)$ the Cauchy problem

$$
\begin{cases}
  u(0) = u^0 \\
  \frac{du}{dt} + \partial\varphi(u(t)) \ni f(t), \quad t \in [0,T]
\end{cases}
$$

has a unique strong solution $u \in C(0,T;H)$ such that

1. $\varphi \circ u \in L^1(0,T;H)$
2. $\varphi \circ u$ is absolutely continuous on $[\delta,T]$ for any $\delta \in ]0,T[$
3. $\sqrt{t}|\frac{du}{dt}| \in L^2(0,T;H)$.

Moreover, if $u_0 \in D(\varphi)$ then

4. $\varphi \circ u$ is absolutely continuous on $[0,T]$.
5. $|\frac{du}{dt}| \in L^2(0,T;H)$.

In the case of the simplified friction model, the convex function is given by

$$
J(v) = \int_{\partial\Omega} |v(x)| \, dS(x), \quad v \in V.
$$

Since $J(v) \geq 0 = J(0)$ for any $v \in V$ we have $0 \in \partial j(0)$. We consider the non empty closed convex set

$$
D_0 = \partial J(0) = \{w \in V : \int_{\partial\Omega} |v(x)| \, dS(x) \geq \int_{\Omega} \{w(x)v(x) + \nabla w \cdot \nabla v \} \, dx, \forall v \in V \}.
$$

Notice that $J$ is homogeneous and we check easily that in this case the conjugate function $J^*$ is given by

$$
J^*(w) = \begin{cases} 
0, & \text{if } w \in D_0 \\
+\infty, & \text{if } w \in V \setminus D_0
\end{cases}
$$

and thus $D(J^*) = D_0$. Observing that $J$ is a seminorm we deduce immediately that $\partial J(v) \subseteq \partial J(0)$ for any $v \in V$ implying that

$$
D(\partial J^*) = D((\partial J)^{-1}) = R(\partial J) = \bigcup_{v \in V} \partial J(v) = \partial J(0) = D_0.
$$
Applying Theorem 1.1 to the simplified friction model (with \( A = Id \)) we obtain

**Proposition 1.1** For any \( F \in W^{1,2}_{\text{loc}}(\mathbb{R}_+;V) \), \( \varepsilon > 0 \) and initial condition \( u_\varepsilon^0 \) such that \( \frac{u_\varepsilon^0 - F(0)}{\varepsilon} \in D_0 \) there is a unique solution \( u_\varepsilon \in W^{1,2}_{\text{loc}}(\mathbb{R}_+;V) \) satisfying

\[
\int_0^T \| \dot{u}_\varepsilon(t) \|^2 \, dt \leq \int_0^T \| \dot{F}(t) \|^2 \, dt, \quad \forall \ T > 0. \tag{12}
\]

**Proof.** We only justify the estimate (12). Multiplying (11) by \( \dot{y}_\varepsilon \) one gets after integration on \([0,T]\)

\[ \varepsilon \int_0^T \| \dot{y}_\varepsilon(t) \|^2 \, dt + J^*(y_\varepsilon(T)) - J^*(y_\varepsilon(0)) = -\int_0^T (\dot{F}(t), \dot{y}_\varepsilon(t)) \, dt. \]

By the hypothesis \( y_\varepsilon^0 = \frac{u_\varepsilon^0 - F(0)}{\varepsilon} \in D_0 = D(J^*) \) and thus

\[ \varepsilon \int_0^T \| \dot{y}_\varepsilon(t) \|^2 \, dt = -\int_0^T (\dot{F}(t), \dot{y}_\varepsilon(t)) \, dt \]

implying that

\[ \int_0^T (\dot{u}_\varepsilon(t), \dot{u}_\varepsilon(t) - \dot{F}(t)) \, dt = 0. \]

Our estimate comes easily by Cauchy-Schwarz inequality.

**Remark 1.1** Notice that the hypothesis on the initial condition in Proposition 1.1 coincides with (8), since \( D_0 = -D_0 \).

