Some remarks on time-dependent variational problems and their asymptotic behaviour

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Abstract

The subject matter of this paper deals with asymptotic behaviour for quasi-static variational inequalities, with respect to physical parameters like friction coefficient, compliance coefficient, etc. By convex duality the quasi-static problems can be recast into standard evolution problems, whose study rely on well-known methods. In this framework the stability with respect to small friction coefficients reduces to long time behaviour for evolution problems.

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1 Introduction

Many problems in mechanics are formulated in terms of variational inequalities. We deal with such models when studying obstacle problems, visco-plastic Bingham fluids, elasto-plastic torsion problems, Signorini or contact problems, Coulomb friction law, etc. The solution of these models depend on some physical parameters, like friction coefficient, torsion, compliance coefficient. A natural question concerns the stability of the solutions with respect to these coefficients. For example we want to identify the limit of solutions u_ε when the physical parameter, let say $\varepsilon > 0$ becomes very small. In that case if $\varepsilon \sim 0$, then u_ε behaves like the limit solution $u^0 = \lim_{\varepsilon \searrow 0} u_\varepsilon$. But in many situations the approximation $u_\varepsilon \sim u^0$ is not satisfactory; we also need to compute the first order corrections in the formal expansion

$$u_\varepsilon = u^0 + \varepsilon u^1 + \varepsilon o(\varepsilon).$$

In other words we have to compute the limit $\lim_{\varepsilon \searrow 0}(u_\varepsilon - u^0)/\varepsilon$.

A simplified friction model was introduced in [8] (see also [7], [9]). Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be any bounded open set with smooth boundary and $V = H^1(\Omega)$ endowed with the norm $\|v\| = \left(\int_\Omega |v(x)|^2 + |\nabla v|^2 \,\mathrm{d}x\right)^{1/2}$ for any $v \in V$. For any $\varepsilon > 0$ consider the variational inequality

$$u_\varepsilon \in V \ : \ \int_\Omega \{u_\varepsilon(v - u_\varepsilon) + \nabla u_\varepsilon \cdot (\nabla v - \nabla u_\varepsilon)\} \,\mathrm{d}x \ + \ \varepsilon \int_{\partial\Omega} \{|v| - |u_\varepsilon|\} \,\mathrm{d}S(x) \quad (1)$$
$$\geq \int_\Omega \mathcal{F}(v - u_\varepsilon) \,\mathrm{d}x, \ \forall\, v \in V$$

where $\mathcal{F} \in L^2(\Omega)$. Surely the continuous linear form $v \in V \to \int_\Omega \mathcal{F} v \,\mathrm{d}x$ can be written as the scalar product $v \to \int_\Omega \{Fv + \nabla F \cdot \nabla v\} \,\mathrm{d}x$ for some element $F \in V$. Here ε is a small parameter corresponding to the friction coefficient. Clearly the above problem can be formulated in abstract form: if $(H, (\cdot, \cdot))$ is a Hilbert space find $u_\varepsilon \in H$ such that

$$a(u_\varepsilon, v - u_\varepsilon) + \varepsilon j(v) - \varepsilon j(u_\varepsilon) \geq (F, v - u_\varepsilon), \ \forall\, v \in H. \qquad (2)$$

where $a : H \times H \to \mathbb{R}$ is a bilinear coercive form, $j : H \to]-\infty, +\infty]$ is a proper, convex, l.s.c. function on H and $F \in H$. Therefore for any $\varepsilon > 0$ Lions-Stampacchia's theorem ensures the well-posedness of (2) cf. [8], [12]. We inquire about the asymptotic behaviour of the family of solutions $(u_\varepsilon)_{\varepsilon>0}$ for small ε. For example we are looking for expansion like

$$u_\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ... \tag{3}$$

Plugging this ansatz in (2) yields

$$a(u^0+\varepsilon u^1+..., v-u^0-\varepsilon u^1-...)+\varepsilon j(v)-\varepsilon j(u^0+\varepsilon u^1+...) \geq (F, v-u^0-\varepsilon u^1-...) \tag{4}$$

and passing, at least formally, to the limit when $\varepsilon \searrow 0$ leads to

$$u^0 \in H \ : \ a(u^0, v-u^0) \geq (F, v-u^0), \ \forall\, v \in H$$

which is equivalent to

$$u^0 \in H \ : \ a(u_0, v) = (F, v), \ \forall\, v \in H. \tag{5}$$

Not surprising, the dominant term in (3) solves the elliptic problem (5). The computation of the first order correction term u^1 follows by combining (4), (5). We obtain

$$\varepsilon a(u^1 + \varepsilon u^2 + ..., v - u^0 - \varepsilon u^1 - ...) + \varepsilon j(v) - \varepsilon j(u^0 + \varepsilon u^1 + ...) \geq 0.$$

Simplifying by ε and replacing v by $u^0 + \varepsilon v$ yield

$$a(u^1 + \varepsilon u^2 + ..., \varepsilon(v - u^1) - \varepsilon^2 u^2 - ...) + j(u^0 + \varepsilon v) - j(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...) \geq 0$$

which is equivalent to

$$a(u^1 + \varepsilon u^2 + ..., v - u^1 - \varepsilon u^2 - ...) + \frac{j(u^0 + \varepsilon v) - j(u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...)}{\varepsilon} \geq 0.$$

