BOUNDARY VALUE PROBLEM FOR THE THREE DIMENSIONAL TIME PERIODIC VLASOV-MAXWELL SYSTEM

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Abstract. *In this work we study the existence of time periodic weak solution for the three dimensional Vlasov-Maxwell system with boundary conditions. The main idea consists of using the mass, momentum and energy conservation laws which allow to obtain a priori estimates in the case of a star-shaped bounded spatial domain. First of all time periodic solutions are constructed for a regularized system. The existence for the Vlasov-Maxwell system follows by weak stability under uniform estimates. These results apply for both classical and relativistic cases and for systems with several species of particles.*

Key words. Vlasov-Maxwell equations, weak/mild formulation, regularization.

AMS subject classifications. 35F30, 35L40.

1. Introduction.

The coupled non linear system given by the Vlasov-Maxwell equations is a classical model in the kinetic theory of plasma. The main assumption underlying the model is that collisions are so rare that they may be neglected.

Consider Ω an open bounded subset of $\mathbb{R}^3_x$, with boundary $\partial\Omega$ regular. We introduce the notations $\Sigma = \partial\Omega \times \mathbb{R}^3_p$ and :

$$\Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^3_p \mid \pm (v(p) \cdot n(x)) > 0\}, \tag{1.1}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x and $v(p)$ is the velocity associated to some energy function $\mathcal{E}(p)$ by $v(p) = \nabla_p \mathcal{E}(p),\ p \in \mathbb{R}^3_p$. The functions to be considered are :

$$\mathcal{E}(p) = \frac{|p|^2}{2m}, \ \ v(p) = \frac{p}{m}, \tag{1.2}$$

for the classical case and :

$$\mathcal{E}(p) = mc_0^2 \left(\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{1/2} - 1 \right), \ \ v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{-1/2}, \tag{1.3}$$

for the relativistic case, where m is the mass of particles, c_0 is the light speed in the vacuum. We denote by $f(t,x,p)$ the particles distribution depending on the time t, the position $x \in \Omega$ and the momentum $p \in \mathbb{R}^3_p$ and by $(E(t,x), B(t,x))$ the electro-magnetic field depending on t and x. If we note by $F(t,x,p) = q \cdot (E(t,x) + v(p) \wedge B(t,x))$ the electro-magnetic force, the Vlasov problem is given by :

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x,p) \cdot \nabla_p f = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \tag{1.4}$$

$$f(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-, \tag{1.5}$$

where q is the charge of particles and g represents the distribution of the incoming particles, which is a given T periodic function. Some other boundary conditions can be considered as we will see later on. The problem $(1.4), (1.5)$ is coupled with the Maxwell equations :

$$\partial_t E - c_0^2 \cdot \mathrm{rot}\ B = -\frac{j(t,x)}{\varepsilon_0}, \ \ \partial_t B + \mathrm{rot}\ E = 0, \ \ \mathrm{div}\ E = \frac{\rho(t,x)}{\varepsilon_0}, \ \ \mathrm{div}\ B = 0, \ \ (t,x) \in \mathbb{R}_t \times \Omega, \tag{1.6}$$

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with the boundary condition :

$$n(x) \wedge E(t,x) + c_0 \cdot n(x) \wedge (n(x) \wedge B(t,x)) = h(t,x), \ \ (t,x) \in \mathbb{R}_t \times \partial\Omega, \tag{1.7}$$

where ε_0 is the permittivity of the vacuum, $j(t,x) = q\int_{\mathbb{R}^3_p} f(t,x,p)v(p)dp$ is the current density and h is a given T periodic function on the boundary $\mathbb{R}_t \times \partial\Omega$ such that $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$. We introduce also the permeability of the vacuum, μ_0, given by $\varepsilon_0 \cdot \mu_0 \cdot c_0^2 = 1$.

When the magnetic field is neglected, the electric field derives from a potential $E = -\nabla_x \Phi$ and we obtain the Vlasov-Poisson system :

$$\partial_t f + v(p) \cdot \nabla_x f - q\nabla_x \Phi \cdot \nabla_p f = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \ \ f(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-,$$

$$-\Delta_x \Phi = \frac{\rho}{\varepsilon_0}, \ \ (t,x) \in \mathbb{R}_t \times \Omega, \ \ \Phi(t,x) = \varphi_0(t,x), \ \ (t,x) \in \mathbb{R}_t \times \partial\Omega,$$

where φ_0 is the potential on the boundary $\mathbb{R}_t \times \partial\Omega$. This model can be justified for large light speed ($c_0 \to +\infty$), see Degond [12].

The aim of this paper is to prove the existence of time T periodic solution for the three dimensional Vlasov-Maxwell system (1.4), (1.5), (1.6), (1.7) when the boundary conditions are supposed T periodic, with $T > 0$ fixed. The hypothesis on the boundary data will be detailed later on, typically we suppose that the incoming energy is bounded $\int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| g(t,x,p)\mathcal{E}(p) dt d\sigma dp < +\infty$ and $\int_0^T \int_{\partial\Omega} |h|^2(t,x)\ dt d\sigma < +\infty$.

Various results were obtained for the free space system of Vlasov-Poisson. Weak solutions were constructed by Arseneev [1], Horst and Hunze [24]. The existence of classical solutions has been studied by Ukai and Okabe [32], Horst [23], Batt [2], Pfaffelmoser [28]. The existence of global classical solutions for the Vlasov-Poisson equations with small initial data is a result of Bardos and Degond [5], see also Schaeffer [30], [31]. The propagation of the moments for the three dimensional Vlasov-Poisson system was studied by Lions and Perthame in [27]. The existence of global weak solution for the Vlasov-Maxwell system in three dimensions was obtained by DiPerna and Lions [14], one of the key points being the compactness result of velocity averages (see also [19]). Results for the relativistic case were obtained by Glassey and Schaeffer [17], Glassey and Strauss [18].

Results for the initial-boundary value problem were obtained by Ben Abdallah [7] for the Vlasov-Poisson system in three dimensions and Guo [21] for the Vlasov-Maxwell system. The stationary problem for the Vlasov-Poisson equations was studied by Greengard and Raviart [20] in one dimension and by Poupaud [29] in three dimensions for the Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system was done by Degond and Raviart [13] in the case of the plane diode. The regularity of the solutions for the Vlasov-Maxwell system in a half line has been studied by Guo [22]. Results for the one dimensional time periodic case can be found in [9] for the Vlasov-Poisson system and in [8] for the Vlasov-Maxwell system. The N dimensional $N \geq 2$ time periodic Vlasov-Poisson system is studied in [10]. General results for transport theory were obtained by Bardos [4], DiPerna and Lions [15], Beals and Protopopescu [6].

As usual we start by analyzing a regularized system for which the existence of solution follows by fixed point methods (the Schauder theorem). As a second step we need to establish a priori estimates for the regularized solutions, namely to find uniform bounds for the total (kinetic and electro-magnetic) energy. First, by using the conservation laws of the mass and the energy we obtain as usual :

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} & f(t,x,p)(1+\mathcal{E}(p))\ dxdp + \frac{\varepsilon_0}{2}\frac{d}{dt}\int_\Omega (|E(t,x)|^2 + c_0^2|B(t,x)|^2)\ dx \\
& + \int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)(1+\mathcal{E}(p))\ d\sigma dp + \frac{\varepsilon_0 c_0}{2}\int_{\partial\Omega} (|n \wedge E|^2 + c_0^2 |n \wedge B|^2)\ d\sigma \\
& = \int_{\Sigma^-} |(v(p)\cdot n(x))| g(t,x,p)(1+\mathcal{E}(p))\ d\sigma dp + \frac{\varepsilon_0 c_0}{2}\int_\Omega |h(t,x)|^2\ dx, \ \ t \in \mathbb{R}_t.
\end{aligned} \tag{1.8}$$

If for the initial-boundary value problems the equation (1.8) provides immediately bounds for the total energy after integration on $[0,t]$, $t>0$, the situation is different for the time periodic case, since in this case initial data are not available. Nevertheless, after integration of (1.8) over a period we obtain :

$$\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)(1+\mathcal{E}(p))\;dtd\sigma dp+\frac{\varepsilon_0 c_0}{2}\int_0^T\int_{\partial\Omega}(|n\wedge E|^2+c_0^2|n\wedge B|^2)\;dtd\sigma$$
$$=\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)(1+\mathcal{E}(p))\;dtd\sigma dp+\frac{\varepsilon_0 c_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma. \quad (1.9)$$

In order to estimate the total energy we use also the momentum conservation law. We suppose that $\partial\Omega$ is strictly star-shaped with respect to some point $x_0\in\Omega$ *i.e.*, $\exists r>0$ such that $(n(x)\cdot(x-x_0))\geq r$, $\forall x\in\partial\Omega$, and we multiply the Vlasov equation by the test function $(p\cdot(x-x_0))$:

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(p\cdot(x-x_0))\;dxdp+\int_\Sigma(v(p)\cdot n(x))\gamma f(t,x,p)(p\cdot(x-x_0))\;d\sigma dp \quad (1.10)$$
$$=\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(v(p)\cdot p)\;dxdp+\int_\Omega\int_{\mathbb{R}^3_p} qf(t,x,p)(E+v(p)\wedge B)\cdot(x-x_0)\;dxdp.$$

By using the Maxwell equations the last integral in the above equation can be written :

$$\int_\Omega\int_{\mathbb{R}^3_p} qf(t,x,p)(E+v(p)\wedge B)\cdot(x-x_0)\;dxdp=\int_\Omega(\rho E+j\wedge B)\cdot(x-x_0)\;dx$$
$$=\varepsilon_0\int_\Omega[(E\mathrm{div}\;E-E\wedge\mathrm{rot}\;E)+c_0^2(B\mathrm{div}\;B-B\wedge\mathrm{rot}\;B)]\cdot(x-x_0)\;dx$$
$$-\varepsilon_0\int_\Omega\partial_t(E\wedge B)\cdot(x-x_0)\;dx. \quad (1.11)$$

We use also the identity $(u\mathrm{div}\;u-u\wedge\mathrm{rot}\;u)_i=\sum_{j=1}^3\frac{\partial}{\partial x_j}(u_iu_j)-\frac12\frac{\partial}{\partial x_i}|u|^2$, $1\leq i\leq 3$ and the inequality $(v(p)\cdot p)\geq\mathcal{E}(p)$, $\forall p\in\mathbb{R}^3_p$. After integration by parts and direct computations the equations (1.10), (1.11) yield :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f\mathcal{E}(p)\;dtdxdp+\frac{\varepsilon_0}{2}\int_0^T\int_\Omega(|E|^2+c_0^2|B|^2)\;dtdx+\frac{\varepsilon_0 r}{2}\int_0^T\int_{\partial\Omega}[(n\cdot E)^2+c_0^2(n\cdot B_0)^2]\;dtd\sigma$$
$$\leq R\int_0^T\int_\Sigma|(v(p)\cdot n(x))|\cdot|p|\gamma f\;dtd\sigma dp+\frac{\varepsilon_0 R}{2}\int_0^T\int_{\partial\Omega}[|n\wedge E|^2+c_0^2|n\wedge B|^2]\;dtd\sigma$$
$$+\varepsilon_0 R\int_0^T\int_{\partial\Omega}[|(n\cdot E)|\cdot|n\wedge E|+c_0^2|(n\cdot B)|\cdot|n\wedge B|]\;dtd\sigma, \quad (1.12)$$

where $R=\sup_{x\in\partial\Omega}|x-x_0|$. The estimate (1.12) together with (1.9) clearly give the desired bounds. There is another important point to be clarified : in the above computations we used the divergence equations $\mathrm{div}\;E=\frac{\rho}{\varepsilon_0}$, $\mathrm{div}\;B=0$. In the case of initial value problem these equations hold true as soon as they are verified by the initial data. In the time periodic case the idea is to regularize the Vlasov-Maxwell equations and to use the time periodicity. Indeed we consider T periodic solutions for the perturbed equations :

$$\alpha f+\partial_t f+v(p)\cdot\nabla_x f+q(E+v(p)\wedge B)\cdot\nabla_p f=0,\;\;(t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

$$\alpha E+\partial_t E-c_0^2\cdot\mathrm{rot}\;B=-\frac{1}{\varepsilon_0}j(t,x),\;\;\alpha B+\partial_t B+\mathrm{rot}\;E=0,\;\;(t,x)\in\mathbb{R}_t\times\Omega,$$

where $\alpha>0$ is a small parameter. Note that the continuity equation is in this case $\alpha\rho+\partial_t\rho+\mathrm{div}\;j=0$. As usual, by taking the divergence in the perturbed Maxwell equations we find $(\alpha+\partial_t)(\mathrm{div}\;E-$

$\frac{\rho}{\varepsilon_0}) = 0$, $(\alpha + \partial_t)\mathrm{div}\ B = 0$ and by time periodicity we deduce that div $E = \frac{\rho}{\varepsilon_0}$, div $B = 0$. Once we have obtained uniform estimates for the total energy of the solutions for the regularized Vlasov-Maxwell system, the existence of T periodic weak solution for the non perturbed Vlasov-Maxwell system follows easily by weak stability results (cf. [14]).

The content of this paper is organized as follows: first we recall some basic definitions and results concerning the Vlasov problem. The time periodic Maxwell equations are studied in section 3. In section 4 we prove the existence for the regularized Vlasov-Maxwell system by using a fixed point technique. In the next section we obtain the a priori estimates by using the conservation laws of the mass, momentum and energy. In section 6 we perform the passing to the limit for the sequence of regularized solutions. We end with some remarks concerning the system with specular boundary condition.

2. The Vlasov equation.

The Vlasov equation describes the evolution of a population of charged particles under the action of the electro-magnetic force. In this section we suppose that the electro-magnetic field is a given T periodic function (E, B). The time periodic Vlasov problem is given by :

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x,p) \cdot \nabla_p f = 0,\ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \tag{2.1}$$

$$f(t,x,p) = g(t,x,p),\ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-. \tag{2.2}$$

By taking into account that $\nabla_{(x,p)} \cdot (v(p), F(t,x,p)) = 0$, the equation (2.1) can be written also :

$$\partial_t f + \nabla_x \cdot (v(p) f) + \nabla_p \cdot (F(t,x,p) f) = 0,\ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p.$$

Since there is no uniqueness for the Vlasov problem (2.1), (2.2) (because the distribution function can take arbitrary constant values on the characteristics which remain in the domain), it is convenient to consider also the perturbed problem :

$$\alpha f + \partial_t f + v(p) \cdot \nabla_x f + F(t,x,p) \cdot \nabla_p f = 0,\ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \tag{2.3}$$

with the boundary condition (2.2), where $\alpha > 0$ is fixed. We introduce the definitions of weak/mild solution for the perturbed Vlasov problem :

Definition 2.1. *Assume that $E, B \in L^\infty(\mathbb{R}_t \times \Omega)^3$ and $(v(p) \cdot n(x))g \in L^1_{loc}(\mathbb{R}_t \times \Sigma^-)$ are T periodic. We say that $f \in L^1_{loc}(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$ is a T periodic weak solution for the perturbed Vlasov problem (2.3), (2.2) iff f is T periodic and :*

$$\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p) \cdot \nabla_x\varphi - F(t,x,p) \cdot \nabla_p\varphi) dt dx dp$$
$$= -\int_0^T\!\!\int_{\Sigma^-} (v(p) \cdot n(x)) g(t,x,p) \varphi(t,x,p) dt d\sigma dp, \tag{2.4}$$

for all test function which belongs to :

$$\mathcal{T}_w = \{\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p) \mid \exists R > 0 : \varphi = \varphi \cdot \mathbf{1}_{\{|p| \le R\}}, \varphi|_{\mathbb{R}_t \times \Sigma^+} = 0, \varphi(\cdot + T) = \varphi\}.$$

Remark 2.2. *In the above definition we can assume that E, B are only in $L^s(]0,T[\times\Omega)^3$ by requiring more regularity on f, namely $f \in L^r_{loc}(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, where s is the conjugate exponent of r.*

Suppose now that E, B are T periodic and belong to $L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$. In this case we can define the notion of solution by characteristics or mild solution. First of all let us introduce the characteristics : for $(t,x,p) \in \mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p$ we denote by $(X(s), P(s)) = (X(s;t,x,p), P(s;t,x,p))$ the unique solution of the system :

$$\frac{dX}{ds} = v(P(s;t,x,p)), \quad \frac{dP}{ds} = F(s, X(s;t,x,p), P(s;t,x,p)), \quad s_{in}(t,x,p) \le s \le s_{out}(t,x,p), \tag{2.5}$$

with the conditions $X(s=t;t,x,p) = x, P(s=t;t,x,p) = p$. Here s_{in}, s_{out} represent the incoming, respectively outgoing time given by :

$$s_{in}(t,x,p) = \sup\{s \le t \mid X(s;t,x,p) \in \partial\Omega\},$$

$$s_{out}(t,x,p) = \inf\{s \ge t \mid X(s;t,x,p) \in \partial\Omega\}.$$

The mild formulation follows formally by solving :

$$\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi = \psi(t,x,p), \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p,$$

with the boundary condition $\varphi|_{\mathbb{R}_t\times\Sigma^+} = 0$. By integration along the characteristic curves we obtain :

$$\varphi^\alpha_\psi(t,x,p) = \int_t^{s_{out}(t,x,p)} e^{-\alpha(s-t)}\psi(s, X(s;t,x,p), P(s;t,x,p))ds,$$

and we define the mild solution by :

Definition 2.3. *Assume that $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ and $(v(p)\cdot n(x))g \in L^1_{loc}(\mathbb{R}_t \times \Sigma^-)$ are T periodic, $\alpha > 0$. We say that $f \in L^1_{loc}(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$ is a T periodic mild solution for the perturbed Vlasov problem* (2.3), (2.2) *iff f is T periodic and :*

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi^\alpha_\psi(t,x,p)dtd\sigma dp, \tag{2.6}$$

for all test function which belongs to :

$$\mathcal{T}_m = \{\psi \in C^0(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p) \mid \exists R > 0 : \psi = \psi \cdot \mathbf{1}_{\{|p|\le R\}}, \ \ \psi(\cdot + T) = \psi\}.$$

For $\alpha = 0$ one gets the definitions of the weak/mild solution for the Vlasov problem (2.1), (2.2). The existence of the T periodic mild solution is a standard result and follows by change of variables along characteristics (see also the *Remark* 2.6).

Proposition 2.4. *Assume that $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ and $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ are T periodic, $\alpha > 0$. Then the perturbed Vlasov problem* (2.3), (2.2) *has an unique T periodic mild solution $f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, verifying $\|f\|_\infty \le \|g\|_\infty$. Moreover, if $g \ge 0$ then $f \ge 0$.*

Remark 2.5. *It is easy to check that all T periodic mild solution is also T periodic weak solution.*

Remark 2.6. *It is well known that the T periodic mild solution is given by $f(t,x,p) = e^{-\alpha(t - s_{in}(t,x,p))}g(s_{in}, X(s_{in};t,x,p), P(s_{in};t,x,p))$ if $s_{in}(t,x,p) > -\infty$ and $f(t,x,p) = 0$ otherwise.*

Remark 2.7. *Under the same hypothesis as in Proposition* 2.4, *the T periodic mild solution f has a trace $\gamma^+ f \in L^\infty(\mathbb{R}_t \times \Sigma^+)$ verifying the following Green formula :*

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp \tag{2.7}$$

$$= -\int_0^T\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp - \int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)\varphi(t,x,p)dtd\sigma dp,$$

for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$ *with compact support in momentum and T periodic in time. The trace* $\gamma^+ f$ *is given by the same formula as in the Remark* 2.6 *and we have* $\|\gamma^+ f\|_\infty \le \|g\|_\infty$*. Moreover, if* $g \ge 0$ *then* $\gamma^+ f \ge 0$.