## 2 Asymptotic behaviour of the simplified friction model

We investigate the behaviour of the simplified friction model (7) for small values of \( \varepsilon > 0 \). Observe that the convex function \( J \) is a bounded seminorm on \( V \). Indeed, by trace theorem, we have

\[ J(v) = \| v \|_{L^1(\partial\Omega)} \leq C(\Omega) \| v \|, \quad \forall \ v \in V. \]

In particular we have for any \( v \in V \)

\[ \partial J(v) \subset \partial J(0) \subset \{ w \in V : \| w \| \leq C(\Omega) \}. \]
2.1 A priori estimates

We establish several uniform estimates with respect to the parameter $\varepsilon > 0$. It is easily seen that $(u_\varepsilon(t))_{\varepsilon > 0}$ converges towards $F(t)$ in $V$ uniformly in time. Since we want to determine the first order term in asymptotic expansion like (3) we need to estimate the oscillations $y_\varepsilon = \frac{u_\varepsilon - F}{\varepsilon}$ of $u_\varepsilon$ around the limit function $F$.

**Proposition 2.1** Let $F \in W^{1,p}_{\text{loc}}(\mathbb{R}^+; V)$ and $u^0_\varepsilon \in V$ such that $\frac{u^0_\varepsilon - F(0)}{\varepsilon} \in D_0$. Then we have

i) $\sup_{\varepsilon > 0} \left\| \frac{u_\varepsilon - F}{\varepsilon} \right\|_{C(\mathbb{R}^+; V)} \leq C(\Omega)$.

ii) $\sup_{\varepsilon > 0} \left\| \dot{u_\varepsilon} \right\|_{L^p(0,T; V)} \leq \| \dot{F} \|_{L^p(0,T; V)}$, $\forall T > 0$.

In particular if $\ddot{F} = 0$ then

iii) $\sup_{\varepsilon > 0} \left\| \frac{\sqrt{\varepsilon}}{\varepsilon} (\dot{u_\varepsilon} - \dot{F}) \right\|_{L^2(0,T; V)} \leq \sqrt{\frac{5}{2}} C(\Omega)$.

**Proof.** Here $(\cdot, \cdot)$ stands for the standard scalar product of $V$. Using (7) one gets

$$(u_\varepsilon(t) - F(t), v - \dot{u_\varepsilon}(t)) + \varepsilon J(v) - \varepsilon J(\dot{u_\varepsilon}(t)) \geq 0, \forall \, v \in V.$$ 

Taking $v$ such that $v - \dot{u_\varepsilon}(t) = F(t) - u_\varepsilon(t)$ we obtain

$$\|u_\varepsilon(t) - F(t)\|^2 \leq \varepsilon J(\dot{u_\varepsilon}(t) + F(t) - u_\varepsilon(t)) - \varepsilon J(\dot{u_\varepsilon}(t))$$

$$\leq \varepsilon J(F(t) - u_\varepsilon(t))$$

$$\leq \varepsilon C(\Omega) \|u_\varepsilon(t) - F(t)\|$$

saying that

$$\left\| \frac{u_\varepsilon - F}{\varepsilon} \right\|_{C(\mathbb{R}^+; V)} \leq C(\Omega), \forall \, \varepsilon > 0.$$ 

For a.a. $t > 0$ and $h > 0$ we have by (7) written in $t + h$ with $v = 0$

$$(u_\varepsilon(t + h), \dot{u_\varepsilon}(t + h)) \leq (F(t + h), \dot{u_\varepsilon}(t + h)) - \varepsilon J(\dot{u_\varepsilon}(t + h)). \quad (13)$$