Again, a formal passing to the limit when $\varepsilon \searrow 0$ leads to

$$a(u_1, v - u_1) + (\partial j(u^0), v - u_1) \geq 0, \ \forall\, v \in H.$$

If we denote by $A : H \to H$ the linear operator associated to the bilinear form $a(\cdot,\cdot)$, the previous inequality says that u^1 belongs to the closed convex set $K = -A^{-1}\partial j(u^0)$. Actually it is not very hard to see that u^1 solves the variational problem

$$u^1 \in K \ : \ a(u^1, v - u^1) \geq 0, \ \forall\, v \in K. \tag{6}$$

Surely, once we have determined the terms $u^0, u^1, \ldots$ we need to check the validity of the asymptotic expansion (3), for example that $u_\varepsilon - u^0 = o(\varepsilon)$, $u_\varepsilon - u^0 - \varepsilon u^1 = \varepsilon o(\varepsilon)$, etc. Such kind of results have been obtained in [4].

Here we intend to perform similar asymptotic analysis for quasistatic variational inequalities associated to (1): find $u_\varepsilon \in W^{1,p}_{\text{loc}}(\mathbb{R}_+; V)$ such that

$$\begin{cases} u_\varepsilon(0) = u^0_\varepsilon \\ u_\varepsilon(t) \in V \ : \quad \int_\Omega \{u_\varepsilon(t)(v - \dot{u}_\varepsilon(t)) + \nabla u_\varepsilon \cdot (\nabla v - \nabla \dot{u}_\varepsilon)\} \,\mathrm{d}x + \varepsilon \int_{\partial\Omega} \{|v| - |\dot{u}_\varepsilon|\} \,\mathrm{d}S(x) \\ \qquad\qquad\qquad \geq \int_\Omega \{F(t)(v - \dot{u}_\varepsilon(t)) + \nabla F(t) \cdot (\nabla v - \nabla \dot{u}_\varepsilon(t))\} \,\mathrm{d}x, \ \forall\, v \in V \end{cases} \tag{7}$$

where $F \in W^{1,p}_{\text{loc}}(\mathbb{R}_+; V)$ for some $p \in]1, +\infty]$. Notice that by the inclusion $W^{1,p}(0,T;V) \subset C(0,T;V)$ we have $F, u_\varepsilon \in C(0,T;V)$ for any $T > 0$ and thus the variational inequality in (7) is meaningful for any $t \in \mathbb{R}_+$. The well-posedness of quasistatic variational inequalities has been established in [10], [1], [2], [3], [11]. The existence can be obtained by using Euler backward finite difference approximation. Taking as usual $v = 0$ and $v = 2\dot{u}_\varepsilon(t)$ in (7) we obtain that (7) is equivalent to

$$\begin{cases} u_\varepsilon(0) = u^0_\varepsilon, \ \ u_\varepsilon(t) \in V, \ \forall\, t \in \mathbb{R}_+ \\ \int_\Omega \{u_\varepsilon(t)\dot{u}_\varepsilon(t) + \nabla u_\varepsilon \cdot \nabla \dot{u}_\varepsilon\} \,\mathrm{d}x + \varepsilon \int_{\partial\Omega} |\dot{u}_\varepsilon| \,\mathrm{d}S(x) = \int_\Omega \{F(t)\dot{u}_\varepsilon(t) + \nabla F(t) \cdot \nabla \dot{u}_\varepsilon(t)\} \,\mathrm{d}x \\ \int_\Omega \{u_\varepsilon(t)v + \nabla u_\varepsilon \cdot \nabla v\} \,\mathrm{d}x + \varepsilon \int_{\partial\Omega} |v| \,\mathrm{d}S(x) \geq \int_\Omega \{F(t)v + \nabla F(t) \cdot \nabla v\} \,\mathrm{d}x, \ \forall\, v \in V. \end{cases}$$

In particular the initial condition u^0_ε should satisfy for any $v \in V$

$$\int_\Omega \{u^0_\varepsilon(x)v(x) + \nabla u^0_\varepsilon \cdot \nabla v\} \,\mathrm{d}x + \varepsilon \int_{\partial\Omega} |v(x)| \,\mathrm{d}S(x) \geq \int_\Omega \{F_0(x)v(x) + \nabla F_0 \cdot \nabla v\} \,\mathrm{d}x. \tag{8}$$

It is possible to transform the quasistatic problem (7) into a standard evolution problem. Indeed, the problem (7) is equivalent to

$$A\left(\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}\right)+\partial j(\dot u_\varepsilon(t))\ni 0,\ \ t\in\mathbb{R}_+$$

and thus to

$$\left[\dot u_\varepsilon(t),-A\left(\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}\right)\right]\in\partial j,\ \ t\in\mathbb{R}_+. \tag{9}$$

