PERMANENT REGIMES FOR THE 1D VLASOV-POISSON SYSTEM WITH BOUNDARY CONDITIONS

M. BOSTAN †

Abstract. *We prove the existence of weak solutions for the Vlasov-Poisson problem with time periodic boundary conditions in one dimension. We consider boundary data with finite charge and current. This analysis is based upon the mild formulation for the regularized Vlasov-Poisson equations.*

Key words. Vlasov-Poisson equations, weak/mild formulation, regularization.

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1. Introduction. Many studies in physics and applied physics are modeled by the kinetic equations (Vlasov, Boltzmann, etc.) coupled with the equations of electromagnetism (Poisson, Maxwell). A few domains of application are semiconductors, particle accelerators, electron guns, etc. Various results were shown for free space systems. Weak solutions of the Vlasov-Poisson equations were constructed by Arseneev [2], Illner and Neunzert [16], Horst and Hunze [15]. The existence of weak solutions of the Vlasov-Maxwell system was shown by DiPerna and Lions [11].

There are few mathematical works on boundary value problems. For the stationary case results have been obtained by Greengard and Raviart [13] for the one dimensional Vlasov-Poisson system and by Poupaud [17] for the multidimensional Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system has been performed by Degond and Raviart in [10] in the case of the plane diode. Weak solutions of the initial-boundary value problem for the Vlasov-Poisson system are obtained by Abdallah in [1]. The regularity of the solution for the Vlasov-Maxwell system in a half line has been analysed by Guo in [14].

The periodic case has been studied as well (see [5], [6], [7]), but existence results are available only under some restrictive hypothesis concerning the velocity support of the boundary incoming particle distribution and the potential data. Basically the above model doesn't handle charge flows with small incoming velocities. The main idea was to keep only the particles which are travelling through the domain in finite time, which makes possible to get estimates for the charge and current densities.

In this paper we study the existence for the $1D$ Vlasov-Poisson problem with time periodic boundary conditions :

$$\partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = 0, \;\; (t,x,v) \in \mathbb{R}_t \times]0,1[\times \mathbb{R}_v,$$

$$f(t, x = 0, v > 0) = g_0(t, v > 0), \;\; f(t, x = 1, v < 0) = g_1(t, v < 0), \; t \in \mathbb{R}_t$$

$$E(t,x) = -\partial_x U, \;\; -\partial_x^2 U = \rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv, \;\; (t,x) \in \mathbb{R}_t \times]0,1[,$$

$$U(t, x = 0) = \varphi_0(t), \;\; U(t, x = 1) = \varphi_1(t), \;\; t \in \mathbb{R}_t.$$

The function $f(t,x,v)$ denotes the particle distribution depending on the time t, the position x and the velocity v. The electric field $E(t,x)$ derives from an electrostatic potential U satisfying the Poisson equation with charge density ρ. The boundary conditions $g_0, g_1, \varphi_0, \varphi_1$ are supposed T-periodic in time, for some $T > 0$.

The main goal of this paper is to establish the existence in the general case, under minimal hypothesis, say for incoming particle distribution with finite charge (as has been shown for the stationary case in [13]). In this case we prove that the solution f belongs to L^1. The major difficulty

†Université de Franche-Comté, 16 route de Gray F-25030 Besançon Cedex, tel : (33).(0)3.81.66.63.38, fax : (33).(0)3.81.66.66.23, mbostan@descartes.univ-fcomte.fr

when studying this problem is the lack of natural a-priori estimate of the solution. In fact, since we are looking for permanent regimes, initial datas are not available and therefore applying directly conservation laws like :

$$\int_0^1 \int_{\mathbb{R}_v} f(t,x,v)dxdv \leq \int_0^1 \int_{\mathbb{R}_v} f(t_0,x,v)dxdv + \int_{t_0}^t \int_{v>0} vg_0(s,v)dsdv$$
$$- \int_{t_0}^t \int_{v<0} vg_1(s,v)dsdv, \;\; t > t_0,$$

doesn't provide any estimate as long we don't have any information on $f(t_0)$. By the other hand, even if there is $t_0 \in \mathbb{R}$ such that $f(t_0) \in L^1$, the previous inequality gives us only an estimate of the charge in terms of the incoming current whereas the natural estimate would be in terms of the incoming charge. In fact we can prove that if the incoming particle distribution has finite current (resp. kinetic energy) then the solution verify $|v|f \in L^1$ (resp. $|v|^2 f \in L^1$).

This work begins with the study of linear time periodic Vlasov equation (the electric field is assumed to be known and T-periodic). We introduce the weak and mild formulations and recall some usual computations for such solutions. We introduce also a perturbed Vlasov equation. In this section we present a very important lemma concerning bounds for the velocity change along the characteristics (see Lemma 2.11) which states that along all characteristics associated to a regular field the following inequality holds :

$$\mid V(s_1) - V(s_2) \mid \leq C \cdot \|E\|_{L^\infty}^{1/2}, \;\; \forall\, s_1, s_2,$$

where C is a constant depending only on the diameter of the spatial domain $\Omega =]0,1[$ here.

In Section 3 the Vlasov-Poisson system is analysed. The existence of a T-periodic solution will be obtained by the application of the Schauder fixed point theorem. The non uniqueness of the solution for the Vlasov problem doesn't allow to directly apply the fixed point method. We need to introduce a perturbed problem by adding an absorption term αf in the Vlasov equation, where $\alpha > 0$ is a small parameter, and to regularize the electric field, which allows to use the mild formulation. The perturbed problem writes :

$$\alpha f(t,x,v) + \partial_t f + v \cdot \partial_x f + E_\varepsilon(t,x) \cdot \partial_v f = 0, \;\; (t,x,v) \in \mathbb{R}_t \times]0,1[\times \mathbb{R}_v,$$

$$f(t, x=0, v>0) = g_0(t, v>0), \;\; f(t, x=1, v<0) = g_1(t, v<0), \; t \in \mathbb{R}_t,$$

$$E_\varepsilon(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E(s,y)dy, \; (t,x) \in \mathbb{R}_t \times]0,1[,$$

where E is the electric field given by the Poisson equation with source $\rho(t,x) = \int_{\mathbb{R}_v} f(t,x,v)dv$:

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t), \; (t,x) \in \mathbb{R}_t \times]0,1[,$$

and $\zeta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\zeta(\frac{\cdot}{\varepsilon}), \; \varepsilon > 0$ is a mollifier sequence. Clearly, the perturbed Vlasov problem has unique T-periodic weak solution and the existence for the non linear perturbed problem follows easily by fixed point argument. Indeed, in this case ($\alpha > 0$ fixed) we obtain immeadeately the following estimate for the T-periodic weak solution of the perturbed Vlasov problem :

$$\int_0^1 \int_{\mathbb{R}_v} f(t,x,v)dxdv \leq \left(\frac{1}{\alpha T} + 1\right) \left(\int_0^T \int_{v>0} vg_0(t,v)dtdv - \int_0^T \int_{v<0} vg_1(t,v)dtdv \right), \; t \in \mathbb{R}_t,$$

which allows to define a fixed point application.

Obviously, the main difficulty consists of finding uniform estimates for the perturbed problems with $\alpha > 0, \varepsilon > 0$. In Section 4 we obtain estimates for the total charge and current and the electric field. The main tool is the Lemma 2.11 combined with the mild formulation. In fact the previous lemma allows to get bounds on the particle lifetimes at least for particles with initial velocities v large enough. Indeed, since along a characteristic we have $\mid V(s)-v \mid \leq C \cdot \|E\|_{L^\infty}^{1/2}, \;\; s_{in} \leq s \leq s_{out}$, we deduce that $|V(s)|$ is bounded from below $|V(s)| \geq |v| - C \cdot \|E\|_{L^\infty}^{1/2}$ and therefore :

$$s_{out} - s_{in} \leq \frac{1}{|v| - C \cdot \|E\|_{L^\infty}^{1/2}}, \;\text{ if }\; |v| > C \cdot \|E\|_{L^\infty}^{1/2}.$$

These arguments work for bounded spatial domains.

In Section 5 we prove the existence of the T-periodic weak solution for the Vlasov-Poisson system by passing $\alpha \to 0$. In order to pass to the limit in the non linear term $E_n \cdot \partial_v f_n$ we can combine the strong convergence in L^1 of E_n with the weak $\star$ convergence in L^∞ of f_n. Some generalizations are analysed as well. Basically, for incoming datas satisfying $|v|^p g \in L^1$ for some integer $p \geq 1$ we prove that $|v|^p f \in L^1$.

We end this paper with several remarks and conclusions. We investigate the Vlasov-Poisson system with several species of particles as well as the case of attractive (gravitational) potentials.

2. The Vlasov equation. The equation which governs the transport of charged particles is called the Vlasov equation and in one dimension is given by :

$$\partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = 0, \quad (t,x,v) \in \mathbb{R}_t \times]0,1[\times \mathbb{R}_v, \tag{2.1}$$

where $f(t,x,v)$ is the density of particles under the action of the electric field $E(t,x) = -\partial_x U$ and $U(t,x)$ is the potential. Charged particles are injected through the boundary :

$$f(t,x,v) = g(t,x,v), \quad (t,x,v) \in \mathbb{R}_t \times \Sigma^-, \tag{2.2}$$

where Σ^- is the subset of the boundary of the phase space $]0,1[\times\mathbb{R}_v$ corresponding to the incoming velocities :

$$\Sigma^- = \{(0,v) \mid v > 0\} \cup \{(1,v) \mid v < 0\} = \Sigma_0^- \cup \Sigma_1^-.$$

Similarly we define also $\Sigma^+ = \{(0,v) \mid v < 0\} \cup \{(1,v) \mid v > 0\} = \Sigma_0^+ \cup \Sigma_1^+$ which corresponds to the outgoing velocities and $\Sigma^0 = \{(0,0),(1,0)\}$. With the notation $g|_{\mathbb{R}_t \times \Sigma_0^-} = g_0, g|_{\mathbb{R}_t \times \Sigma_1^-} = g_1$ the previous boundary condition (2.2) writes :

$$f(t,x=0,v>0) = g_0(t,v>0), \quad f(t,x=1,v<0) = g_1(t,v<0). \tag{2.3}$$

The functions $g_0, g_1 \geq 0$ which describe the emission profiles of the injected charged particles are supposed T-periodic in time, $T > 0$. Now let us briefly recall the definition of weak and mild solutions for the Vlasov problem (2.1), (2.3).

2.1. Weak solution for the Vlasov problem. We introduce the spaces L_i^-, $L_{i,loc}^-$ of incoming data with bounded or locally bounded fluxes :

$$L_i^- = \{g(t,v) \mid v \cdot g(t,v) \in L^1(]0,T[\times\Sigma_i^-)\},$$

$$L_{i,loc}^- = \{g(t,v) \mid v \cdot g(t,v) \in L^1_{loc}(]0,T[\times\Sigma_i^-)\},$$

where $i = 0,1$. We shall use also the following notations :

$$G_p := \frac{1}{T}\int_0^T\!\!\int_{v>0} |v|^p g_0(t,v)dtdv + \frac{1}{T}\int_0^T\!\!\int_{v<0} |v|^p g_1(t,v)dtdv, \;\; 0 \leq p < +\infty,$$

and :

$$G_\infty := \max\{\|g_0\|_{L^\infty(\mathbb{R}_t \times \Sigma_0^-)}, \|g_1\|_{L^\infty(\mathbb{R}_t \times \Sigma_1^-)}\},$$

when g_0, g_1 belong to the corresponding spaces.