Using now (7) in $t$ with $v = \dot{u_\varepsilon}(t) + \dot{u_\varepsilon}(t + h)$ yields

$$-(u_\varepsilon(t), \dot{u_\varepsilon}(t + h)) \leq -(F(t), \dot{u_\varepsilon}(t + h)) + \varepsilon J(\dot{u_\varepsilon}(t) + \dot{u_\varepsilon}(t + h)) - \varepsilon J(\dot{u_\varepsilon}(t)). \quad (14)$$
Combining (13), (14) leads to

\[
(u_\varepsilon(t + h) - u_\varepsilon(t), \dot{u}_\varepsilon(t + h)) \leq (F(t + h) - F(t), \dot{u}_\varepsilon(t + h)) + \varepsilon \{J(\dot{u}_\varepsilon(t) + \dot{u}_\varepsilon(t + h)) - J(\dot{u}_\varepsilon(t)) - J(\dot{u}_\varepsilon(t + h))\}
\]

\[
\leq (F(t + h) - F(t), \dot{u}_\varepsilon(t + h))
\]

since \(J\) is a seminorm. We deduce that

\[
(u_\varepsilon(t + h) - u_\varepsilon(t), \dot{u}_\varepsilon(t + h)) \leq (F(t + h) - F(t), \dot{u}_\varepsilon(t + h))
\]

\[
\leq \|F(t + h) - F(t)\| \|\dot{u}_\varepsilon(t + h)\|
\]

which implies by letting \(h \searrow 0\)

\[
\|\dot{u}_\varepsilon(t)\| \leq \|\dot{F}(t)\|, \text{ for a.a. } t > 0
\]

and therefore

\[
\sup_{\varepsilon > 0} \|\dot{u}_\varepsilon\|_{L^p(0,T;V)} \leq \|\dot{F}\|_{L^p(0,T;V)}, \forall T > 0.
\]

Assume now that \(\dot{F} = 0\), let say \(F(t) = F(0) + tG, F(0), G \in V\). By the previous computations we know that

\[
\|\dot{u}_\varepsilon(t)\| \leq \|\dot{F}(t)\| = \|G\|, \text{ for a.a. } t > 0
\]

and thus the functions \(t \to u_\varepsilon(t)\) and \(t \to J(u_\varepsilon(t))\) are Lipschitz continuous. Indeed

\[
|J(u_\varepsilon(t+h)) - J(u_\varepsilon(t))| \leq J(u_\varepsilon(t+h) - u_\varepsilon(t)) \leq C(\Omega)\|u_\varepsilon(t+h) - u_\varepsilon(t)\| \leq C(\Omega)\|h\|\|G\|.
\]

Therefore in any differentiability point \(t_0\) of \(u_\varepsilon\) and \(J \circ u_\varepsilon\) we have

\[
\frac{dJ(u_\varepsilon)}{dt}|_{t=t_0} = (q, \frac{du_\varepsilon}{dt}|_{t=t_0}), \forall q \in \partial J(u_\varepsilon(t_0)).
\]

We also write \(\frac{d}{dt}J(u_\varepsilon) = (\partial J(u_\varepsilon(t)), \dot{u}_\varepsilon(t))\). We justify the last statement of Proposition 2.1 only for smooth solutions \((u_\varepsilon)_{\varepsilon > 0}\). The general result follows by standard regularization arguments and we skip them. Observe that (7) is equivalent to

\[
u_\varepsilon(t) - F(t) + \varepsilon \partial J(\dot{u}_\varepsilon(t)) \geq 0, \ t \in \mathbb{R}_+.
\]
We intend to multiply (15) by \( \dot{u}_e = \ddot{u}_e - \ddot{F} \). For this notice that
\[
\frac{d}{dt} \left( J(\dot{u}_e) - J(\dot{F}) \right) = (\partial J(\dot{u}_e(t)), \ddot{u}_e(t)).
\] (16)

Putting together (15), (16) implies
\[
(u_e(t) - F(t), \ddot{u}_e - \ddot{F}) + \varepsilon \frac{d}{dt} \left( J(\dot{u}_e) - J(\dot{F}) \right) = 0
\] (17)

and we obtain
\[
\frac{d}{dt}(u_e(t) - F(t), \dot{u}_e(t) - \dot{F}(t)) + \varepsilon \frac{d}{dt} \left( J(\dot{u}_e) - J(\dot{F}) \right) = \|\dot{u}_e(t) - \dot{F}(t)\|_2^2.
\] (18)