For simplicity we assume that the function j is even *i.e.*,

$$D(j)=-D(j),\ \ j(v)=j(-v)\ \text{ for any } v\in D(j). \tag{10}$$

Observe that this is the case of the model (1). Under the assumption (10) it is easily seen that ∂j is odd : $D(\partial j)=-D(\partial j)$ and $\partial j(-x)=-\partial j(x)$ for any $x\in D(\partial j)$ *i.e.*,

$$[x,y]\in\partial j\ \text{ iff }\ [-x,-y]\in\partial j.$$

Therefore (9) becomes

$$\left[-\dot u_\varepsilon(t),A\left(\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}\right)\right]\in\partial j.$$

Consider now the conjugate function $j^\star$ by convexity duality

$$j^\star(w)=\sup_{v\in H}\{(w,v)-j(v)\},\ \ w\in H.$$

It is well known [6] that $j^\star$ is proper, convex, l.s.c. and $\partial j^\star=(\partial j)^{-1}$. Therefore we obtain

$$\left[A\left(\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}\right),-\dot u_\varepsilon(t)\right]\in(\partial j)^{-1}=\partial j^\star$$

saying that u_ε solves the evolution problem

$$\frac{du_\varepsilon}{dt}+\partial j^\star A\left(\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}\right)\ni 0,\ \ t\in\mathbb{R}_+.$$

Introducing the notation $y_\varepsilon=\frac{u_\varepsilon(t)-u^0(t)}{\varepsilon}$ finally we deduce that the quasistatic problem (7) can be written

$$\left\{\begin{array}{l} y_\varepsilon(0)=\dfrac{u_\varepsilon^0-u_0^0}{\varepsilon}=:y_\varepsilon^0\\ \varepsilon\dfrac{dy_\varepsilon}{dt}+\partial j^\star Ay_\varepsilon(t)\ni-\dfrac{du^0}{dt},\ \ t\in\mathbb{R}_+.\end{array}\right. \tag{11}$$

The well-posedness of (11) comes by standard results on evolution problems associated to maximal monotone operators [5]. We use the well-known result [5] pp. 72, which adapts easily to our case, due to the ellipticity of the operator A.

Theorem 1.1 *(Brezis) Let $\varphi : H \to]-\infty, +\infty]$ a proper, convex, l.s.c. function on a Hilbert space and $f \in L^2(0,T;H)$. Then for any $u^0 \in \overline{D(\varphi)} = \overline{D(\partial\varphi)}$ the Cauchy problem*

$$\begin{cases} u(0) = u^0 \\ \dfrac{du}{dt} + \partial\varphi(u(t)) \ni f(t), \quad t \in [0,T[\end{cases}$$

has a unique strong solution $u \in C(0,T;H)$ such that

i) $\varphi \circ u \in L^1(0,T;H)$

ii) $\varphi \circ u$ is absolutely continuous on $[\delta, T]$ for any $\delta \in]0,T[$

iii) $\sqrt{t}\dfrac{du}{dt} \in L^2(0,T;H)$.

Moreover, if $u_0 \in D(\varphi)$ then

iv) $\varphi \circ u$ is absolutely continuous on $[0,T]$

v) $\dfrac{du}{dt} \in L^2(0,T;H)$.

In the case of the simplified friction model, the convex function is given by

$$J(v) = \int_{\partial\Omega} |v(x)| \, \mathrm{d}S(x), \quad v \in V.$$

Since $J(v) \geq 0 = J(0)$ for any $v \in V$ we have $0 \in \partial j(0)$. We consider the non empty closed convex set

$$D_0 = \partial J(0) = \{w \in V \ : \ \int_{\partial\Omega} |v(x)| \, \mathrm{d}S(x) \geq \int_{\Omega} \{w(x)v(x) + \nabla w \cdot \nabla v\} \, \mathrm{d}x, \ \forall \ v \in V\}.$$

Notice that J is homogeneous and we check easily that in this case the conjugate function $J^\star$ is given by

$$J^\star(w) = \begin{cases} 0, & \text{if } w \in D_0 \\ +\infty, & \text{if } w \in V \setminus D_0 \end{cases}$$

and thus $D(J^\star) = D_0$. Observing that J is a seminorm we deduce immediately that $\partial J(v) \subset \partial J(0)$ for any $v \in V$ implying that

$$D(\partial J^\star) = D((\partial J)^{-1}) = R(\partial J) = \cup_{v \in V} \partial J(v) = \partial J(0) = D_0.$$

Applying Theorem 1.1 to the simplified friction model (with $A = Id$) we obtain

Proposition 1.1 *For any* $F \in W^{1,2}_{\rm loc}(\mathbb{R}_+; V)$, $\varepsilon > 0$ *and initial condition* u^0_ε *such that* $\frac{u^0_\varepsilon - F(0)}{\varepsilon} \in D_0$ *there is a unique solution* $u_\varepsilon \in W^{1,2}_{\rm loc}(\mathbb{R}_+; V)$ *satisfying*

$$\int_0^T \|\dot{u}_\varepsilon(t)\|^2 \, {\rm d}t \le \int_0^T \|\dot{F}(t)\|^2 \, {\rm d}t, \ \forall \, T > 0. \tag{12}$$