Remark 2.8. *Under the same hypothesis as in Proposition* 2.4*, a T periodic bounded weak solution of the perturbed Vlasov problem* (2.3), (2.2) *is unique and therefore coincide with the T periodic mild solution.*

Proof. Assume that $f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$ is a T periodic weak solution with boundary data $g = 0$. We have $\partial_t f + v(p) \cdot \nabla_x f + F(t,x,p) \cdot \nabla_p f = -\alpha f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$ and therefore (cf. [4], [15]) we obtain :

$$\frac{1}{2} \cdot (\partial_t f^2 + v(p) \cdot \nabla_x f^2 + F \cdot \nabla_p f^2) = -\alpha f^2.$$

After integration on $]0,T[\times \Omega \times \mathbb{R}^3_p$ we deduce that :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f^2(t,x,p)\; dtdxdp + \frac{1}{2}\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(\gamma^+ f)^2(t,x,p)\; dtd\sigma dp = 0,$$

or $f = 0$, $\gamma^+ f = 0$.
□

Remark 2.9. *Under the same hypothesis as in Proposition* 2.4*, with* $g \ge 0$, $\int_0^T\!\int_{\Sigma^-} |(v(p) \cdot n(x))| g(t,x,p)\; dtd\sigma dp < +\infty$*, the T periodic mild/weak solution belongs to* $L^1(]0,T[\times\Omega\times\mathbb{R}^3_p)$ *and*

$$\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\; dtdxdp \le \frac{1}{\alpha}\int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g(t,x,p)\; dtd\sigma dp.$$

Proof. By applying the mild formulation with the test function $\psi(t,x,p) = \chi_R(|p|)$ where $\chi_R(u) = \chi(\frac{u}{R})$, $\chi \in C_c(\mathbb{R})$, $\chi(u) = 1$ if $|u| \le 1$, $\chi(u) = 0$ if $|u| \ge 2$ and $0 \le \chi \le 1$, we deduce that $0 \le \varphi^\alpha_\psi \le \frac{1}{\alpha}$ and therefore :

$$\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f \mathbf{1}_{\{|p|\le R\}}\; dtdxdp \le \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f\psi\; dtdxdp \le \frac{1}{\alpha}\int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g\; dtd\sigma dp, \forall R > 0.$$

The conclusion follows by letting $R \to +\infty$ and by using the monotone convergence theorem.
□

3. The Maxwell equations.

In this section we assume that the charge and current densities $\rho = q \cdot \int_{\mathbb{R}^3_p} f\; dp$, $j = q \cdot \int_{\mathbb{R}^3_p} v(p) f\; dp$ are given T periodic functions and we study the time periodic Maxwell equations :

$$\partial_t E - c_0^2 \cdot \mathrm{rot}\; B = -\frac{1}{\varepsilon_0} j(t,x),\;\; \partial_t B + \mathrm{rot}\; E = 0,\;\; (t,x) \in \mathbb{R}_t \times \Omega. \tag{3.1}$$

On the boundary we impose the Silver-Müller condition :

$$n(x) \wedge E(t,x) + c_0 \cdot n(x) \wedge (n(x) \wedge B(t,x)) = h(t,x),\;\; (t,x) \in \mathbb{R}_t \times \partial\Omega. \tag{3.2}$$

In order to be consistent to the perturbed Vlasov problem we consider also the perturbed Maxwell equations :

$$\alpha \cdot E(t,x) + \partial_t E - c_0^2 \cdot \mathrm{rot}\; B = -\frac{1}{\varepsilon_0} j(t,x),\;\; \alpha \cdot B(t,x) + \partial_t B + \mathrm{rot}\; E = 0,\;\; (t,x) \in \mathbb{R}_t \times \Omega, \tag{3.3}$$

where $\alpha > 0$ is fixed. Let us introduce the standard Hilbert spaces $H(\text{rot}\,;\Omega), H(\text{div}\,;\Omega)$ defined by :

$$H(\text{rot}\,;\Omega) = \{u \in L^2(\Omega)^3 \mid \text{rot}\; u \in L^2(\Omega)^3\},\;\; H(\text{div}\,;\Omega) = \{u \in L^2(\Omega)^3 \mid \text{div}\; u \in L^2(\Omega)\},$$

endowed with the norms :

$$\left(\|u\|^2_{L^2(\Omega)^3} + \|\text{rot}\; u\|^2_{L^2(\Omega)^3}\right)^{1/2}, \text{ respectively } \left(\|u\|^2_{L^2(\Omega)^3} + \|\text{div}\; u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

It is well known that $C^1(\overline{\Omega})^3$ is dense in $H(\text{rot}\,;\Omega)$ and $H(\text{div}\,;\Omega)$ (see [16]). The application $\varphi \in C^1(\overline{\Omega})^3 \to n\wedge\varphi \in C^1(\partial\Omega)^3$ extends by continuity to a continuous linear map $n\wedge : H(\text{rot}\,;\Omega) \to \left(H^{1/2}(\partial\Omega)^3\right)' = H^{-1/2}(\partial\Omega)^3$ such that :

$$\int_\Omega \text{rot}\; u\cdot\Phi\; dx - \int_\Omega u\cdot \text{rot}\;\Phi\; dx = \langle n\wedge u, \varphi\rangle_{H^{-1/2}(\partial\Omega)^3, H^{1/2}(\partial\Omega)^3},$$

for all functions $u \in H(\text{rot}\,;\Omega)$, $\Phi \in H^1(\Omega)^3$, $\varphi = \Phi|_{\partial\Omega} \in H^{1/2}(\partial\Omega)^3$. By the other hand, the application $\varphi \in C^1(\overline{\Omega})^3 \to n\cdot\varphi \in C^1(\partial\Omega)$ extends by continuity to a continuous linear map $n\cdot : H(\text{div}\,;\Omega) \to \left(H^{1/2}(\partial\Omega)\right)' = H^{-1/2}(\partial\Omega)$ such that :

$$\int_\Omega \text{div}\; u\;\Phi\; dx + \int_\Omega u\cdot \text{grad}\;\Phi\; dx = \langle n\cdot u, \varphi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)},$$

for all functions $u \in H(\text{div}\,;\Omega)$, $\Phi \in H^1(\Omega), \varphi = \Phi|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. We note $\mathcal{H} = L^2(\Omega)^6$ endowed with the norm $\left(\|E\|^2_{L^2(\Omega)^3} + c_0^2\cdot\|B\|^2_{L^2(\Omega)^3}\right)^{1/2}$ and define $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ given by :

$$D(\mathcal{A}) = \{(E,B) \in \mathcal{H} \mid \text{rot}\; E, \text{rot}\; B \in L^2(\Omega)^3, n\wedge E, n\wedge B \in L^2(\partial\Omega)^3, n\wedge E + c_0 n\wedge(n\wedge B)|_{\partial\Omega} = 0\},$$

and :

$$\mathcal{A}(E,B) = (-c_0^2\cdot \text{rot}\; B, \text{rot}\; E),\;\; \forall (E,B) \in D(\mathcal{A}).$$

We check by direct computations that for $(E,B) \in D(\mathcal{A})$ we have :

$$\langle \mathcal{A}(E,B),(E,B)\rangle = -c_0^2\int_\Omega \{\text{rot}\; B\cdot E - \text{rot}\; E\cdot B\}\; dx = c_0\int_{\partial\Omega} |n\wedge E|^2\; d\sigma = c_0^3\int_{\partial\Omega}|n\wedge B|^2\; d\sigma \geq 0.$$

We have the following result (see [26]) :

Proposition 3.1. *The operator $\mathcal{A}$ is maximal monotone.*

Proposition 3.2. *The adjoint of the unbounded operator $\mathcal{A}$ is given by $\mathcal{A}^\star : D(\mathcal{A}^\star) \subset \mathcal{H} \to \mathcal{H}$ where :*

$$D(\mathcal{A}^\star) = \{(E,B) \in \mathcal{H} \mid rot\; E, rot\; B \in L^2(\Omega)^3, n\wedge E, n\wedge B \in L^2(\partial\Omega)^3, n\wedge E - c_0 n\wedge(n\wedge B)|_{\partial\Omega} = 0\},$$

and :

$$\mathcal{A}^\star(E,B) = (c_0^2\cdot rot\; B, -rot\; E),\;\; \forall (E,B) \in D(\mathcal{A}^\star).$$

Remark 3.3. *By direct computation we check that for $(E,B) \in D(\mathcal{A}^\star)$ we have :*

$$\langle \mathcal{A}^\star(E,B),(E,B)\rangle = c_0\int_{\partial\Omega}|n\wedge E|^2\; d\sigma \geq 0,$$

and as before we can prove that $\mathcal{A}^\star$ is maximal monotone (see also [11], p.113).

In order to solve the perturbed periodic Maxwell equations we establish an easy existence and uniqueness result for general perturbed evolution equations. We will need the following lemma :

LEMMA 3.4. *Assume that $g \in L^1_{loc}(\mathbb{R}_t;\mathbb{R})$ is a T periodic function and $\alpha > 0$ is fixed. Then :*

$$\left|\int_0^t e^{-\alpha(t-s)} g(s)\, ds\right| \le \left(\frac{1}{\alpha T} + 4\right) \|g\|_{L^1(]0,T[)}, \ \ \forall t \ge 0.$$

Proof. Consider $G : [0,+\infty[\to \mathbb{R},\ G(t) = \int_0^t \{g(s) - \langle g\rangle\}\ ds$, where $\langle g\rangle := \frac{1}{T}\int_0^T g(t)\ dt$. Obviously G is T periodic and bounded $|G(t)| \le 2\cdot \|g\|_{L^1(]0,T[)},\ \ \forall t \ge 0$. We have :

$$\begin{aligned}\int_0^t e^{-\alpha(t-s)} g(s)\, ds =& \int_0^t e^{-\alpha(t-s)} (g(s) - \langle g\rangle)\, ds + \langle g\rangle \cdot \int_0^t e^{-\alpha(t-s)}\, ds\\ =& \int_0^t e^{-\alpha(t-s)} G'(s)\, ds + \alpha^{-1}(1-e^{-\alpha t})\langle g\rangle\\ =& G(t) + \alpha^{-1}(1-e^{-\alpha t})\langle g\rangle - \alpha\int_0^t e^{-\alpha(t-s)} G(s)\, ds.\end{aligned}$$

Finally we deduce that :

$$\begin{aligned}\left|\int_0^t e^{-\alpha(t-s)} g(s)\, ds\right| \le& |G(t)| + \alpha^{-1}|\langle g\rangle| + \|G\|_{L^\infty}\cdot\int_0^t \alpha e^{-\alpha(t-s)}\, ds\\ \le& \left(\frac{1}{\alpha T} + 4\right)\|g\|_{L^1(]0,T[)}.\end{aligned} \tag{3.4}$$

□

PROPOSITION 3.5. *Assume that $A : D(A) \subset H \to H$ is a linear maximal monotone operator on a Hilbert space H, $f \in C^1(\mathbb{R}_t;H)$ is T periodic, $\alpha > 0$ fixed. Thus there is an unique T periodic solution $\xi \in C(\mathbb{R}_t;D(A)) \cap C^1(\mathbb{R}_t;H)$ for the perturbed evolution equation :*

$$\alpha\cdot x(t) + x' + Ax(t) = f(t),\ \ t \in \mathbb{R}_t. \tag{3.5}$$

Moreover, we have the following estimates :

$$\|\xi\|_{L^\infty(\mathbb{R}_t;H)} \le \left(\frac{1}{\alpha T} + 4\right)\|f\|_{L^1(]0,T[;H)},\ \ \|\xi'\|_{L^\infty(\mathbb{R}_t;H)} \le \left(\frac{1}{\alpha T} + 4\right)\|f'\|_{L^1(]0,T[;H)}.$$

Proof. Consider an arbitrary $x_0 \in D(A)$ and denote by $x(\cdot;0,x_0) \in C([0,+\infty[;D(A)) \cap C^1([0,+\infty[;H)$ the unique solution of (3.5) with the initial condition x_0. We have :

$$\alpha(x(t+T) - x(t)) + x'(t+T) - x'(t) + Ax(t+T) - Ax(t) = 0,\ \ t \ge 0.$$

After multiplication by $x(t+T) - x(t)$, by using the monotonicity of A we obtain :

$$\frac{d}{dt}\{e^{2\alpha t}\|x(t+T) - x(t)\|^2\} \le 0,\ \ t \ge 0,$$

which implies that $\|x(t+T) - x(t)\| \le e^{-\alpha t}\|x(T) - x(0)\| \le e^{-\alpha t}(2\|x_0\| + \|f\|_{L^1(]0,T[;H)})$. If we denote by $(x_n)_n$ the functions $x_n(t) = x(nT+t),\ 0 \le t \le T,\ n \ge 0$ we deduce that :

$$\|x_{n+1}(t) - x_n(t)\| \le e^{-\alpha(nT+t)}(2\|x_0\| + \|f\|_{L^1(]0,T[;H)}) \le e^{-\alpha nT}(2\|x_0\| + \|f\|_{L^1(]0,T[;H)}),$$

and therefore $x_n \to \xi$ in $C([0,T];H)$. We have $\xi(T) = \lim_{n\to+\infty} x_n(T) = \lim_{n\to+\infty} x(nT+T) = \lim_{n\to+\infty} x_{n+1}(0) = \xi(0)$. With the notation $y_h(t) = x(t+h) - x(t)$, we have :

$$\alpha(y_h(t+T) - y_h(t)) + y_h'(t+T) - y_h'(t) + Ay_h(t+T) - Ay_h(t) = 0, \ \ t \geq 0.$$

After multiplication by $y_h(t+T) - y_h(t)$ we deduce as before that $\frac{1}{h}\|y_h(t+T) - y_h(t)\| \leq e^{-\alpha t}\frac{1}{h}\|y_h(T) - y_h(0)\|$, $h > 0, \ t \geq 0$ and by passing $h \searrow 0$ we obtain :

$$\|x'(t+T) - x'(t)\| \leq e^{-\alpha t}\|x'(T) - x'(0)\| \leq e^{-\alpha t}(2\|x'(0)\| + \|f'\|_{L^1(]0,T[;H)}).$$

In particular $\|x_{n+1}'(t) - x_n'(t)\| \leq e^{-\alpha nT}(2\|x'(0)\| + \|f'\|_{L^1(]0,T[;H)})$ and therefore $x_n' \to \eta$ in $C([0,T];H)$. Now, by taking into account that A is closed and $[x_n(t), f(t) - \alpha x_n(t) - x_n'(t)] \in A, \ 0 \leq t \leq T, n \geq 0$ we find by passing to the limit for $n \to +\infty$ that $[\xi(t), f(t) - \alpha\xi(t) - \eta(t)] \in A, \ 0 \leq t \leq T$ or $\xi(t) \in D(A)$ and $\alpha\xi(t) + \eta(t) + A\xi(t) = f(t), \ 0 \leq t \leq T$. It is easy to check that $\eta = \xi'$ and thus $\xi \in C([0,T];D(A)) \cap C^1([0,T];H)$ is a T periodic solution for (3.5). In order to estimate ξ observe that :

$$\alpha\|x(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \|f(t)\| \cdot \|x(t)\|, \ \ t \geq 0,$$

which implies by using the Bellman lemma that :

$$\|x(t)\| \leq e^{-\alpha t}\|x(0)\| + \int_0^t e^{-\alpha(t-s)}\|f(s)\| \, ds.$$

From the *Lemma* 3.4 we deduce that $\|x_n(t)\| \leq e^{-\alpha(nT+t)}\|x_0\| + \left(\frac{1}{\alpha T} + 4\right)\|f\|_{L^1(]0,T[;H)}$, or $\|\xi\|_{L^\infty(\mathbb{R}_t;H)} \leq (\frac{1}{\alpha T} + 4)\|f\|_{L^1(]0,T[;H)}$. In order to estimate ξ' we write $\alpha y_h(t) + y_h'(t) + Ay_h(t) = f(t+h) - f(t)$ and as before we deduce that for $h > 0, t \geq 0$:

$$\frac{1}{h}\|y_h(t)\| \leq e^{-\alpha t}\frac{1}{h}\|y_h(0)\| + \int_0^t e^{-\alpha(t-s)}\frac{1}{h}\|f(s+h) - f(s)\| \, ds.$$

By passing to the limit for $h \searrow 0$ one gets :

$$\|x'(t)\| \leq e^{-\alpha t}\|x'(0)\| + \int_0^t e^{-\alpha(t-s)}\|f'(s)\| \, ds,$$

and thus by using the *Lemma* 3.4 finally we find that $\|\xi'\|_{L^\infty(\mathbb{R}_t;H)} \leq (\frac{1}{\alpha T} + 4)\|f'\|_{L^1(]0,T[;H)}$. The uniqueness of the periodic solution follows easily by standard arguments : consider ξ_1, ξ_2 two periodic solutions. We have as before that :

$$\alpha \cdot \|\xi_1(t) - \xi_2(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\xi_1(t) - \xi_2(t)\|^2 + (A\xi_1(t) - A\xi_2(t), \xi_1(t) - \xi_2(t)) = 0.$$

After integration on $[0,T]$ one gets that $\alpha \cdot \int_0^T \|\xi_1(t) - \xi_2(t)\|^2 \, dt \leq 0$, or $\xi_1 = \xi_2$. □

REMARK 3.6. *The previous result holds for $f \in W^{1,1}_{loc}(\mathbb{R}_t;H)$, T periodic (see also [3], p. 138).*

Proof. Approximate f by f_ε in $W^{1,1}_{loc}(\mathbb{R}_t;H)$ with $f_\varepsilon \in C^1(\mathbb{R}_t;H)$, T periodic and remark that the corresponding solutions $(x_\varepsilon)_{\varepsilon>0}$ converge in $C^1(\mathbb{R}_t;H)$. □

REMARK 3.7. *The equation $\alpha \cdot x(t) - x'(t) + Ax(t) = f(t), \ t \in \mathbb{R}_t$ has also an unique T periodic solution verifying the same estimates (take $x(t) = y(T-t), 0 \leq t \leq T$, where y solves $\alpha \cdot y + y' + Ay = \tilde{f}$, with $\tilde{f}(t) = f(T-t), 0 \leq t \leq T$).*

By transposition we can define the notion of T periodic weak solution as follows :

DEFINITION 3.8. *Assume that* $A : D(A) \subset H \to H$ *is a linear operator densely defined on a Hilbert space,* $f \in L^1_{loc}(\mathbb{R}_t; H)$*,* T *periodic. We say that* $x \in C(\mathbb{R}_t; H)$ *is a* T *periodic weak solution of* (3.5) *iff* x *is* T *periodic and :*

$$\int_0^T \langle x(t), \alpha\varphi(t) - \varphi'(t) + A^\star\varphi(t)\rangle \, dt = \int_0^T \langle f(t), \varphi(t)\rangle \, dt,$$

for all $\varphi \in C(\mathbb{R}_t; D(A^\star)) \cap C^1(\mathbb{R}_t; H)$*,* T *periodic.*

REMARK 3.9. *If* $A : D(A) \subset H \to H$ *is a linear maximal monotone operator on a Hilbert space,* $f \in L^1_{loc}(\mathbb{R}_t; H)$ *is* T *periodic,* $\alpha > 0$ *is fixed, then there is an unique* T *periodic weak solution* $\xi \in C(\mathbb{R}_t; H)$ *of* (3.5) *verifying* $\|\xi\|_{L^\infty(\mathbb{R}_t;H)} \le (\frac{1}{\alpha T} + 4)\|f\|_{L^1(]0,T[;H)}$.

Proof. Consider $f_n \in C^1(\mathbb{R}_t; H)$, T periodic, such that $\lim_{n\to+\infty} f_n = f$ in $L^1(]0, T[; H)$ and denote by $x_n \in C(\mathbb{R}_t; D(A)) \cap C^1(\mathbb{R}_t; H)$ the corresponding strong solution. By the *Proposition* 3.5 we have that $\|x_n - x_m\|_{L^\infty(\mathbb{R}_t;H)} \le \left(\frac{1}{\alpha T} + 4\right)\|f_n - f_m\|_{L^1(]0,T[;H)}$ and thus $(x_n)_n$ converges to some T periodic function $\xi \in C(\mathbb{R}_t; H)$ such that $\|\xi\|_{L^\infty(\mathbb{R}_t;H)} \le \left(\frac{1}{\alpha T} + 4\right)\|f\|_{L^1(]0,T[;H)}$. If $\varphi \in C(\mathbb{R}_t; D(A^\star)) \cap C^1(\mathbb{R}_t; H)$ is T periodic, we have for all n :

$$\int_0^T \langle x_n(t), \alpha\varphi(t) - \varphi'(t) + A^\star\varphi(t)\rangle \, dt = \int_0^T \langle f_n(t), \varphi(t)\rangle \, dt.$$

By passing to the limit in respect to n we deduce that ξ is a T periodic weak solution. In order to prove the uniqueness, consider x_1, x_2 two T periodic weak solutions and therefore we have :

$$\int_0^T \langle x_1(t) - x_2(t), \alpha\varphi(t) - \varphi'(t) + A^\star\varphi(t)\rangle \, dt = 0, \ \forall \varphi \in C(\mathbb{R}_t; D(A^\star)) \cap C^1(\mathbb{R}_t; H), T \text{ periodic} .$$

In particular we deduce that $\int_0^T \langle x_1(t) - x_2(t), g(t)\rangle \, dt = 0$, $\forall g \in C^1(\mathbb{R}_t; H)$, T periodic (take φ the strong T periodic solution for $\alpha\varphi(t) - \varphi' + A^\star\varphi(t) = g(t), \ t \in \mathbb{R}_t$). It follows by density that $\int_0^T \langle x_1(t) - x_2(t), g(t)\rangle \, dt = 0$, $\forall g \in L^1_{loc}(\mathbb{R}_t; H)$, T periodic, or $x_1 = x_2$. □

We consider also the equation :

$$\alpha \cdot (\varphi(t), \psi(t)) - \frac{d}{dt}(\varphi, \psi) + \mathcal{A}^\star(\varphi(t), \psi(t)) = (f(t), g(t)), \ \ t \in \mathbb{R}_t. \tag{3.6}$$

PROPOSITION 3.10. *Assume that* $(f, g) \in C^1(\mathbb{R}_t; \mathcal{H})$ *is* T *periodic,* $\alpha > 0$ *fixed. Then there is an unique* T *periodic solution* $(\varphi, \psi) \in C(\mathbb{R}_t; D(\mathcal{A}^\star)) \cap C^1(\mathbb{R}_t; \mathcal{H})$ *for the equation* (3.6) *which verifies the estimates :*

$$\|(\varphi, \psi)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \le \left(\frac{1}{\alpha T} + 4\right) \|(f, g)\|_{L^1(]0,T[;\mathcal{H})},$$

and :

$$c_0^{1/2}\|n \wedge \varphi\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} = c_0^{3/2}\|n \wedge \psi\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} \le \left(\frac{1}{\alpha T} + 4\right)^{1/2} \|(f, g)\|_{L^1(]0,T[;\mathcal{H})}.$$

Proof. The existence and uniqueness of the T periodic solution as well as the first estimate follow by the *Proposition* 3.5, the *Remarks* 3.3, 3.7. By the other hand observe that :

$$\langle(\varphi, \psi), \mathcal{A}^\star(\varphi, \psi)\rangle = c_0 \int_{\partial\Omega} |n \wedge \varphi(t)|^2 \, d\sigma = c_0^3 \int_{\partial\Omega} |n \wedge \psi(t)|^2 \, d\sigma.$$

After integration on $]0,T[$, equation (3.6) gives :

$$\alpha \cdot \|(\varphi,\psi)\|^2_{L^2(]0,T[;\mathcal{H})} + c_0 \cdot \|n\wedge\varphi\|^2_{L^2(]0,T[;L^2(\partial\Omega)^3)} \leq \int_0^T \|(f(t),g(t))\|_{\mathcal{H}} \cdot \|(\varphi(t),\psi(t))\|_{\mathcal{H}}\, dt$$
$$\leq \|(f,g)\|_{L^1(]0,T[;\mathcal{H})} \cdot \|(\varphi,\psi)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq \left(\frac{1}{\alpha T}+4\right) \cdot \|(f,g)\|^2_{L^1(]0,T[;\mathcal{H})},$$

and thus the second estimate follows.
□

REMARK 3.11. *If $(f,g)\in L^1_{loc}(\mathbb{R}_t;\mathcal{H})$ is T periodic, then the T periodic weak solution of the equation* (3.6) *verifies the same estimates as in the Proposition* 3.10.