DEFINITION 2.1. *Assume that $E \in L^\infty(\mathbb{R}_t\times]0,1[)$ and $g_0 \in L^-_{0,loc}$, $g_1 \in L^-_{1,loc}$ are T-periodic functions in time. We say that $f \in L^1_{loc}(]0,T[\times]0,1[\times\mathbb{R}_v)$ is a T-periodic weak solution for the Vlasov problem* (2.1), (2.3) *iff :*

$$-\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)(\partial_t\varphi + v\cdot\partial_x\varphi + E(t,x)\cdot\partial_v\varphi)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)\varphi(t,1,v)dtdv,$$

for all test function $\varphi \in \mathcal{T}_w$ where :

$$\mathcal{T}_w = \{\varphi \in W^{1,\infty}(\mathbb{R}_t\times]0,1[\times\mathbb{R}_v) \mid \varphi \text{ is } T\text{-periodic in time }, \varphi|_{\mathbb{R}_t\times\Sigma^+} = 0,$$

$$\exists R > 0 : supp(\varphi) \subset \mathbb{R}_t \times [0,1] \times B_R\}.$$

2.2. Mild solution for the Vlasov problem. Throughout this paper we need to consider also some special solutions for (2.1), (2.3), which are called mild solutions or solutions by characteristics. These solutions require more regularity for the electric field and they are particular case of weak solutions. Assure that $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(]0,1[))$ is T-periodic and for $(t,x,v) \in \mathbb{R}_t\times]0,1[\times\mathbb{R}_v$ denote by $(X(s;t,x,v), V(s;t,x,v))$ the unique solution for the ordinary differential system of equations :

$$\frac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \quad \frac{d}{ds}V(s;t,x,v) = E(s,X(s;t,x,v)), \tag{2.4}$$

for $s \in (s_{in}, s_{out})$ which verify the condition :

$$X(s=t;t,x,v) = x, \quad V(s=t;t,x,v) = v. \tag{2.5}$$

Here $s_{in} = s_{in}(t,x,v)$ (resp. $s_{out} = s_{out}(t,x,v)$) represents the incoming (resp. outgoing) time of the particle in the domain $]0,1[$ defined by :

$$s_{in}(t,x,v) = \sup\{s \le t \ : \ (X(s;t,x,v), V(s;t,x,v)) \in \Sigma^-\} \ge -\infty,$$

and :

$$s_{out}(t,x,v) = \inf\{s \ge t \ : \ (X(s;t,x,v), V(s;t,x,v)) \in \Sigma^+ \cup \Sigma^0\} \le +\infty.$$

Using the previous notations, the total travel time through the domain (lifetime) writes $\tau(t,x,v) = s_{out}(t,x,v) - s_{in}(t,x,v) \le +\infty$. Now we replace in the Definition 2.1 the function $\partial_t\varphi + v\cdot\partial_x\varphi + E(t,x)\cdot\partial_v\varphi$ by ψ, which gives after integration along the characteristics curves :

$$\varphi(t,x,v) = -\int_t^{s_{out}(t,x,v)} \psi(s, X(s;t,x,v), V(s;t,x,v))\, ds,$$

and we define the mild solutions as follows :

DEFINITION 2.2. *Assume that $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(]0,1[))$ and $g_0 \in L^-_{0,loc}$, $g_1 \in L^-_{1,loc}$ are T-periodic functions in time. We say that $f \in L^1_{loc}(]0,T[\times]0,1[\times\mathbb{R}_v)$ is a T-periodic mild solution for the Vlasov problem* (2.1), (2.3) *iff :*

$$\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)\psi(t,x,v)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)\int_t^{s_{out}(t,0,v)} \psi(s, X(s;t,0,v), V(s;t,0,v))dsdtdv - \int_0^T\int_{v<0} vg_1(t,v)\int_t^{s_{out}(t,1,v)} \psi(s, X(s;t,1,v), V(s;t,1,v))dsdtdv,$$

for all test function $\psi \in \mathcal{T}_m$ *where :*

$$\mathcal{T}_m = \{\psi \in L^\infty(\mathbb{R}_t\times]0,1[\times\mathbb{R}_v) \mid \psi \text{ is } T\text{-periodic in time },$$

$$\exists R > 0 : supp(\psi) \subset \mathbb{R}_t \times [0,1] \times B_R\}.$$

Sometimes we shall use the notations :

$$(X(s), V(s)) = (X(s;t,x,v), V(s;t,x,v)),$$

$$(X^0(s), V^0(s)) = (X(s;t,0,v), V(s;t,0,v)),$$

$$(X^1(s), V^1(s)) = (X(s;t,1,v), V(s;t,1,v)),$$

and :

$$s_{out} = s_{out}(t,x,v),\;\; s^0_{out} = s_{out}(t,0,v),\;\; s^1_{out} = s_{out}(t,1,v),$$

$$s_{in} = s_{in}(t,x,v),\;\; s^0_{in} = s_{in}(t,0,v),\;\; s^1_{in} = s_{in}(t,1,v).$$

Remark 2.3. *In fact the mild solution is given by* $f(t,x,v) = g_i(s_{in}, V(s_{in};t,x,v))$ *if* $s_{in}(t,x,v) > -\infty$ *and* $X(s_{in};t,x,v) = i$*, where* $i = 0,1$ *and* $f(t,x,v) = 0$ *otherwise.*

Remark 2.4. *Since* E *is* T*-periodic we have* $X(s+T;t+T,x,v) = X(s;t,x,v)$*,* $V(s+T;t+T,x,v) = V(s;t,x,v)$*,* $s_{in}(t+T,x,v) = s_{in}(t,x,v)+T$ *for all* $(s,t,x,v) \in \mathbb{R}_s \times \mathbb{R}_t\times]0,1[\times\mathbb{R}_v$ *and thus, by the periodicity of* g_0, g_1 *it follows that the mild solution is* T*-periodic.*

Remark 2.5. *There is in general no uniqueness for the weak solution because* f *can take arbitrarily values on the characteristics such that* $s_{in} = -\infty$*. But it is possible to prove that the mild solution is the unique minimal solution for the transport equation (see [17] and [4] for definitions and proofs).*

2.3. Weak and mild solutions for the perturbed Vlasov problem. We intend to apply a fixed point procedure on the electric field. For example let us define the following map :

$$E \to f_E \text{ solution of the Vlasov problem } \to \rho_E \text{ charge density of } f_E$$

$$\to E_1 \text{ solution of the Poisson problem whith source } \rho_E.$$

Unfortunately the above map is not well defined since we have no uniqueness for the Vlasov problem. In order to recover uniqueness property we need to introduce an absorption term $\alpha f,\ \alpha > 0$. The perturbed Vlasov equation writes now :

$$\alpha f(t,x,v) + \partial_t f + v\cdot\partial_x f + E(t,x)\cdot\partial_v f = 0, \quad (t,x,v) \in \mathbb{R}_t\times]0,1[\times\mathbb{R}_v. \tag{2.6}$$

Obviously, the weak and mild formulations previously introduced for the Vlasov problem still hold for the perturbed problem with the corresponding modifications due to the term αf (when $\alpha = 0$ we recover the Definitions 2.1 and 2.2) :

Definition 2.6. *Under the same hypothesis as in Definition* 2.1 *we say that* f *is a* T*-periodic weak solution for the perturbed Vlasov problem* (2.6)*,* (2.3) *iff :*

$$-\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)(-\alpha\varphi + \partial_t\varphi + v\cdot\partial_x\varphi + E(t,x)\cdot\partial_v\varphi)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)\varphi(t,1,v)dtdv,$$

for all test function $\varphi \in \mathcal{T}_w$.

REMARK 2.7. *After multiplication by* f *and integration on* $]0,T[\times]0,1[\times\mathbb{R}_v$ *we can easily check that there is a unique weak solution for the perturbed Vlasov problem (see [6] pp. 657, [3],[12]).*

DEFINITION 2.8. *Under the same hypothesis as in Definition* 2.2 *we say that* f *is a* T*-periodic mild solution for the perturbed Vlasov problem* (2.6), (2.3) *iff :*

$$\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)\psi(t,x,v)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s_{out}^0} e^{-\alpha(s-t)}\psi(s,X(s;t,0,v),V(s;t,0,v))ds$$
$$-\int_0^T\int_{v<0} vg_1(t,v)dtdv \int_t^{s_{out}^1} e^{-\alpha(s-t)}\psi(s,X(s;t,1,v),V(s;t,1,v))ds,$$

for all test function $\psi \in \mathcal{T}_m$.

REMARK 2.9. *We can easily check that if* $g_0 \in L_0^-$, $g_1 \in L_1^-$ *then the mild solution belongs to* $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$. Indeed, let us consider $\chi \in C^1(\mathbb{R})$, $0 \le \chi \le 1$, $supp(\chi) \subset [-2,2]$, $\chi|_{[-1,1]} = 1$. By taking $\psi_R(t,x,v) = \chi(v/R) \in \mathcal{T}_m$ as test function we have :

$$\begin{aligned}\int_0^T\int_0^1\int_{|v|<R} f(t,x,v)dtdxdv &\le \int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)\psi_R(t,x,v)dtdxdv\\
&= \int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s_{out}^0} e^{-\alpha(s-t)}\chi(\frac{V^0(s)}{R})ds\\
&- \int_0^T\int_{v<0} vg_1(t,v)dtdv \int_t^{s_{out}^1} e^{-\alpha(s-t)}\chi(\frac{V^1(s)}{R})ds\\
&\le \frac{1}{\alpha}\left(\int_0^T\int_{v>0} vg_0(t,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)dtdv\right),\ R>0.\end{aligned}$$

Thus by passing $R \to +\infty$ we deduce that f belongs to $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and that :

$$\frac{1}{T}\|f\|_{L^1} \le \frac{G_1}{\alpha}, \quad \alpha > 0. \tag{2.7}$$

REMARK 2.10. *Moreover, under the same hypothesis as in the previous remark, if* $\psi \in L^\infty$ *is* T*-periodic with unbounded velocity support, then the mild formulation still holds.*

For this let us formulate a lemma concerning bounds for the velocity change along the characteristics. This result is the key point of our analysis and it will be used several times throughout this paper :

LEMMA 2.11. *Assume that* $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(]0,1[))$ *is a regular electric field. Then for all characteristics* $(X(s),V(s))$, $s_{in} \le s \le s_{out}$ *we have :*

$$|V(s_1) - V(s_2)| \le 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}, \quad s_{in} \le s_1 \le s_2 \le s_{out}.$$

Proof. If $|V(s_{1,2})| \le \sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$ or $\|E\|_{L^\infty} = 0$ the conclusion follows trivially. Suppose that $\|E\|_{L^\infty} > 0$, $|V(s_1)| > \sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$, for the other case the same argument applies. By integration along the characteristics curves we find :

$$V(s) \ge V(s_1) - (s-s_1)\|E\|_{L^\infty},\ s \in [s_1,s_2],$$

$$V(s_1) \ge V(s) - (s-s_1)\|E\|_{L^\infty},\ s \in [s_1,s_2],$$

and also :

$$1 \ge X(s) - X(s_1) \ge (s-s_1)V(s_1) - \frac{1}{2}(s-s_1)^2\|E\|_{L^\infty},\ s \in [s_1,s_2],$$

$$1 \geq X(s_1) - X(s) \geq -(s-s_1)V(s_1) - \frac{1}{2}(s-s_1)^2\|E\|_{L^\infty},\ s \in [s_1, s_2].$$

Therefore $F(s) := \frac{1}{2}(s-s_1)^2\|E\|_{L^\infty} - |V(s_1)|(s-s_1)+1 \geq 0,\ s \in [s_1,s_2]$ and since the discriminant $\Delta = |V(s_1)|^2 - 2\|E\|_{L^\infty}$ is positive it follows that the quadratic function F has two real roots $s_1 < r_1 < r_2$ given by :

$$r_{1,2} = s_1 + \frac{|V(s_1)| \mp \sqrt{|V(s_1)|^2 - 2\|E\|_{L^\infty}}}{\|E\|_{L^\infty}}.$$

By the other hand we have :

$$F(s_2) = \frac{\|E\|_{L^\infty}}{2}\left(s_2 - s_1 - \frac{|V(s_1)|}{\|E\|_{L^\infty}}\right)^2 + 1 - \frac{|V(s_1)|^2}{2\|E\|_{L^\infty}} \geq 0,$$

and therefore we deduce that :

$$\left|s_2 - s_1 - \frac{|V(s_1)|}{\|E\|_{L^\infty}}\right| > \frac{\sqrt{\Delta}}{\|E\|_{L^\infty}}.$$

If $s_2 - s_1 - |V(s_1)|/\|E\|_{L^\infty} < -\sqrt{\Delta}/\|E\|_{L^\infty}$ thus by using that $|V(s_1)| > \sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$ we have :

$$|V(s_1) - V(s_2)| \leq (s_2 - s_1)\|E\|_{L^\infty} \leq |V(s_1)| - \sqrt{|V(s_1)|^2 - 2\|E\|_{L^\infty}} < \sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}.$$

Now let us consider the case when $s_2 - s_1 - |V(s_1)|/\|E\|_{L^\infty} > \sqrt{\Delta}/\|E\|_{L^\infty}$ which implies that $s_2 > s_1 + (|V(s_1)| + \sqrt{\Delta})/\|E\|_{L^\infty} = r_2$. Therefore we have $s_1 < r_1 < r_2 < s_2$ which is in contradiction with $F(s) \geq 0,\ s \in [s_1, s_2]$, since $F(s) < 0$ for $s \in (r_1, r_2) \subset [s_1, s_2]$. □