Proof. We only justify the estimate (12). Multiplying (11) by $\dot{y}_\varepsilon$ one gets after integration on $[0, T]$

$$\varepsilon \int_0^T \|\dot{y}_\varepsilon(t)\|^2 \, {\rm d}t + J^\star(y_\varepsilon(T)) - J^\star(y_\varepsilon(0)) = - \int_0^T (\dot{F}(t), \dot{y}_\varepsilon(t)) \, {\rm d}t.$$

By the hypothesis $y^0_\varepsilon = \frac{u^0_\varepsilon - F(0)}{\varepsilon} \in D_0 = D(J^\star)$ and thus

$$\varepsilon \int_0^T \|\dot{y}_\varepsilon(t)\|^2 \, {\rm d}t = - \int_0^T (\dot{F}(t), \dot{y}_\varepsilon(t)) \, {\rm d}t$$

implying that

$$\int_0^T (\dot{u}_\varepsilon(t), \dot{u}_\varepsilon(t) - \dot{F}(t)) \, {\rm d}t = 0.$$

Our estimate comes easily by Cauchy-Schwarz inequality. □

Remark 1.1 *Notice that the hypothesis on the initial condition in Proposition 1.1 coincides with (8), since* $D_0 = -D_0$.

2 Asymptotic behaviour of the simplified friction model

We investigate the behaviour of the simplified friction model (7) for small values of $\varepsilon > 0$. Observe that the convex function J is a bounded seminorm on V. Indeed, by trace theorem, we have

$$J(v) = \|v\|_{L^1(\partial\Omega} \le C(\Omega)\|v\|, \ \forall \, v \in V.$$

In particular we have for any $v \in V$

$$\partial J(v) \subset \partial J(0) \subset \{w \in V \ : \ \|w\| \le C(\Omega)\}.$$

2.1 A priori estimates

We establish several uniform estimates with respect to the parameter $\varepsilon > 0$. It is easily seen that $(u_\varepsilon(t))_{\varepsilon>0}$ concerges towards $F(t)$ in V uniformly in time. Since we want to determine the first order term in asymptotic expansion like (3) we need to estimate the oscillations $y_\varepsilon = \frac{u_\varepsilon - F}{\varepsilon}$ of u_ε around the limit function F.

Proposition 2.1 *Let $F \in W^{1,p}_{\text{loc}}(\mathbb{R}_+; V)$ and $u^0_\varepsilon \in V$ such that $\frac{u^0_\varepsilon - F(0)}{\varepsilon} \in D_0$. Then we have*
i) $\sup_{\varepsilon>0} \left\| \frac{u_\varepsilon - F}{\varepsilon} \right\|_{C(\mathbb{R}_+;V)} \leq C(\Omega)$.
ii) $\sup_{\varepsilon>0} \|\dot{u}_\varepsilon\|_{L^p(0,T;V)} \leq \|\dot{F}\|_{L^p(0,T;V)}, \ \forall\, T > 0$.
In particular if $\ddot{F} = 0$ then
iii) $\sup_{\varepsilon>0} \left\| \frac{\sqrt{t}}{\varepsilon} (\dot{u}_\varepsilon - \dot{F}) \right\|_{L^2(0,T;V)} \leq \sqrt{\frac{5}{2}} C(\Omega)$.

Proof. Here $(\cdot,\cdot)$ stands for the standard scalar product of V. Using (7) one gets

$$(u_\varepsilon(t) - F(t), v - \dot{u}_\varepsilon(t)) + \varepsilon J(v) - \varepsilon J(\dot{u}_\varepsilon(t)) \geq 0, \ \forall\, v \in V.$$

Taking v such that $v - \dot{u}_\varepsilon(t) = F(t) - u_\varepsilon(t)$ we obtain

$$\begin{aligned} \|u_\varepsilon(t) - F(t)\|^2 &\leq \varepsilon J(\dot{u}_\varepsilon(t) + F(t) - u_\varepsilon(t)) - \varepsilon J(\dot{u}_\varepsilon(t)) \\ &\leq \varepsilon J(F(t) - u_\varepsilon(t)) \\ &\leq \varepsilon C(\Omega) \|u_\varepsilon(t) - F(t)\| \end{aligned}$$

saying that

$$\left\| \frac{u_\varepsilon - F}{\varepsilon} \right\|_{C(\mathbb{R}_+;V)} \leq C(\Omega), \ \forall\, \varepsilon > 0.$$

For a.a. $t > 0$ and $h > 0$ we have by (7) written in $t + h$ with $v = 0$

$$(u_\varepsilon(t+h), \dot{u}_\varepsilon(t+h)) \ \leq \ (F(t+h), \dot{u}_\varepsilon(t+h)) - \varepsilon J(\dot{u}_\varepsilon(t+h)). \tag{13}$$

Using now (7) in t with $v = \dot{u}_\varepsilon(t) + \dot{u}_\varepsilon(t+h)$ yields

$$-(u_\varepsilon(t), \dot{u}_\varepsilon(t+h)) \leq -(F(t), \dot{u}_\varepsilon(t+h)) + \varepsilon J(\dot{u}_\varepsilon(t) + \dot{u}_\varepsilon(t+h)) - \varepsilon J(\dot{u}_\varepsilon(t)). \tag{14}$$