We can prove now the existence and uniqueness of the T periodic solution for the perturbed Maxwell equations. Let us start with a result for strong solutions.

PROPOSITION 3.12. *Assume that $j\in C^1(\mathbb{R}_t;L^2(\Omega)^3)$ is T periodic and that there is $\tilde{h}\in C^1(\mathbb{R}_t;H^1(\Omega)^3)\cap C^2(\mathbb{R}_t;L^2(\Omega)^3)$ T periodic such that $n\wedge\tilde{h}|_{\mathbb{R}_t\times\partial\Omega}=h$. Then for $\alpha>0$ fixed there is an unique T periodic solution $(E,B)\in C(\mathbb{R}_t;H(rot\,;\Omega)^2)\cap C^1(\mathbb{R}_t;\mathcal{H})$ for the perturbed Maxwell problem* (3.3), (3.2). *Moreover we have :*

$$\alpha\cdot\int_0^T\!\!\int_\Omega \{|E(t,x)|^2 + c_0^2\cdot|B(t,x)|^2\}\,dtdx + \frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega}\{|n\wedge E(t,x)|^2 + c_0^2\cdot|n\wedge B(t,x)|^2\}\,dtd\sigma$$
$$= -\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot E(t,x)\,dtdx + \frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\,dtd\sigma, \tag{3.7}$$

$$\|(E,B)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq c_0^{1/2}\left(\frac{1}{\alpha T}+4\right)^{1/2}\|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} + \left(\frac{1}{\alpha T}+4\right)\frac{\|j\|_{L^1(]0,T[;L^2(\Omega)^3)}}{\varepsilon_0}, \tag{3.8}$$

and :

$$\int_0^T \langle (E(t),B(t)),(f(t),g(t))\rangle_{\mathcal{H}} dt = c_0\int_0^T\!\!\int_{\partial\Omega}(n\wedge\varphi)h\,dtd\sigma - \frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot\varphi(t,x)\,dtdx, \tag{3.9}$$

for all $(f,g)\in L^1_{loc}(\mathbb{R}_t;\mathcal{H})$, T periodic, where (φ,ψ) is the T periodic weak solution for (3.6).

Proof. In order to prove the uniqueness consider $(E_1,B_1),(E_2,B_2)$ two periodic solutions and observe that, since $n\wedge(E_1-E_2)+c_0 n\wedge(n\wedge(B_1-B_2))|_{\mathbb{R}_t\times\partial\Omega}=h-h=0$, then $(E,B)=(E_1-E_2,B_1-B_2)\in C(\mathbb{R}_t;D(\mathcal{A}))\cap C^1(\mathbb{R}_t;\mathcal{H})$ is also T periodic solution for the perturbed evolution equation $\alpha(E,B)+\frac{d}{dt}(E,B)+\mathcal{A}(E,B)=0$. After multiplication by (E,B) and integration on $]0,T[$ we deduce that $(E,B)=(0,0)$. In order to prove the existence let us take $(E_1,B_1)\in C(\mathbb{R}_t;D(\mathcal{A}))\cap C^1(\mathbb{R}_t;\mathcal{H})$ the unique T periodic solution for :

$$\alpha\cdot(E_1,B_1)+\frac{d}{dt}(E_1,B_1)+\mathcal{A}(E_1,B_1) = \left(-\frac{j}{\varepsilon_0}-\alpha\tilde{h}-\frac{d}{dt}\tilde{h}, -\text{rot}\,\tilde{h}\right)\in C^1(\mathbb{R}_t;\mathcal{H}).$$

We verify that $(E,B)=(E_1+\tilde{h},B)$ is a T periodic solution for (3.3), (3.2) with $(n\wedge E(t), n\wedge B(t))\in L^2(\partial\Omega)^6$. By multiplying (3.3) by (E,B) and by using (3.2) we obtain :

$$\alpha\cdot\|(E,B)\|^2 + \frac{1}{2}\frac{d}{dt}\|(E,B)\|^2 + c_0\int_{\partial\Omega}(n\wedge E)\cdot(n\wedge E-h)\,d\sigma = -\frac{1}{\varepsilon_0}\int_\Omega j(t,x)\cdot E(t,x)\,dx.$$

Using again (3.2) we find that $(n\wedge E)\cdot(n\wedge E-h)=\frac{1}{2}(|n\wedge E|^2+c_0^2\cdot|n\wedge B|^2-|h|^2)$ and therefore:

$$\alpha\cdot\int_\Omega\{|E(t,x)|^2+c_0^2\cdot|B(t,x)|^2\}\;dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\{|E(t,x)|^2+c_0^2\cdot|B(t,x)|^2\}\;dx$$
$$+\frac{c_0}{2}\int_{\partial\Omega}\{|n\wedge E(t,x)|^2+c_0^2|n\wedge B(t,x)|^2\}\;d\sigma$$
$$=-\frac{1}{\varepsilon_0}\int_\Omega j(t,x)\cdot E(t,x)\;dx+\frac{c_0}{2}\int_{\partial\Omega}|h(t,x)|^2\;d\sigma. \quad (3.10)$$

Finally, after integration on $]0,T[$ we deduce :

$$\alpha\cdot\int_0^T\!\!\int_\Omega\{|E(t,x)|^2+c_0^2\cdot|B(t,x)|^2\}\;dtdx+\frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega}\{|n\wedge E(t,x)|^2+c_0^2\cdot|n\wedge B(t,x)|^2\}\;dtd\sigma$$
$$=-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot E(t,x)\;dtdx+\frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma.$$

For $(f,g)\in C^1(\mathbb{R}_t;\mathcal{H})$, T periodic, consider (φ,ψ) the unique T periodic solution of (3.6). After multiplication by (φ,ψ) of the perturbed Maxwell equations and integration on $]0,T[$ we find that :

$$\int_0^T\langle(E(t),B(t)),(f(t),g(t))\rangle_{\mathcal{H}}\;dt=\int_0^T\langle(E(t),B(t)),\alpha\cdot(\varphi(t),\psi(t))-\frac{d}{dt}(\varphi,\psi)+\mathcal{A}^\star(\varphi(t),\psi(t))\rangle_{\mathcal{H}}\;dt$$
$$=c_0\cdot\int_0^T\!\!\int_{\partial\Omega}(n\wedge\varphi)\cdot h(t,x)\;dtd\sigma-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot\varphi(t,x)\;dtdx.$$

By using the *Remarks* 3.9, 3.11 we verify easily that the previous equality holds for $(f,g)\in L^1_{loc}(\mathbb{R}_t;\mathcal{H})$, T periodic, with (φ,ψ) the associated T periodic weak solution. By the *Proposition* 3.10 we deduce that for all $(f,g)\in L^1_{loc}(\mathbb{R}_t;\mathcal{H})$, T periodic, we have :

$$\left|\int_0^T\langle(E(t),B(t)),(f(t),g(t))\rangle_{\mathcal{H}}\;dt\right|\le c_0^{1/2}\left(\frac{1}{\alpha T}+4\right)^{1/2}\|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)}\|(f,g)\|_{L^1(]0,T[;\mathcal{H})}$$
$$+\frac{1}{\varepsilon_0}\left(\frac{1}{\alpha T}+4\right)\|j\|_{L^1(]0,T[;L^2(\Omega)^3)}\|(f,g)\|_{L^1(]0,T[;\mathcal{H})},$$

and thus the estimate (3.8) of our proposition follows.
□

REMARK 3.13. *In particular the solution constructed above verifies :*

$$\int_0^T\!\!\int_\Omega\{E(t,x)\cdot(\alpha\varphi-\partial_t\varphi)-c_0^2B(t,x)\cdot rot\;\varphi\}\;dtdx-c_0^2\int_0^T\!\!\int_{\partial\Omega}(n\wedge B)\cdot\varphi\;dtd\sigma=-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j\cdot\varphi\;dtdx,$$

and :

$$\int_0^T\!\!\int_\Omega\{B(t,x)\cdot(\alpha\psi-\partial_t\psi)+E(t,x)\cdot rot\;\psi\}\;dtdx+\int_0^T\!\!\int_{\partial\Omega}(n\wedge E)\cdot\psi\;dtd\sigma=0,$$

for all $\varphi,\psi\in C^1(\mathbb{R}_t\times\overline{\Omega})^3$, *$T$ periodic.*

DEFINITION 3.14. *Assume that* $j\in L^1_{loc}(\mathbb{R}_t;L^2(\Omega)^3)$ *and* $h\in L^2_{loc}(\mathbb{R}_t;L^2(\partial\Omega)^3)$ *are T periodic,* $(n\cdot h)|_{\mathbb{R}_t\times\partial\Omega}=0$, $\alpha>0$ *fixed. We say that* $(E,B)\in C(\mathbb{R}_t;\mathcal{H})$ *is a T periodic weak solution for the perturbed Maxwell problem* (3.3), (3.2) *iff* (E,B) *is T periodic and :*

$$\int_0^T\langle(E(t),B(t)),\alpha(\varphi(t),\psi(t))-\frac{d}{dt}(\varphi,\psi)+\mathcal{A}^\star(\varphi(t),\psi(t))\rangle_{\mathcal{H}}\;dt=c_0\int_0^T\!\!\int_{\partial\Omega}(n\wedge\varphi)\cdot h(t,x)\;dtd\sigma$$
$$-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot\varphi(t,x)\;dtdx,$$

for all $(\varphi, \psi) \in C(\mathbb{R}_t; D(\mathcal{A}^\star)) \cap C^1(\mathbb{R}_t; \mathcal{H})$, *$T$ periodic.*

Proposition 3.15. *Assume that* $j \in L^1_{loc}(\mathbb{R}_t; L^2(\Omega)^3)$ *and* $h \in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$ *are T periodic,* $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$. *Then, for* $\alpha > 0$ *fixed there is an unique T periodic weak solution for the perturbed Maxwell problem* (3.3), (3.2) *verifying the equalities* (3.7), (3.9) *and the estimate* (3.8).

Proof. Consider $j_k \in C^1(\mathbb{R}_t; L^2(\Omega)^3)$ and $\tilde{h}_k \in C^1(\mathbb{R}_t; H^1(\Omega)^3) \cap C^2(\mathbb{R}_t; L^2(\Omega)^3)$ T periodic such that $j_k \to j$ in $L^1(]0,T[; L^2(\Omega)^3)$ and $n \wedge \tilde{h}_k \to h$ in $L^2(]0,T[; L^2(\partial\Omega)^3)$. Denote by (E_k, B_k) the corresponding T periodic strong solutions. By (3.8), (3.7) we deduce that $(E_k, B_k) \to (E, B)$ in $C(\mathbb{R}_t; \mathcal{H})$ and $(n \wedge E_k, n \wedge B_k)$ converges in $L^2(]0,T[; L^2(\partial\Omega)^3)$ to some function denoted $(n \wedge E, n \wedge B)$. Finally, by passing to the limit for $k \to +\infty$ we deduce that (E, B) verifies (3.7), (3.9), (3.8). Observe also that by the *Remark* 3.13 we have :

$$\int_0^T\!\!\int_\Omega \{E_k(t,x)\cdot(\alpha\varphi - \partial_t\varphi) - c_0^2 B_k(t,x)\cdot \mathrm{rot}\,\varphi\}\, dtdx - c_0^2 \int_0^T\!\!\int_{\partial\Omega} (n \wedge B_k)\cdot\varphi\, dtd\sigma = -\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j_k \cdot \varphi\, dtdx,$$

for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega})^3$, T periodic, and by passing to the limit for $k \to +\infty$ we deduce that :

$$\int_0^T\!\!\int_\Omega \{E(t,x)\cdot(\alpha\varphi - \partial_t\varphi) - c_0^2 B(t,x)\cdot \mathrm{rot}\,\varphi\}\, dtdx - c_0^2 \int_0^T\!\!\int_{\partial\Omega} (n \wedge B)\cdot\varphi\, dtd\sigma = -\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j \cdot \varphi\, dtdx.$$

Similarly we obtain that :

$$\int_0^T\!\!\int_\Omega \{B(t,x)\cdot(\alpha\psi - \partial_t\psi) + E(t,x)\cdot \mathrm{rot}\,\psi\}\, dtdx + \int_0^T\!\!\int_{\partial\Omega} (n \wedge E)\cdot\psi\, dtd\sigma = 0,$$

for all $\psi \in C^1(\mathbb{R}_t \times \overline{\Omega})^3$.
In order to prove the uniqueness, consider $(E_1, B_1), (E_2, B_2)$ two T periodic weak solutions and observe that $(E_1 - E_2, B_1 - B_2)$ is a T periodic weak solution corresponding to $j = 0$, $h = 0$. Thus, by the *Remark* 3.9 we obtain that $E_1 - E_2 = 0, B_1 - B_2 = 0$. □

4. The perturbed Vlasov-Maxwell system.

We study now the full perturbed Vlasov-Maxwell system (2.3), (2.2), (3.3), (3.2). We prove the existence of a T periodic solution by using the Schauder fixed point theorem. Let us start by the relativistic case. In this case we have $|j(t,x)| \le |q| \cdot \int_{\mathbb{R}^3_p} |v(p)| f(t,x,p)\, dp \le c_0 \cdot |\rho(t,x)|$, $(t,x) \in \mathbb{R}_t \times \Omega$. We consider the set :

$$\mathcal{X} = \{(E,B) \in L^\infty(\mathbb{R}_t; \mathcal{H}) \mid (E,B)(\cdot + T) = (E,B)(\cdot)\},$$

and define the application $\mathcal{F} : \mathcal{X} \to \mathcal{X}$, $\mathcal{F}(E,B) = (\tilde{E}, \tilde{B})$ by :

$$(E,B) \to (E_\varepsilon, B_\varepsilon) \to f \to j = q \cdot \int_{\mathbb{R}^3_p} v(p) f(t,x,p) dp \to j_\varepsilon \to (\tilde{E}, \tilde{B}),$$

where :
(i) $(E_\varepsilon, B_\varepsilon)$ is the convolution of $(\overline{E}, \overline{B})$ (the extension of (E,B) by 0 outside Ω) by $\zeta_\varepsilon(t,x)$:

$$(E_\varepsilon, B_\varepsilon)(t,x) = ((\overline{E}, \overline{B}) \star \zeta_\varepsilon)(t,x) = \int_0^T\!\!\int_\Omega (E(s,y), B(s,y))\zeta_\varepsilon(t-s, x-y)\, dsdy, (t,x) \in \mathbb{R}_t \times \Omega,$$

with $\zeta_\varepsilon(t,x) = \zeta_{\varepsilon_1,\varepsilon_2}(t,x) = \zeta_{1,\varepsilon_1}(t)\cdot\zeta_{2,\varepsilon_2}(x) = \left[\frac{1}{\varepsilon_1}\sum_{k\in Z}\zeta_3\left(\frac{t-kT}{\varepsilon_1}\right)\right]\frac{1}{\varepsilon_2^3}\zeta_2\left(\frac{x}{\varepsilon_2}\right)$, $\zeta_2 \in C^\infty_c(\mathbb{R}^3)$, $\zeta_3 \in C^\infty_c(\mathbb{R})$, $\zeta_2, \zeta_3 \ge 0$, $\mathrm{supp}\,\zeta_2 \subset B(0,1)$, $\mathrm{supp}\,\zeta_3 \subset [-1,1]$, $\int_{\mathbb{R}^3}\zeta_2(u)\, du = \int_{\mathbb{R}}\zeta_3(u) du = 1$ (note that $(E_\varepsilon, B_\varepsilon)$ is also T periodic);

(*ii*) f is the T periodic mild solution of (2.3), (2.2) corresponding to the regular force $F_\varepsilon(t,x,p) = q(E_\varepsilon(t,x) + v(p) \wedge B_\varepsilon(t,x))$ (cf. *Proposition* 2.4) ;
(*iii*) j_ε is the convolution of $\overline{j}$ (the extension of j by 0 outside Ω) by $\zeta_\varepsilon^{\vee}(t,x) = \zeta_\varepsilon(-t,-x)$:

$$j_\varepsilon(t,x) = (\overline{j} \star \zeta_\varepsilon^{\vee})(t,x) = \int_0^T \int_\Omega j(s,y)\zeta_\varepsilon(s-t,y-x)\, dsdy, \quad (t,x) \in \mathbb{R}_t \times \Omega \ ;$$

(*iv*) $(\tilde{E},\tilde{B})$ is the T periodic weak solution of (3.3), (3.2) (see *Proposition* 3.15) associated to the current density j_ε.

Let us consider $M_{\alpha,\varepsilon} := c_0^{1/2} \cdot \left(\frac{1}{\alpha T} + 4\right)^{1/2} \|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} + \left(\frac{1}{\alpha T} + 4\right) \frac{\|\zeta_2\|_{L^2(\mathbb{R}^3)}}{\varepsilon_0 \cdot \varepsilon_2^{3/2}} \cdot \frac{c_0|q|}{\alpha} \int_0^T\int_{\Sigma^-} |(v(p) \cdot n(x))| g \, dtd\sigma dp$ and $\mathcal{C} = \{(E,B) \in \mathcal{X} \mid \|(E,B)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \le M_{\alpha,\varepsilon}\}$ which is a convex compact subset in respect to the weak $\star$ topology of $L^\infty(\mathbb{R}_t;\mathcal{H})$.

Proposition 4.1. *Assume that g, h are T periodic such that $g \ge 0$, $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$ and*

$$\int_0^T\int_{\Sigma^-} |(v(p) \cdot n(x))| g(t,x,p)\, dtd\sigma dp + \int_0^T\int_{\partial\Omega} |h(t,x)|^2\, dtd\sigma < +\infty.$$

Then $\mathcal{F}(\mathcal{X}) \subset \mathcal{C}$.

Proof. By the estimate (3.8) we have :

$$\|(\tilde{E},\tilde{B})\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \le c_0^{1/2} \cdot \left(\frac{1}{\alpha T} + 4\right)^{1/2} \|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} + \left(\frac{1}{\alpha T} + 4\right) \frac{1}{\varepsilon_0} \|j_\varepsilon\|_{L^1(]0,T[;L^2(\Omega)^3)}.$$

On the other hand we have :

$$\|j_\varepsilon\|_{L^1(]0,T[;L^2(\Omega)^3)} \le \int_0^T \|j(t)\|_{L^1} \cdot \|\zeta_{2,\varepsilon}^{\vee}\|_{L^2}\, dt = \frac{1}{\varepsilon_2^{3/2}} \|\zeta_2\|_{L^2} \|j\|_{L^1(]0,T[;L^1(\Omega)^3)}.$$

But from the *Remark* 2.9 we deduce that :

$$\|j\|_{L^1(]0,T[;L^1(\Omega)^3)} \le |q| \int_0^T\int_\Omega \int_{\mathbb{R}^3_p} |v(p)| f(t,x,p)\, dtdxdp \le \frac{c_0|q|}{\alpha} \int_0^T\int_{\Sigma^-} |(v(p) \cdot n(x))| g(t,x,p)\, dtd\sigma dp,$$

and therefore $\|(\tilde{E},\tilde{B})\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \le M_{\alpha,\varepsilon}$. □

Proposition 4.2. *Assume that $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, h are T periodic, with $g \ge 0$, $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$ and*

$$\int_0^T\int_{\Sigma^-} |(v(p) \cdot n(x))| \mathcal{E}(p) g(t,x,p)\, dtd\sigma dp + \int_0^T\int_{\partial\Omega} |h(t,x)|^2\, dtd\sigma < +\infty.$$

Then the application $\mathcal{F}$ is continuous in respect to the weak $\star$ topology of $L^\infty(\mathbb{R}_t;\mathcal{H})$.