Now let us consider the mild test function $\psi_R(t,x,v) = \psi(t,x,v)\cdot\chi(v/R) \in \mathcal{T}_m$. In order to simplify the calculation we treat only the terms of the left boundary located in $x=0$. Exactly the same calculus apply for the right boundary in $x=1$. We have :

$$\begin{aligned}\int_0^T\!\!\int_0^1\!\!\int_{|v|<R} f\psi dtdxdv + \int_0^T\!\!\int_0^1\!\!\int_{|v|>R} f\psi\chi(\frac{v}{R})dtdxdv &= \int_0^T\!\!\int_{0<v<R_1} vg_0(t,v)dtdv\int_t^{s^0_{out}} e^{-\alpha(s-t)}\psi\chi(\frac{V^0(s)}{R})ds \\ &+ \int_0^T\!\!\int_{v>R_1} vg_0(t,v)dtdv\int_t^{s^0_{out}} e^{-\alpha(s-t)}\psi\chi(\frac{V^0(s)}{R})ds + \{\text{right boundary terms}\} \\ &= \mathcal{I}_1(R) + \mathcal{I}_2(R) + \mathcal{I}_3(R) + \mathcal{I}_4(R),\end{aligned} \tag{2.8}$$

where $R_1 = R - 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$. By the previous lemma we deduce that for $0 < v < R_1$ we have $|V^0(s)| \leq 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2} + v \leq R$ and therefore $\chi(V^0(s)/R) = 1$ for $s \in (t, s^0_{out})$ which implies that:

$$\lim_{R\to+\infty}\mathcal{I}_1(R) = \int_0^T\!\!\int_{v>0} vg_0(t,v)dtdv\int_t^{s^0_{out}} e^{-\alpha(s-t)}\psi(s, X^0(s), V^0(s))ds.$$

On the other hand, since $\left|\int_t^{s^0_{out}} e^{-\alpha(s-t)}\psi\chi(V^0(s)/R)ds\right| \leq \|\psi\|_{L^\infty}/\alpha$ and $g_0 \in L^-_0$ we have also the convergence :

$$\mathcal{I}_2(R) \leq \frac{1}{\alpha}\|\psi\|_{L^\infty}\int_0^T\!\!\int_{v>R_1} vg_0(t,v)dtdv \to 0,$$

when $R \to +\infty$. Since f belongs to $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$, $\psi \in L^\infty,\ 0 \leq \chi \leq 1$ we can pass to the limit in (2.8) for $R \to +\infty$ and the mild formulation holds. In particular for $\psi_R = 1_{\{|v|>R\}}$ we have:

$$\int_0^T\int_0^1\int_{|v|>R} f(t,x,v)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s^0_{out}} e^{-\alpha(s-t)} 1_{\{|V^0(s)|>R\}}ds + \{...\}$$
$$= \int_0^T\int_{v>R_1} vg_0(t,v)dtdv \int_t^{s^0_{out}} e^{-\alpha(s-t)} 1_{\{|V^0(s)|>R\}}ds + \{...\}$$
$$\leq \frac{1}{\alpha}\left(\int_0^T\int_{v>R_1} vg_0(t,v)dtdv - \int_0^T\int_{v<-R_1} vg_1(t,v)dtdv\right).$$

REMARK 2.12. *If $g_0 \in L^-_0$, $g_1 \in L^-_1$ and $E \in L^\infty$ then all T-periodic weak solution for* (2.6), (2.3) *belongs to $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and verifies the same estimate* (2.7).

REMARK 2.13. *If $g_0 \in L^-_0$, $g_1 \in L^-_1$ and $E \in L^\infty$ then the weak formulation holds also for test function $\varphi \in W^{1,\infty}$ with unbounded support in velocity (take as test function $\varphi_R = \varphi \cdot \chi(v/R)$ and pass $R \to +\infty$).*

3. The Vlasov-Poisson system. The electric field is due to the charge of particles (self-consistent-field) :

$$\partial_x E = -\partial_x^2 U = \rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv,\ (t,x) \in \mathbb{R}_t\times]0,1[, \tag{3.1}$$

and to the applied voltage on the boundary :

$$U(t,x=0) = \varphi_0(t),\ U(t,x=1) = \varphi_1(t),\ t \in \mathbb{R}_t. \tag{3.2}$$

As above, the electrostatic potentials φ_0, φ_1 are supposed T-periodic in time. The system formed by the equations (2.1), (3.1) and the boundary conditions (2.3) and (3.2) is called the Vlasov-Poisson problem (in one dimension). Obviously, in one dimension the Poisson electric field writes :

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t),\ (t,x) \in \mathbb{R}_t\times]0,1[,$$

and therefore we can give the following definition :

DEFINITION 3.1. *Assume that $g_0 \in L^-_{0,loc}$, $g_1 \in L^-_{1,loc}$, $\varphi_1 - \varphi_0 \in L^\infty(\mathbb{R}_t)$ are T-periodic functions. We say that $(f,E) \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)\times L^\infty(\mathbb{R}_t\times]0,1[)$ is a T-periodic weak solution for the Vlasov-Poisson problem iff f is a T-periodic weak solution for the Vlasov problem* (2.1), (2.3) *corresponding to the electric field E given by the Poisson problem :*

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t),\ (t,x) \in \mathbb{R}_t\times]0,1[,$$

with $\rho(t,x) := \int_{\mathbb{R}_v} f(t,x,v)dv,\ (t,x) \in \mathbb{R}_t\times]0,1[$.

As it was explained in the paragraph 2.3, we need to consider also a perturbed system. Let us introduce the notion of T-periodic mild solution for the perturbed Vlasov-Poisson problem. For this we have to regularize the electric field ; we consider mollifiers $\zeta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\zeta(\frac{\cdot}{\varepsilon}),\ \varepsilon > 0$, where $\zeta \in C_0^\infty(\mathbb{R}),\ \zeta \geq 0,\ supp(\zeta) \subset [-1,+1],\ \int_{\mathbb{R}} \zeta(u)du = 1$.

DEFINITION 3.2. *Under the same hypothesis we say that $(f,E) \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)\times L^\infty(\mathbb{R}_t\times]0,1[)$ is a T-periodic mild solution for the perturbed Vlasov-Poisson problem iff f is the T-periodic mild solution for the perturbed Vlasov problem* (2.6), (2.3) *corresponding to the regularized electric field $E_\varepsilon(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E(s,y)dy$ and E is given by the Poisson problem :*

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t),\ (t,x) \in \mathbb{R}_t\times]0,1[.$$

3.1. Existence for the perturbed Vlasov-Poisson problem. As a first step in the study of periodic weak solutions for the Vlasov-Poisson problem we prove the existence for the perturbed problem. In this section the parameters $\alpha, \varepsilon > 0$ are fixed and we use the Schauder fixed point theorem. For the moment consider that the electric field is given and let us deduce some bounds for f in the L^1 norm :

PROPOSITION 3.3. *Assume that $E \in L^\infty(\mathbb{R}_t\times]0,1[)$, $g_0 \in L_0^-, g_1 \in L_1^-$ are T-periodic and that f is the T-periodic weak solution for* (2.1), (2.3). *Then $f \in L^\infty(\mathbb{R}_t; L^1(]0,1[\times\mathbb{R}_v))$ and :*

$$\|f\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[\times\mathbb{R}_v))} \le \left(\frac{1}{\alpha} + T\right) G_1.$$

Proof. As we saw in the previous section, $f \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and $\|f\|_{L^1} \le \frac{T}{\alpha} G_1,\ \alpha > 0$. Thus there is $t_1 \in]0,T[$ such that :

$$\int_0^1\int_{\mathbb{R}_v} f(t_1,x,v)dxdv \le \frac{G_1}{\alpha}.$$

Now by integration of the perturbated Vlasov equation on $]t_1,t[\times]0,1[\times\mathbb{R}_v$ where $t_1 \le t \le t_1 + T$ we find :

$$\begin{aligned}\|f(t)\|_{L^1} =& \int_0^1\int_{\mathbb{R}_v} f(t,x,v)dxdv \le \int_0^1\int_{\mathbb{R}_v} f(t_1,x,v)dxdv + \int_{t_1}^t\int_{v>0} vg_0(s,v)dsdv \\ &- \int_{t_1}^t\int_{v<0} vg_1(s,v)dsdv \le \left(\frac{1}{\alpha}+T\right) G_1,\end{aligned}$$

and therefore the conclusion follows by periodicity. □

THEOREM 3.4. *Assume that $\varphi_1 - \varphi_0 \in L^\infty(\mathbb{R}_t), g_0, g_1$ are T-periodic functions such that :*

$$\begin{aligned}(H_1)\quad G_1 =& \frac{1}{T}\int_0^T\int_{v>0} vg_0(t,v)dtdv - \frac{1}{T}\int_0^T\int_{v<0} vg_1(t,v)dtdv < +\infty,\\ (H_\infty)\quad G_\infty =& \max\{\|g_0\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)}, \|g_1\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} < +\infty.\end{aligned}$$

Then, for every $\alpha, \varepsilon > 0$ there is a T-periodic mild solution for the perturbed Vlasov-Poisson problem.

Proof. Let us define $R_\alpha = \left(T + \frac{1}{\alpha}\right) G_1 + \|\varphi_1 - \varphi_0\|_{L^\infty}$ and consider the set :

$$X_{\alpha,\varepsilon} = \{E \in L^\infty(\mathbb{R}_t\times]0,1[) \mid E(t,x) = E(t+T,x),\ (t,x) \in \mathbb{R}_t\times]0,1[,\ \|E\|_{L^\infty} \le R_\alpha\},$$

which is convex and compact in respect to the weak $\star$ topology of L^∞. As fixed point application we define $F_\alpha(E), E \in X_{\alpha,\varepsilon}$ as follows :

$$F_{\alpha,\varepsilon}(E)(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t),\ (t,x) \in \mathbb{R}_t\times]0,1[, \tag{3.3}$$

where f is the mild T-periodic solution of (2.6), (2.3) corresponding to the regularized field $E_\varepsilon(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E(s,y)dy$. By the previous Proposition 3.3 we deduce that ρ belongs to $L^\infty(\mathbb{R}_t; L^1(]0,1[))$ and from (3.3) it comes that $\|F_{\alpha,\varepsilon}(E)\|_{L^\infty} \le R_\alpha,\ E \in X_{\alpha,\varepsilon}$. Since $F_{\alpha,\varepsilon}(E)$ is also T-periodic it follows that $F_{\alpha,\varepsilon}(X_{\alpha,\varepsilon}) \subset X_{\alpha,\varepsilon}$. Now let us prove the continuity of the application $F_{\alpha,\varepsilon}$. For this, consider a sequence $(E_n)_n \subset X_{\alpha,\varepsilon}$ such that $E_n \rightharpoonup E$, weakly $\star$ in $L^\infty(]0,T[\times]0,1[)$ which implies the pointwise convergence for $(t,x) \in \mathbb{R}_t\times]0,1[$:

$$E_{n,\varepsilon}(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E_n(s,y)dy \to \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E(s,y)dy = E_\varepsilon(t,x).$$

Thus, by the dominated convergence theorem we obtain that $(E_{n,\varepsilon})_n$ converges strongly to E_ε in $L^2(]0,T[\times]0,1[)$ when $n \to +\infty$. Denote by $(f_n)_n$ the sequence of T-periodic mild solutions associated to $(E_{n,\varepsilon})_n$. Since $\|f_n\|_{L^\infty} \le \|g\|_{L^\infty}$ we have, at least for a subsequence, that :

$$f_n \rightharpoonup f, \text{ weak } \star \text{ in } L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v).$$

As $(f_n)_n$ are mild solutions, they are also weak solutions and therefore we have for all $\varphi \in \mathcal{T}_w,\ n$:

$$-\int_0^T\int_0^1\int_{\mathbb{R}_v} f_n(-\alpha\varphi + \partial_t\varphi + v\cdot\partial_x\varphi + E_{n,\varepsilon}(t,x)\cdot\partial_v\varphi)dtdxdv = \int_0^T\int_{v>0} vg_0(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)\varphi(t,1,v)dtdv.$$

Obviously the following convergence holds :

$$\lim_{n\to+\infty}\int_0^T\int_0^1\int_{\mathbb{R}_v} f_n(-\alpha\varphi + \partial_t\varphi + v\cdot\partial_x\varphi)dtdxdv = \int_0^T\int_0^1\int_{\mathbb{R}_v} f(-\alpha\varphi + \partial_t\varphi + v\cdot\partial_x\varphi)dtdxdv.$$