Taking \( v = \dot{F}(t) \) in (7) we deduce that
\[
-(u_e(t) - F(t), \dot{u}_e(t) - \dot{F}(t)) - \varepsilon \left\{ J(u_e(t)) - J(F(t)) \right\} \geq 0, \ t \in \mathbb{R}_+.
\]

We consider the non negative function \( b_e : \mathbb{R}_+ \to \mathbb{R}_+ \) given by
\[
b_e(t) = -(u_e(t) - F(t), \dot{u}_e(t) - \dot{F}(t)) - \varepsilon \left\{ J(u_e(t)) - J(F(t)) \right\}, \ t \in \mathbb{R}_+
\]
and therefore (18) becomes
\[
\|\dot{u}_e(t) - \dot{F}(t)\|_2^2 + \dot{b}_e(t) = 0, \ t \in \mathbb{R}_+.
\] (19)

Let us consider \( T > 0 \) and integrate over \([s, T]\) for any \( s \in [0, T]\)
\[
\int_s^T \|\dot{u}_e(t) - \dot{F}(t)\|_2^2 \, dt = b_e(s) - b_e(T) \leq b_e(s).
\]

Integrating now for \( s \in [0, T]\) yields
\[
\int_0^T \|\dot{u}_e(t) - \dot{F}(t)\|_2^2 \, dt \leq \frac{1}{2} \|u_e^0 - F(0)\|_2^2 - \varepsilon \int_0^T \{J(u_e(t)) - J(F(t))\} \, dt
\]
\[
\leq \frac{1}{2} (C(\Omega) \varepsilon)^2 - \varepsilon \int_0^T \{J(u_e(t)) - J(F(t))\} \, dt.
\] (20)

For estimating the last term, take any element \( q \) in \( \partial J(\dot{F}) \) and notice that
\[
\int_0^T \{J(u_e(t)) - J(F(t))\} \, dt \geq \int_0^T (q, \dot{u}_e(t) - \dot{F}(t)) \, dt
\]
\[
= (q, u_e(T) - F(T)) - (q, u_e^0 - F(0))
\]
\[
\geq -2C(\Omega)^2 \varepsilon.
\] (21)
since \( \|q\| \leq C(\Omega) \) (\( J \) being Lipschitz continuous of constant \( C(\Omega) \)) and \( \|u_\varepsilon - F\|_{C(\mathbb{R}^+;V)} \leq C(\Omega)\varepsilon \). Finally combining (20), (21) we deduce for any \( T > 0 \)
\[
\int_0^T t \|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2 \, dt \leq \frac{5}{2} C(\Omega)^2 \varepsilon^2
\]
saying that \( \|\sqrt{t} \left( \dot{u}_\varepsilon - \dot{F} \right) / \varepsilon \|_{L^2(\mathbb{R}^+;V)} \leq \sqrt{\frac{5}{2}} C(\Omega) \).

2.2 Limit of first order fluctuations

Based on the previous estimates we deduce that \( \lim_{\varepsilon \downarrow 0} u_\varepsilon = F \) in \( C(\mathbb{R}^+;V) \) and \( \left( (u_\varepsilon - F) / \varepsilon \right)_{\varepsilon > 0} \) is bounded in \( L^\infty(\mathbb{R}^+;V) \). Therefore there is a sequence \( (\varepsilon_k)_k \) converging towards 0 and a function \( y \in L^\infty(\mathbb{R}^+;V) \) such that
\[
y_{\varepsilon_k} := \frac{u_{\varepsilon_k} - F}{\varepsilon_k} \rightharpoonup y \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}^+;V) \text{ and weakly in } L^2_{\text{loc}}(\mathbb{R}^+;V).
\]
Since \( (\sqrt{t} y_{\varepsilon_k})_k \) is bounded in \( L^2(\mathbb{R}^+;V) \) we may assume that \( \sqrt{t} y_{\varepsilon_k} \rightharpoonup \sqrt{t} z \) weakly in \( L^2(\mathbb{R}^+;V) \). It is easily seen that \( z \) coincides with the distribution derivative of \( y \).

For any \( t \in \mathbb{R}^+ \) we introduce the non empty closed convex set \( K(t) = -\partial J(\dot{F}(t)) \).