Proof. Consider $(E_k,B_k) \rightharpoonup (E,B)$ weakly $\star$ in $L^\infty(\mathbb{R}_t;\mathcal{H})$. Denote by f_k, f the T periodic mild solutions of (2.3) corresponding to the regularized forces $F_{k,\varepsilon} = q(\overline{E}_k \star \zeta_\varepsilon + v(p) \wedge (\overline{B}_k \star \zeta_\varepsilon))$:

$$\alpha f_k + \partial_t f_k + v(p) \cdot \nabla_x f_k + q(\overline{E}_k \star \zeta_\varepsilon + v(p) \wedge (\overline{B}_k \star \zeta_\varepsilon)) \cdot \nabla_p f_k = 0, \quad (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p,$$

respectively $F_\varepsilon = q(\overline{E} \star \zeta_\varepsilon + v(p) \wedge (\overline{B} \star \zeta_\varepsilon))$:

$$\alpha f + \partial_t f + v(p) \cdot \nabla_x f + q(\overline{E} \star \zeta_\varepsilon + v(p) \wedge (\overline{B} \star \zeta_\varepsilon)) \cdot \nabla_p f = 0, \quad (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p,$$

with the boundary conditions (2.2). Since (E_k,B_k) converges weakly $\star$ in $L^\infty(\mathbb{R}_t;\mathcal{H})$, we have the pointwise convergence $(\overline{E}_k \star \zeta_\varepsilon, \overline{B}_k \star \zeta_\varepsilon)(t,x) \to (\overline{E} \star \zeta_\varepsilon, \overline{B} \star \zeta_\varepsilon)(t,x)$, $(t,x) \in \mathbb{R}_t \times \Omega$, as $k \to +\infty$.

Moreover, $(\overline{E}_k\star\zeta_\varepsilon, \overline{B}_k\star\zeta_\varepsilon)$ is bounded in $L^\infty(\mathbb{R}_t\times\Omega)^6$ and by the dominated convergence theorem we deduce that $(\overline{E}_k\star\zeta_\varepsilon, \overline{B}_k\star\zeta_\varepsilon)\to(\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$ in $L^2(]0,T[\times\Omega)$ as k goes to $+\infty$. Since $(f_k)_k$ is bounded in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)$, $\|f_k\|_\infty\le\|g\|_\infty$, we can suppose (after extraction of a subsequence) that $f_k\rightharpoonup\tilde f$ weakly $\star$ in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)$. In order to prove that $\tilde f$ is T periodic weak solution of (2.3), (2.2) corresponding to the field $(\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$, take $\varphi\in\mathcal{T}_w$ and observe that :

$$\int_{\mathbb{R}^3_p} f_k(t,x,p)\,\nabla_p\varphi\,dp\rightharpoonup\int_{\mathbb{R}^3_p}\tilde f(t,x,p)\,\nabla_p\varphi\,dp,\ \text{weakly in}\ \ L^2(]0,T[\times\Omega),$$

$$\int_{\mathbb{R}^3_p} f_k(t,x,p)(\nabla_p\varphi\wedge v(p))\,dp\rightharpoonup\int_{\mathbb{R}^3_p}\tilde f(t,x,p)(\nabla_p\varphi\wedge v(p))\,dp,\ \text{weakly in}\ \ L^2(]0,T[\times\Omega).$$

By combining with the strong convergence of $(\overline{E}_k\star\zeta_\varepsilon, \overline{B}_k\star\zeta_\varepsilon)$ it is easy to prove that $\tilde f$ is T periodic weak solution for (2.3), (2.2) associated to the field $(\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$. In fact, by the uniqueness of the T periodic weak solution (see *Remark* 2.8) we deduce that $\tilde f$ is the T periodic mild solution $\tilde f=f$. Moreover we can prove that all the sequence $(f_k)_k$ converges to f weakly $\star$ in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)$. Now we want to pass to the limit in the perturbed Maxwell equations. We need to establish the convergence for the regularized current densities $(\overline{j}_k\star\overset{\vee}{\zeta_\varepsilon})_k$. By using the Green formula (2.7) with the test function $\varphi(t,x,p)=|p|\cdot\chi_R(|p|)$ we find that :

$$\alpha\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f_k(t,x,p)|p|\chi_R(|p|)\,dtdxdp\le\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)|p|\chi_R(|p|)\,dtd\sigma dp$$
$$+\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} qf_k(t,x,p)\{(\overline{E}_k\star\zeta_\varepsilon)+v(p)\wedge(\overline{B}_k\star\zeta_\varepsilon)\}\cdot\frac{p}{|p|}\left(\chi_R(|p|)+\frac{|p|}{R}\chi'(\frac{|p|}{R})\right)\,dtdxdp.$$

By passing to the limit for $R\to+\infty$ we deduce that :

$$\alpha\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f_k(t,x,p)\cdot|p|\,dtdxdp\le\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)\cdot|p|\,dtd\sigma dp \tag{4.1}$$
$$+\frac{|q|}{\alpha}\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|g\,dtd\sigma dp\cdot\frac{1}{\varepsilon_2^{3/2}}\|\zeta_2\|_{L^2(\mathbb{R}^3)}\|E_k\|_{L^\infty(\mathbb{R}_t;L^2(\Omega)^3)},$$

and therefore, since $\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\,dtd\sigma dp<+\infty$, we deduce that $\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f_k\cdot|p|\,dtdxdp$ is bounded by some constant C, uniformly in k. Similarly we obtain that $\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f\cdot|p|\,dtdxdp\le C$. We can prove that $j_k\rightharpoonup j$ weakly in $L^1(]0,T[\times\Omega)$. Indeed, for $\varphi\in L^\infty(]0,T[\times\Omega)$ we have :

$$\left|\int_0^T\!\!\int_\Omega j_k\varphi\,dtdx-\int_0^T\!\!\int_\Omega j\varphi\,dtdx\right|\le\left|\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} q(f_k-f)v(p)\varphi\cdot\mathbf{1}_{\{|p|\le R\}}\,dtdxdp\right|$$
$$+\frac{|q|}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}(f_k+f)|v(p)|\cdot|\varphi|\cdot|p|\,dtdxdp$$
$$\le\left|\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} q(f_k-f)v(p)\varphi\cdot\mathbf{1}_{\{|p|\le R\}}\,dtdxdp\right|+\frac{2C\cdot c_0\cdot|q|\cdot\|\varphi\|_\infty}{R}.$$

We take R_δ large enough such that $2C\cdot c_0\cdot|q|\cdot\|\varphi\|_\infty/R_\delta<\delta/2$ and since $f_k\rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)$ we have :

$$\left|\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} q(f_k-f)v(p)\varphi\cdot\mathbf{1}_{\{|p|\le R_\delta\}}\,dtdxdp\right|<\frac{\delta}{2},\ \ \forall k\ge k_\delta,$$

which implies that $\left|\int_0^T\int_\Omega (j_k - j)\varphi \, dtdx\right| \leq \delta, \ \forall k \geq k_\delta$. We deduce the pointwise convergence $(\overline{j}_k \star \zeta_\varepsilon^\surd)(t,x) \to (\overline{j} \star \zeta_\varepsilon^\surd)(t,x), \ \forall (t,x) \in \mathbb{R}_t \times \Omega$. Moreover, since $(\overline{j}_k \star \zeta_\varepsilon^\surd)_k$ is bounded we have $\overline{j}_k \star \zeta_\varepsilon^\surd \to \overline{j} \star \zeta_\varepsilon^\surd$ in $L^2(]0,T[\times\Omega)^3$. Consider now $(\varphi,\psi) \in C(\mathbb{R}_t, D(\mathcal{A}^\star)) \cap C^1(\mathbb{R}_t;\mathcal{H})$ T periodic and note $(f,g) = \alpha(\varphi,\psi) - \frac{d}{dt}(\varphi,\psi) + \mathcal{A}^\star(\varphi,\psi)$. We have :

$$\int_0^T \langle (\tilde{E}_k(t), \tilde{B}_k(t)), (f(t), g(t))\rangle_{\mathcal{H}} \, dt = c_0 \int_0^T\int_{\partial\Omega} (n \wedge \varphi) \cdot h(t,x) \, dtd\sigma - \frac{1}{\varepsilon_0} \int_0^T\int_\Omega (\overline{j}_k \star \zeta_\varepsilon^\surd)(t,x) \cdot \varphi(t,x) \, dtdx. \tag{4.2}$$

Since $\|(\tilde{E}_k, \tilde{B}_k)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq M_{\alpha,\varepsilon}, \ \forall k$ we can suppose that, at least for a subsequence, we have $(\tilde{E}_k, \tilde{B}_k) \rightharpoonup (e,b)$ weakly $\star$ in $L^\infty(\mathbb{R}_t;\mathcal{H})$. By passing to the limit in (4.2) for $k \to +\infty$ one gets :

$$\int_0^T \langle (e(t), b(t)), (f(t), g(t))\rangle_{\mathcal{H}} \, dt = c_0 \int_0^T\int_{\partial\Omega} (n\wedge\varphi)\cdot h(t,x) \, dtd\sigma - \frac{1}{\varepsilon_0} \int_0^T\int_\Omega (\overline{j}\star\zeta_\varepsilon^\surd)(t,x)\cdot\varphi(t,x) \, dtdx,$$

and thus $(e,b) = (\tilde{E}, \tilde{B}) = \mathcal{F}(E,B)$ (the unique T periodic weak solution of (3.3), (3.2) associated to $\overline{j} \star \zeta_\varepsilon^\surd$). By the uniqueness of the T periodic weak solution of (3.3), (3.2) we deduce also that all the sequence $\mathcal{F}(E_k, B_k) = (\tilde{E}_k, \tilde{B}_k)$ converges to $(\tilde{E}, \tilde{B}) = \mathcal{F}(E,B)$ weakly $\star$ in $L^\infty(\mathbb{R}_t;\mathcal{H})$, or the application $\mathcal{F}$ is continuous in respect to the weak $\star$ topology of $L^\infty(\mathbb{R}_t;\mathcal{H})$. □

Theorem 4.3. *Assume that* $g \in L^\infty(\mathbb{R}_t \times \Sigma^-), h$ *are* T *periodic,* $g \geq 0$, $\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g \, dtd\sigma dp + \int_0^T\int_{\partial\Omega} |h|^2 \, dtd\sigma < +\infty$ *and there is* $\tilde{h} \in C^1(\mathbb{R}_t; H^1(\Omega)^3) \cap C^2(\mathbb{R}_t; L^2(\Omega)^3)$ T *periodic such that* $n \wedge \tilde{h}|_{\mathbb{R}_t\times\partial\Omega} = h$. *Then, for every* $\alpha, \varepsilon_1, \varepsilon_2 > 0$ *there is a* T *periodic solution for the perturbed relativistic Vlasov-Maxwell system :*

$$\alpha f + \partial_t f + v(p)\cdot\nabla_x f + q((\overline{E}\star\zeta_\varepsilon) + v(p)\wedge(\overline{B}\star\zeta_\varepsilon))\cdot\nabla_p f = 0, \ (t,x,p) \in \mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

$$\alpha E + \partial_t E - c_0^2 \cdot rot\ B = -\frac{\overline{j}\star\zeta_\varepsilon^\surd}{\varepsilon_0}, \ \ \alpha B + \partial_t B + rot\ E = 0, \ \ (t,x) \in \mathbb{R}_t\times\Omega, \tag{4.3}$$

$$f(t,x,p) = g(t,x,p), \ (t,x,p) \in \mathbb{R}_t\times\Sigma^-, \ \ n\wedge E(t,x) + c_0\cdot n\wedge(n\wedge B(t,x)) = h(t,x), \ \ (t,x) \in \mathbb{R}_t\times\partial\Omega,$$

with $j = q\int_{\mathbb{R}^3_p} f(t,x,p) v(p) \, dp$. *Moreover* $\|(E,B)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq M_{\alpha,\varepsilon}$, $(E,B) \in C(\mathbb{R}_t; H(rot\,;\Omega)^2) \cap C^1(\mathbb{R}_t;\mathcal{H})$ *and :*

$$\alpha\int_0^T\int_\Omega \{|E(t,x)|^2 + c_0^2\cdot|B(t,x)|^2\} \, dtdx + \frac{c_0}{2}\int_0^T\int_{\partial\Omega} \{|n\wedge E(t,x)|^2 + c_0^2\cdot|n\wedge B(t,x)|^2\} \, dtd\sigma = -\frac{1}{\varepsilon_0}\int_0^T\int_\Omega (\overline{j}\star\zeta_\varepsilon^\surd)(t,x)\cdot E(t,x) \, dtdx + \frac{c_0}{2}\int_0^T\int_{\partial\Omega} |h(t,x)|^2 \, dtd\sigma. \tag{4.4}$$

Proof. By the *Propositions* 4.1, 4.2, the fixed point theorem of Schauder applies and thus we deduce the existence of a T periodic solution for the perturbed Vlasov-Maxwell system. The other statements follow by the *Proposition* 3.12.
□

The following proposition establish an a priori estimate for boundary terms.

PROPOSITION 4.4. *Under the hypothesis of Theorem 4.3, consider (f, E, B) the T periodic solution constructed above. Then we have :*

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)(1+\mathcal{E}(p))\, dtdxdp + \alpha \int_0^T\!\!\int_\Omega \{\varepsilon_0 |E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\, dtdx$$
$$+ \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}(p))\, dtd\sigma dp + \frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 |n\wedge E|^2 + \frac{1}{\mu_0}|n \wedge B|^2\}\, dtd\sigma$$
$$= \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g(1+\mathcal{E}(p))\, dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega} |h(t,x)|^2\, dtd\sigma. \tag{4.5}$$

Proof. Consider the test function $\varphi(t,x,p) = \mathcal{E}(p)\chi_R(|p|)$. By using the Green formula (2.7) we have :

$$\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\mathcal{E}(p)\chi_R(|p|)\, dtd\sigma dp + \int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x)) g\mathcal{E}(p)\chi_R(|p|)\, dtd\sigma dp$$
$$+\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\mathcal{E}(p)\chi_R(|p|)\, dtdxdp$$
$$= \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} qf((\overline{E}\star\zeta_\varepsilon) + v(p)\wedge(\overline{B}\star\zeta_\varepsilon))\cdot \left(v(p)\chi_R(|p|) + \mathcal{E}(p)\chi'(\frac{|p|}{R})\cdot\frac{1}{R}\cdot\frac{p}{|p|}\right)\, dtdxdp.$$

Observe that we can pass to the limit for $R\to+\infty$ in the last integral since $\overline{E}\star\zeta_\varepsilon \in L^\infty(\mathbb{R}_t\times\Omega)$ and

$$f\cdot\left|v(p)\chi_R(|p|) + \mathcal{E}(p)\chi'(\frac{|p|}{R})\cdot\frac{1}{R}\cdot\frac{p}{|p|}\right| \le f\cdot\left(|v(p)| + 2\frac{\mathcal{E}(p)}{|p|}\cdot\|\chi'\|_\infty\right)$$
$$\le f\cdot|v(p)|\cdot(1+2\cdot\|\chi'\|_\infty)$$
$$\le f\cdot c_0\cdot(1+2\cdot\|\chi'\|_\infty) \in L^1(]0,T[\times\Omega\times\mathbb{R}^3_p).$$

By passing to the limit for $R\to+\infty$ and by using the monotone convergence theorem we deduce that :

$$\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\mathcal{E}(p)\, dtd\sigma dp + \int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x)) g\mathcal{E}(p)\, dtd\sigma dp + \alpha\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f\mathcal{E}(p)\, dtdxdp$$
$$= \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} qf(\overline{E}\star\zeta_\varepsilon)v(p)\, dtdxdp$$
$$= \int_0^T\!\!\int_\Omega E(t,x)\cdot(\overline{j}\star\zeta_\varepsilon^{\checkmark})(t,x)\, dtdx.$$

Similarly, by using the test function $\varphi(t,x,p) = \chi_R(|p|)$ and by letting $R\to+\infty$ we obtain that :

$$\alpha\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f\, dtdxdp + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\, dtd\sigma dp = \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g\, dtd\sigma dp.$$

By combining with (4.4), finally one gets (4.5). $\square$

REMARK 4.5. *Under the hypothesis of Theorem 4.3 we have for a.e. $t \in \mathbb{R}_t$:*

$$\alpha \int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f(t,x,p)\, dxdp + \alpha \int_\Omega \{\varepsilon_0 |E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\, dx$$
$$+\frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)(1+\mathcal{E}(p))\, dxdp + \frac{1}{2}\frac{d}{dt}\int_\Omega \{\varepsilon_0 |E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\, dx$$
$$+\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f(t,x,p)\, d\sigma dp + \frac{c_0}{2}\int_{\partial\Omega} \{\varepsilon_0 |n \wedge E(t,x)|^2 + \frac{1}{\mu_0}|n\wedge B(t,x)|^2\}\, d\sigma$$
$$= \int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g(t,x,p)\, d\sigma dp + \frac{c_0\varepsilon_0}{2}\int_{\partial\Omega} |h(t,x)|^2\, d\sigma.$$

Proof. By using the Green formula (2.7) with the test function $\varphi(t,x,p) = \theta(t)\cdot\mathcal{E}(p)\cdot\chi_R(|p|)$, $\theta \in C^1_c(]0,T[)$ and by letting $R \to +\infty$ we obtain :

$$\int_0^T \theta(t) \int_\Omega \int_{\mathbb{R}^3_p} \alpha f \mathcal{E}(p)\, dxdp\, dt + \int_0^T \theta(t) \int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f\, d\sigma dp\, dt$$
$$= \int_0^T \theta'(t) \int_\Omega \int_{\mathbb{R}^3_p} f\mathcal{E}(p)\, dxdp\, dt + \int_0^T \theta(t)\int_\Omega E(t,x)\cdot(\overline{j}\star\zeta_\varepsilon^{\checkmark})(t,x)\, dx\, dt, \qquad (4.6)$$

or :

$$\alpha \int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\mathcal{E}(p)\, dxdp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\mathcal{E}(p)\, dxdp + \int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)\mathcal{E}(p)\, d\sigma dp$$
$$= -\int_{\Sigma^-} (v(p)\cdot n(x)) g(t,x,p)\mathcal{E}(p)\, d\sigma dp + \int_\Omega E(t,x)\cdot(\overline{j}\star\zeta_\varepsilon^{\checkmark})(t,x)\, dx, \quad \text{a.e. } t\in\mathbb{R}_t. \qquad (4.7)$$

Similarly we obtain for a.e. $t \in \mathbb{R}_t$:

$$\alpha \int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\, dxdp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\, dxdp + \int_\Sigma (v(p)\cdot n(x))\gamma f(t,x,p)\, d\sigma dp = 0. \qquad (4.8)$$

The conclusion follows by combining (3.10) (written for the current density $\overline{j}\star\zeta_\varepsilon^{\checkmark}$), (4.6), (4.8). □

PROPOSITION 4.6. *Under the hypothesis of the Theorem 4.3 we have $\alpha\rho + \partial_t\rho + \text{div}\, j = 0$, in $\mathcal{D}'(]0,T[\times\Omega)$, $\text{div}\, E = (\overline{\rho}\star\zeta_\varepsilon^{\checkmark})/\varepsilon_0$, $\text{div}\, B = 0$.*

Proof. Since $\alpha\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f\, dtdxdp + \int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\, dtd\sigma dp = \int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))| g\, dtd\sigma dp$, we can consider as test function in the weak formulation all function $\varphi \in C^1_c(]0,T[\times\Omega)$. We have :

$$\langle \alpha\cdot\rho + \partial_t\rho + \text{div}\, j, \varphi\rangle = \int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(\alpha\cdot\varphi(t,x) - \partial_t\varphi - v(p)\cdot\nabla_x\varphi)\, dtdxdp = 0,$$

or $\alpha\rho + \partial_t\rho + \text{div}\, j = 0$ in $\mathcal{D}'(]0,T[\times\Omega)$ (in fact the above equality holds for all function $\varphi \in C^1_c([0,T]\times\Omega)$, T periodic). We verify easily that we have also $\alpha\cdot(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}) + \partial_t(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}) + \text{div}\,(\overline{j}\star\zeta_\varepsilon^{\checkmark}) = 0$ in $\mathcal{D}'(]0,T[\times\Omega)$. Now, by taking the divergence of the perturbed Maxwell equations we have :

$$\alpha\cdot\text{div}\, E + \partial_t\text{div}\, E = -\frac{1}{\varepsilon_0}\text{div}\,(\overline{j}\star\zeta_\varepsilon^{\checkmark}) = \frac{\alpha}{\varepsilon_0}(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}) + \frac{1}{\varepsilon_0}\partial_t(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}),$$

and thus :

$$\alpha\left(\text{div}\, E - \frac{1}{\varepsilon_0}(\overline{\rho}\star\zeta_\varepsilon^{\checkmark})\right) + \partial_t\left(\text{div}\, E - \frac{1}{\varepsilon_0}(\overline{\rho}\star\zeta_\varepsilon^{\checkmark})\right) = 0,$$

and we deduce by periodicity that div $E = \frac{1}{\varepsilon_0}(\overline{\rho} \star \zeta_\varepsilon^{\checkmark})$. In the same manner we have $\alpha \cdot$ div $B +$ ∂_tdiv $B = 0$ which implies that div $B = 0$. □

Let us analyze now the classical case, with $\mathcal{E}(p) = \frac{|p|^2}{2m}$, $v(p) = \frac{p}{m}$, $\forall p \in \mathbb{R}^3_p$. We introduce also the energy and the velocity functions :

$$\mathcal{E}_c(p) = mc^2 \left(\left(1 + \frac{|p|^2}{m^2c^2}\right)^{1/2} - 1\right), \;\; v_c(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2c^2}\right)^{-1/2},$$

with $c > 0$. Observe that $\lim_{c\to+\infty} \mathcal{E}_c(p) = \mathcal{E}(p)$, $\lim_{c\to+\infty} v_c(p) = v(p)$ uniformly on bounded subsets of $\mathbb{R}^3_p$. The idea is to get the existence for the perturbed classical Vlasov-Maxwell system by letting $c \to +\infty$ in the perturbed relativistic Vlasov-Maxwell system (but keeping c_0 fixed in the perturbed Maxwell equations).