In order to pass to the limit the other term we remark that since $(f_n)_n$ are uniformly bounded in L^∞ and φ has bounded support in velocity, we have that $\int_{\mathbb{R}_v} f_n(t,x,v)\partial_v\varphi dv$ converges to $\int_{\mathbb{R}_v} f(t,x,v)\partial_v\varphi dv$ weakly in $L^2(]0,T[\times]0,1[)$. Finally, by combining with the strong convergence of $(E_{n,\varepsilon})_n$ in $L^2(]0,T[\times]0,1[)$ we deduce that :

$$\lim_{n\to+\infty}\int_0^T\int_0^1 E_{n,\varepsilon}(t,x)\int_{\mathbb{R}_v} f_n(t,x,v)\partial_v\varphi dvdtdx = \int_0^T\int_0^1 E_\varepsilon(t,x)\int_{\mathbb{R}_v} f(t,x,v)\partial_v\varphi dvdtdx,$$

and thus f is a T- periodic weak solution for :

$$\alpha f + \partial_t f + v\cdot\partial_x f + E_\varepsilon(t,x)\cdot\partial_v f = 0, (t,x,v)\in\mathbb{R}_t\times]0,1[\times\mathbb{R}_v,\ f|_{\mathbb{R}_t\times\Sigma^-} = g.$$

Since for the perturbed Vlasov problem we have uniqueness for the T-periodic weak solution, it follows that f is the T-periodic mild solution corresponding to E_ε. Note also that, from the uniqueness it comes also that the whole sequence $(f_n)_n$ converges weakly $\star$ in L^∞. Let us analyse now the term $\int_0^x \rho_n(t,y)dy$. We have :

$$\left|\int_0^x \rho_n(t,y)dy\right| \le \int_0^1\int_{\mathbb{R}_v} f_n(t,x,v)dxdv \le \left(\frac{1}{\alpha}+T\right)G_1,\ (t,x)\in\mathbb{R}_t\times]0,1[,\ \forall n,$$

and thus $(\int_0^x \rho_n(t,y)dy)_n$ converges weakly $\star$ in $L^\infty(]0,T[\times]0,1[)$. In order to identify the weak $\star$ limit let us calculate for $\theta\in C_c(]0,T[\times]0,1[)$:

$$\int_0^T\int_0^1\int_0^x dy\int_{\mathbb{R}_v} f_n(t,y,v)\theta(t,x)dvdtdx = \int_0^T\int_0^1\int_{\mathbb{R}_v} f_n(t,x,v)\int_x^1\theta(t,y)dydtdxdv$$
$$= \int_0^T\int_0^1\int_{|v|<R} f_n\int_x^1\theta(t,y)dydtdxdv + \int_0^T\int_0^1\int_{|v|>R} f_n\int_x^1\theta(t,y)dydtdxdv = \mathcal{I}_1^n(R) + \mathcal{I}_2^n(R).$$

Taking into account the remark 2.10 we deduce that $\lim_{R\to+\infty}\mathcal{I}_2^n(R) = 0$ uniformly in respect to n. On the other hand, since $\int_x^1\theta(t,y)dy\cdot 1_{\{|v|<R\}}$ belongs to $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ we have also the convergence :

$$\lim_{n\to+\infty}\mathcal{I}_1^n(R) = \mathcal{I}_1(R) = \int_0^T\int_0^1\int_{|v|<R} f(t,x,v)\int_x^1\theta(t,y)dydtdxdv.$$

Finally, by combining the above convergences one gets that :

$$\int_0^x \rho_n(t,y)dy \rightharpoonup \int_0^x \rho(t,y)dy, \text{ weak } \star \text{ in } L^\infty(]0,T[\times]0,1[),$$

where $\rho(t,x) = \int_{\mathbb{R}_v} f(t,x,v)dv$. Exactly in the same manner we find that :

$$\int_0^1 (1-y)\rho_n(t,y)dy \rightharpoonup \int_0^1 (1-y)\rho(t,y)dy, \text{ weak } \star \text{ in } L^\infty(]0,T[),$$

and therefore :

$$\begin{aligned} F_{\alpha,\varepsilon}(E_n) = & \int_0^x \rho_n(t,y)dy - \int_0^1 (1-y)\rho_n(t,y)dy - \varphi_1(t) + \varphi_0(t) \\ \rightharpoonup & \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t) \\ = & F_{\alpha,\varepsilon}(E), \text{ weak } \star \text{ in } L^\infty(]0,T[\times]0,1[), \end{aligned}$$

which proves the continuity of the application $F_{\alpha,\varepsilon}$.

By using the Schauder fixed point theorem we deduce that there is a T-periodic mild solution for the perturbed Vlasov-Poisson problem. □

4. Estimates for the perturbed T-periodic mild solutions. In order to simplify the formulas, in this section we shall systematically skip the indexes α, ε. Generally (f, E) stands for T-periodic mild solutions of the perturbed Vlasov-Poisson problem which writes :

$$\alpha f(t,x,v) + \partial_t f + v \cdot \partial_x f + \widetilde{E}(t,x) \cdot \partial_v f = 0, \ (t,x,v) \in \mathbb{R}_t \times]0,1[\times \mathbb{R}_v,$$

$$f(t,x,v) = g(t,x,v), \ (t,x,v) \in \mathbb{R}_t \times \Sigma^-,$$

$$\widetilde{E}(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)ds \int_0^1 \zeta_\varepsilon(x-y)E(s,y)dy, \ (t,x) \in \mathbb{R}_t \times]0,1[,$$

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t), \ (t,x) \in \mathbb{R}_t \times]0,1[.$$

As usual we use the notations :

$$\rho(t,x) = \int_{\mathbb{R}_v} f(t,x,v)dv, \ \ j(t,x) = \int_{\mathbb{R}_v} v f(t,x,v)dv, \ (t,x) \in \mathbb{R}_t \times]0,1[.$$

In this section we are looking for uniform estimates of the charge, current and electric field. It is convenient to introduce also :

$$(M_\rho, M_{|j|}) := \sup_{\alpha,\varepsilon>0} \frac{1}{T} \int_0^T \int_0^1 \int_{\mathbb{R}_v} f_{\alpha,\varepsilon}(t,x,v)(1,|v|)dtdxdv,$$

$$(C_\rho, C_{|j|}) := \sup_{\alpha,\varepsilon>0, t\in\mathbb{R}_t} \int_0^1 \int_{\mathbb{R}_v} f_{\alpha,\varepsilon}(t,x,v)(1,|v|)dxdv,$$

and :

$$C_E := \sup_{\alpha,\varepsilon>0} \|E_{\alpha,\varepsilon}\|_{L^\infty},$$

with $M_\rho, M_{|j|}, C_\rho, C_{|j|}, C_E \in [0,+\infty]$.

At the begining we assume that :

$$(H_0')\ \ G_0' := \int_{v>0} \sup_{t\in\mathbb{R}_t}\{g_0(t,v)\}dv + \int_{v<0} \sup_{t\in\mathbb{R}_t}\{g_1(t,v)\}dv < +\infty,$$
$$(H_1)\ \ G_1 := \frac{1}{T}\int_0^T\int_{v>0} vg_0(t,v)dtdv - \frac{1}{T}\int_0^T\int_{v<0} vg_1(t,v)dtdv < +\infty,$$
$$(H_\infty)\ \ G_\infty := \max\{\|g_0\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)}, \|g_1\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} < +\infty,$$

which assure the existence of the T-periodic mild solutions (see Theorem 3.4), but later on we shall see that only H_0', H_∞ are sufficient.

Remember that the T-periodic mild solutions satisfy :

$$\int_0^1\int_{\mathbb{R}_v} f(t,x,v)dxdv \le \left(\frac{1}{\alpha}+T\right)G_1,\ t\in\mathbb{R}_t,$$

and :

$$\|E\|_{L^\infty(\mathbb{R}_t\times]0,1[)} \le \left(\frac{1}{\alpha}+T\right)G_1 + \|\varphi_1-\varphi_0\|_{L^\infty}.$$

4.1. Estimate of the total charge. Let us consider as test function in the mild formulation $\psi(t,x,v) = 1_{\{|v|>R_2\}}$ with $R_2 = 6\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$. We have :

$$\begin{aligned}\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)dtdxdv =& \int_0^T\int_0^1\int_{|v|<R_2} f(t,x,v)dtdxdv + \int_0^T\int_0^1\int_{|v|>R_2} f(t,x,v)dtdxdv\\ \le& \int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s^0_{out}} e^{-\alpha(s-t)}1_{\{|V^0(s)|>R_2\}}ds\\ &-\int_0^T\int_{v<0} vg_1(t,v)dtdv \int_t^{s^1_{out}} e^{-\alpha(s-t)}1_{\{|V^1(s)|>R_2\}}ds + 2TR_2G_\infty.\end{aligned}$$

Now, by using Lemma 2.11 we deduce that if $R_3 = 4\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}$ we have :

$$\begin{aligned}\int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s^0_{out}} e^{-\alpha(s-t)}1_{\{|V^0(s)|>R_2\}}ds =& \int_0^T\int_{v>R_3} vg_0(t,v)dtdv \int_t^{s^0_{out}} e^{-\alpha(s-t)}1_{\{|V^0(s)|>R_2\}}ds\\ \le& \int_0^T\int_{v>R_3} vg_0(t,v)\frac{1}{v-2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}}dtdv\\ \le& 2\int_0^T\int_{v>0} g_0(t,v)dtdv.\end{aligned}$$

Finally one gets that :

$$\frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)dtdxdv \le 12\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}G_\infty + 2G_0 \le 12\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}G_\infty + 2G_0'.$$

We need also to estimate f in $L^\infty(\mathbb{R}_t; L^1(]0,1[\times\mathbb{R}_v))$. First of all notice that from the previous estimate it follows that there is $t_1\in]0,T[$ such that :

$$\int_0^1\int_{\mathbb{R}_v} f(t_1,x,v)dxdv \le \frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)dtdxdv \le 12\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}G_\infty + 2G_0'.$$

On the other way, by integration of the perturbed Vlasov equation on $]t_1,t[\times]0,1[\times\mathbb{R}_v,\ t_1 \le t \le t_1+T$, we have :

$$
\begin{aligned}
e^{\alpha t}\int_0^1\int_{\mathbb{R}_v} f(t,x,v)dxdv = & e^{\alpha t_1}\int_0^1\int_{\mathbb{R}_v} f(t_1,x,v)dxdv + \int_{t_1}^t e^{\alpha\tau}\int_{\mathbb{R}_v} v(f(\tau,0,v)-f(\tau,1,v))d\tau dv \\
\le & e^{\alpha t_1}(12\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}G_\infty + 2G_0') + \int_{t_1}^t e^{\alpha\tau}\int_{|v|>R_2} v(f(\tau,0,v)-f(\tau,1,v))d\tau dv \\
& + \int_{t_1}^t e^{\alpha\tau}\int_{0<v<R_2} vg_0(\tau,v)d\tau dv - \int_{t_1}^t e^{\alpha\tau}\int_{0>v>-R_2} vg_1(\tau,v)d\tau dv. \qquad (4.1)
\end{aligned}
$$

We need to estimate the integral $\mathcal{I}(t_1,t_2) = \int_{t_1}^{t_2} e^{\alpha\tau}\int_{|v|>R_2} v(f(\tau,0,v)-f(\tau,1,v))d\tau dv$, for $0 \le t_2 - t_1 \le T$. We shall consider the applications :

$$F_0 : \mathbb{R}_t \times [R_3,+\infty[\to \mathbb{R}^2,\ F_0(t,v) = (s_{out}(t,0,v), V(s_{out}(t,0,v);t,0,v)),$$

and :

$$F_1 : \mathbb{R}_t \times]-\infty,-R_3] \to \mathbb{R}^2,\ F_1(t,v) = (s_{out}(t,1,v), V(s_{out}(t,1,v);t,1,v)).$$

By using one more time Lemma 2.11 it is clear that F_0, F_1 are well defined and we have :

$$s_{out}^0 \le \frac{1}{v - 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}} \le \frac{1}{2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}},\ v \ge R_3,\quad X(s_{out}^0;t,0,v) = 1,$$

$$s_{out}^1 \le \frac{1}{-v - 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}} \le \frac{1}{2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}},\ v \le -R_3,\quad X(s_{out}^1;t,1,v) = 0.$$