Combining (13), (14) leads to

$$\begin{aligned}(u_\varepsilon(t+h)-u_\varepsilon(t),\dot u_\varepsilon(t+h)) &\le (F(t+h)-F(t),\dot u_\varepsilon(t+h))\\ &+ \varepsilon\{J(\dot u_\varepsilon(t)+\dot u_\varepsilon(t+h))-J(\dot u_\varepsilon(t))-J(\dot u_\varepsilon(t+h))\}\\ &\le (F(t+h)-F(t),\dot u_\varepsilon(t+h))\end{aligned}$$

since J is a seminorm. We deduce that

$$\left(\frac{u_\varepsilon(t+h)-u_\varepsilon(t)}{h},\dot u_\varepsilon(t+h)\right) \le \left\|\frac{F(t+h)-F(t)}{h}\right\| \|\dot u_\varepsilon(t+h)\|$$

which implies by letting $h \searrow 0$

$$\|\dot u_\varepsilon(t)\| \le \|\dot F(t)\|, \ \text{ for a.a. } t>0$$

and therefore

$$\sup_{\varepsilon>0}\|\dot u_\varepsilon\|_{L^p(0,T;V)} \le \|\dot F\|_{L^p(0,T;V)},\ \forall\, T>0.$$

Assume now that $\ddot F = 0$, let say $F(t) = F(0)+tG,\ F(0), G \in V$. By the previous computations we know that

$$\|\dot u_\varepsilon(t)\| \le \|\dot F(t)\| = \|G\|, \ \text{ for a.a. } t>0$$

and thus the functions $t \to u_\varepsilon(t)$ and $t \to J(u_\varepsilon(t))$ are Lipschitz continuous. Indeed

$$|J(u_\varepsilon(t+h))-J(u_\varepsilon(t))| \le J(u_\varepsilon(t+h)-u_\varepsilon(t)) \le C(\Omega)\|u_\varepsilon(t+h)-u_\varepsilon(t)\| \le C(\Omega)|h|\|G\|.$$

Therefore in any differentiability point t_0 of u_ε and $J\circ u_\varepsilon$ we have

$$\frac{dJ(u_\varepsilon)}{dt}|_{t=t_0} = \left(q, \frac{du_\varepsilon}{dt}|_{t=t_0}\right),\ \forall\, q \in \partial J(u_\varepsilon(t_0)).$$

We also write $\frac{d}{dt}J(u_\varepsilon) = (\partial J(u_\varepsilon(t)), \dot u_\varepsilon(t))$. We justify the last statement of Proposition 2.1 only for smooth solutions $(u_\varepsilon)_{\varepsilon>0}$. The general result follows by standard regularization arguments and we skip them. Observe that (7) is equivalent to

$$u_\varepsilon(t) - F(t) + \varepsilon\partial J(\dot u_\varepsilon(t)) \ni 0, \ \ t \in \mathbb{R}_+. \tag{15}$$

We intend to multiply (15) by $\ddot{u}_\varepsilon = \ddot{u}_\varepsilon - \ddot{F}$. For this notice that

$$\frac{d}{dt}\Big(J(\dot{u}_\varepsilon) - J(\dot{F})\Big) = (\partial J(\dot{u}_\varepsilon(t)), \ddot{u}_\varepsilon(t)). \tag{16}$$

Putting together (15), (16) implies

$$(u_\varepsilon(t) - F(t), \ddot{u}_\varepsilon - \ddot{F}) + \varepsilon \frac{d}{dt}\Big(J(\dot{u}_\varepsilon) - J(\dot{F})\Big) = 0 \tag{17}$$

and we obtain

$$\frac{d}{dt}(u_\varepsilon(t) - F(t), \dot{u}_\varepsilon(t) - \dot{F}(t)) + \varepsilon \frac{d}{dt}\Big(J(\dot{u}_\varepsilon) - J(\dot{F})\Big) = \|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2. \tag{18}$$

Taking $v = \dot{F}(t)$ in (7) we deduce that

$$-(u_\varepsilon(t) - F(t), \dot{u}_\varepsilon(t) - \dot{F}(t)) - \varepsilon\Big\{J(\dot{u}_\varepsilon(t)) - J(\dot{F}(t))\Big\} \geq 0,\ t \in \mathbb{R}_+.$$

We consider the non negative function $b_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$b_\varepsilon(t) = -(u_\varepsilon(t) - F(t), \dot{u}_\varepsilon(t) - \dot{F}(t)) - \varepsilon\Big\{J(\dot{u}_\varepsilon(t)) - J(\dot{F}(t))\Big\},\ t \in \mathbb{R}_+$$

and therefore (18) becomes

$$\|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2 + \dot{b}_\varepsilon(t) = 0,\ t \in \mathbb{R}_+. \tag{19}$$

Let us consider $T > 0$ and integrate over $[s, T]$ for any $s \in [0, T]$

$$\int_s^T \|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2 \,\mathrm{d}t = b_\varepsilon(s) - b_\varepsilon(T) \leq b_\varepsilon(s).$$