Theorem 4.7. *Assume that $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, h are T periodic, with $g \geq 0$ and*

$$\int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g(t,x,p)\; dt d\sigma dp + \int_0^T\!\!\int_{\partial\Omega} |h(t,x)|^2\; dt d\sigma < +\infty,$$

and there is $\tilde{h} \in C^1(\mathbb{R}_t; H^1(\Omega)^3) \cap C^2(\mathbb{R}_t; L^2(\Omega)^3)$ T periodic such that $n \wedge \tilde{h}|_{\mathbb{R}_t\times\partial\Omega} = h$. Then for every $\alpha, \varepsilon_1, \varepsilon_2 > 0$ there is a T periodic solution for the perturbed classical Vlasov-Maxwell system

$$\alpha f + \partial_t f + v(p)\cdot\nabla_x f + q((\overline{E}\star\zeta_\varepsilon) + v(p)\wedge(\overline{B}\star\zeta_\varepsilon))\cdot\nabla_p f = 0,\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

$$\alpha E + \partial_t E - c_0^2 \cdot rot\; B = -\frac{\overline{j}\star\zeta_\varepsilon^{\checkmark}}{\varepsilon_0},\;\; \alpha B + \partial_t B + rot\; E = 0,\;\; (t,x)\in\mathbb{R}_t\times\Omega, \tag{4.9}$$

$$f(t,x,p) = g(t,x,p),\; (t,x,p)\in\mathbb{R}_t\times\Sigma^-,\;\; n\wedge E(t,x) + c_0\cdot n\wedge(n\wedge B(t,x)) = h(t,x),\;\; (t,x)\in\mathbb{R}_t\times\partial\Omega,$$

with $j = q\int_{\mathbb{R}^3_p} f(t,x,p)v(p)\; dp$. Moreover $(E,B)\in C(\mathbb{R}_t; H(rot\,;\Omega)^2)\cap C^1(\mathbb{R}_t;\mathcal{H})$ and the solution verifies :

$$\begin{aligned}\alpha \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} & f(t,x,p)(1+\mathcal{E}(p))\; dt dx dp + \alpha\int_0^T\!\!\int_\Omega \{\varepsilon_0|E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\; dt dx \\ &+ \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}(p))\; dt d\sigma dp + \frac{c_0}{2}\int_0^T\!\!\int_{\partial\Omega}\{\varepsilon_0|n\wedge E|^2 + \frac{1}{\mu_0}|n\wedge B|^2\}\; dt d\sigma \\ &= \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g(1+\mathcal{E}(p))\; dt d\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\; dt d\sigma, \end{aligned} \tag{4.10}$$

$$\alpha\rho + \partial_t\rho + div\; j = 0,\;\; div\; E = \frac{1}{\varepsilon_0}(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}),\;\; div\; B = 0\;\; in\; \mathcal{D}'(]0,T[\times\Omega).$$

Proof. Since $\int_0^T\!\int_{\Sigma^-} |(v_c(p)\cdot n(x))|\mathcal{E}_c(p) g\; dt d\sigma dp \leq \int_0^T\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g\; dt d\sigma dp < +\infty$, by the *Theorem* 4.3 we deduce that there is a T periodic solution (f_c, E_c, B_c) for the system :

$$\alpha f_c + \partial_t f_c + v_c(p)\cdot\nabla_x f_c + q((\overline{E}_c\star\zeta_\varepsilon) + v_c(p)\wedge(\overline{B}_c\star\zeta_\varepsilon))\cdot\nabla_p f_c = 0,\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

$$\alpha E_c + \partial_t E_c - c_0^2\cdot \text{rot}\; B_c = -\frac{\overline{j}_c\star\zeta_\varepsilon^{\checkmark}}{\varepsilon_0},\;\; \alpha B_c + \partial_t B_c + \text{rot}\; E_c = 0,\;\; (t,x)\in\mathbb{R}_t\times\Omega,$$

$$f_c(t,x,p) = g(t,x,p),\ (t,x,p) \in \mathbb{R}_t \times \Sigma^-,\ \ n \wedge E_c(t,x) + c_0 \cdot n \wedge (n \wedge B_c(t,x)) = h,\ \ (t,x) \in \mathbb{R}_t \times \partial\Omega,$$

with $j_c = q \int_{\mathbb{R}^3_p} f_c(t,x,p) v_c(p)\ dp$. Indeed the *Propositions* 4.1, 4.2 hold true by defining the application $\mathcal{F}_c$ as before and by taking $\mathcal{C}_c = \{(E,B) \in \mathcal{X} \mid \|(E,B)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq M_{\alpha,\varepsilon,c}\}$, where

$$M_{\alpha,\varepsilon,c} = c_0^{1/2} \left(\frac{1}{\alpha T} + 4 \right)^{1/2} \|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} + \left(\frac{1}{\alpha T} + 4 \right) \frac{c|q| \cdot \|\zeta_2\|_{L^2}}{\alpha \cdot \varepsilon_0 \cdot \varepsilon_2^{3/2}} \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| g\ dt d\sigma dp.$$

Moreover, the *Proposition* 4.4 applies as well and thus we have :

$$\begin{aligned}
\alpha \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} & f_c(t,x,p)(1 + \mathcal{E}_c(p))\ dt dx dp + \alpha \int_0^T \int_\Omega \{\varepsilon_0 |E_c(t,x)|^2 + \frac{1}{\mu_0} |B_c(t,x)|^2\}\ dt dx \\
& + \int_0^T \int_{\Sigma^+} (v_c(p) \cdot n(x)) \gamma^+ f_c (1 + \mathcal{E}_c(p))\ dt d\sigma dp + \frac{c_0}{2} \int_0^T \int_{\partial\Omega} \{\varepsilon_0 |n \wedge E_c|^2 + \frac{1}{\mu_0} |n \wedge B_c|^2\}\ dt d\sigma \\
& = \int_0^T \int_{\Sigma^-} |(v_c(p) \cdot n(x))| g (1 + \mathcal{E}_c(p))\ dt d\sigma dp + \frac{c_0 \varepsilon_0}{2} \int_0^T \int_{\partial\Omega} |h(t,x)|^2\ dt d\sigma \\
& \leq \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| g (1 + \mathcal{E}(p))\ dt d\sigma dp + \frac{c_0 \varepsilon_0}{2} \int_0^T \int_{\partial\Omega} |h(t,x)|^2\ dt d\sigma.
\end{aligned}$$

We deduce that E_c, B_c and $n \wedge E_c, n \wedge B_c$ are uniformly bounded in $L^2(]0,T[;L^2(\Omega)^3)$, respectively in $L^2(]0,T[;L^2(\partial\Omega)^3)$ in respect to $c > 0$. Observe also that $(E_c, B_c)_c$ is bounded in $L^\infty(\mathbb{R}_t;\mathcal{H})$. Indeed, by observing that $(v_c(p) \cdot p) = |v_c(p)| \cdot |p| \in [\mathcal{E}_c(p), 2\mathcal{E}_c(p)]$, we have :

$$\begin{aligned}
|q|^{-1} \|j_c\|_{L^1(]0,T[;L^1(\Omega)^3)} & \leq \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} |v_c(p)| \cdot f_c(t,x,p)\ dt dx dp \\
& = \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} |v_c(p)| \cdot f_c \cdot \mathbf{1}_{\{|p| \leq 1\}}\ dt dx dp + \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} |v_c(p)| \cdot f_c \cdot \mathbf{1}_{\{|p| > 1\}}\ dt dx dp \\
& \leq \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} \frac{1}{m} f_c(t,x,p)\ dt dx dp + \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} 2\mathcal{E}_c(p) f_c(t,x,p)\ dt dx dp \\
& \leq \frac{1}{\alpha} \left(\int_0^T \int_{\Sigma^-} (m^{-1} + 2\mathcal{E}(p)) |(v(p) \cdot n(x))| g\ dt d\sigma dp + c_0 \cdot \varepsilon_0 \int_0^T \int_{\partial\Omega} |h|^2\ dt d\sigma \right) \\
& = C_1.
\end{aligned}$$

Therefore, as in the proof of *Proposition* 4.1 we deduce that :

$$\|(E_c, B_c)\|_{L^\infty(\mathbb{R}_t;\mathcal{H})} \leq c_0^{1/2} \left(\frac{1}{\alpha T} + 4 \right)^{1/2} \|h\|_{L^2(]0,T[;L^2(\partial\Omega)^3)} + \left(\frac{1}{\alpha T} + 4 \right) \frac{\|\zeta_2\|_{L^2}}{\varepsilon_0 \cdot \varepsilon_2^{3/2}} \cdot |q| \cdot C_1 = C_2.$$

Take $(c_k)_k$ with $c_k > 0, \forall k$ and $\lim_{k \to +\infty} c_k = +\infty$. We denote by (f_k, E_k, B_k) the corresponding solution. After extraction of subsequences we can assume that :

$$f_k \rightharpoonup f \ \ \text{weakly} \ \star \ \text{in} \ L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p), \ \ \gamma^+ f_k \rightharpoonup \gamma^+ f \ \ \text{weakly} \ \star \ \text{in} \ L^\infty(\mathbb{R}_t \times \Sigma^+),$$

$$(E_k, B_k) \rightharpoonup (E, B) \ \ \text{weakly} \ \star \ \text{in} \ L^\infty(\mathbb{R}_t; \mathcal{H}),$$

$$(n \wedge E_k, n \wedge B_k) \rightharpoonup (n \wedge E, n \wedge B), \ \ \text{weakly} \ \ \text{in} \ L^2(]0,T[;L^2(\partial\Omega)^3).$$

By standard arguments we deduce that :

$$\alpha \int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(1+\mathcal{E}(p))\,dtdxdp + \alpha\int_0^T\int_\Omega \{\varepsilon_0|E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\,dtdx$$
$$+\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}(p))\,dtd\sigma dp + \frac{c_0}{2}\int_0^T\int_{\partial\Omega}\{\varepsilon_0|n\wedge E|^2 + \frac{1}{\mu_0}|n\wedge B|^2\}\,dtd\sigma$$
$$\le \int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|g(1+\mathcal{E}(p))\,dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\,dtd\sigma = C(g,h).$$

As usual we prove that $\lim_{k\to+\infty}(\overline{E}_k\star\zeta_\varepsilon, \overline{B}_k\star\zeta_\varepsilon) = (\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$ in $L^2(]0,T[\times\Omega)$ and by combining with the weak $\star$ convergence of $f_k, \gamma^+ f_k$, after passing to the limit for $k\to+\infty$ in the Green formula (2.7), we deduce that $f, \gamma^+ f$ verify the weak formulation of the perturbed T periodic Vlasov problem corresponding to the electro-magnetic field $(\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$ and to the energy and velocity functions $\mathcal{E}(p), v(p)$ (since $(\overline{E}\star\zeta_\varepsilon, \overline{B}\star\zeta_\varepsilon)$ is regular, by the uniqueness of the T periodic weak solution when $\alpha>0$ we deduce that f is also the T periodic mild solution). Now we want to pass to the limit in the perturbed Maxwell equations. By observing that :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}|v_k(p)|\cdot f_k(t,x,p)\cdot\mathbf{1}_{\{|p|>R\}}\,dtdxdp \le \frac{1}{R}\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}|v_k(p)|\cdot|p|\cdot f_k(t,x,p)\,dtdxdp$$
$$\le\frac{2}{R}\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}_k(p)f_k(t,x,p)\,dtdxdp$$
$$\le\frac{2}{R}\cdot\frac{C(g,h)}{\alpha},$$

and :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}|v(p)|\cdot f(t,x,p)\cdot\mathbf{1}_{\{|p|>R\}}\,dtdxdp \le \frac{1}{R}\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}|v(p)|\cdot|p|\cdot f(t,x,p)\,dtdxdp$$
$$=\frac{2}{R}\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f(t,x,p)\,dtdxdp$$
$$\le\frac{2}{R}\cdot\frac{C(g,h)}{\alpha},$$

we deduce as in the proof of *Proposition* (4.2) that $\lim_{k\to+\infty} j_k = j$ weakly in $L^1(]0,T[\times\Omega)^3$ and that $\lim_{k\to+\infty}(\overline{j}_k\star\zeta_\varepsilon^{\checkmark}) = \overline{j}\star\zeta_\varepsilon^{\checkmark}$ strongly in $L^2(]0,T[\times\Omega)^3$. Consider $(\varphi,\psi)\in C(\mathbb{R}_t; D(\mathcal{A}^\star))\cap C^1(\mathbb{R}_t;\mathcal{H})$ T periodic and note $(f,g) = \alpha(\varphi,\psi) - \frac{d}{dt}(\varphi,\psi) + \mathcal{A}^\star(\varphi,\psi)$. We have :

$$\int_0^T\langle(E_k(t),B_k(t)),(f(t),g(t))\rangle_{\mathcal{H}}\,dt = c_0\int_0^T\int_{\partial\Omega}(n\wedge\varphi)\cdot h(t,x)\,dtd\sigma$$
$$-\frac{1}{\varepsilon_0}\int_0^T\int_\Omega(\overline{j}_k\star\zeta_\varepsilon^{\checkmark})(t,x)\cdot\varphi(t,x)\,dtdx,$$

and after passing to the limit for $k\to+\infty$ we deduce that (E,B) is a T periodic weak solution for the perturbed Maxwell equations, corresponding to the current density $\overline{j}\star\zeta_\varepsilon^{\checkmark}$. By the *Proposition* 3.12 we deduce that (E,B) is T periodic strong solution. The other statements follow exactly as in the relativistic case. □

5. A priori estimates.

In the following we want to establish a priori estimates for the normal trace of the electro-magnetic field as well as for the total (kinetic and electro-magnetic) energy. For this we suppose that $\partial\Omega$ is strictly star-shaped in respect to some point $x_0\in\Omega$ (i.e., $\exists r>0$ such that

$(x-x_0)\cdot n(x) \geq r, \forall x \in \partial\Omega$). After translation we can assume that $x_0 = 0 \in \Omega$ and thus $x \cdot n(x) \geq r$, $\forall x \in \partial\Omega$. This hypothesis was already used in order to estimate the solutions of the Maxwell equations by using the multiplier method (see [25]). We will use also the following lemma, whose proof is immediate and is left to the reader.

LEMMA 5.1. *Assume that $\partial\Omega$ is regular (C^1), $u \in H(div\ ;\Omega) \cap H(rot\ ;\Omega)$. Then we have the following equality in $\mathcal{D}'(\Omega)$:*

$$u_i div\ u - (u \wedge rot\ u)_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2, \ \ \forall 1 \leq i \leq 3. \tag{5.1}$$

In order to estimate the normal trace of the electro-magnetic field we need the following lemma. We begin with the stationary case.

LEMMA 5.2. *Assume that Ω is bounded with $\partial\Omega$ regular and strictly star-shaped in respect to $0 \in \Omega$. Consider $u \in H(div\ ;\Omega) \cap H(rot\ ;\Omega)$ with $n \wedge u \in L^2(\partial\Omega)^3$ such that*

$$\int_\Omega (u_i div\ u - (u \wedge rot\ u)_i)\theta(x)\ dx + \int_\Omega w_i \cdot \nabla_x \theta\ dx = \int_\Omega v_i \theta(x)\ dx + \int_{\partial\Omega} (n(x)\cdot w_i(x))\theta(x)\ d\sigma, \tag{5.2}$$

for all function $\theta \in C^1(\overline{\Omega})^3$, $1 \leq i \leq 3$, where v_i, $w_i = (w_{ij})_{1\leq j\leq 3}$, $n \cdot w_i$ are some given functions verifying $v_i \in L^1(\Omega), w_i \in L^1(\Omega)^3$, $w_{ii} \geq 0$, $n \cdot w_i \in L^1(\partial\Omega), 1 \leq i \leq 3$. Then u has a normal trace in $L^2(\partial\Omega)$ i.e., there is $n \cdot u \in L^2(\partial\Omega)$ such that $\int_\Omega div\ u\ \theta(x)\ dx + \int_\Omega u \cdot \nabla_x\theta\ dx = \int_{\partial\Omega}(n(x)\cdot u(x))\theta(x)\ d\sigma$, $\forall \theta \in C^1(\overline{\Omega})$. Moreover we have the estimate :

$$\|n\cdot u\|^2_{L^2(\partial\Omega)} + \sum_{i=1}^{3}\|w_{ii}\|_{L^1(\Omega)} + \|u\|^2_{L^2(\Omega)^3} \leq C\{\|n\wedge u\|^2_{L^2(\partial\Omega)^3} + \|v\|_{L^1(\Omega)^3} + \sum_{i=1}^{3}\|n\cdot w_i\|_{L^1(\partial\Omega)}\}.$$

Proof. Since $u \in H(\text{div}\ ;\Omega) \cap H(\text{rot}\ ;\Omega)$, $n \wedge u \in L^2(\partial\Omega)^3$, we can approximate u by smooth functions $u^k \in C^1(\overline{\Omega})^3$ such that $u^k \to u, \text{rot}\ u^k \to \text{rot}\ u$ in $L^2(\Omega)^3$, div $u^k \to$ div u in $L^2(\Omega)$, $n \wedge u^k \to n \wedge u$ in $L^2(\partial\Omega)^3$. By using the *Lemma* 5.1 we have :

$$\begin{aligned}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(u_i^k u_j^k) - \frac{1}{2}\frac{\partial}{\partial x_i}|u^k|^2 =& u_i^k \text{div}\ u^k - (u^k \wedge \text{rot}\ u^k)_i \\ =& u_i \text{div}\ u - (u \wedge \text{rot}\ u)_i + r_i^k, \ \ \forall 1 \leq i \leq 3,\end{aligned} \tag{5.3}$$

where $r_i^k \to 0$ in $L^1(\Omega)$ as $k \to +\infty$. By using (5.3) and the hypothesis (5.2) with the test function $\theta(x) = x_i$ we obtain for $1 \leq i \leq 3$:

$$\begin{aligned}\int_{\partial\Omega}\sum_{j=1}^{3} u_i^k u_j^k x_i n_j\ d\sigma -& \int_\Omega \sum_{j=1}^{3} u_i^k u_j^k \delta_{ij}\ dx - \frac{1}{2}\int_{\partial\Omega}|u^k|^2 x_i n_i\ d\sigma + \frac{1}{2}\int_\Omega |u^k|^2\ dx \\ =& \int_\Omega x_i v_i(x)\ dx + \int_{\partial\Omega}(n(x)\cdot w_i(x))x_i\ d\sigma - \int_\Omega \sum_{j=1}^{3} w_{ij}\delta_{ij}\ dx + \int_\Omega r_i^k x_i\ dx.\end{aligned} \tag{5.4}$$

By taking the sum for $1 \leq i \leq 3$ in (5.4) we obtain :

$$\begin{aligned}\int_\Omega \sum_{i=1}^{3} w_{ii}\ dx +& \int_{\partial\Omega}(u^k\cdot x)(u^k\cdot n)\ d\sigma + \frac{1}{2}\int_\Omega |u^k|^2\ dx - \frac{1}{2}\int_{\partial\Omega}|u^k|^2(n\cdot x)\ d\sigma \\ =& \int_\Omega x\cdot v(x)\ dx + \int_{\partial\Omega}\sum_{i=1}^{3}(n(x)\cdot w_i(x))x_i\ d\sigma + \int_\Omega r^k\cdot x\ dx.\end{aligned} \tag{5.5}$$

By using the decomposition $u^k = (n \cdot u^k)n - n \wedge (n \wedge u^k)$ after easy computations we obtain the identity :

$$\int_\Omega \sum_{i=1}^3 w_{ii}\ dx + \frac{1}{2}\int_\Omega |u^k|^2\ dx + \frac{1}{2}\int_{\partial\Omega} (u^k \cdot n)^2 (n(x)\cdot x)\ d\sigma = \frac{1}{2}\int_{\partial\Omega} |u^k \wedge n|^2 (n(x)\cdot x)\ d\sigma \tag{5.6}$$
$$+\int_{\partial\Omega} (u^k\cdot n)[(n\wedge(n\wedge u^k))\cdot x]\ d\sigma + \int_\Omega x\cdot v(x)\ dx + \int_{\partial\Omega}\sum_{i=1}^3 (n(x)\cdot w_i(x))x_i\ d\sigma + \int_\Omega r^k\cdot x\ dx.$$

By our hypothesis there is $0 < r \le R$ such that $r \le (n(x)\cdot x) \le |x| \le R$, $\forall x \in \partial\Omega$ and thus we have :

$$\int_\Omega \sum_{i=1}^3 w_{ii}\ dx + \frac{1}{2}\|u^k\|^2_{L^2(\Omega)^3} + \frac{r}{2}\|n\cdot u^k\|^2_{L^2(\partial\Omega)} \le \frac{R}{2}\|n\wedge u^k\|^2_{L^2(\partial\Omega)^3} + R\sum_{i=1}^3 \|n\cdot w_i\|_{L^1(\partial\Omega)}$$
$$+R\cdot\|v\|_{L^1(\Omega)^3} + R\|r^k\|_{L^1(\Omega)^3} + R\|n\cdot u^k\|_{L^2(\partial\Omega)}\|n\wedge u^k\|_{L^2(\partial\Omega)^3}, \tag{5.7}$$

which implies that :

$$\sum_{i=1}^3 \|w_{ii}\|_{L^1(\Omega)} + \|u^k\|^2_{L^2(\Omega)^3} + \|(n\cdot u^k)\|^2_{L^2(\partial\Omega)} \le C(r,R)(\|n\wedge u^k\|^2_{L^2(\partial\Omega)^3} + \|v\|_{L^1(\Omega)^3}$$
$$+\sum_{i=1}^3 \|n\cdot w_i\|_{L^1(\partial\Omega)} + \|r^k\|_{L^1(\Omega)^3}). \tag{5.8}$$