Moreover F_0, F_1 are one to one maps since $\|\widetilde{E}\|_{L^\infty(\mathbb{R}_t;W^{1,\infty}(]0,1[))} \le \|E\|_{L^\infty}(1+\int_{\mathbb{R}}|\zeta'(u)|du/\varepsilon)$ and therefore the uniqueness of the characteristics holds. By standard calculations we get :

$$\left|\frac{\partial F_0}{\partial(t,v)}\right| = \frac{v}{V^0(s_{out}^0)} \in \left[\frac{2}{3},2\right],\ (t,v)\in\mathbb{R}_t\times[R_3,+\infty[,$$

$$\left|\frac{\partial F_1}{\partial(t,v)}\right| = \frac{-v}{V^1(s_{out}^1)} \in \left[\frac{2}{3},2\right],\ (t,v)\in\mathbb{R}_t\times]-\infty,-R_3].$$

We have :

$$
\begin{aligned}
\mathcal{I}(t_1,t_2) = & \int_{t_1}^{t_2} e^{\alpha t}\int_{v>R_2} vg_0(t,v)dtdv - \int_{t_1}^{t_2} e^{\alpha\tau}\int_{u>R_2} uf(\tau,1,u)d\tau du \\
& - \int_{t_1}^{t_2} e^{\alpha t}\int_{v<-R_2} vg_1(t,v)dtdv + \int_{t_1}^{t_2} e^{\alpha\tau}\int_{u<-R_2} uf(\tau,0,u)d\tau du \\
= & \mathcal{I}^+(t_1,t_2) + \mathcal{I}^-(t_1,t_2).
\end{aligned}
$$

But with the change of variables $(\tau,u) = (s_{out}(t,0,v), V(s_{out}(t,0,v);t,0,v)) = F_0(t,v)$ we have :

$$\int_{t_1}^{t_2} e^{\alpha\tau}\int_{u>R_2} uf(\tau,1,u)d\tau du = \int\int_{F_0^{-1}(]t_1,t_2[\times]R_2,+\infty[)} e^{\alpha t} vg_0(t,v)dtdv.$$

If we denote by $R_4 = \max\{8\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2}, 2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2} + 1/(t_2-t_1)\}$ we can easily check that $\bigcup_{v\ge R_4}([t_1,t_2-\delta(v)]\times\{v\}) \subset F_0^{-1}([t_1,t_2]\times[R_2,+\infty[)$ with $\delta(v) = 1/(v-2\sqrt{2}\cdot\|E\|_{L^\infty}^{1/2})$. Therefore, by taking into account that $v \cdot \delta(v) \leq 4/3$ for $v \geq R_4$ we get :

$$\begin{aligned}\mathcal{I}^+(t_1,t_2) \leq & \int_{t_1}^{t_2} e^{\alpha t} \int_{R_2<v<R_4} v g_0(t,v)dtdv + \int_{v>R_4} \int_{t_2-\delta(v)}^{t_2} e^{\alpha t} v g_0(t,v)dvdt \\ \leq & e^{\alpha t_2} \left(\int_{t_1}^{t_2} \int_{R_2<v<R_4} v g_0(t,v)dtdv + \frac{4}{3} \int_{v>0} \sup_{t\in\mathbb{R}_t} \{g_0(t,v)\}dv \right).\end{aligned}$$

On the other hand since $R_4 \leq 8\sqrt{2} \cdot \|E\|_{L^\infty}^{1/2} + 1/(t_2-t_1) = R_5 + 1/(t_2-t_1)$ we have also :

$$\begin{aligned}\int_{t_1}^{t_2} \int_{R_2<v<R_4} v g_0(t,v)dtdv = & \int_{t_1}^{t_2} \int_{R_2<v<R_5} v g_0(t,v)dtdv + \int_{t_1}^{t_2} \int_{R_5<v<R_5+1/(t_2-t_1)} v g_0(t,v)dtdv \\ \leq & R_5 \int_0^T \int_{v>0} g_0(t,v)dtdv + \left(R_5 + \frac{1}{t_2-t_1} \right) \int_{t_1}^{t_2} \int_{v>0} g_0(t,v)dtdv \\ \leq & 16\sqrt{2} \cdot \|E\|_{L^\infty}^{1/2} \int_0^T \int_{v>0} g_0(t,v)dtdv + \int_{v>0} \sup_{t\in\mathbb{R}_t} \{g_0(t,v)\}dv.\end{aligned}$$

The right boundary term $\mathcal{I}^-(t_1,t_2)$ can be estimated in the same manner and finally one gets :

$$\mathcal{I}(t_1,t_2) \leq e^{\alpha t_2} \left(16\sqrt{2} \cdot \|E\|_{L^\infty}^{1/2} T G_0 + \frac{7}{3} G_0' \right),$$

and therefore we deduce from (4.1) that :

$$\int_0^1 \int_{\mathbb{R}_v} f(t,x,v)dxdv \leq (12 \cdot G_\infty + 22 \cdot T G_0)\sqrt{2} \cdot \|E\|_{L^\infty}^{1/2} + \frac{13}{3} G_0', \ t \in \mathbb{R}_t.$$

From the Poisson equation we deduce that :

$$\|E(t)\|_{L^\infty(]0,1[)} \leq |\varphi_1(t) - \varphi_0(t)| + \int_0^1 \int_{\mathbb{R}_v} f(t,x,v)dxdv, \ t \in \mathbb{R}_t,$$

which combined with the previous inequality implies that $\|E\|_{L^\infty} \leq A \cdot \|E\|_{L^\infty}^{1/2} + B$, with $A = 12 \cdot 2^{1/2} G_\infty + 22 \cdot 2^{1/2} T G_0, \ B = \|\varphi_1 - \varphi_0\|_{L^\infty} + \frac{13}{3} G_0'$, and therefore :

$$\|E\|_{L^\infty(\mathbb{R}_t\times]0,1[)} \leq A^2 + 2B,$$

$$\|\rho\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[))} \leq A(A + B^{1/2}) + \frac{13}{3} G_0',$$

$$\frac{1}{T} \|f\|_{L^1(]0,T[\times]0,1[\times\mathbb{R}_v)} \leq 12\sqrt{2}(A + B^{1/2}) G_\infty + 2G_0',$$

which can be written :

$$M_\rho \leq 12 \cdot 2^{1/2}(A + B^{1/2}) G_\infty + 2G_0', \ \ C_\rho \leq A(A + B^{1/2}) + \frac{13}{3} G_0', \ \ C_E \leq A^2 + 2B. \tag{4.2}$$

4.2. Estimate of the rest of charge ($\int_{|v|>R} f dv$). We shall need also to estimate integrals like $\int_0^T \int_0^1 \int_{|v|>R} f(t,x,v)dtdxdv$ or $\int_0^1 \int_{|v|>R} f(t,x,v)dxdv, \ t \in \mathbb{R}_t$. In fact, since we know that $C_E < +\infty$, we have, by taking $\psi = 1_{\{|v|>R\}}$ as test function in the mild formulation, with R large enough, $R_6 = R - 2\sqrt{2} \cdot C_E^{1/2}$:

$$\begin{aligned}
\int_0^T\int_0^1\int_{|v|>R} f dtdxdv =& \int_0^T\int_{v>0} vg_0(t,v)dtdv \int_t^{s_{out}^0} e^{-\alpha(s-t)} 1_{\{|V^0(s)|>R\}} ds \\
&- \int_0^T\int_{v<0} vg_1(t,v)dtdv \int_t^{s_{out}^1} e^{-\alpha(s-t)} 1_{\{|V^1(s)|>R\}} ds \\
\leq& \int_0^T\int_{v>R_6} vg_0(t,v)(s_{out}(t,0,v)-t)dtdv - \int_0^T\int_{v<-R_6} vg_1(t,v)(s_{out}(t,1,v)-t)dtdv \\
\leq& \int_0^T\int_{v>R_6} vg_0(t,v)\frac{1}{v-2\sqrt{2}\cdot C_E^{1/2}}dtdv - \int_0^T\int_{v<-R_6} vg_1(t,v)\frac{1}{-v-2\sqrt{2}\cdot C_E^{1/2}}dtdv \\
\leq& \frac{R-2\sqrt{2}\cdot C_E^{1/2}}{R-4\sqrt{2}\cdot C_E^{1/2}} \left(\int_0^T\int_{v>R_6} g_0(t,v)dtdv - \int_0^T\int_{v<-R_6} g_1(t,v)dtdv \right),
\end{aligned} \tag{4.3}$$

and thus :

$$\lim_{R\to+\infty} \int_0^T\int_0^1\int_{|v|>R} f(t,x,v)dtdxdv = 0,$$

uniformly in respect to $\alpha, \varepsilon > 0$. Moreover, in order to estimate $\int_0^1\int_{|v|>R} f(t,x,v)dxdv$, let us remark that there is $t_1 \in]0,T[$ such that :

$$\begin{aligned}
\int_0^1\int_{|v|>R} f(t_1,x,v)dxdv \leq& \frac{1}{T}\int_0^T\int_0^1\int_{|v|>R} f(t,x,v)dtdxdv \\
\leq& \frac{R-2\sqrt{2}\cdot C_E^{1/2}}{R-4\sqrt{2}\cdot C_E^{1/2}} \cdot \frac{1}{T} \left(\int_0^T\int_{v>R_6} g_0(t,v)dtdv - \int_0^T\int_{v<-R_6} g_1(t,v)dtdv \right).
\end{aligned} \tag{4.4}$$

After multiplication of the perturbed Vlasov equation by $1-\chi_R(v) = 1-\chi(v/R)$ we have :

$$\partial_t(e^{\alpha t} f(1-\chi_R(v))) + v\cdot\partial_x(e^{\alpha t} f(1-\chi_R(v))) + \widetilde{E}(t,x)\cdot\partial_v(e^{\alpha t} f(1-\chi_R(v))) = -e^{\alpha t}\widetilde{E} f \chi'(v/R)\frac{1}{R},$$

and after integration on $]t_1,t[\times]0,1[\times\mathbb{R}_v$ one gets :

$$\begin{aligned}
e^{\alpha t}\int_0^1\int_{|v|>2R} f(t,x,v)dxdv \leq& e^{\alpha t_1}\int_0^1\int_{|v|>R} f(t_1,x,v)dxdv \\
&+ \int_{t_1}^t e^{\alpha\tau}\int_{|v|>R} v(f(\tau,0,v)-f(\tau,1,v))(1-\chi_R(v))d\tau dv \\
&- \int_{t_1}^t e^{\alpha\tau}\int_0^1\int_{R<|v|<2R} \widetilde{E} f(\tau,x,v)\chi'(v/R)\frac{1}{R}d\tau dxdv, \; t\in[t_1,t_1+T].
\end{aligned} \tag{4.5}$$

The first term in the right hand side of the previous inequality can be estimated by using (4.4). For the third one we have :

$$\left| \int_{t_1}^t e^{\alpha\tau}\int_0^1\int_{R<|v|<2R} \widetilde{E} f(\tau,x,v)\chi'(v/R)\frac{1}{R}d\tau dxdv \right| \leq e^{\alpha t} C_E \|\chi'\|_{L^\infty} \frac{TM_\rho}{R} \to 0, \tag{4.6}$$

when R goes to $+\infty$, uniformly in $\alpha, \varepsilon > 0$. In order to estimate integrals like $\mathcal{I}_R(t_1,t_2) = \int_{t_1}^{t_2} e^{\alpha\tau}\int_{|v|>R} v(f(\tau,0,v)-f(\tau,1,v))(1-\chi_R(v))d\tau dv$ as before, remark that :

$$\begin{aligned}
\mathcal{I}_R(t_1,t_2) =& \int_{t_1}^{t_2} e^{\alpha t}\int_{v>R} vg_0(t,v)(1-\chi_R(v))dtdv - \int_{t_1}^{t_2} e^{\alpha\tau}\int_{u>R} uf(\tau,1,u)(1-\chi_R(u))d\tau du \\
&- \int_{t_1}^{t_2} e^{\alpha t}\int_{v<-R} vg_1(t,v)(1-\chi_R(v))dtdv + \int_{t_1}^{t_2} e^{\alpha\tau}\int_{u<-R} uf(\tau,0,u)(1-\chi_R(u))d\tau du \\
=& \mathcal{I}_R^+(t_1,t_2) + \mathcal{I}_R^-(t_1,t_2).
\end{aligned}$$