**Theorem 2.1** Assume that \( F(t) = F_0 + tG \) with \( F_0, G \in V \) and that \( (u^0_\varepsilon - F_0) / \varepsilon \in D_0 \). Then there is a sequence \( (\varepsilon_k)_k \) converging towards 0 and an element \( y \in -\partial J(G) \) such that
\[
\lim_{k \to +\infty} \frac{u_{\varepsilon_k} - F}{\varepsilon_k} = y \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}^+;V) \text{ and strongly in } L^2_{\text{loc}}([0, +\infty[;V).
\]

**Proof.** Take \( \eta \in C^1_c([0, +\infty[;\mathbb{R}) \) a non negative function. Multiplying (7) by \( \eta(t) \) one gets
\[
\int_{\mathbb{R}^+} \left( \frac{u_{\varepsilon_k}(t) - F(t)}{\varepsilon_k}, v - \dot{u}_{\varepsilon_k}(t) \right) \eta(t) \, dt + \int_{\mathbb{R}^+} J(v) \eta(t) \, dt \geq \int_{\mathbb{R}^+} \dot{J}(\dot{u}_{\varepsilon_k}(t)) \eta(t) \, dt. \tag{22}
\]
Since $\eta$ has compact support in $]0, +\infty[$, one gets by Proposition 2.1
\[
\left| \int_{\mathbb{R}_+} \{ J(\dot{u}_\varepsilon(t)) - J(\dot{F}) \} \eta(t) \, dt \right| \leq C(\Omega) \int_{\mathbb{R}_+} \| \dot{u}_\varepsilon(t) - \dot{F} \| \eta(t) \, dt \\
\leq C(\Omega) \| \sqrt{t}(\dot{u}_\varepsilon - \dot{F}) \|_{L^2(\mathbb{R}_+; V)} \left\| \frac{\eta}{\sqrt{t}} \right\|_{L^2(\mathbb{R}_+; \mathbb{R})} \\
\leq \sqrt{\frac{5}{2}} \varepsilon C(\Omega) \left\| \frac{\eta}{\sqrt{t}} \right\|_{L^2(\mathbb{R}_+; \mathbb{R})}
\]
and therefore we have the convergence
\[
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}_+} J(\dot{u}_\varepsilon(t)) \eta(t) \, dt = \int_{\mathbb{R}_+} J(\dot{F}) \eta(t) \, dt.
\]
Similarly, combining weak and strong convergences one gets
\[
\int_{\mathbb{R}_+} (y_\varepsilon(t), v - \dot{u}_\varepsilon(t)) \eta(t) \, dt = \int_{\mathbb{R}_+} (y_\varepsilon(t), v - \dot{F}) \eta(t) \, dt \\
+ \int_{\mathbb{R}_+} (y_\varepsilon(t), \dot{F} - \dot{u}_\varepsilon(t)) \eta(t) \, dt \\
\rightarrow \int_{\mathbb{R}_+} (y(t), v - \dot{F}) \eta(t) \, dt, \text{ as } k \rightarrow +\infty.
\]
Passing to the limit in (22) when $k \rightarrow +\infty$ yields
\[
\int_{\mathbb{R}_+} \{ (y(t), v - \dot{F}) + J(v) - J(\dot{F}) \} \eta(t) \, dt \geq 0
\]
for any non negative function $\eta \in C^1_c([0, +\infty[; \mathbb{R})$ and therefore one gets for a.a. $t > 0$
\[
(y(t), v - \dot{F}) + J(v) - J(\dot{F}) \geq 0, \ v \in V
\]
(23)
saying that $y(t) \in K$ for a.a. $t > 0$. Observe that the set $K$ does not depend on $t$ and we have $K = -\partial J(G)$. Take $v$ an arbitrary element of $K$, $[\dot{F}, -v] \in \partial J$. By (7) we know that $y_\varepsilon + \partial J(\dot{u}_\varepsilon(t)) \ni 0$ which is equivalent to $[\dot{u}_\varepsilon(t), -y_\varepsilon(t)] \in \partial J$.
Therefore by the monotonicity of $\partial J$ one gets
\[
(\dot{u}_\varepsilon(t) - \dot{F}, -(y_\varepsilon(t) - v)) \geq 0
\]
and after multiplication by $\varepsilon_k^{-1} \eta \geq 0$, $\eta \in C_c([0, +\infty[; \mathbb{R})$ and integration on $\mathbb{R}_+$ we deduce
\[
\int_{\mathbb{R}_+} (\dot{y}_\varepsilon(t), v - y_\varepsilon(t)) \eta(t) \, dt \geq 0.
\]
(24)
Observe that \((y_{\varepsilon_k}), (\dot{y}_{\varepsilon_k})_k\) are bounded in \(L^2(\text{supp}(\eta); V)\) and thus (after extraction eventually) we have the convergence \(\lim_{k \to +\infty} y_{\varepsilon_k} = y\) strongly in \(L^2(\text{supp}(\eta); V)\).