Since $n\wedge u^k \to n\wedge u$ in $L^2(\partial\Omega)^3$ and $r^k \to 0$ in $L^1(\Omega)^3$ we deduce that $(n\cdot u^k)_k$ is bounded in $L^2(\partial\Omega)$. In fact we can prove that $(n\cdot u^k)_k$ converges in $L^2(\partial\Omega)$. For this let us introduce the bilinear application :

$$a_i(f,g) = f_i \mathrm{div}\ g - (f\wedge \mathrm{rot}\ g)_i,\ f\in L^2(\Omega)^3, g\in H(\mathrm{div}\ ;\Omega)\cap H(\mathrm{rot}\ ;\Omega).$$

We have :

$$a_i(f-g,f-g) = a_i(f,f) - a_i(g,g) - a_i(g,f-g) - a_i(f-g,g),\ \forall f,g\in H(\mathrm{div}\ ;\Omega)\cap H(\mathrm{rot}\ ;\Omega).$$

Observe that $a_i(u^k,u^k) = a_i(u,u) + r_i^k$, $a_i(u^l,u^l) = a_i(u,u) + r_i^l$. By taking into account that :

$$\|a_i(u^l,u^k-u^l)\|_{L^1(\Omega)} \le \|u_i^l\|_{L^2(\Omega)}\|\mathrm{div}\ u^k - \mathrm{div}\ u^l\|_{L^2(\Omega)} + \|u^l\|_{L^2(\Omega)^3}\|\mathrm{rot}\ u^k - \mathrm{rot}\ u^l\|_{L^2(\Omega)^3} \to 0,$$

when $k,l\to+\infty$ and :

$$\|a_i(u^k-u^l,u^l)\|_{L^1(\Omega)} \le \|u_i^k-u_i^l\|_{L^2(\Omega)}\|\mathrm{div}\ u^l\|_{L^2(\Omega)} + \|u^k-u^l\|_{L^2(\Omega)^3}\|\mathrm{rot}\ u^l\|_{L^2(\Omega)^3} \to 0,$$

when $k,l\to+\infty$ we deduce that for $1\le i\le 3$ we have :

$$\sum_{j=1}^3 \frac{\partial}{\partial x_j}\{(u_i^k-u_i^l)(u_j^k-u_j^l)\} - \frac{1}{2}\frac{\partial}{\partial x_i}|u^k-u^l|^2 = a_i(u^k-u^l,u^k-u^l)$$
$$= a_i(u^k,u^k) - a_i(u^l,u^l) - a_i(u^l,u^k-u^l) - a_i(u^k-u^l,u^l)$$
$$= r_i^k - r_i^l - a_i(u^l,u^k-u^l) - a_i(u^k-u^l,u^l) = r_i^{kl},$$

where $r^{kl}\to 0$ in $L^1(\Omega)^3$ when $k,l\to+\infty$. Now, by the previous computations (this time with $v=0, w=0, n\cdot w=0$) we deduce that :

$$\|n\cdot u^k - n\cdot u^l\|_{L^2(\partial\Omega)} \le C(r,R)\{\|n\wedge u^k - n\wedge u^l\|_{L^2(\partial\Omega)^3} + \|r^{kl}\|^{1/2}_{L^1(\Omega)^3}\},$$

and thus $(n\cdot u^k)_k$ is a Cauchy sequence in $L^2(\partial\Omega)$, or $(n\cdot u^k)_k$ converges in $L^2(\partial\Omega)$. Moreover, the limit doesn't depend on the approximation sequence $(u^k)_k$ and we can associate to u the normal trace $n\cdot u := \lim_{k\to+\infty} n\cdot u^k$ in $L^2(\partial\Omega)$. By passing to the limit in (5.8) we find :

$$\sum_{i=1}^{3}\|w_{ii}\|_{L^1(\Omega)} + \|u\|^2_{L^2(\Omega)^3} + \|(n\cdot u)\|^2_{L^2(\partial\Omega)} \le C(r,R)(\|n\wedge u\|^2_{L^2(\partial\Omega)^3} + \|v\|_{L^1(\Omega)^3} + \sum_{i=1}^{3}\|n\cdot w_i\|_{L^1(\partial\Omega)}). \tag{5.9}$$

Moreover we have $\int_\Omega \mathrm{div}\ u^k\ \theta(x)\ dx + \int_\Omega u^k\cdot\nabla_x\theta\ dx = \int_{\partial\Omega}(n(x)\cdot u^k(x))\theta(x)\ d\sigma,\ \forall\theta\in C^1(\overline{\Omega})$, and by passing to the limit for $k\to+\infty$ we find that :

$$\int_\Omega \mathrm{div}\ u\ \theta(x)\ dx + \int_\Omega u\cdot\nabla_x\theta\ dx = \int_{\partial\Omega}(n(x)\cdot u(x))\theta(x)\ d\sigma,\ \forall\theta\in C^1(\overline{\Omega}).$$

□

Once we have defined $n\cdot u\in L^2(\partial\Omega)$ we can define the trace of u on $\partial\Omega$ by $\gamma u = (n\cdot u)n - n\wedge(n\wedge u)\in L^2(\partial\Omega)^3$. By construction we have $(n\cdot\gamma u) = (n\cdot u)$ and $n\wedge\gamma u = n\wedge u$ and therefore :

$$\begin{aligned}\|\gamma u\|^2_{L^2(\partial\Omega)^3} &= \|n\cdot\gamma u\|^2_{L^2(\partial\Omega)} + \|n\wedge\gamma u\|^2_{L^2(\partial\Omega)^3}\\ &\le C\left\{\|n\wedge u\|^2_{L^2(\partial\Omega)^3} + \|v\|_{L^1(\Omega)^3} + \sum_{i=1}^{3}\|n\cdot w_i\|_{L^1(\partial\Omega)}\right\}.\end{aligned}$$

Moreover, the equality $\sum_{j=1}^3 \frac{\partial}{\partial x_j}(u_iu_j) - \frac12\frac{\partial}{\partial x_i}|u|^2 = v_i + \mathrm{div}\ w_i$ holds in $\mathcal{D}'(\overline{\Omega}), \forall 1\le i\le 3$. Indeed, for $\theta\in C^1(\overline{\Omega})$ we can write for $1\le i\le 3$:

$$\begin{aligned}\int_\Omega\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(u_i^ku_j^k) - \frac12\frac{\partial}{\partial x_i}|u^k|^2\right)\theta(x)\ dx &= \int_\Omega\left(u_i\mathrm{div}\ u - (u\wedge\mathrm{rot}\ u)_i + r_i^k\right)\theta(x)\ dx\\ &= \int_\Omega v_i(x)\theta(x)\ dx - \int_\Omega w_i(x)\cdot\nabla_x\theta\ dx\\ &+ \int_{\partial\Omega}(n(x)\cdot w_i(x))\theta(x)\ d\sigma + \int_\Omega r_i^k\theta(x)\ dx.\end{aligned}$$

After integration by parts we deduce that :

$$\begin{aligned}\int_{\partial\Omega}u_i^k\sum_{j=1}^{3}u_j^k\ n_j\theta(x)\ d\sigma - \frac12\int_{\partial\Omega}|u^k|^2n_i\ \theta(x)\ d\sigma - \int_\Omega u_i^k\sum_{j=1}^{3}u_j^k\frac{\partial\theta}{\partial x_j}\ dx + \frac12\int_\Omega|u^k|^2\frac{\partial\theta}{\partial x_i}\ dx\\ = \int_\Omega v_i(x)\theta(x)\ dx - \int_\Omega w_i(x)\cdot\nabla_x\theta\ dx\\ + \int_{\partial\Omega}(n(x)\cdot w_i(x))\theta(x)\ d\sigma + \int_\Omega r_i^k\theta(x)\ dx,\end{aligned}$$

and by passing to the limit for $k\to+\infty$ we obtain :

$$\begin{aligned}\int_{\partial\Omega}(\gamma u)_i\sum_{j=1}^{3}(\gamma u)_jn_j\ \theta(x)\ d\sigma - \frac12\int_{\partial\Omega}|\gamma u|^2n_i\ \theta(x)\ d\sigma - \int_\Omega u_i\sum_{j=1}^{3}u_j\frac{\partial\theta}{\partial x_j}\ dx + \frac12\int_\Omega|u|^2\frac{\partial\theta}{\partial x_i}\ dx\\ = \int_\Omega v_i(x)\theta(x)\ dx - \int_\Omega w_i(x)\cdot\nabla_x\theta\ dx + \int_{\partial\Omega}(n(x)\cdot w_i(x))\theta(x)\ d\sigma.\end{aligned}$$

We have similar results for the time dependent case. The proof is left to the reader.

LEMMA 5.3. *Assume that Ω is a bounded domain with $\partial\Omega$ regular (C^1) and strictly star-shaped in respect to the origin $0 \in \Omega$. Consider $u \in L^2(]0,T[;H(div\ ;\Omega)) \cap L^2(]0,T[;H(rot\ ;\Omega))$ with $n \wedge u \in L^2(]0,T[;L^2(\partial\Omega)^3)$ such that :*

$$\int_0^T\!\!\int_\Omega \{u_i div\ u - (u \wedge rot\ u)_i\}\theta(t,x)\ dtdx + \int_0^T\!\!\int_\Omega z_i(t,x)\partial_t\theta\ dtdx + \int_0^T\!\!\int_\Omega w_i \cdot \nabla_x\theta\ dtdx$$
$$= \int_0^T\!\!\int_\Omega v_i(t,x)\theta(t,x)\ dtdx + \int_0^T\!\!\int_{\partial\Omega} (n(x)\cdot w_i(x))\theta(t,x)\ dtd\sigma,\ 1 \le i \le 3,$$

for all function $\theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$ T periodic in time, where $z, v, w_i = (w_{ij})_{1\le j\le 3}, n \cdot w_i$ are some given functions verifying $z, v \in L^1(]0,T[;L^1(\Omega)^3), w_i \in L^1(]0,T[;L^1(\Omega)^3), w_{ii} \ge 0, n\cdot w_i \in L^1(]0,T[\times\partial\Omega), 1 \le i \le 3$. Then u has a normal trace $n \cdot u \in L^2(]0,T[\times\partial\Omega)$:

$$\int_\Omega div\ u(t)\ \theta(x)\ dx + \int_\Omega u(t,x)\cdot\nabla_x\theta\ dx = \int_{\partial\Omega}(n\cdot u(t))\theta(x)\ d\sigma, \forall\, \theta \in C^1(\overline{\Omega})\ \ a.e.\ t \in \mathbb{R}_t,$$

and we have the estimate :

$$\|n\cdot u\|^2_{L^2(]0,T[\times\partial\Omega)} + \sum_{i=1}^3 \|w_{ii}\|_{L^1(]0,T[\times\Omega)} + \|u\|^2_{L^2(]0,T[;L^2(\Omega)^3)} \le C\cdot(\|n\wedge u\|^2_{L^2(]0,T[;L^2(\partial\Omega)^3)}$$
$$+ \|v\|_{L^1(]0,T[;L^1(\Omega)^3)} + \sum_{i=1}^3 \|n\cdot w_i\|_{L^1(]0,T[\times\partial\Omega)}).$$

REMARK 5.4. *We can define the trace $\gamma u = (n\cdot u)n - n\wedge(n\wedge u) \in L^2(]0,T[;L^2(\partial\Omega)^3)$ and we have $\sum_{j=1}^3 \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2 = v_i + \partial_t z_i + div\ w_i$ in $\mathcal{D}'(]0,T[\times\overline{\Omega})$, $1 \le i \le 3$.*

REMARK 5.5. *The previous results adapt easily when replacing $u div\, u - u\wedge rot\, u$ by $\varepsilon_0\{E div\, E - E\wedge rot\ E\} + \frac{1}{\mu_0}\{B div\ B - B \wedge rot\ B\}$.*

PROPOSITION 5.6. *Assume that Ω is bounded, with $\partial\Omega$ smooth and strictly star-shaped (in respect to $0 \in \Omega$). Under the hypothesis of the Theorems 4.7, 4.3, consider (f,E,B) the T periodic solution of the perturbed Vlasov-Maxwell system (classical case (4.9), or relativistic case (4.3)) with fixed $0 < \alpha, \varepsilon_1, \varepsilon_2 < 1$. Then the electro-magnetic field has normal trace $(n\cdot E, n\cdot B) \in L^2(]0,T[\times\partial\Omega)^2$:*

$$\int_\Omega div\ E(t)\ \theta(x)\ dx + \int_\Omega E(t,x)\cdot\nabla_x\theta\ dx = \int_{\partial\Omega}(n(x)\cdot E(t,x))\theta(x)\ d\sigma, \forall\,\theta \in C^1(\overline{\Omega}),\ a.e.\ t\in\mathbb{R}_t,$$

and :

$$\int_\Omega B(t,x)\cdot\nabla_x\theta\ dx = \int_{\partial\Omega}(n(x)\cdot B(t,x))\theta(x)\ d\sigma, \forall\,\theta\in C^1(\overline{\Omega}),\ a.e.\ t\in\mathbb{R}_t.$$

Moreover, the solution satisfies the following estimate :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\mathcal{E}(p)\ dtdxdp + \int_0^T\!\!\int_\Omega \{\varepsilon_0|E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\ dtdx$$
$$+ \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f(t,x,p)\ dtd\sigma dp$$
$$+ \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0[(n\cdot E)^2 + |n\wedge E|^2] + \frac{1}{\mu_0}[(n\cdot B)^2 + |n\wedge B|^2]\}\ dtd\sigma$$
$$\le C\cdot\left\{\int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\ dtd\sigma dp + \int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\ dtd\sigma\right\} + \frac{C\cdot\varepsilon_2^s}{\alpha^q\cdot\varepsilon_1^r},$$

for some constant C and exponents $q, r, s > 0$ and the total energy is uniformly bounded in time :

$$\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\, dxdp + \int_\Omega \{\varepsilon_0 |E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\, dx \le \frac{C \cdot \varepsilon_2^s}{\alpha^q \cdot \varepsilon_1^r}$$
$$+C \cdot \left\{ \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g(t,x,p)\, dtd\sigma dp + \int_0^T\!\!\int_{\partial\Omega} |h(t,x)|^2\, dtd\sigma \right\},\ \forall t \in \mathbb{R}_t.$$

Proof. Consider the test function $\varphi_i(t,x,p) = p_i\theta(t,x),\ 1 \le i \le 3,\ \theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$, T periodic. By using the Green formula (remark that this is possible since $\int_0^T\!\int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f\ dtdxdp < +\infty$, $\int_0^T\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\ dtd\sigma dp < +\infty$ and thus in both classical and relativistic cases we have $\int_0^T\!\int_\Omega \int_{\mathbb{R}^3_p} |p| f\ dtdxdp < +\infty$, $\int_0^T\!\int_{\Sigma^+} (v(p)\cdot n(x))|p|\gamma^+ f\ dtd\sigma dp < +\infty$) we deduce that :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p) p_i \theta(t,x)\, dtdxdp + \int_0^T\!\!\int_\Sigma (v(p)\cdot n(x)) p_i \gamma f(t,x,p)\theta(t,x)\, dtd\sigma dp$$
$$= \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p) p_i (\partial_t\theta + v(p)\cdot\nabla_x\theta)\, dtdxdp$$
$$+ \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} q\theta(t,x) f(t,x,p)\{(\overline{E}\star\zeta_\varepsilon) + v(p)\wedge(\overline{B}\star\zeta_\varepsilon)\}_i\, dtdxdp.$$

Let us consider $i = 1$ and compute the term involving the electro-magnetic field :

$$\begin{aligned}
\mathcal{I}_1 =& \int_0^T\!\!\int_\Omega \theta(t,x)\{\rho(t,x)(\overline{E}_1\star\zeta_\varepsilon) + j_2(t,x)(\overline{B}_3\star\zeta_\varepsilon) - j_3(t,x)(\overline{B}_1\star\zeta_\varepsilon)\}\, dtdx \\
=& \int_0^T\!\!\int_\Omega \{E_1(t,x)((\overline{\theta\rho})\star\zeta_\varepsilon^{\checkmark}) + B_3(t,x)((\overline{\theta j_2})\star\zeta_\varepsilon^{\checkmark}) - B_2(t,x)((\overline{\theta j_3})\star\zeta_\varepsilon^{\checkmark})\}\, dtdx \\
=& \int_0^T\!\!\int_\Omega \theta\{E_1(t,x)(\overline{\rho}\star\zeta_\varepsilon^{\checkmark}) + B_3(t,x)(\overline{j}_2\star\zeta_\varepsilon^{\checkmark}) - B_2(t,x)(\overline{j}_3\star\zeta_\varepsilon^{\checkmark})\}\, dtdx \\
&+ \int_0^T\!\!\int_\Omega E_1(t,x) \int_0^T\!\!\int_\Omega (\theta(s,y)-\theta(t,x))\rho(s,y)\zeta_\varepsilon^{\checkmark}(t-s,x-y)\, dsdy\, dtdx \\
&+ \int_0^T\!\!\int_\Omega B_3(t,x) \int_0^T\!\!\int_\Omega (\theta(s,y)-\theta(t,x)) j_2(s,y)\zeta_\varepsilon^{\checkmark}(t-s,x-y)\, dsdy\, dtdx \\
&- \int_0^T\!\!\int_\Omega B_2(t,x) \int_0^T\!\!\int_\Omega (\theta(s,y)-\theta(t,x)) j_3(s,y)\zeta_\varepsilon^{\checkmark}(t-s,x-y)\, dsdy\, dtdx \\
=& M_1 + R_1(\theta).
\end{aligned}$$

By using the perturbed Maxwell equations as well as the *Proposition* 4.6 the main term M_1 can be written as :

$$\begin{aligned}
M_1 =& \int_0^T\!\!\int_\Omega \theta(t,x)\{E_1\varepsilon_0 \mathrm{div}\ E + \varepsilon_0 B_2(\alpha E_3 + \partial_t E_3 - c_0^2(\mathrm{rot}\ B)_3) - \varepsilon_0 B_3(\alpha E_2 + \partial_t E_2 - c_0^2(\mathrm{rot}\ B)_2)\}\, dtdx \\
=& \int_0^T\!\!\int_\Omega \theta(t,x)\{\varepsilon_0(E_1 \mathrm{div}\ E - (E\wedge \mathrm{rot}\ E)_1) + \frac{1}{\mu_0}(B_1 \mathrm{div}\ B - (B\wedge\mathrm{rot}\ B)_1) - 2\alpha\varepsilon_0(E\wedge B)_1\}\, dtdx \\
&+ \int_0^T\!\!\int_\Omega \varepsilon_0 (E\wedge B)_1 \partial_t\theta\, dtdx.
\end{aligned}$$

Finally we obtain for $1 \leq i \leq 3$ that :

$$\int_0^T\!\!\int_\Omega \{\varepsilon_0(E_i \mathrm{div}\ E - (E \wedge \mathrm{rot}\ E)_i) + \frac{1}{\mu_0}(B_i \mathrm{div}\ B - (B \wedge \mathrm{rot}\ B)_i)\}\theta(t,x)\ dtdx + \int_0^T\!\!\int_\Omega \varepsilon_0 (E \wedge B)_i \partial_t \theta\ dtdx$$
$$+ \int_0^T\!\!\int_\Omega \left(\int_{\mathbb{R}^3_p} p_i f\ dp \right) \partial_t \theta\ dtdx + \int_0^T\!\!\int_\Omega \left(\int_{\mathbb{R}^3_p} p_i v(p) f\ dp \right) \cdot \nabla_x \theta\ dtdx$$
$$= \alpha \int_0^T\!\!\int_\Omega \left(2\varepsilon_0 (E \wedge B)_i + \int_{\mathbb{R}^3_p} p_i f\ dp \right) \theta(t,x)\ dtdx$$
$$+ \int_0^T\!\!\int_{\partial\Omega} \theta(t,x) \left(\int_{\mathbb{R}^3_p} (v(p) \cdot n(x)) p_i \gamma f\ dp \right)\ dtd\sigma - R_i(\theta),$$

for all $\theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$, T periodic. Let us introduce the notations :

$$v_i = 2\alpha\varepsilon_0 (E \wedge B)_i + \alpha \int_{\mathbb{R}^3_p} p_i f\ dp \in L^1(]0,T[\times\Omega),\ 1 \leq i \leq 3,$$

$$w_i = \int_{\mathbb{R}^3_p} p_i v(p) f\ dp \in L^1(]0,T[; L^1(\Omega)^3),\ 1 \leq i \leq 3,$$

$$n \cdot w_i = \int_{\mathbb{R}^3_p} (v(p) \cdot n(x)) p_i \gamma f\ dp \in L^1(]0,T[\times\partial\Omega),\ 1 \leq i \leq 3,$$

$$z_i = \varepsilon_0 (E \wedge B)_i + \int_{\mathbb{R}^3_p} p_i f\ dp \in L^1(]0,T[\times\Omega),\ 1 \leq i \leq 3.$$

