The linear Boltzmann equation with space periodic electric field

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Abstract

We investigate the well posedness of the stationary linear Boltzmann equation with space periodic electric field. The existence follows by standard perturbation techniques and stability properties under uniform a priori estimates. The uniqueness (up to a multiplicative constant) of the weak solution holds for space periodic electric fields with non vanishing average, one of the main ingredients being the dissipation properties of the relaxation operator.

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1 Introduction

The subject matter of this paper concerns the free space linear Boltzmann equation

$$v(p)\partial_x f + F(x)\partial_p f = \frac{1}{\tau} Q(f), \quad (x,p) \in \mathbb{R}^2 \tag{1}$$

where $\tau > 0$ is the relaxation time and Q is the relaxation operator defined for any integrable function $g \in L^1(\mathbb{R})$ by

$$Q(g)(p) = \langle g \rangle M_\theta(p) - g(p), \quad \langle g \rangle = \int_{\mathbb{R}} g(p)\,\mathrm{d}p.$$

The function $f = f(x,p)$ represents the number density of a population of charged particles. We are looking for stationary states and therefore f depends on the space $x \in \mathbb{R}$ and the momentum $p \in \mathbb{R}$. The notation $v(p)$ stands for the relativistic velocity

$$v(p) = \frac{p}{m}\left(1 + \frac{p^2}{m^2 c_0^2}\right)^{-1/2}$$

where m is the mass of the particles and c_0 is the light speed in the vacuum. The kinetic energy associated to $v(p)$ is given by

$$\mathcal{E}(p) = m c_0^2 \left(\left(1 + \frac{p^2}{m^2 c_0^2}\right)^{1/2} - 1 \right)$$

and we have $\mathcal{E}\,'(p) = v(p), p \in \mathbb{R}$. The relativistic Maxwellian $M_\theta(p)$ entering the relaxation operator Q is given by

$$M_\theta(p) = \exp\left(-\frac{\mathcal{E}(p)}{\theta}\right)\left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(q)}{\theta}\right)\,\mathrm{d}q\right)^{-1}.$$

The equation (1) models charge transport phenomena, with applications in semiconductor theory or plasma physics [4]. The force field is given by $F = qE$ where q is the charge of the particles and E is the electric field. The boundary value problem of (1) has been studied in [6] by using comparison principles. One of the crucial points was to observe that

$$\mathcal{M}_{\theta,\phi}(x,p) = \exp\left(-\frac{\mathcal{E}(p) + q\phi(x)}{\theta}\right), \quad \phi\,' = -E$$

is a particular solution for (1), vanishing both the transport operator $v(p)\partial_x + F(x)\partial_p$ and the collision operator Q. Using the distribution $\mathcal{M}_{\theta,\phi}$ the author of [6] obtained existence results for the boundary value problem associated to (1) with incoming data comparable with $\mathcal{M}_{\theta,\phi}$. Recently the same problem has been investigated for general integrable data cf. [3].

The aim of this article is to analyze the free space problem (1). As said before, the function $\mathcal{M}_{\theta,\phi}$, and obviously all the multiple of $\mathcal{M}_{\theta,\phi}$, are solutions for (1). But we will see that there are other solutions for the same equation. For example, if the electric field is constant with respect to x and $E \neq 0$ we can find particle densities f depending only on the momentum by solving analytically the ordinary differential equation

$$\frac{1}{\tau} f(p) + qE\frac{df}{dp} = \frac{1}{\tau} M_\theta(p), \quad \int_{\mathbb{R}} f(p) \, \mathrm{d}p = 1. \tag{2}$$