Integrating now for $s \in [0, T]$ yields

$$\begin{aligned}\int_0^T t\, \|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2 \,\mathrm{d}t &\leq \frac{1}{2}\|u_\varepsilon^0 - F(0)\|^2 - \varepsilon \int_0^T \{J(\dot{u}_\varepsilon(t)) - J(\dot{F}(t))\} \,\mathrm{d}t \\ &\leq \frac{1}{2}(C(\Omega)\varepsilon)^2 - \varepsilon \int_0^T \{J(\dot{u}_\varepsilon(t)) - J(\dot{F}(t))\} \,\mathrm{d}t. \end{aligned} \tag{20}$$

For estimating the last term, take any element q in $\partial J(\dot{F})$ and notice that

$$\begin{aligned}\int_0^T \{J(\dot{u}_\varepsilon(t)) - J(\dot{F}(t))\} \,\mathrm{d}t &\geq \int_0^T (q, \dot{u}_\varepsilon(t) - \dot{F}(t)) \,\mathrm{d}t \\ &= (q, u_\varepsilon(T) - F(T)) - (q, u_\varepsilon^0 - F(0)) \\ &\geq -2C(\Omega)^2\varepsilon \end{aligned} \tag{21}$$

since $\|q\| \le C(\Omega)$ (J being Lipschitz continuous of constant $C(\Omega)$) and $\|u_\varepsilon - F\|_{C(\mathbb{R}_+;V)} \le C(\Omega)\varepsilon$. Finally combining (20), (21) we deduce for any $T > 0$

$$\int_0^T t\ \|\dot{u}_\varepsilon(t) - \dot{F}(t)\|^2\ \mathrm{d}t \le \frac{5}{2} C(\Omega)^2 \varepsilon^2$$

saying that $\|\sqrt{t}\ (\dot{u}_\varepsilon - \dot{F})/\varepsilon\|_{L^2(\mathbb{R}_+;V)} \le \sqrt{\frac{5}{2}} C(\Omega)$. □

2.2 Limit of first order fluctuations

Based on the previous estimates we deduce that $\lim_{\varepsilon \searrow 0} u_\varepsilon = F$ in $C(\mathbb{R}_+;V)$ and $((u_\varepsilon - F)/\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_+;V)$. Therefore there is a sequence $(\varepsilon_k)_k$ converging towards 0 and a function $y \in L^\infty(\mathbb{R}_+;V)$ such that

$$y_{\varepsilon_k} := \frac{u_{\varepsilon_k} - F}{\varepsilon_k} \rightharpoonup y \ \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+;V) \text{ and weakly in } L^2_{\mathrm{loc}}(\mathbb{R}_+;V).$$

Since $(\sqrt{t}\ \dot{y}_{\varepsilon_k})_k$ is bounded in $L^2(\mathbb{R}_+;V)$ we may assume that $\sqrt{t}\ \dot{y}_{\varepsilon_k} \rightharpoonup \sqrt{t}\ z$ weakly in $L^2(\mathbb{R}_+;V)$. It is easily seen that z coincides with the distribution derivative of y. For any $t \in \mathbb{R}_+$ we introduce the non empty closed convex set $K(t) = -\partial J(\dot{F}(t))$.

Theorem 2.1 *Assume that $F(t) = F_0 + tG$ with $F_0, G \in V$ and that $(u^0_\varepsilon - F_0)/\varepsilon \in D_0$. Then there is a sequence $(\varepsilon_k)_k$ converging towards 0 and an element $y \in -\partial J(G)$ such that*

$$\lim_{k\to+\infty} \frac{u_{\varepsilon_k} - F}{\varepsilon_k} = y \ \ \textit{weakly } \star \ \textit{ in } L^\infty(\mathbb{R}_+;V) \textit{ and strongly in } L^2_{\mathrm{loc}}(]0,+\infty[;V).$$

Proof. Take $\eta \in C^1_c(]0,+\infty[;\mathbb{R})$ a non negative function. Multiplying (7) by $\eta(t)$ one gets

$$\int_{\mathbb{R}_+} \left(\frac{u_{\varepsilon_k}(t) - F(t)}{\varepsilon_k}, v - \dot{u}_{\varepsilon_k}(t) \right) \eta(t)\ \mathrm{d}t + \int_{\mathbb{R}_+} J(v)\eta(t)\ \mathrm{d}t \ge \int_{\mathbb{R}_+} J(\dot{u}_{\varepsilon_k}(t))\eta(t)\ \mathrm{d}t. \quad (22)$$

Since η has compact support in $]0,+\infty[$, one gets by Proposition 2.1

$$\begin{aligned}\left|\int_{\mathbb{R}_+} \{J(\dot u_\varepsilon(t)) - J(\dot F)\}\eta(t)\ \mathrm{d}t\right| &\leq C(\Omega)\int_{\mathbb{R}_+} \|\dot u_\varepsilon(t) - \dot F\|\eta(t)\ \mathrm{d}t \\ &\leq C(\Omega)\|\sqrt{t}(\dot u_\varepsilon - \dot F)\|_{L^2(\mathbb{R}_+;V)}\left\|\frac{\eta}{\sqrt{t}}\right\|_{L^2(\mathbb{R}_+;\mathbb{R})} \\ &\leq \sqrt{\frac{5}{2}}\ \varepsilon\ C(\Omega)^2\left\|\frac{\eta}{\sqrt{t}}\right\|_{L^2(\mathbb{R}_+;\mathbb{R})}\end{aligned}$$

and therefore we have the convergence

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}_+} J(\dot u_\varepsilon(t))\eta(t)\ \mathrm{d}t = \int_{\mathbb{R}_+} J(\dot F)\eta(t)\ \mathrm{d}t.$$