Taking into account that for R large enough such that $\delta(R) \leq t_2 - t_1$ and $\eta = C_E \cdot \delta(R)$ we have :

$$\underset{v \geq R+\eta}{\cup}([t_1, t_2 - \delta(v)] \times \{v\}) \subset F_0^{-1}([t_1, t_2] \times [R, +\infty[),$$

where $\delta(v) = 1/(v - 2\sqrt{2} \cdot C_E^{1/2})$, by the same change of variables it follows that :

$$\begin{aligned}\mathcal{I}_R^+(t_1, t_2) \leq & e^{\alpha t_2} \int_{t_1}^{t_2} \int_{R<v<R+\eta} v g_0(t, v)(1 - \chi_R(v)) dt dv \\ & + e^{\alpha t_2} \int_{t_2 - \delta(v)}^{t_2} \int_{v>R+\eta} v g_0(t, v)(\chi_R(V(s^0_{out}; t, 0, v)) - \chi_R(v)) dt dv.\end{aligned}$$

But for $R < v < R + \eta$ we have $1 - \chi_R(v) = |\chi_R(R) - \chi_R(v)| \leq \frac{\eta}{R} \|\chi'\|_{L^\infty}$. We have also :

$$|\chi_R(V(s^0_{out}; t, 0, v)) - \chi_R(v)| \leq \frac{|V^0(s^0_{out}) - v|}{R} \|\chi'\|_{L^\infty} \leq \frac{2\|\chi'\|_{L^\infty} \sqrt{2} \cdot C_E^{1/2}}{R},$$

and thus :

$$\begin{aligned}\mathcal{I}_R^+(t_1, t_2) \leq & e^{\alpha t_2} \frac{\eta(R+\eta)}{R} \|\chi'\|_{L^\infty} \int_{t_1}^{t_2} \int_{v>R} g_0(t, v) dt dv + \frac{e^{\alpha t_2}}{R} \int_{v>R} v\delta(v) \sup_{t \in \mathbb{R}_t} \{g_0(t, v)\} \|\chi'\|_{L^\infty} 2\sqrt{2} \cdot C_E^{1/2} \\ \leq & e^{\alpha t_2} \|\chi'\|_{L^\infty} const(C_E) \delta(R) \left(\int_0^T \int_{v>0} g_0(t, v) dt dv + \int_{v>0} \sup_{t \in \mathbb{R}_t} \{g_0(t, v)\} dv \right),\end{aligned} \quad (4.7)$$

for $R \geq 2\sqrt{2} \cdot C_E^{1/2} + 1/(t_2 - t_1)$. The same arguments apply for the right boundary term $\mathcal{I}_R^-(t_1, t_2)$ and therefore we have :

$$\mathcal{I}_R(t_1, t_2) = \mathcal{I}_R^+(t_1, t_2) + \mathcal{I}_R^-(t_1, t_2) \leq e^{\alpha t_2} \|\chi'\|_{L^\infty} const(C_E) \delta(R)(TG_0 + G_0'), \quad (4.8)$$

where $R \geq 2\sqrt{2} \cdot C_E^{1/2} + 1/(t_2 - t_1)$, $0 \leq t_2 - t_1 \leq T$. Finally, by using (4.5), (4.4), (4.6) and (4.8) we find that :

$$\begin{aligned}\int_0^1 \int_{|v|>2R} f(t, x, v) dx dv \leq & \frac{R - 2\sqrt{2} \cdot C_E^{1/2}}{R - 4\sqrt{2} \cdot C_E^{1/2}} \cdot \frac{1}{T} \left(\int_0^T \int_{v>R_6} g_0(t, v) dt dv - \int_0^T \int_{v<-R_6} g_1(t, v) dt dv \right) \\ & + \frac{1}{R} C_E \|\chi'\|_{L^\infty} T M_\rho + \delta(R) \|\chi'\|_{L^\infty} const(C_E)(TG_0 + G_0'),\end{aligned} \quad (4.9)$$

which implies that $\int_0^1 \int_{|v|>2R} f(t, x, v) dx dv \to 0$ when $R \to +\infty$, uniformly for $\alpha, \varepsilon > 0$ and $|t_2 - t_1| \geq \beta > 0$. By periodicity, we deduce that the convergence is uniform for $\alpha, \varepsilon > 0$, $t \in \mathbb{R}_t$. Notice that all these estimates don't require any information about G_1. As we shall see, the hypothesis H_1 is not necessary for the existence. In conclusion we proved the following proposition :

Proposition 4.1. *Assume that $g_0, g_1, \varphi_0, \varphi_1$ are T-periodic functions satisfying $\varphi_1 - \varphi_0 \in L^\infty(\mathbb{R}_t)$ and the hypothesis H_0', H_1 and H_∞. Denote by $(f_{\alpha,\varepsilon}, E_{\alpha,\varepsilon})$ T-periodic mild solutions of the perturbed Vlasov-Poisson problem with $\alpha > 0, \varepsilon > 0$ (see Theorem 3.4). Then the following estimates hold uniformly in respect to $\alpha > 0, \varepsilon > 0$:*

$$\|f_{\alpha,\varepsilon}\|_{L^1(]0,T[\times]0,1[\times\mathbb{R}_v)} \leq C,$$

$$\|f_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}_t; L^1(]0,1[\times\mathbb{R}_v))} = \|\rho_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}_t; L^1(]0,1[))} \leq C,$$

$$\|E_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}_t \times]0,1[)} \leq C,$$

where the constant C depends only on $T, \|\varphi_1-\varphi_0\|_{L^\infty(\mathbb{R}_t)}, G_0', G_\infty$ (and not on G_1). Moreover the following convergences hold :

$$\lim_{R\to+\infty}\int_0^T\int_0^1\int_{|v|>R} f_{\alpha,\varepsilon}(t,x,v)dtdxdv = 0, \text{ uniformly in respect to } \alpha>0,\varepsilon>0,G_1,$$

$$\lim_{R\to+\infty}\int_0^1\int_{|v|>R} f_{\alpha,\varepsilon}(t,x,v)dxdv = 0, \text{ uniformly in respect to } \alpha>0,\varepsilon>0,t\in\mathbb{R}_t,G_1.$$

5. Existence for the Vlasov-Poisson problem. Now we can prove the following existence result :

THEOREM 5.1. *Assume that $\varphi_1-\varphi_0\in L^\infty(\mathbb{R}_t)$, g_0, g_1 are T-periodic functions such that :*

$$(H_0')\ \ G_0' := \int_{v>0}\sup_{t\in\mathbb{R}_t}\{g_0(t,v)\}dv + \int_{v<0}\sup_{t\in\mathbb{R}_t}\{g_1(t,v)\}dv < +\infty,$$
$$(H_\infty)\ \ G_\infty := \max\{\|g_0\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)}, \|g_1\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} < +\infty.$$

Then there is a T-periodic weak solution (f,E) for the Vlasov-Poisson problem such that :

$$f\in L^1(]0,T[\times]0,1[\times\mathbb{R}_v),\ \rho\in L^\infty(\mathbb{R}_t;L^1(]0,1[)),\ E\in L^\infty(\mathbb{R}_t\times]0,1[).$$

Proof. For $\alpha>0$ we consider the perturbed boundary datas defined by :

$$g_0^\alpha(t,v) = \frac{g_0(t,v)}{1+\alpha v},\ t\in\mathbb{R}_t, v>0,$$

and :

$$g_1^\alpha(t,v) = \frac{g_1(t,v)}{1-\alpha v},\ t\in\mathbb{R}_t, v<0.$$

We have for $\alpha>0$:

$$G_0^{\alpha\prime} := \int_{v>0}\sup_{t\in\mathbb{R}_t}\{g_0^\alpha(t,v)\}dv + \int_{v<0}\sup_{t\in\mathbb{R}_t}\{g_1^\alpha(t,v)\}dv \le G_0' < +\infty,$$

$$G_1^\alpha := \frac{1}{T}\int_0^T\int_{v>0} vg_0^\alpha(t,v)dtdv - \frac{1}{T}\int_0^T\int_{v<0} vg_1^\alpha(t,v)dtdv \le \frac{1}{\alpha}G_0 \le \frac{1}{\alpha}G_0' < +\infty.$$

and :

$$G_\infty^\alpha := \max\{\|g_0^\alpha\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)}, \|g_1^\alpha\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} \le G_\infty,$$

and therefore there is a T-periodic mild solution for the perturbed Vlasov-Poisson problem with $\alpha=\varepsilon>0$. Moreover, since $G_0^{\alpha\prime}\le G_0'$, $G_\infty^\alpha\le G_\infty$ we have the following estimates for $\alpha>0$:

$$\frac{1}{T}\|f_\alpha\|_{L^1(]0,T[\times]0,1[\times\mathbb{R}_v)} = \frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v} f_\alpha(t,x,v)dtdxdv \le M_\rho,$$

$$\|\rho_\alpha\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[))} = \sup_{t\in\mathbb{R}_t}\int_0^1\int_{\mathbb{R}_v} f_\alpha(t,x,v)dxdv \le C_\rho,$$

$$\|f_\alpha\|_{L^\infty}\le G_\infty,\quad \|E_\alpha\|_{L^\infty(\mathbb{R}_t\times]0,1[)}\le C_E,$$

where M_ρ, C_ρ, C_E verify the inequalities (4.2). Therefore there are $f \in L^\infty(\mathbb{R}_t \times]0,1[\times \mathbb{R}_v), E \in L^\infty(\mathbb{R}_t \times]0,1[)$ T-periodic functions such that :

$$f_{\alpha_n} \rightharpoonup f, \text{ weak } \star \text{ in } L^\infty(\mathbb{R}_t \times]0,1[\times \mathbb{R}_v),$$
$$E_{\alpha_n} \rightharpoonup E, \text{ weak } \star \text{ in } L^\infty(\mathbb{R}_t \times]0,1[),$$

where $\alpha_n \to 0$ when $n \to +\infty$. Moreover we can easily check that we have also the convergence $\widetilde{E}_{\alpha_n} \rightharpoonup E$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times]0,1[)$, when $n \to +\infty$. Since f_{α_n} are mild solutions, they are also weak solutions and thus :

$$\begin{aligned} -&\int_0^T\int_0^1\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)(-\alpha_n\varphi(t,x,v) + \partial_t\varphi + v\cdot\partial_x\varphi + \widetilde{E}_{\alpha_n}(t,x)\cdot\partial_v\varphi)dtdxdv \\ =&\int_0^T\int_{v>0} vg_0^{\alpha_n}(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1^{\alpha_n}(t,v)\varphi(t,1,v)dtdv, \end{aligned}$$

for all $\varphi \in \mathcal{T}_w$. Obviously we have :

$$\lim_{n\to+\infty}\int_0^T\int_0^1\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)(-\alpha_n\varphi + \partial_t\varphi + v\cdot\partial_x\varphi)dtdxdv = \int_0^T\int_0^1\int_{\mathbb{R}_v} f(t,x,v)(\partial_t\varphi + v\cdot\partial_x\varphi)dtdxdv.$$

On the other hand, since φ has bounded support in velocity, by the dominated convergence theorem we deduce that :

$$\begin{aligned} \lim_{n\to+\infty} &\int_0^T\int_{v>0} vg_0^{\alpha_n}(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1^{\alpha_n}(t,v)\varphi(t,1,v)dtdv \\ &= \int_0^T\int_{v>0} vg_0(t,v)\varphi(t,0,v)dtdv - \int_0^T\int_{v<0} vg_1(t,v)\varphi(t,1,v)dtdv. \end{aligned}$$