Finally we deduce easily that

\[
\int_{\mathbb{R}_+} (\dot{y}(t), v - y(t))\eta(t) \, dt \geq 0
\]

implying that

\[
y(t) \in K : (\dot{y}(t), v - y(t)) \geq 0, \text{ for a.a. } t > 0, \forall v \in K.
\]

The previous variational inequality says that for any \(v \in K\) the function \(t \to \frac{1}{2}\|y(t) - v\|^2\) is non increasing i.e.,

\[
\frac{1}{2}\|y(t + h) - v\|^2 \leq \frac{1}{2}\|y(t) - v\|^2, \quad t > 0, h > 0.
\]

Taking \(v = y(t) \in K\) one gets \(y(t + h) = y(t), t, h > 0\) and thus \(y(\cdot)\) is a constant function.

Actually we can show that \(\lim_{\varepsilon \searrow 0} y_{\varepsilon}(t) = y\) strongly in \(V\) and uniformly for \(t \in [\delta, +\infty[\) for any \(\delta > 0\). The proof relies on long time behaviour of semigroup generated by maximal monotone operators. Indeed, by (11) we know that the simplified friction model (7) is equivalent to

\[
\begin{cases}
\varepsilon \frac{dy_{\varepsilon}}{dt} + \partial J^*(y_{\varepsilon}(t)) \ni -G, t \in \mathbb{R}_+ \\
y_{\varepsilon}(0) = \frac{u_0^\varepsilon - F_0}{\varepsilon} \in D_0.
\end{cases}
\] (25)

We introduce the fast variable \(s = \frac{t}{\varepsilon}\) and the new unknown \(z_{\varepsilon}(s) = y_{\varepsilon}(t)\). We obtain the problem

\[
\begin{cases}
\frac{dz_{\varepsilon}}{ds} + \partial J^*(z_{\varepsilon}(s)) \ni -G, s \in \mathbb{R}_+ \\
z_{\varepsilon}(0) = \frac{u_0^\varepsilon - F_0}{\varepsilon} =: z_0^\varepsilon \in D_0.
\end{cases}
\] (26)

Assume that \((z_{\varepsilon}^0)_{\varepsilon > 0}\) converges as \(\varepsilon \searrow 0\) to some element \(z^0 \in \overline{D_0} = D_0\) and let us consider \(z \in C(\mathbb{R}_+; V)\) the unique strong solution of (26) corresponding to the initial condition \(z^0\). Using the monotonicity of \(\partial J^*\) we easily check that

\[
\|z_{\varepsilon}(s) - z(s)\| \leq \|z_{\varepsilon}^0 - z^0\| \to 0 \quad \text{as } \varepsilon \searrow 0
\]
and therefore we can write
\[ \| y_\varepsilon(t) - y \| \leq \| z(t/\varepsilon) - z(t/\varepsilon) \| + \| z(t/\varepsilon) - y \| \leq \| z^0_\varepsilon - z^0 \| + \| z(t/\varepsilon) - y \| \]
from which we deduce that \( \lim_{\varepsilon \searrow 0} y_\varepsilon(t) = y \) strongly in \( V \) and uniformly for \( t \in [\delta, +\infty[ \) for any \( \delta > 0 \) provided that \( z \) converges for large time towards \( y \). We are done if we justify such a long time behaviour for \( z \). This is a direct consequence of well-known results concerning the stability theory of semigroups. More precisely we appeal to Baillon theorem and Bruck comparison result. For the sake of completeness we recall here these results.