For the explicit formula of this solution in the non relativistic case ($i.e.,$ $v(p) = p/m$) and existence results for boundary value and Milne problems involving this solution we refer to [5], [2]. The natural questions are: what is the physical relevant solution for (1); what is the criterion for selecting the appropriate solution? In the specific case of constant non vanishing electric field the right solution seems to be that given by (2) which remains bounded on $\mathbb{R}^2$, while the distribution $\mathcal{M}_{\theta,\phi}$ becomes unbounded (as $x \to -\infty$ if $qE > 0$ and as $x \to +\infty$ if $qE < 0$). Therefore our selection criterion could be related to the uniform behavior of the solution with respect to the space variable, provided that the electric field is bounded for $x \in \mathbb{R}$. We call such solutions permanent regimes (with respect to $x \in \mathbb{R}$). Surely, one of the main difficulties when dealing with permanent regimes for (1) is the lack of boundary conditions; the absence of these informations has to be compensated with the uniform behavior of the solution with respect to the space variable $x \in \mathbb{R}$. Observe also that since (1) is linear we only can expect uniqueness up to a multiplicative constant. Eventually this constant can be determined by imposing the current $j = q\int_{\mathbb{R}} v(p) f \, \mathrm{d}p$ which it is easily seen to be constant with respect to the space $x \in \mathbb{R}$

$$\frac{d}{dx} \int_{\mathbb{R}} v(p) f \, \mathrm{d}p = 0, \quad x \in \mathbb{R}.$$

Therefore a legitim uniqueness result for permanent regimes could be

Uniqueness. *Consider f, g two permanent solutions for (1) having the same current*

$$q \int_{\mathbb{R}} v(p) f \, \mathrm{d}p = q \int_{\mathbb{R}} v(p) g \, \mathrm{d}p.$$

Then the solutions f, g coincide.

And of course we are left with the difficult task concerning the existence o such permanent solutions. This paper is devoted to the particular case of space periodic solutions. We prove the well posedness of (1) when the electric field is space periodic. Up to our knowledge it is the first work on this direction. Besides the physical relevance of these cases, their study is very interesting from the mathematical point of view. Moreover we expect that similar results could be established for more general cases, as the almost periodic one, by adapting the same techniques. Our main result is the following

Theorem 1.1 *Assume that $E \in L^\infty(\mathbb{R})$ is a bounded L-periodic electric field.*
a) If $\int_0^L E(x) \, \mathrm{d}x = 0$ then all the periodic solutions for the linear Boltzmann equation are of the form $k\mathcal{M}_{\theta,\phi}$ with $k \in \mathbb{R}$.
b) If $\int_0^L E(x) \, \mathrm{d}x \neq 0$ then for any $j \in \mathbb{R}$ there is a unique periodic weak solution f for the linear Boltzmann equation verifying $q \int_{\mathbb{R}} v(p) f \, \mathrm{d}p = j$. Moreover the solution satisfies

$$\mathrm{sgn} f = \mathrm{sgn} \frac{j}{\int_0^L E \, \mathrm{d}x}, \quad (1+\mathcal{E}(p)) f \in L^1([0,L]\times\mathbb{R}), \quad f \in L^\infty([0,L]\times\mathbb{R}), \quad \langle f \rangle \in L^\infty([0,L])$$

and

$$\frac{1}{\tau} \int_0^L \int_{\mathbb{R}} (f - \langle f \rangle M_\theta) \ln\left(\frac{f}{\langle f \rangle M_\theta}\right) \mathrm{d}p \, \mathrm{d}x = \frac{q}{\theta} \int_0^L E(x) \, \mathrm{d}x \int_{\mathbb{R}} v(p) f \, \mathrm{d}p.$$

In particular the solution f is non negative iff $\frac{j}{\int_0^L E \, \mathrm{d}x} \geq 0$.

Our paper is organized as follows. In Section 2 we recall the notion of periodic weak solution and some immediate properties. Section 3 is devoted to the uniqueness result, based on new dissipative properties for the relaxation operator. In the next section we discuss the existence of periodic weak solution: we analyze a penalized periodic problem, we establish a priori estimates and we conclude by stability results.

2 Weak solutions

We assume that $F(x) = qE(x)$ is a given L-periodic bounded force field and we introduce the notion of weak solution (or solution in the sense of distributions) for (1). We consider the spatial periodic domain $\mathbb{T} = \mathbb{R}/(L\mathbb{Z})$.

Definition 2.1 *Assume that F belongs to $L^\infty(\mathbb{T})$. We say that $f \in L^1(\mathbb{T} \times \mathbb{R})$ is a periodic weak solution for (1) iff*

$$-\int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p)(v(p)\partial_x\varphi + F(x)\partial_p\varphi)\ \mathrm{d}p\ \mathrm{d}x = \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} Q(f)\varphi(x,p)\ \mathrm{d}p\ \mathrm{d}x \qquad (3)$$

for any function $\varphi \in C^1_c(\mathbb{T} \times \mathbb{R})$.