In order to pass to the limit the other term, we shall prove that $(E_{\alpha_n}(t))_n$ is relatively compact in $L^1(]0,1[)$, $t \in \mathbb{R}_t$ (see [9] pp. 73 for compactness results in L^1). Indeed, first of all $(E_{\alpha_n}(t))_n$ is bounded in $L^\infty(]0,1[)$ and thus in $L^1(]0,1[)$. Moreover it is clear that for all $\varepsilon > 0$ there is ω open set such that $\overline{\omega} \subset]0,1[$ and $\int_{]0,1[-\omega} |E_{\alpha_n}(t,x)|dx < \varepsilon, \ \forall n$. Let us consider now $\varepsilon > 0, \omega =]x_1, x_2[\subset]0,1[$ and $|h| < \min\{x_1, 1-x_2\}$. We have :

$$\begin{aligned} \int_{x_1}^{x_2} |E_{\alpha_n}(t,x+h) - E_{\alpha_n}(t,x)|dx \le& \int_{x_1}^{x_2} \left| \int_x^{x+h} \rho_{\alpha_n}(t,y)dy \right| dx \\ \le& |h| \int_0^1 \rho_{\alpha_n}(t,x)dx \\ \le& C_\rho |h| \to 0, \ h \to 0, \end{aligned}$$

which implies that $(E_{\alpha_n}(t))_n$ is relatively compact in $L^1(]0,1[)$. Since $E_{\alpha_n}(t) \rightharpoonup E(t)$ weakly $\star$ in $L^\infty(]0,1[)$ we deduce that all the sequence $(E_{\alpha_n}(t))_n$ converges to $E(t)$ in $L^1(]0,1[)$, $t \in \mathbb{R}_t$ and by the dominated convergence theorem it follows that $E_{\alpha_n} \to E$ strongly in $L^1(]0,T[\times]0,1[)$. Now, since φ has bounded support in velocity, we can write :

$$\begin{aligned} &\left| \int_0^T\int_0^1 E_{\alpha_n}(t,x)\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)\partial_v\varphi dvdtdx - \int_0^T\int_0^1 E(t,x)\int_{\mathbb{R}_v} f(t,x,v)\partial_v\varphi dvdtdx \right| \\ \le& \left| \int_0^T\int_0^1 (E_{\alpha_n} - E)\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)\partial_v\varphi dvdtdx \right| + \left| \int_0^T\int_0^1\int_{\mathbb{R}_v} (f_{\alpha_n}(t,x,v) - f(t,x,v))E\partial_v\varphi dtdxdv \right| \to 0, \end{aligned}$$

and thus f is a T-periodic weak solution for the Vlasov problem corresponding to the field E. Moreover, since $f_{\alpha_n} \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times]0,1[\times \mathbb{R}_v)$ we have that $f_{\alpha_n} \rightharpoonup f$ weakly in

$L^1(]0,T[\times]0,1[\times B_R),\ R>0$ and therefore :

$$\begin{aligned}\frac{1}{T}\int_0^T\!\!\int_0^1\!\!\int_{|v|<R} f(t,x,v)dtdxdv &\leq \frac{1}{T}\liminf_{n\to+\infty}\int_0^T\!\!\int_0^1\!\!\int_{|v|<R} f_{\alpha_n}(t,x,v)dtdxdv\\ &\leq \frac{1}{T}\liminf_{n\to+\infty}\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}dtdxdv \leq M_\rho,\ R>0,\end{aligned}$$

which implies that $f\in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and $\frac{1}{T}\int_0^T\!\int_0^1\!\int_{\mathbb{R}_v} f(t,x,v)dtdxdv \leq M_\rho$. We can prove that $f_{\alpha_n} \rightharpoonup f$ weakly in $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$. Indeed, for $\theta\in L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$ we can write :

$$\begin{aligned}\left|\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v}(f_{\alpha_n}-f)\theta(t,x,v)dtdxdv\right| &\leq \left|\int_0^T\!\!\int_0^1\!\!\int_{|v|<R}(f_{\alpha_n}-f)\theta(t,x,v)dtdxdv\right|\\ &+\|\theta\|_{L^\infty}\left(\int_0^T\!\!\int_0^1\!\!\int_{|v|>R} f_{\alpha_n}dtdxdv+\int_0^T\!\!\int_0^1\!\!\int_{|v|>R} f dtdxdv\right).\end{aligned}$$

From (4.3) it follows that we can take $R=R(\varepsilon)$ large enough such that :

$$\int_0^T\!\!\int_0^1\!\!\int_{|v|>R} f_{\alpha_n}(t,x,v)dtdxdv \leq \frac{\varepsilon}{4\|\theta\|_{L^\infty}},\ n>0,$$

$$\int_0^T\!\!\int_0^1\!\!\int_{|v|>R} f(t,x,v)dtdxdv \leq \frac{\varepsilon}{4\|\theta\|_{L^\infty}},$$

and since $f_{\alpha_n} \rightharpoonup f$ weakly $\star$ in $L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$ we have also :

$$\left|\int_0^T\!\!\int_0^1\!\!\int_{|v|<R}(f_{\alpha_n}-f)\theta(t,x,v)dtdxdv\right| < \frac{\varepsilon}{2},\ n\geq n_\varepsilon,$$

which allows to conclude. In particular $\rho_{\alpha_n}\rightharpoonup\rho$ weakly in $L^1(]0,T[\times]0,1[)$. Now, for all $t\in\mathbb{R}_t$ we have also the convergence $f_{\alpha_n}(t)\rightharpoonup f(t)$ weakly $\star$ in $L^\infty(]0,1[\times\mathbb{R}_v)$. In particular we have $f_{\alpha_n}(t)\rightharpoonup f(t)$ weakly in $L^1(]0,1[\times B_R),\ R>0$, and therefore :

$$\begin{aligned}\int_0^1\!\!\int_{|v|<R} f(t,x,v)dxdv &\leq \liminf_{n\to+\infty}\int_0^1\!\!\int_{|v|<R} f_{\alpha_n}(t,x,v)dxdv\\ &\leq \liminf_{n\to+\infty}\int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)dxdv \leq C_\rho,\ t\in\mathbb{R}_t,\end{aligned}$$

which implies that $f(t)\in L^1(]0,1[\times\mathbb{R}_v)$ and $\|f\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[\times\mathbb{R}_v))} = \|\rho\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[))}\leq C_\rho$. By using (4.9), we can prove that $f_{\alpha_n}(t)\rightharpoonup f(t)$ weakly in $L^1(]0,1[\times\mathbb{R}_v),\ t\in\mathbb{R}_t$. We have also the convergence $\rho_{\alpha_n}(t)\rightharpoonup\rho(t)$ weakly in $L^1(]0,1[)$ for all $t\in\mathbb{R}_t$ and therefore :

$$\int_0^x \rho_{\alpha_n}(t,y)dy \to \int_0^x \rho(t,y)dy,\ (t,x)\in\mathbb{R}_t\times[0,1],$$

and :

$$\int_0^1 (1-y)\rho_{\alpha_n}(t,y)dy \to \int_0^1 (1-y)\rho(t,y)dy,\ t\in\mathbb{R}_t.$$

Now, by using the Poisson equation we deduce that there is E_1 such that :

$$\begin{aligned}E_1(t,x) &= \lim_{n\to+\infty} E_{\alpha_n}(t,x)\\ &= \lim_{n\to+\infty}\left(\int_0^x \rho_{\alpha_n}(t,y)dy - \int_0^1(1-y)\rho_{\alpha_n}(t,y)dy - \varphi_1(t)+\varphi_0(t)\right)\\ &= \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t)+\varphi_0(t),\ (t,x)\in\mathbb{R}_t\times[0,1],\end{aligned}$$

with $\|E_1\|_{L^\infty} \le C_E$ which implies also that $E_{\alpha_n} \to E_1$ in $L^1(]0,T[\times]0,1[)$ and therefore the field $E = E_1$ verifies also the Poisson equation :

$$E(t,x) = \int_0^x \rho(t,y)dy - \int_0^1 (1-y)\rho(t,y)dy - \varphi_1(t) + \varphi_0(t),\ (t,x) \in \mathbb{R}_t \times]0,1[.$$

□

Let us state now another existence result. This time we suppose that H_1 and H_∞ hold, but not H_0' and we shall prove that the solution has more regularity.

THEOREM 5.2. *Assume that $\varphi_1 - \varphi_0 \in L^\infty(\mathbb{R}_t), g_0, g_1$ are T-periodic functions such that :*

$$(H_1)\ \ G_1 := \frac{1}{T}\int_0^T\!\!\int_{v>0} vg_0(t,v)dtdv - \frac{1}{T}\int_0^T\!\!\int_{v<0} vg_1(t,v)dtdv < +\infty,$$
$$(H_\infty)\ \ G_\infty := \max\{\|g_0\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)}, \|g_1\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} < +\infty.$$

Then there is a T-periodic weak solution (f,E) for the Vlasov-Poisson problem which verifies :

$$f \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v), \rho \in L^\infty(\mathbb{R}_t; L^1(]0,1[)), |v|f \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v), E \in L^\infty(\mathbb{R}_t\times]0,1[).$$

Moreover, if H_1' holds :

$$(H_1')\ \ G_1' := \int_{v>0} v \cdot \sup_{t\in\mathbb{R}_t}\{g_0(t,v)\}dv - \int_{v<0} v \cdot \sup_{t\in\mathbb{R}_t}\{g_1(t,v)\}dv < +\infty,$$

then $|v|f$ belongs to $L^\infty(\mathbb{R}_t; L^1(]0,1[\times\mathbb{R}_v))$; in particular $j = \int_{\mathbb{R}_v} vf(t,x,v)dv \in L^\infty(\mathbb{R}_t; L^1(]0,1[))$.

The proof is quite similar with the previous one. We don't go into details, but we only sketch below the different arguments. This time, since H_1, H_∞ are verified, we can apply the Theorem 3.4 with $\alpha = \varepsilon > 0$ for the boundary datas g_0, g_1. Exactly as in Section 4.1 we have :

$$\frac{1}{T}\int_0^T\!\!\int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)dtdxdv \le 12\sqrt{2}\cdot\|E_{\alpha_n}\|_{L^\infty}^{1/2}G_\infty + 2G_0,$$

and there is $t_1 = t_1^\alpha \in]0,T[$ such that :

$$\int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}(t_1,x,v)dxdv \le 12\sqrt{2}\cdot\|E_{\alpha_n}\|_{L^\infty}^{1/2}G_\infty + 2G_0.$$

By integration of the perturbed Vlasov equation on $]t_1,t[\times]0,1[\times\mathbb{R}_v,\ t \in [t_1, t_1+T]$ we have :

$$\begin{aligned}\int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)dxdv \le& \int_0^1\!\!\int_{\mathbb{R}_v} f_{\alpha_n}(t_1,x,v)dxdv \\ &+ \int_{t_1}^t\int_{v>0} vg_0(\tau,v)d\tau dv - \int_{t_1}^t\int_{v<0} vg_1(\tau,v)d\tau dv \\ \le& 12\sqrt{2}\cdot\|E_{\alpha_n}\|_{L^\infty}^{1/2}G_\infty + 2G_0 + TG_1.\end{aligned}$$

From the Poisson equation we have :

$$\|E_{\alpha_n}\|_{L^\infty} \le \|\varphi_1 - \varphi_0\|_{L^\infty} + \|\rho_{\alpha_n}\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[))},$$

and therefore we obtain that :

$$\|E_{\alpha_n}\|_{L^\infty} \le C\cdot\|E_{\alpha_n}\|_{L^\infty} + D,\ \alpha > 0,$$

with $C = 12 \cdot 2^{1/2} G_\infty$, $D = \|\varphi_1 - \varphi_0\|_{L^\infty} + 2G_0 + TG_1$. Finally one gets for $\alpha > 0$:

$$\begin{aligned}\|E_{\alpha_n}\|_{L^\infty} \leq & C_E \leq C^2 + 2D,\\ \|\rho_{\alpha_n}\|_{L^\infty(\mathbb{R}_t;L^1(]0,1[))} \leq & C_\rho \leq C(C + D^{1/2}) + 2G_0 + TG_1,\\ \frac{1}{T}\|f\|_{L^1(]0,T[\times]0,1[\times\mathbb{R}_v)} \leq & M_\rho \leq C(C + D^{1/2}) + 2G_0.\end{aligned}$$

In order to estimate the charge outside a ball B_R in $\mathbb{R}_v$ this time it is easy to calculate integrals like $\mathcal{I}_R(t_1, t_2)$ by :

$$\begin{aligned}\mathcal{I}_R(t_1,t_2) = & \int_{t_1}^{t_2} e^{\alpha t} \int_{|v|>R} v(f(t,0,v) - f(t,1,v))(1 - \chi_R(v)) dtdv\\ \leq & e^{\alpha t_2}\left(\int_{t_1}^{t_2}\int_{v>R} vg_0(t,v)dtdv - \int_{t_1}^{t_2}\int_{v<-R} vg_1(t,v)dtdv\right)\\ \leq & e^{\alpha t_2}\left(\int_0^T\int_{v>R} vg_0(t,v)dtdv - \int_0^T\int_{v<-R} vg_1(t,v)dtdv\right),\ 0 \leq t_2 - t_1 \leq T,\end{aligned}$$

and the proof follows exactly as before.