Then we have for $1 \leq i \leq 3$, $\theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$, T periodic :

$$\int_0^T\!\!\int_\Omega \{\varepsilon_0(E_i \mathrm{div}\ E - (E \wedge \mathrm{rot}\ E)_i) + \frac{1}{\mu_0}(B_i \mathrm{div}\ B - (B \wedge \mathrm{rot}\ B)_i)\}\theta(t,x)\ dtdx$$
$$+ \int_0^T\!\!\int_\Omega \{z_i(t,x)\partial_t\theta + w_i(t,x) \cdot \nabla_x \theta\}\ dtdx \tag{5.10}$$
$$= \int_0^T\!\!\int_\Omega v_i(t,x)\theta(t,x)\ dtdx + \int_0^T\!\!\int_{\partial\Omega} (n \cdot w_i)\theta(t,x)\ dtd\sigma - R_i(\theta).$$

We can not apply directly the *Lemma* 5.3 since we have the extra term R_i in (5.10). By performing the same computations as in the proof of *Lemma* 5.2 we deduce that there is a constant C such that :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} (p \cdot v(p)) f\ dtdxdp + \int_0^T\!\!\int_\Omega \{\varepsilon_0 |E|^2 + \frac{1}{\mu_0}|B|^2\}\ dtdx + \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 (n \cdot E)^2 + \frac{1}{\mu_0}(n \cdot B)^2\}\ dtd\sigma$$
$$\leq C\{\int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 |n \wedge E|^2 + \frac{1}{\mu_0}|n \wedge B|^2\}\ dtd\sigma + \alpha\varepsilon_0 \int_0^T\!\!\int_\Omega |E \wedge B|\ dtdx$$
$$+ \alpha \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p} |p| f\ dtdxdp + \int_0^T\!\!\int_\Sigma |(v(p) \cdot n(x))| \cdot |p| \cdot \gamma f\ dtd\sigma dp + \sum_{i=1}^3 |R_i|\}, \tag{5.11}$$

where R_i is the corresponding term to the test function $\theta(x) = x_i$, for example :

$$R_1 = \int_0^T\int_\Omega E_1(t,x)\int_0^T\int_\Omega (y_1 - x_1)\rho(s,y)\zeta_\varepsilon^\vee(t-s,x-y)\;dsdy\;dtdx$$
$$+\int_0^T\int_\Omega B_3(t,x)\int_0^T\int_\Omega (y_1 - x_1)j_2(s,y)\zeta_\varepsilon^\vee(t-s,x-y)\;dsdy\;dtdx$$
$$-\int_0^T\int_\Omega B_2(t,x)\int_0^T\int_\Omega (y_1 - x_1)j_3(s,y)\zeta_\varepsilon^\vee(t-s,x-y)\;dsdy\;dtdx.$$

Note that from the *Proposition* 4.4 and the *Theorem* 4.7 we already have an uniform estimate for the tangential traces $(n\wedge E, n\wedge B)$ in $L^2(]0,T[;L^2(\partial\Omega)^3)$. On the other hand, by using one more time (4.5), (4.10) one gets :

$$2\alpha\varepsilon_0 c_0\int_0^T\int_\Omega |E\wedge B|\;dtdx \le 2\alpha\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_0^T\int_\Omega |E(t,x)|\cdot|B(t,x)|\;dtdx \tag{5.12}$$
$$\le\alpha\int_0^T\int_\Omega\{\varepsilon_0|E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\;dtdx$$
$$\le\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|\mathcal{E}(p)g(t,x,p)\;dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma.$$

We have also :

$$\alpha\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}|p|f(t,x,p)\;dtdxdp \le \alpha C\cdot\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f(t,x,p)\;dtdxdp \tag{5.13}$$
$$\le C\cdot\left(\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\;dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma\right),$$

and :

$$\int_0^T\int_\Sigma |(v(p)\cdot n(x))|\cdot|p|\gamma f(t,x,p)\;dtd\sigma dp \le C\cdot\int_0^T\int_\Sigma|(v(p)\cdot n(x))|(1+\mathcal{E}(p))\gamma f(t,x,p)\;dtd\sigma dp$$
$$\le C\cdot\left(\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\;dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma\right). \tag{5.14}$$

Now we need to estimate the term $R_i,\ 1\le i\le 3$. We have :

$$|R_1| \le \varepsilon_2\int_0^T\int_\Omega\{(|\overline{E}_1|\star\zeta_\varepsilon)\cdot|\rho| + (|\overline{B}_3|\star\zeta_\varepsilon)\cdot|j_2| + (|\overline{B}_2|\star\zeta_\varepsilon)\cdot|j_3|\}\;dtdx.$$

Note that for both classical or relativistic case we have :

$$|\rho(t,x)| = |q|\int_{\mathbb{R}^3_p} f\cdot\mathbf{1}_{\{|p|\le R\}}\;dp + |q|\int_{\mathbb{R}^3_p} f\cdot\mathbf{1}_{\{|p|>R\}}\;dp \le C\cdot R^3\|g\|_\infty + \frac{C}{R}\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f\;dp$$
$$\le C\cdot\|g\|_\infty^{1/4}\cdot\left(\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f\;dp\right)^{3/4},$$

and thus $\|\rho\|_{L^{\frac{4}{3}}(]0,T[\times\Omega)} \le C\cdot\|g\|_\infty^{1/4}\cdot\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f\;dtdxdp\right)^{3/4} \le \frac{C}{\alpha^{3/4}}$, where for the last inequality we used (4.5), (4.10). Thus, by using the Hölder and Young inequalities and (4.5), (4.10) we deduce that :

$$\int_0^T\!\!\int_\Omega (|\overline{E}_1|\star\zeta_\varepsilon)\cdot|\rho(t,x)|\;dtdx \le \|(|\overline{E}_1|\star\zeta_\varepsilon)\|_{L^4}\cdot\|\rho\|_{L^{\frac43}} \le \|\overline{E}_1\|_{L^2}\cdot\|\zeta_\varepsilon\|_{L^{\frac43}}\cdot\|\rho\|_{L^{\frac43}(]0,T[\times\Omega)}$$
$$\le \frac{C}{\alpha^{1/2}\cdot\varepsilon_1^{1/4}\cdot{\varepsilon_2}^{3/4}}\|\zeta_3\|_{L^{\frac43}}\cdot\|\zeta_2\|_{L^{\frac43}}\cdot\frac{C}{\alpha^{3/4}}$$
$$\le \frac{C}{\alpha^{5/4}\cdot\varepsilon_1^{1/4}\cdot\varepsilon_2^{3/4}}.$$

In the relativistic case we have also $\|j\|_{L^{\frac43}(]0,T[\times\Omega)}\le C\cdot\alpha^{-3/4}$ and thus we obtain similar bounds for the terms $\int_0^T\!\!\int_\Omega(|\overline{B}_3|\star\zeta_\varepsilon)\cdot|j_2|\;dtdx$, $\int_0^T\!\!\int_\Omega(|\overline{B}_2|\star\zeta_\varepsilon)\cdot|j_3|\;dtdx$, which implies that $|R_1|\le C\cdot\alpha^{-5/4}\cdot\varepsilon_2^{1/4}\cdot{\varepsilon_1}^{-1/4}$. In the classical case we can estimate j by interpolation :

$$|j(t,x)|\le|q|\int_{\mathbb{R}^3_p}\frac{|p|}{m}f\;dp\le C\cdot\|g\|_\infty\cdot R^4+\frac{2|q|}{R}\int_{\mathbb{R}^3_p}\frac{|p|^2}{2m}f(t,x,p)\;dp\le C\cdot\|g\|_\infty^{1/5}\cdot\left(\int_{\mathbb{R}^3_p}\mathcal{E}(p)f\;dp\right)^{4/5},$$

and thus $\|j\|_{L^{\frac54}(]0,T[\times\Omega)}\le C\cdot\|g\|_\infty^{1/5}\cdot\left(\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f\;dtdxdp\right)^{4/5}\le C\cdot\alpha^{-4/5}$. Now, by using the Hölder and Young inequalities we deduce that :

$$\int_0^T\!\!\int_\Omega (|\overline{B}_3|\star\zeta_\varepsilon)\cdot|j_2(t,x)|\;dtdx \le \|(|\overline{B}_3|\star\zeta_\varepsilon)\|_{L^5}\cdot\|j\|_{L^{\frac54}} \le \|\overline{B}_3\|_{L^2}\cdot\|\zeta_\varepsilon\|_{L^{\frac{10}{7}}}\cdot\|j\|_{L^{\frac54}(]0,T[\times\Omega)}$$
$$\le \frac{C}{\alpha^{1/2}\cdot\varepsilon_1^{3/10}\cdot{\varepsilon_2}^{9/10}}\|\zeta_3\|_{L^{\frac{10}{7}}}\cdot\|\zeta_2\|_{L^{\frac{10}{7}}}\cdot\frac{C}{\alpha^{4/5}}$$
$$\le \frac{C}{\alpha^{13/10}\cdot\varepsilon_1^{3/10}\cdot\varepsilon_2^{9/10}}.$$

Therefore, in the classical case we have $|R_1|\le C\cdot\alpha^{-5/4}\cdot\varepsilon_2^{1/4}\cdot{\varepsilon_1}^{-1/4}+C\cdot\alpha^{-13/10}\cdot\varepsilon_2^{1/10}\cdot{\varepsilon_1}^{-3/10}$. The conclusion follows easily by observing that $(p\cdot v(p))\ge\mathcal{E}(p),\ \forall p\in\mathbb{R}^3_p$ and by combining (4.5), (4.10), (5.11), (5.12), (5.13), (5.14) we can take $q=\frac{13}{10}, r=\frac{3}{10}, s=\frac{1}{10}$. We deduce that there is $t_0\in]0,T[$ such that :

$$\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f(t_0,x,p)\;dxdp+\int_\Omega\{\varepsilon_0|E(t_0,x)|^2+\frac{1}{\mu_0}|B(t_0,x)|^2\}\;dx$$
$$\le\frac{C}{T}\left\{\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\;dtd\sigma dp+\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma\right\}+\frac{C}{T}\frac{\varepsilon_2^s}{\alpha^q\cdot\varepsilon_1^r}.$$

By the *Remark* 4.5 we deduce that for $t\in]t_0,t_0+T[$ we have :

$$\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f(t,x,p)\;dxdp+\frac12\int_\Omega\{\varepsilon_0|E(t,x)|^2+\frac{1}{\mu_0}|B(t,x)|^2\}\;dx$$
$$\le\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f(t_0,x,p)\;dxdp+\frac12\int_\Omega\{\varepsilon_0|E(t_0,x)|^2+\frac{1}{\mu_0}|B(t_0,x)|^2\}\;dx$$
$$+\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|\mathcal{E}(p)g(s,x,p)\;dsd\sigma dp+\frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(s,x)|^2\;dsd\sigma,$$

and the last conclusion follows by time periodicity. □

6. The Vlasov-Maxwell system.

In this section we prove our main result concerning the existence of T periodic weak solution for the Vlasov-Maxwell system.

THEOREM 6.1. *Assume that Ω is bounded, with $\partial\Omega$ smooth and strictly star-shaped, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-), h$ are T periodic such that $g \geq 0$, $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$ and :*

$$W_0 := \int_0^T\!\!\int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p))g(t,x,p)\ dtd\sigma dp + \int_0^T\!\!\int_{\partial\Omega} |h(t,x)|^2\ dtd\sigma < +\infty.$$

Then there is a T periodic solution (in distributions) $(f, E, B) \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p) \times L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$ for the Vlasov-Maxwell system (classical or relativistic case) :

$$\partial_t f + v(p) \cdot \nabla_x f + q(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0,\ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p,$$

$$\partial_t E - c_0^2 \cdot rot\ B = -\frac{j}{\varepsilon_0},\ \ \partial_t B + rot\ E = 0,\ \ div\ E = \frac{\rho}{\varepsilon_0},\ \ div\ B = 0,\ \ (t,x) \in \mathbb{R}_t \times \Omega,$$

$$f(t,x,p) = g(t,x,p),\ (t,x,p) \in \mathbb{R}_t \times \Sigma^-,\ \ n \wedge E(t,x) + c_0 \cdot n \wedge (n \wedge B(t,x)) = h(t,x),\ \ (t,x) \in \mathbb{R}_t \times \partial\Omega.$$

Moreover the continuity equation is satisfied $\partial_t \rho + div\, j = 0$ in $\mathcal{D}'(]0,T[\times\Omega)$, there is trace functions $\gamma^+ f$, $\|\gamma^+ f\|_\infty \leq \|g\|_\infty$, normal and tangential traces $(n \cdot E, n \cdot B), (n \wedge E, n \wedge B)$ and for some constant C depending on $m, \varepsilon_0, \mu_0, \Omega$ we have :

$$\begin{aligned} ess\ sup_{s\in\mathbb{R}} \{ & \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(s,x,p)\ dxdp + \int_\Omega \{\varepsilon_0 |E(s,x)|^2 + \frac{1}{\mu_0}|B(s,x)|^2\}\ dx\} \\ & + \int_0^T\!\!\int_{\Sigma^+} (v(p) \cdot n(x))(1 + \mathcal{E}(p))\gamma^+ f(t,x,p)\ dtd\sigma dp \\ & + \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 \left[(n \cdot E)^2 + |n \wedge E|^2\right] + \frac{1}{\mu_0}\left[(n \cdot B)^2 + |n \wedge B|^2\right]\}\ dtd\sigma \\ & \leq C \cdot W_0. \end{aligned}$$

Proof. We take $\eta > 0$ a small parameter and we consider $0 < \alpha, \varepsilon_1, \varepsilon_2 < 1$ such that $\alpha^q = \eta^{1/4}, \varepsilon_1^r = \eta^{1/4}$ and $\varepsilon_2^s = \eta$ where $q, r, s > 0$ are given in *Proposition* 5.6. We regularize also the boundary data h by taking $h_\eta \to h$ in $L^2(]0,T[; L^2(\partial\Omega)^3)$ such that there is $\tilde{h}_\eta \in C^1(\mathbb{R}_t; H^1(\Omega)^3) \cap C^2(\mathbb{R}_t; L^2(\Omega)^3)$ T periodic with $n \wedge \tilde{h}_\eta|_{\mathbb{R}_t \times \partial\Omega} = h_\eta$. From the *Theorems* 4.3, 4.7 applied to the boundary data (g, h_η) we deduce the existence of a T periodic solution for the perturbed Vlasov-Maxwell system (f_η, E_η, B_η). By the *Proposition* 5.6 we deduce that for all $s \in \mathbb{R}_t$:

$$\begin{aligned} & \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f_\eta(s,x,p)\ dxdp + \int_\Omega \{\varepsilon_0 |E_\eta(s,x)|^2 + \frac{1}{\mu_0}|B_\eta(s,x)|^2\}\ dx \\ & + \int_0^T\!\!\int_{\Sigma^+} (v(p) \cdot n(x))(1 + \mathcal{E}(p))\gamma^+ f_\eta(t,x,p)\ dtd\sigma dp \\ & + \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 \left[(n \cdot E_\eta)^2 + |n \wedge E_\eta|^2\right] + \frac{1}{\mu_0}\left[(n \cdot B_\eta)^2 + |n \wedge B_\eta|^2\right]\}\ dtd\sigma \\ & \leq C \cdot W_{0,\eta} + C \cdot \eta^{1/2}, \end{aligned} \tag{6.1}$$

where $W_{0,\eta} = \int_0^T\!\int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p))g(t,x,p)\ dtd\sigma dp + \int_0^T\!\int_{\partial\Omega} |h_\eta(t,x)|^2\ dtd\sigma$. After extraction of subsequences we can suppose that there is $f, \gamma^+ f, E, B, (n \cdot E), (n \cdot B), n \wedge E, n \wedge B$ such that

$f_k := f_{\eta_k} \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, $\gamma^+ f_k := \gamma^+ f_{\eta_k} \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$, $(E_k, B_k) := (E_{\eta_k}, B_{\eta_k}) \rightharpoonup (E, B)$ weakly in $L^2(]0,T[; L^2(\Omega)^3)^2$, $(n \cdot E_k, n \cdot B_k) := (n \cdot E_{\eta_k}, n \cdot B_{\eta_k}) \rightharpoonup (n \cdot E, n \cdot B)$ weakly in $L^2(]0,T[\times\partial\Omega)^2$, $(n \wedge E_k, n \wedge B_k) := (n \wedge E_{\eta_k}, n \wedge B_{\eta_k}) \rightharpoonup (n \wedge E, n \wedge B)$ weakly in $L^2(]0,T[; L^2(\partial\Omega)^3)^2$, where $\eta_k \to 0$, as $k \to +\infty$. We note also by $\alpha_k, \varepsilon_{1,k}, \varepsilon_{2,k}$ the values given by $\alpha_k^q = \eta_k^{1/4}, \varepsilon_{1,k}^r = \eta_k^{1/4}, \varepsilon_{2,k}^s = \eta_k$. Obviously, by weak $\star$ convergence we have :

$$\max(\|f\|_\infty, \|\gamma^+ f\|_\infty) \le \max(\liminf_{k\to+\infty} \|f_k\|_\infty, \liminf_{k\to+\infty} \|\gamma^+ f_k\|_\infty) \le \|g\|_\infty.$$

For $R > 0$, $\theta \in C^0(\mathbb{R}_t)$, $\theta \ge 0$ we have :

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} \theta(t)\mathcal{E}(p)\cdot \mathbf{1}_{\{|p|\le R\}} f(t,x,p)\; dtdxdp + \int_0^T\!\!\int_\Omega \theta(t)\{\varepsilon_0 |E(t,x)|^2 + \frac{1}{\mu_0}|B(t,x)|^2\}\; dtdx \\
&\quad\le \liminf_{k\to+\infty} \left\{ \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} \theta(t)\mathcal{E}(p)\cdot \mathbf{1}_{\{|p|\le R\}} f_k\; dtdxdp + \int_0^T\!\!\int_\Omega \theta(t)\{\varepsilon_0 |E_k|^2 + \frac{1}{\mu_0}|B_k|^2\}\; dtdx \right\} \\
&\quad\le \liminf_{k\to+\infty} \int_0^T \theta(t)(C\cdot W_{0,k} + C\cdot \eta_k^{1/2})\; dt \\
&\quad= C\cdot W_0 \int_0^T \theta(t)\; dt.
\end{aligned}$$

By letting $R \to +\infty + \infty$ we deduce that :

$$\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(s,x,p)\; dxdp + \int_\Omega \{\varepsilon_0 |E(s,x)|^2 + \frac{1}{\mu_0}|B(s,x)|^2\}\; dx \le C\cdot W_0, \quad \text{a.e. } s \in \mathbb{R}_t.$$

Similarly we have :

$$\begin{aligned}
&\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f(t,x,p)\mathbf{1}_{\{|p|\le R\}}\; dtd\sigma dp \\
&\qquad + \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 \left[(n\cdot E)^2 + |n\wedge E|^2\right] + \frac{1}{\mu_0}\left[(n\cdot B)^2 + |n\wedge B|^2\right]\}\; dtd\sigma \\
&\qquad \le \liminf_{k\to+\infty} (\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f_k(t,x,p)\mathbf{1}_{\{|p|\le R\}}\; dtd\sigma dp \\
&\qquad + \int_0^T\!\!\int_{\partial\Omega} \{\varepsilon_0 \left[(n\cdot E_k)^2 + |n\wedge E_k|^2\right] + \frac{1}{\mu_0}\left[(n\cdot B_k)^2 + |n\wedge B_k|^2\right]\}\; dtd\sigma) \\
&\qquad \le C\cdot W_0, \quad \forall R > 0.
\end{aligned}$$

We check that (f, E, B) is a T periodic solution of the Vlasov-Maxwell system. Indeed, by the Green formula we have for all function $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$ T periodic, with compact support in momentum :

$$\begin{aligned}
&-\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f_k(-\alpha_k\cdot\varphi(t,x,p) + \partial_t\varphi + v(p)\cdot\nabla_x\varphi + q((\overline{E}_k \star \zeta_{\varepsilon_k}) + v(p)\wedge(\overline{B}_k\star\zeta_{\varepsilon_k}))\cdot\nabla_p\varphi)\; dtdxdp \\
&\qquad = -\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f_k\varphi\; dtd\sigma dp - \int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x)) g\varphi\; dtd\sigma dp.
\end{aligned}$$

Observe that $\alpha_k \int_0^T\!\int_\Omega \int_{\mathbb{R}^3_p} \varphi f_k\; dtdxdp \to 0$ since $\|f_k\|_\infty \le \|g\|_\infty$, and $(\overline{E}_k \star \zeta_{\varepsilon_k}, \overline{B}_k \star \zeta_{\varepsilon_k}) \rightharpoonup (E, B)$ weakly in $L^2(]0,T[; L^2(\Omega)^3)$, as $k \to +\infty$. Note that the compactness average result of DiPerna and Lions [14] adapts easily in the time periodic case and still holds true for bounded spatial domains (use a cut-off function $\eta \in C^\infty_c(\Omega)$ and write $f_k = \eta f_k + (1-\eta) f_k$ as it was done in [21] page 256).