Similarly, combining weak and strong convergences one gets

$$\begin{aligned}\int_{\mathbb{R}_+}(y_{\varepsilon_k}(t), v - \dot u_{\varepsilon_k}(t))\eta(t)\ \mathrm{d}t &= \int_{\mathbb{R}_+}(y_{\varepsilon_k}(t), v - \dot F)\eta(t)\ \mathrm{d}t \\ &+ \int_{\mathbb{R}_+}(y_{\varepsilon_k}(t), \dot F - \dot u_{\varepsilon_k}(t))\eta(t)\ \mathrm{d}t \\ &\to \int_{\mathbb{R}_+}(y(t), v - \dot F)\eta(t)\ \mathrm{d}t,\ \text{as } k\to+\infty.\end{aligned}$$

Passing to the limit in (22) when $k\to+\infty$ yields

$$\int_{\mathbb{R}_+}\{(y(t), v - \dot F) + J(v) - J(\dot F)\}\eta(t)\ \mathrm{d}t \geq 0$$

for any non negative function $\eta\in C_c^1(]0,+\infty[;\mathbb{R})$ and therefore one gets for a.a. $t>0$

$$(y(t), v - \dot F) + J(v) - J(\dot F) \geq 0,\ \ v\in V \tag{23}$$

saying that $y(t)\in K$ for a.a. $t>0$. Observe that the set K does not depend on t and we have $K = -\partial J(G)$. Take v an arbitrary element of K, $[\dot F, -v]\in\partial J$. By (7) we know that $y_{\varepsilon_k} + \partial J(\dot u_{\varepsilon_k}(t)) \ni 0$ which is equivalent to $[\dot u_{\varepsilon_k}(t), -y_{\varepsilon_k}(t)]\in\partial J$. Therefore by the monotonicity of ∂J one gets

$$(\dot u_{\varepsilon_k}(t) - \dot F, -(y_{\varepsilon_k}(t) - v)) \geq 0$$

and after multiplication by $\varepsilon_k^{-1}\eta\geq 0$, $\eta\in C_c(]0,+\infty[;\mathbb{R})$ and integration on $\mathbb{R}_+$ we deduce

$$\int_{\mathbb{R}_+}(\dot y_{\varepsilon_k}(t), v - y_{\varepsilon_k}(t))\eta(t)\ \mathrm{d}t \geq 0. \tag{24}$$

Observe that $(y_{\varepsilon_k})_k, (\dot{y}_{\varepsilon_k})_k$ are bounded in $L^2(\mathrm{supp}(\eta); V)$ and thus (after extraction eventually) we have the convergence $\lim_{k\to+\infty} y_{\varepsilon_k} = y$ strongly in $L^2(\mathrm{supp}(\eta); V)$. Finally we deduce easily that

$$\int_{\mathbb{R}_+} (\dot{y}(t), v - y(t))\eta(t)\ \mathrm{d}t \geq 0$$

implying that

$$y(t) \in K\ :\ (\dot{y}(t), v - y(t)) \geq 0, \quad \text{for a.a. } t > 0,\ \forall\, v \in K.$$

The previous variational inequality says that for any $v \in K$ the function $t \to \frac{1}{2}\|y(t) - v\|^2$ is non increasing *i.e.,*

$$\frac{1}{2}\|y(t+h) - v\|^2 \leq \frac{1}{2}\|y(t) - v\|^2, \quad t > 0, h > 0.$$

Taking $v = y(t) \in K$ one gets $y(t+h) = y(t), t, h > 0$ and thus $y(\cdot)$ is a constant function. □

Actually we can show that $\lim_{\varepsilon\searrow 0} y_\varepsilon(t) = y$ strongly in V and uniformly for $t \in [\delta, +\infty[$ for any $\delta > 0$. The proof relies on long time behaviour of semigroup generated by maximal monotone operators. Indeed, by (11) we know that the simplified friction model (7) is equivalent to

$$\begin{cases} \varepsilon \dfrac{dy_\varepsilon}{dt} + \partial J^\star(y_\varepsilon(t)) \ni -G, t \in \mathbb{R}_+ \\ y_\varepsilon(0) = \dfrac{u_\varepsilon^0 - F_0}{\varepsilon} \in D_0. \end{cases} \tag{25}$$

We introduce the fast variable $s = \frac{t}{\varepsilon}$ and the new unknown $z_\varepsilon(s) = y_\varepsilon(t)$. We obtain the problem

$$\begin{cases} \dfrac{dz_\varepsilon}{ds} + \partial J^\star(z_\varepsilon(s)) \ni -G, s \in \mathbb{R}_+ \\ z_\varepsilon(0) = \dfrac{u_\varepsilon^0 - F_0}{\varepsilon} =: z_\varepsilon^0 \in D_0. \end{cases} \tag{26}$$