Now, in order to prove that $|v|f_{\alpha_n} \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$, let us multiply the perturbed Vlasov equation by $|v|$:

$$\alpha(|v|f_{\alpha_n}) + \partial_t(|v|f_{\alpha_n}) + v\cdot\partial_x(|v|f_{\alpha_n}) + \widetilde{E}_{\alpha_n}(t,x)\cdot\partial_v(|v|f_{\alpha_n}) = \widetilde{E}_{\alpha_n}\frac{v}{|v|}f_{\alpha_n}, (t,x,v) \in \mathbb{R}_t\times]0,1[\times\mathbb{R}_v.$$

The mild formulation writes this time :

$$\begin{aligned}\int_0^T\int_0^1\int_{\mathbb{R}_v}|v|f_{\alpha_n}\psi dtdxdv = & \int_0^T\int_0^1\int_{\mathbb{R}_v}\widetilde{E}_{\alpha_n}\frac{v}{|v|}f_{\alpha_n}\int_t^{s_{out}} e^{-\alpha(s-t)}\psi(s,X(s),V(s))dsdtdxdv\\ & + \int_0^T\int_{v>0}|v|^2 g_0(t,v)\int_t^{s^0_{out}} e^{-\alpha(s-t)}\psi(s,X^0(s),V^0(s))dsdtdv\\ & + \int_0^T\int_{v<0}|v|^2 g_1(t,v)\int_t^{s^1_{out}} e^{-\alpha(s-t)}\psi(s,X^1(s),V^1(s))dsdtdv,\end{aligned}$$

for all $\psi \in \mathcal{T}_m$, and thus, for $\psi = 1_{\{|v|>R\}}$ (in fact take $\psi = \chi_{R'} - \chi_R \in \mathcal{T}_m$ with $R' > 2R$ and pass $R' \to +\infty$), R large enough such that $R_1 = R - 2\sqrt{2}\cdot C_E^{1/2} > 0$ we get :

$$\begin{aligned}&\int_0^T\int_0^1\int_{|v|>R}|v|f_{\alpha_n}dtdxdv \leq C_E\int_0^T\int_0^1\int_{|v|>R_1} f_{\alpha_n}(t,x,v)\frac{1}{|v| - 2\sqrt{2}\cdot C_E^{1/2}}dtdxdv\\ &+\int_0^T\int_{v>R_1} vg_0(t,v)\frac{v}{v - 2\sqrt{2}\cdot C_E^{1/2}}dtdv + \int_0^T\int_{v<-R_1} vg_1(t,v)\frac{v}{-v - 2\sqrt{2}\cdot C_E^{1/2}}dtdv \to 0,\end{aligned}$$

when $R \to +\infty$, uniformly in respect to $\alpha > 0$. By taking for example $R = 6\sqrt{2}\cdot C_E^{1/2}$ one gets :

$$\begin{aligned}\frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v}|v|f_{\alpha_n}dtdxdv = & \frac{1}{T}\int_0^T\int_0^1\int_{|v|<R}|v|f_{\alpha_n}dtdxdv + \frac{1}{T}\int_0^T\int_0^1\int_{|v|>R}|v|f_{\alpha_n}dtdxdv\\ \leq & M_{|j|} \leq (6\cdot 2^{1/2} + \frac{1}{2\cdot 2^{1/2}})\cdot C_E^{1/2}M_\rho + 2G_1,\end{aligned}$$

and thus $|v|f_{\alpha_n} \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$. Now, if $f_{\alpha_n} \rightharpoonup f$ weakly $\star$ in $L^\infty(]0,T[\times]0,1[\times\mathbb{R}_v)$ we have also that $|v|f_{\alpha_n} \rightharpoonup |v|f$ weakly in $L^1(]0,T[\times]0,1[\times B_R)$, $R > 0$ and thus :

$$\begin{aligned}\frac{1}{T}\int_0^T\int_0^1\int_{|v|<R}|v|f(t,x,v)dtdxdv \leq & \frac{1}{T}\liminf_{n\to+\infty}\int_0^T\int_0^1\int_{|v|<R}|v|f_{\alpha_n}(t,x,v)dtdxdv\\ \leq & \frac{1}{T}\liminf_{n\to+\infty}\int_0^T\int_0^1\int_{\mathbb{R}_v}|v|f_{\alpha_n}(t,x,v)dtdxdv \leq M_{|j|},\ R > 0.\end{aligned}$$

It comes that $|v|f \in L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and $\frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v}|v|f(t,x,v)dtdxdv \le M_{|j|}$. In fact we can prove that $|v|f_{\alpha_n} \rightharpoonup |v|f$ weakly in $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$ and $j_{\alpha_n} \rightharpoonup j$ weakly in $L^1(]0,T[\times]0,1[)$. Assume now that H_1' holds. In order to estimate j_{α_n} and j in $L^\infty(\mathbb{R}_t;L^1(]0,1[))$ we can apply the same arguments as for the estimates of ρ_{α_n},ρ in $L^\infty(\mathbb{R}_t;L^1(]0,1[))$. This time we have an extra term which writes :

$$\left|\int_{t_1}^{t_2}e^{\alpha t}\int_0^1\int_{\mathbb{R}_v}\widetilde{E}_{\alpha_n}\frac{v}{|v|}f_{\alpha_n}dtdxdv\right| \le e^{\alpha t_2}\|E_{\alpha_n}\|_{L^\infty}\int_0^T\int_0^1\int_{\mathbb{R}_v}f_{\alpha_n}(t,x,v)dtdxdv,\ 0\le t_2-t_1\le T.$$

In order to estimate the rest of the current, remark that we have :

$$\left|\int_{t_1}^{t_2}e^{\alpha t}\int_0^1\int_{|v|>R}\widetilde{E}_{\alpha_n}\frac{v}{|v|}f_{\alpha_n}(1-\chi_R(v))dtdxdv\right| \le e^{\alpha t_2}\|E_{\alpha_n}\|_{L^\infty}\int_0^T\int_0^1\int_{|v|>R}f_{\alpha_n}(t,x,v)dtdxdv,$$

for $0\le t_2-t_1\le T$ and therefore $\int_0^1\int_{|v|>R}|v|f_{\alpha_n}(t,x,v)dxdv \to 0$ when $R\to+\infty$, uniformly for $\alpha>0,\ t\in\mathbb{R}_t$. Finally we prove that there is $C_{|j|}<+\infty$ (which depends on G_1') such that :

$$\int_0^1\int_{\mathbb{R}_v}|v|f_{\alpha_n}(t,x,v)dxdv \le C_{|j|},\ t\in\mathbb{R}_t,\ \alpha>0,$$
$$\int_0^1\int_{\mathbb{R}_v}|v|f(t,x,v)dxdv \le C_{|j|},\ t\in\mathbb{R}_t,$$

$$|v|f_{\alpha_n}(t) \rightharpoonup |v|f(t), \text{weak in } L^1(]0,1[\times\mathbb{R}_v),$$
$$j_{\alpha_n}(t) \rightharpoonup j(t), \text{weak in } L^1(]0,1[).$$

Obviously this result can be generalized as follows :

Theorem 5.3. *Assume that $\varphi_1-\varphi_0\in L^\infty(\mathbb{R}_t)$, g_0,g_1 are T-periodic functions such that :*

$$(H_p)\ \ G_p := \frac{1}{T}\int_0^T\int_{v>0}|v|^p g_0(t,v)dtdv + \frac{1}{T}\int_0^T\int_{v<0}|v|^p g_1(t,v)dtdv < +\infty,$$
$$(H_\infty)\ \ G_\infty := \max\{\|g_0\|_{L^\infty(\mathbb{R}_t\times\Sigma_0^-)},\|g_1\|_{L^\infty(\mathbb{R}_t\times\Sigma_1^-)}\} < +\infty,$$

for some integer $p\ge1$. Then there is a T-periodic weak solution (f,E) for the Vlasov-Poisson problem which verifies :

$$|v|^p f\in L^1(]0,T[\times]0,1[\times\mathbb{R}_v),\ |v|^{p-1}f\in L^\infty(\mathbb{R}_t;L^1(]0,1[\times\mathbb{R}_v)),\ E\in L^\infty(\mathbb{R}_t\times]0,1[).$$

Moreover, if H_p' holds :

$$(H_p')\ \ G_p' := \int_{v>0}|v|^p\sup_{t\in\mathbb{R}_t}\{g_0(t,v)\}dv + \int_{v<0}|v|^p\sup_{t\in\mathbb{R}_t}\{g_1(t,v)\}dv < +\infty,$$

then $|v|^p f$ belongs to $L^\infty(\mathbb{R}_t;L^1(]0,1[\times\mathbb{R}_v))$.

6. Remarks. First of all notice that the estimates of f on the outgoing boundary follows immeadeately. For example, under the hypothesis of the last theorem, after multiplication by $|v|^{p-1}, p\ge1$ (in fact $|v|^{p-1}\chi_R(v)$, with $R\to+\infty$) and integration of the Vlasov equation we deduce that :

$$\frac{1}{T}\int_0^T\int_{v>0}|v|^p f(t,1,v)dtdv + \frac{1}{T}\int_0^T\int_{v<0}|v|^p f(t,0,v)dtdv \le G_p + \frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v}|E|f(p-1)|v|^{p-2}dtdxdv$$
$$\le G_p + (p-1)\cdot\|E\|_{L^\infty}\frac{1}{T}\int_0^T\int_0^1\int_{\mathbb{R}_v}f(t,x,v)|v|^{p-2}dtdxdv.$$

On the other way, it is possible to pass to the limit the non linear term $E_{\alpha_n} \cdot \partial_v f_{\alpha_n}$ in the perturbed Vlasov equation by using the velocity average lemma of Diperna and Lions (see [11]). In fact once that we have proved that $(f_{\alpha_n})_n, (E_{\alpha_n})_n$ are uniformly bounded in $L^1(]0,T[\times]0,1[\times\mathbb{R}_v)$, respectively $L^\infty(\mathbb{R}_t\times]0,1[)$ we deduce that $\partial_t f_{\alpha_n} + v \cdot \partial_x f_{\alpha_n} = -\alpha_n f_{\alpha_n} - \widetilde{E}_{\alpha_n}(t,x) \cdot \partial_v f_{\alpha_n}$ are uniformly bounded in $L^2(]0,T[\times]0,1[\times H^{-1}(\mathbb{R}_v))$. This implies that $\left(\int_{\mathbb{R}_v} f_{\alpha_n}(t,x,v)\partial_v \varphi dv\right)_n$ are uniformly bounded in $H^{1/4}(]0,T[\times]0,1[)$ and therefore converges to $\int_{\mathbb{R}_v} f(t,x,v)\partial_v\varphi dv$ strongly in $L^2(]0,T[\times]0,1[)$. The conclusion follows by combining with the weak convergence of $(E_{\alpha_n})_n$.

All these results are easily adapted for the Vlasov-Poisson problem (in one dimension) involving several densities f_e, f_i where for example f_e represents the density of electrons, f_i the density of ions.

Obviously, let us remark that changing the sign of the right hand side of the Poisson equation $-\partial^2 U/\partial x^2 = -\rho(t,x)$, which corresponds to an attractive (gravitational) potential doesn't affect any argument, so that all the previous results still hold in this case.

It would be interesting to see if the same kind of arguments apply for studying the multidimensional case. This analysis will be the topic of future related works, [8]. We point out that Lemma 2.11 can be easily generalized for a bounded domain $\Omega \subset \mathbb{R}^N$. Indeed, if $(X(s), V(s)),\ s_{in} \leq s \leq s_{out}$ is an arbitrary characteristic associated to a regular field and $u \in \mathbb{R}^N$ with $\|u\| = 1$, then we have :

$$\frac{d}{ds}x(s) = v(s),\ \ \frac{d}{ds}v(s) = e(s),\ s_{in} \leq s \leq s_{out},$$

where $x(s) = (X(s), u),\ v(s) = (V(s), u),\ e(s) = (E(s, X(s)), u)$ for $s_{in} \leq s \leq s_{out}$. Obviously, $x(s)$ belongs to a bounded interval $\omega \subset \mathbb{R}$ of length $diam(\omega) \leq diam(\Omega)$ and $\|e\|_{L^\infty} \leq \|E\|_{L^\infty}$. After performing the same computations as in Section 2 we get :

$$\mid v(s_1) - v(s_2) \mid \leq 2 \cdot (2 \cdot diam(\omega))^{1/2} \cdot \|e\|_{L^\infty}^{1/2},\ s_{in} \leq s_1 \leq s_2 \leq s_{out},$$

which writes also :

$$|(V(s_1) - V(s_2), u)| \leq 2 \cdot (2 \cdot diam(\Omega))^{1/2} \cdot \|E\|_{L^\infty}^{1/2},\ s_{in} \leq s_1 \leq s_2 \leq s_{out}, \forall\ u \in \mathbb{R}^N, \|u\| = 1,$$

and the conclusion follows.

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