**Theorem 2.2 (Baillon)** Let \( A : D(A) \subset H \to H \) be a maximal monotone and odd operator (i.e., \( D(A) = -D(A) \) and \( A(-x) = \{ -y : y \in Ax \} \)). We denote by \( (S(t))_{t \geq 0} \) the semigroup generated by \( -A \). Then \( A^{-1}0 \neq \emptyset \) and for any \( x \in \overline{D(A)} \) there is an element \( y \in A^{-1}0 \) such that

i) \( \lim_{t \to +\infty} \text{Proj}_{A^{-1}0} S(t)x = y \), strongly in \( H \).

ii) \( \lim_{t \to +\infty} \sigma(t)x = y \), strongly in \( H \), \( \sigma(t)x = \frac{1}{t} \int_0^t S(s)x \, ds, t > 0 \).

**Theorem 2.3 (Bruck)** Assume that the proper, convex, l.s.c. function \( \varphi : H \to [-\infty, +\infty] \) has a minimum point. Let us denote by \( (S(t))_{t \geq 0} \) the semigroup generated by \( -\partial \varphi \). Then for any \( x \in \overline{D(\partial \varphi)} \) we have

\[ \lim_{t \to +\infty} \| S(t)x - \sigma(t)x \| = 0, \quad \text{where} \quad \sigma(t)x = \frac{1}{t} \int_0^t S(s)x \, ds, \quad t > 0. \]

Based on the previous results we obtain

**Theorem 2.4** Assume that \( F(t) = F_0 + tG \), with \( F_0, G \in V \) and that the family \( (\varepsilon^{-1}(u^0_\varepsilon - F_0))_{\varepsilon > 0} \subset D_0 \) converges as \( \varepsilon \searrow 0 \) to some element \( z^0 \in \overline{D_0} = D_0 \). If \( G \in D^*_0 \) then there is \( y \in -\partial J(G) \) such that

\[ \lim_{\varepsilon \searrow 0} \frac{u_\varepsilon(t) - (F_0 + tG)}{\varepsilon} = y, \quad \text{uniformly for} \quad t \in [\delta, +\infty[, \forall \delta > 0. \]
**Proof.** Let $z$ be the unique solution of
\[
\begin{cases}
\frac{dz}{ds} + \partial J^*(z(s)) \ni -G, \quad s \in \mathbb{R}_+
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
\frac{dz}{ds} + \partial \phi_G(z(s)) \ni 0, \quad s \in \mathbb{R}_+
\end{cases}
\]
where $\phi_G(z) = J^*(z) + (G, z)$. It is easily seen that $\phi_G$ is even (since $J^*$ is even and $G \in D_0^\perp$). Therefore $\partial \phi_G$ is odd and we have $(\partial \phi_G)^{-1}0 = -\partial J(G) \neq \emptyset$. We deduce by Baillon theorem and Bruck comparison result that there is a minimum point $y$ for $\phi_G$, $y \in -\partial J(G)$ such that
\[
\lim_{s \to +\infty} \text{Proj}_{-\partial J(G)} z(s) = \lim_{s \to +\infty} z(s) = \lim_{s \to +\infty} \frac{1}{s} \int_0^s z(\tau) \, d\tau = y.
\]
Finally one gets
\[
\|y_\varepsilon(t) - y\| \leq \|z_\varepsilon(t/\varepsilon) - z(t/\varepsilon)\| + \|z(t/\varepsilon) - y\| \leq \|z_\varepsilon^0 - z^0\| + \|z(t/\varepsilon) - y\| \to 0
\]
as $\varepsilon \searrow 0$, uniformly for $t \in [\delta, +\infty[$, for any $\delta > 0$.

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**References**