Assume that $(z_\varepsilon^0)_{\varepsilon>0}$ converges as $\varepsilon \searrow 0$ to some element $z^0 \in \overline{D_0} = D_0$ and let us consider $z \in C(\mathbb{R}_+; V)$ the unique strong solution of (26) corresponding to the initial condition z^0. Using the monotonicity of $\partial J^\star$ we easily check that

$$\|z_\varepsilon(s) - z(s)\| \leq \|z_\varepsilon^0 - z^0\| \to 0 \ \text{ as } \varepsilon \searrow 0$$

and therefore we can write

$$\|y_\varepsilon(t) - y\| \leq \|z_\varepsilon(t/\varepsilon) - z(t/\varepsilon)\| + \|z(t/\varepsilon) - y\| \leq \|z^0_\varepsilon - z^0\| + \|z(t/\varepsilon) - y\|$$

from which we deduce that $\lim_{\varepsilon \searrow 0} y_\varepsilon(t) = y$ strongly in V and uniformly for $t \in [\delta, +\infty[$ for any $\delta > 0$ provided that z converges for large time towards y. We are done if we justify such a long time behaviour for z. This is a direct consequence of well-known results concerning the stability theory of semigroups. More precisely we appeal to Baillon theorem and Bruck comparison result. For the sake of completeness we recall here these results.

Theorem 2.2 *(Baillon) Let $A : D(A) \subset H \to H$ be a maximal monotone and odd operator (i.e., $D(A) = -D(A)$ and $A(-x) = \{-y \ : \ y \in Ax\}$). We denote by $(S(t))_{t\geq 0}$ the semigroup generated by $-A$. Then $A^{-1}0 \neq \emptyset$ and for any $x \in \overline{D(A)}$ there is an element $y \in A^{-1}0$ such that*
i) $\lim_{t\to+\infty} \mathrm{Proj}_{A^{-1}0} S(t)x = y$, strongly in H.
ii) $\lim_{t\to+\infty} \sigma(t)x = y$, strongly in H, $\sigma(t)x = \frac{1}{t}\int_0^t S(s)x \,\mathrm{d}s, t > 0$.

Theorem 2.3 *(Bruck) Assume that the proper, convex, l.s.c. function $\varphi : H \to]-\infty, +\infty]$ has a minimum point. Let us denote by $(S(t))_{t\geq 0}$ the semigroup generated by $-\partial\varphi$. Then for any $x \in \overline{D(\partial\varphi)}$ we have*

$$\lim_{t\to+\infty} \|S(t)x - \sigma(t)x\| = 0, \quad \text{where } \sigma(t)x = \frac{1}{t}\int_0^t S(s)x \,\mathrm{d}s, \ t > 0.$$

Based on the previous results we obtain

Theorem 2.4 *Assume that $F(t) = F_0 + tG$, with $F_0, G \in V$ and that the family $(\varepsilon^{-1}(u^0_\varepsilon - F_0))_{\varepsilon>0} \subset D_0$ converges as $\varepsilon \searrow 0$ to some element $z^0 \in \overline{D_0} = D_0$. If $G \in D_0^\perp$ then there is $y \in -\partial J(G)$ such that*

$$\lim_{\varepsilon \searrow 0} \frac{u_\varepsilon(t) - (F_0 + tG)}{\varepsilon} = y, \quad \text{uniformly for } t \in [\delta, +\infty[, \ \forall\, \delta > 0.$$

Proof. Let z be the unique solution of

$$\begin{cases} \dfrac{dz}{ds} + \partial J^\star(z(s)) \ni -G, \quad s \in \mathbb{R}_+ \\ z(0) = z^0 \in D_0 \end{cases}$$

which is equivalent to

$$\begin{cases} \dfrac{dz}{ds} + \partial \phi_G(z(s)) \ni 0, \quad s \in \mathbb{R}_+ \\ z(0) = z^0 \in D_0 \end{cases}$$

where $\phi_G(z) = J^\star(z) + (G, z)$. It is easily seen that ϕ_G is even (since $J^\star$ is even and $G \in D_0^\perp$). Therefore $\partial\phi_G$ is odd and we have $(\partial\phi_G)^{-1}0 = -\partial J(G) \neq \emptyset$. We deduce by Baillon theorem and Bruck comparison result that there is a minimum point y for ϕ_G, $y \in -\partial J(G)$ such that

$$\lim_{s\to+\infty} \mathrm{Proj}_{-\partial J(G)} z(s) = \lim_{s\to+\infty} z(s) = \lim_{s\to+\infty} \frac{1}{s}\int_0^s z(\tau)\,\mathrm{d}\tau = y.$$

Finally one gets

$$\|y_\varepsilon(t) - y\| \leq \|z_\varepsilon(t/\varepsilon) - z(t/\varepsilon)\| + \|z(t/\varepsilon) - y\| \leq \|z_\varepsilon^0 - z^0\| + \|z(t/\varepsilon) - y\| \to 0$$

as $\varepsilon \searrow 0$, uniformly for $t \in [\delta, +\infty[$, for any $\delta > 0$. □

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