For test functions $\varphi(t,x,p) = \varphi_1(t,x)\cdot\varphi_2(p)$, $\varphi_1 \in C^1(\mathbb{R}_t \times \overline{\Omega})$ T periodic, $\varphi_2 \in C^1_c(\mathbb{R}^3_p)$, by using the velocity average result we have :

$$\lim_{k\to+\infty} \int_{\mathbb{R}^3_p} \nabla_p\varphi_2 f_k \; dp = \int_{\mathbb{R}^3_p} \nabla_p\varphi_2 f \; dp, \text{ in } L^2(]0,T[;L^2(\Omega)^3),$$

$$\lim_{k\to+\infty} \int_{\mathbb{R}^3_p} (\nabla_p\varphi_2 \wedge v(p)) f_k \; dp = \int_{\mathbb{R}^3_p} (\nabla_p\varphi_2 \wedge v(p)) f \; dp, \text{ in } L^2(]0,T[;L^2(\Omega)^3),$$

and thus by combining strong and weak convergence we can pass to the limit in the non linear term :

$$\begin{aligned}\lim_{k\to+\infty} \quad & \int_0^T\!\!\int_\Omega \{\varphi_1(\overline{E}_k \star \zeta_{\varepsilon_k})\cdot \int_{\mathbb{R}^3_p} \nabla_p\varphi_2 f_k \; dp + \varphi_1(\overline{B}_k \star \zeta_{\varepsilon_k})\cdot \int_{\mathbb{R}^3_p} (\nabla_p\varphi_2 \wedge v(p)) f_k \; dp\} \; dtdx \\ = & \int_0^T\!\!\int_\Omega \{\varphi_1 E \cdot \int_{\mathbb{R}^3_p} \nabla_p\varphi_2 f \; dp + \varphi_1 B \cdot \int_{\mathbb{R}^3_p} (\nabla_p\varphi_2 \wedge v(p)) f \; dp\} \; dtdx. \end{aligned} \tag{6.2}$$

Finally we deduce that :

$$\begin{aligned} -\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} \quad & f(t,x,p)(\partial_t\varphi + v(p)\cdot\nabla_x\varphi + q(E(t,x) + v(p)\wedge B(t,x))\cdot\nabla_p\varphi) \; dtdxdp \\ = & -\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\varphi \; dtd\sigma dp - \int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x)) g\varphi \; dtd\sigma dp, \; \forall \varphi = \varphi_1\cdot\varphi_2, \end{aligned}$$

and by density the previous equality holds for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$ T periodic and compactly supported in momentum. Now take $\varphi, \psi \in C^1(\mathbb{R}_t \times \overline{\Omega})^3$, T periodic with $n \wedge \varphi - c_0 n \wedge (n \wedge \psi) = 0$ on $\mathbb{R}_t \times \partial\Omega$. We have :

$$\begin{aligned} & \int_0^T\!\!\int_\Omega \{E_k\cdot(\alpha_k\varphi - \partial_t\varphi) - c_0^2 B_k \cdot \text{rot } \varphi\} \; dtdx + c_0^2 \int_0^T\!\!\int_\Omega \{B_k\cdot(\alpha_k\psi - \partial_t\psi) + E_k\cdot \text{rot } \psi\} \; dtdx \\ & = c_0 \int_0^T\!\!\int_{\partial\Omega} (n\wedge\varphi)\cdot h_k \; dtd\sigma - \frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega (\overline{j}_k \star \check{\zeta}_{\varepsilon_k})\cdot\varphi \; dtdx. \end{aligned}$$

Since $(E_k, B_k)_k$ is bounded in $L^2(]0,T[;L^2(\Omega)^3)$ we have :

$$\lim_{k\to+\infty} \alpha_k \int_0^T\!\!\int_\Omega E_k(t,x)\cdot\varphi(t,x) \; dtdx = \lim_{k\to+\infty} \alpha_k \int_0^T\!\!\int_\Omega B_k(t,x)\cdot\psi(t,x) \; dtdx = 0.$$

We have also :

$$\begin{aligned} \left| \int_0^T\!\!\int_\Omega (\overline{j}_k \star \check{\zeta}_{\varepsilon_k})\cdot\varphi \; dtdx - \int_0^T\!\!\int_\Omega j\cdot\varphi \; dtdx \right| \le & \left| \int_0^T\!\!\int_\Omega j_k \cdot(\overline{\varphi}\star\zeta_{\varepsilon_k} - \varphi) \; dtdx \right| \\ & + \left| \int_0^T\!\!\int_\Omega (j_k - j)\cdot\varphi \; dtdx \right| = I_1^k + I_2^k. \end{aligned}$$

For the first term I_1^k we write :

$$
\begin{aligned}
|q|^{-1}\cdot I_1^k \leq & \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}|v(p)|\cdot f_k(t,x,p)\cdot|(\overline{\varphi}\star\zeta_{\varepsilon_k})(t,x)-\varphi(t,x)|\cdot\mathbf{1}_{\{|p|\leq R\}}\,dtdxdp\\
&+\frac{1}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}|v(p)|\cdot|p|\cdot f_k(t,x,p)\cdot 2\cdot\|\varphi\|_\infty\cdot\mathbf{1}_{\{|p|>R\}}\,dtdxdp\\
\leq & \int_0^T\!\!\int_\Omega|(\overline{\varphi}\star\zeta_{\varepsilon_k})(t,x)-\varphi(t,x)|\left(\int_{\mathbb{R}^3_p}|v(p)|\cdot\|g\|_\infty\cdot\mathbf{1}_{\{|p|\leq R\}}\,dp\right)\,dtdx\\
&+\frac{4\|\varphi\|_\infty}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f_k(t,x,p)\,dtdxdp.
\end{aligned}
$$

For $\delta>0$ we take R_δ large enough such that $\frac{4\|\varphi\|_\infty}{R_\delta}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f_k(t,x,p)\,dtdxdp<\frac{\delta}{2}$, uniformly in k. By using the dominated convergence theorem we have also :

$$
\lim_{k\to+\infty}\int_0^T\!\!\int_\Omega|(\overline{\varphi}\star\zeta_{\varepsilon_k})(t,x)-\varphi(t,x)|\left(\int_{\mathbb{R}^3_p}|v(p)|\cdot\|g\|_\infty\cdot\mathbf{1}_{\{|p|\leq R_\delta\}}\,dp\right)\,dtdx=0,
$$

and therefore $\lim_{k\to+\infty}I_1^k=0$. For the second term we have :

$$
|q|^{-1}I_2^k\leq\left|\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}v(p)\varphi(f_k-f)\mathbf{1}_{\{|p|\leq R\}}\,dtdxdp\right|+\left|\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}v(p)\varphi(f_k-f)\mathbf{1}_{\{|p|>R\}}\,dtdxdp\right|=I_3^k+I_4^k.
$$

By using the inequality $|p|\cdot|v(p)|=(p\cdot v(p))\leq 2\mathcal{E}(p)$ we have for R large enough :

$$
\begin{aligned}
I_4^k\leq & \frac{1}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}f_k\cdot\|\varphi\|_\infty\cdot|p|\cdot|v(p)|\,dtdxdp+\frac{1}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}f\cdot\|\varphi\|_\infty\cdot|p|\cdot|v(p)|\,dtdxdp\\
\leq & \frac{2\|\varphi\|_\infty}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}f_k\mathcal{E}(p)\,dtdxdp+\frac{2\|\varphi\|_\infty}{R}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}f\mathcal{E}(p)\,dtdxdp<\frac{\delta}{2},
\end{aligned}
$$

for $R\geq R_\delta$, since $\left(\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f_k\,dtdxdp\right)_k$ is bounded. Now, for $R=R_\delta$ we can use the weak $\star$ convergence $f_k\rightharpoonup f$ in order to obtain $I_3^k<\frac{\delta}{2}$, for $k\geq k_\delta$. Finally we proved the convergence $\lim_{k\to+\infty}\int_0^T\!\!\int_\Omega(\overline{j}_k\star\zeta_{\varepsilon_k}^{\surd})\cdot\varphi\,dtdx=\int_0^T\!\!\int_\Omega j\cdot\varphi\,dtdx$. By using the weak convergence $(E_k,B_k)\rightharpoonup(E,B)$ in $L^2(]0,T[;L^2(\Omega)^3)^2$ and $h_k\to h$ in $L^2(]0,T[;L^2(\partial\Omega)^3)$ we deduce that :

$$
\begin{aligned}
\int_0^T\!\!\int_\Omega\{-E(t,x)\cdot\partial_t\varphi-c_0^2B(t,x)\cdot\text{rot }\varphi\}\,dtdx+c_0^2\int_0^T\!\!\int_\Omega\{-B(t,x)\cdot\partial_t\psi+E(t,x)\cdot\text{rot }\psi\}\,dtdx\\
=c_0\int_0^T\!\!\int_{\partial\Omega}(n\wedge\varphi)\cdot h(t,x)\,dtd\sigma-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j(t,x)\cdot\varphi(t,x)\,dtdx.
\end{aligned}
$$

In fact, by using the *Remark* 3.13, we can prove that :

$$
\int_0^T\!\!\int_\Omega\{-E(t,x)\cdot\partial_t\varphi-c_0^2B(t,x)\cdot\text{rot }\varphi\}\,dtdx-c_0^2\int_0^T\!\!\int_{\partial\Omega}(n\wedge B)\cdot\varphi(t,x)\,dtd\sigma=-\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega j\cdot\varphi\,dtdx,
$$

and :

$$
\int_0^T\!\!\int_\Omega\{-B(t,x)\cdot\partial_t\psi+E(t,x)\cdot\text{rot }\psi\}\,dtdx+\int_0^T\!\!\int_{\partial\Omega}(n\wedge E)\cdot\psi(t,x)\,dtd\sigma=0,
$$

for all $\varphi, \psi \in C^1(\mathbb{R}_t \times \overline{\Omega})^3$, T periodic. By the *Proposition* 5.6 and by using that div $E_k = \frac{1}{\varepsilon_0}(\overline{\rho}_k \star \zeta_{\varepsilon_k}^{\checkmark})$ we have for $\theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$:

$$\int_0^T\int_\Omega \frac{1}{\varepsilon_0}(\overline{\rho}_k \star \zeta_{\varepsilon_k}^{\checkmark})\theta(t,x)\ dtdx + \int_0^T\int_\Omega E_k(t,x)\cdot\nabla_x\theta\ dtdx = \int_0^T\int_{\partial\Omega}(n\cdot E_k)\theta(t,x)\ dtd\sigma.$$

As previous we have $\lim_{k\to+\infty}\int_0^T\int_\Omega(\overline{\rho}_k \star \zeta_{\varepsilon_k}^{\checkmark})\theta\ dtdx = \int_0^T\int_\Omega \rho\theta\ dtdx$ and thus, by passing to the limit for $k \to +\infty$ one gets :

$$\int_0^T\int_\Omega \frac{1}{\varepsilon_0}\rho\theta\ dtdx + \int_0^T\int_\Omega E\cdot\nabla_x\theta\ dtdx = \int_0^T\int_{\partial\Omega}(n\cdot E)\theta\ dtd\sigma,$$

or div $E = \frac{1}{\varepsilon_0}\rho$ and $(n\cdot E)$ is the normal trace of E. Similarly we deduce that div $B = 0$ and $(n\cdot B)$ is the normal trace of B. By using the Green formula (2.7) with the test function $\theta = \theta(t,x)$, $\theta \in C^1(\mathbb{R}_t \times \overline{\Omega})$, T periodic, we have :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}\alpha_k f_k\theta\ dtdxdp - \int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f_k(\partial_t\theta + v(p)\cdot\nabla_x\theta)\ dtdxdp + \int_0^T\int_\Sigma(v(p)\cdot n(x))\theta\gamma f_k\ dtd\sigma dp = 0.$$

By using that $\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f_k\ dtdxdp\right)_k$, $\left(\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f_k\ dtd\sigma dp\right)_k$ are bounded, after passing to the limit for $k \to +\infty$ we deduce that $-\int_0^T\int_\Omega \rho\partial_t\theta\ dtdx - \int_0^T\int_\Omega j\cdot\nabla_x\theta\ dtdx + \int_0^T\int_\Sigma(v(p)\cdot n(x))\theta q\gamma f\ dtd\sigma dp = 0$ and in particular $\partial_t\rho + \text{div}\ j = 0$ in $\mathcal{D}'$.
□

7. Final remarks.

By using basically the same arguments, it is possible to analyze also the time periodic Vlasov-Maxwell system when the boundary condition (2.2) is replaced by :

$$f(t,x,p) = g(t,x,p) + a(t,x,p)\cdot f(t,x,R(t,x)p),\ \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{7.1}$$

with $R(t,x) : \mathbb{R}^3_p \to \mathbb{R}^3_p$, $R(t,x)p = p - 2(p\cdot n(x))\cdot n(x), \forall(t,x,p)\in\mathbb{R}_t\times\Sigma$ and $0 \le a(t,x,p) \le a_0 < 1$, $\forall(t,x,p)\in\mathbb{R}_t\times\Sigma^-$ (for the definition of T periodic weak solution for the problem (2.1), (7.1) consider in the weak formulation (2.4) test functions $\varphi \in C^1(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}^3_p)$, T periodic, compactly supported in momentum such that $\varphi(t,x,Rp) = a(t,x,p)\varphi(t,x,p)$, $\forall(t,x,p)\in\mathbb{R}_t\times\Sigma^-$). We assume that g, h are T periodic, $g \ge 0$, $g \in L^\infty(\mathbb{R}_t\times\Sigma^-)$, $(n\cdot h)|_{\mathbb{R}_t\times\partial\Omega} = 0$ such that :

$$W_0 = \int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp + \int_0^T\int_{\partial\Omega}|h(t,x)|^2\ dtd\sigma < +\infty.$$

The proofs are quite similar and we don't detail them. We only indicate how a priori estimates can be obtained in this case by formal computations. For example, by using the weak formulation of the Vlasov problem with the test function 1 we get :

$$\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)\ dtd\sigma dp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))\gamma^- f(t,x,p)\ dtd\sigma dp,$$

and by using the boundary condition (7.1) we deduce that :

$$\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1-a(t,x,Rp))\gamma^+ f(t,x,p)\ dtd\sigma dp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\ dtd\sigma dp.$$

Since $a(t,x,p) \le a_0 < 1$, $\forall(t,x,p)\in\mathbb{R}_t\times\Sigma^-$, finally one gets :

$$\int_0^T\int_{\Sigma^\pm}|(v(p)\cdot n(x))|\gamma^\pm f(t,x,p)\ dtd\sigma dp \le \frac{1}{1-a_0}\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)\ dtd\sigma dp.$$

Similarly, by using the weak formulation of the Vlasov problem with the test function $\mathcal{E}(p)$ and by combining with the conservation of the electro-magnetic energy (obtained by multiplication of the Maxwell equations by (E, B)), we deduce as before that :

$$\int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f(t,x,p)\; dtd\sigma dp + \frac{c_0}{2}\int_0^T\int_{\partial\Omega}\{\varepsilon_0|n\wedge E|^2 + \frac{1}{\mu_0}|n\wedge B|^2\}\; dtd\sigma$$
$$= -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^- f(t,x,p)\; dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\; dtd\sigma,$$

and by using the boundary condition (7.1) we obtain as above that :

$$(1-a_0)\int_0^T\int_{\Sigma^+} |(v(p)\cdot n(x))|\mathcal{E}(p)\gamma^+ f(t,x,p)\; dtd\sigma dp + \frac{c_0}{2}\int_0^T\int_{\partial\Omega}\{\varepsilon_0|n\wedge E|^2 + \frac{1}{\mu_0}|n\wedge B|^2\}\; dtd\sigma$$
$$\le -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x))\mathcal{E}(p)g(t,x,p)\; dtd\sigma dp + \frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\; dtd\sigma,$$

and :

$$(1-a_0)\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p)\gamma^- f(t,x,p)\; dtd\sigma dp + a_0\frac{c_0}{2}\int_0^T\int_{\partial\Omega}\{\varepsilon_0|n\wedge E|^2 + \frac{1}{\mu_0}|n\wedge B|^2\}\; dtd\sigma$$
$$\le -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x))\mathcal{E}(p)g(t,x,p)\; dtd\sigma dp + a_0\frac{c_0\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}|h(t,x)|^2\; dtd\sigma.$$

From this point the computations follow exactly as for the case of absorbing conditions.
Note also that all these results apply for the Vlasov-Maxwell system with several species of particles. We point out that similar a priori estimates can be established by formal computations for the T periodic solutions of the Vlasov-Maxwell-Fokker-Planck system, which is obtained from the Vlasov-Maxwell system by replacing the Vlasov equation by :

$$\partial_t f + v(p)\cdot\nabla_x f + q(E(t,x) + v(p)\wedge B(t,x))\cdot\nabla_p f = \mathrm{div}_p(\sigma\nabla_p f + \beta v(p) f),\;\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

where $\beta \ge 0, \sigma > 0$ are fixed parameters.

REFERENCES

[1] A. Arseneev, *Global existence of a weak solution of the Vlasov system of equations*, U.R.S.S. Comp. and Math. Phys. 15(1975), pp. 131-143.

[2] J. Batt, *Global symmetric solutions of the initial value problem of stellar dynamics*, J. Diff. Eq. 25(1977), pp. 342-364.

[3] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff(1976).

[4] C. Bardos, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre*, Ann. Sci. Ecole Norm. Sup. 3(1969), pp. 185-233.

[5] C. Bardos and P. Degond, *Global existence for the Vlasov-Poisson equation in three space variables with small initial data*, Ann. Inst. H. Poincaré, Anal. non linéaire 2(1985), pp. 101-118.

[6] R. Beals and V. Protopopescu, *Abstract time-dependent transport equations*, J. Math. Anal. Appl. (1987), pp. 370-405.

[7] N. Ben Abdallah, *Weak solutions of the initial-boundary value problem for the Vlasov-Poisson system*, Math. Meth. Appl. Sci. 17(1994), no.6, pp. 451-476.

[8] M. Bostan and F. Poupaud, *Periodic solutions of the 1D Vlasov-Maxwell system with boundary conditions*, Math. Methods Appl. Sci. 23(2000), pp. 1195-1221.

[9] M. Bostan, *Permanent regimes for the 1D Vlasov-Poisson system with boundary conditions*, SIAM J. Math. Anal., Vol. 35, No 4(2003), pp.922-948.

[10] M. Bostan, *Boundary value problem for the N dimensional time periodic Vlasov-Poisson system*, submitted.

[11] H. Brezis, *Analyse fonctionnelle*, Masson (1998).

[12] P. Degond, *Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity*, Math. Methods Appl. Sci. 8 (1986), pp. 533-558.

[13] P. Degond and P.-A. Raviart, *An asymptotic analysis of the one-dimensional Vlasov-Poisson system : the Child-Langmuir law*, Asymptotic Anal. 4 (1991) no.3, pp. 187-214.

[14] R. J. DiPerna and P. L. Lions, *Global weak solutions of the Vlasov-Maxwell system*, Comm. Pure Appl. Math. XVII (1989), pp. 729-757.
[15] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98 (1989), pp. 511-547.
[16] G. Duvaut et J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod (1972).
[17] R. Glassey and J. Schaeffer, *On symetric solutions to the relativistic Vlasov-Poisson system*, Comm. Math. Phy. 101(1985), pp. 459-473.
[18] R. Glassey and W. Strauss, *Singularity formation in a collisionless plasma could only occur at high velocities*, Arch. Rat. Mech. Anal., 92(1986), pp. 56-90.
[19] F. Golse, P.-L. Lions, B. Perthame and R. Sentis, *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal. 88(1988), pp. 110-125.
[20] C. Greengard and P.-A. Raviart, *A boundary value problem for the stationary Vlasov-Poisson equations : the plane diode*, Comm. Pure and Appl. Math. vol. XLIII (1990), pp. 473-507.
[21] Y. Guo, *Global weak solutions of the Vlasov-Maxwell system with boundary conditions*, Commun. Math. Phys. 154, pp. 245-263 (1993).
[22] Y. Guo, *Singular solutions to the Vlasov-Maxwell system in a half line*, Arch. Rational Mech. Anal. 131(1995), pp. 241-304.
[23] E. Horst, *On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation*, Math. Meth. Appl. Sci. 3(1981), pp. 229-248.
[24] E. Horst and R. Hunze, *Weak solutions of the initial value problem for the unmodified nonlinear Vlasov equation*, Math. Meth. in the Appl. Sci. 6(1984), pp. 262-279.
[25] V. Komornik, *Exact controllability and stabilization. The multiplier method*, RAM, Masson, Paris, John Wiley & Sons, Ltd., Chichester (1994).
[26] J. E. Lagnese, *A singular perturbation problem in exact controllability of the Maxwell system*, ESAIM, Control. Optim. Calc. Var., 6(2001), pp. 275-289.
[27] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*, Invent. Math. 105(1991), pp. 415-430.
[28] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, J. Diff. Eq. 95(1992), pp. 281-303.
[29] F. Poupaud, *Boundary value problems for the stationary Vlasov-Maxwell system*, Forum Math. 4(1992), pp. 499-527.
[30] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*, Comm. P.D.E. 16(1991), no. 8-9, pp. 1313-1335.
[31] J. Schaeffer, *Global existence for the Vlasov-Poisson system with nearly symetric data*, J. Diff. Eq. 69(1987), pp. 111-148.
[32] T. Ukai and S. Okabe, *On the classical solution in the large time of the two dimensional Vlasov equations*, Osaka J. Math. 15(1978), pp. 245-261.