It is easily seen that the formulation (3) holds true for any test function $\varphi \in C^1_b(\mathbb{T} \times \mathbb{R})$ (*i.e.,* the set of bounded C^1 functions with bounded partial derivatives). Since f belongs to $L^1(\mathbb{T} \times \mathbb{R})$ and the relativistic velocity is bounded $|v(p)| < c_0$, the function $v(p)f \in L^1(\mathbb{T} \times \mathbb{R})$ and therefore the current $j(x) = q\int_{\mathbb{R}} v(p)f\ \mathrm{d}p$ is well defined for a.a. $x \in \mathbb{T}$. In particular taking $\varphi = \varphi(x) \in C^1(\mathbb{T})$ in (3) yields

$$\int_{\mathbb{T}} \varphi'(x)j(x)\ \mathrm{d}x = 0$$

implying that the current is preserved along $x \in \mathbb{T}$.

In order to construct a periodic solution for the linear Boltzmann equation we appeal to perturbation techniques. For any $\alpha > 0$ we consider the penalized problem

$$\alpha f(x,p) + v(p)\partial_x f + F(x)\partial_p f = \frac{1}{\tau}Q(f) + S(x,p), \quad (x,p) \in \mathbb{T} \times \mathbb{R}. \qquad (4)$$

Definition 2.2 *Assume that $F \in L^\infty(\mathbb{T}), S \in L^1(\mathbb{T} \times \mathbb{R})$ and $\alpha > 0$. We say that $f \in L^1(\mathbb{T} \times \mathbb{R})$ is a periodic weak solution for (4) iff*

$$\begin{aligned}
-\int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p)(-\alpha\varphi(x,p) + v(p)\partial_x\varphi + F(x)\partial_p\varphi) \,\mathrm{d}p \,\mathrm{d}x &= \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} Q(f)\varphi(x,p) \,\mathrm{d}p \,\mathrm{d}x \\
&+ \int_{\mathbb{T}}\int_{\mathbb{R}} S(x,p)\varphi(x,p) \,\mathrm{d}p \,\mathrm{d}x
\end{aligned}$$

for any function $\varphi \in C^1_c(\mathbb{T} \times \mathbb{R})$.

As before we check easily that the above formulation holds true for any $\varphi \in C^1_b(\mathbb{T}\times\mathbb{R})$ and we have

$$\alpha \int_{\mathbb{R}} f(x,p) \,\mathrm{d}p + \frac{d}{dx}\int_{\mathbb{R}} v(p)f(x,p) \,\mathrm{d}p = \int_{\mathbb{R}} S(x,p) \,\mathrm{d}p, \quad x \in \mathbb{T}.$$

3 Uniqueness of the periodic weak solution

Consider $f, g \in L^1(\mathbb{T} \times \mathbb{R})$ two periodic weak solutions for (1). By linearity we have

$$v(p)\partial_x(f-g) + F(x)\partial_p(f-g) = \frac{1}{\tau}Q(f-g), \quad (x,p) \in \mathbb{T} \times \mathbb{R} \tag{5}$$

and by standard computations one gets in $\mathcal{D}'(\mathbb{T} \times \mathbb{R})$

$$v(p)\partial_x|f-g| + F(x)\partial_p|f-g| - \frac{1}{\tau}\mathrm{sgn}(f-g)Q(f-g) = 0. \tag{6}$$

After integration with respect to momentum we have as usual

$$\frac{d}{dx}\int_{\mathbb{R}} v(p)|f-g| \,\mathrm{d}p - \frac{1}{\tau}\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g) \,\mathrm{d}p = 0, \text{ in } \mathcal{D}'(\mathbb{T}). \tag{7}$$

Following the idea in [1] we can write

$$-\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g) \,\mathrm{d}p = \int_{\mathbb{R}} \{f-g-M_\theta\langle f-g\rangle\}\{\mathrm{sgn}(f-g)-\mathrm{sgn}(M_\theta\langle f-g\rangle)\} \,\mathrm{d}p \geq 0 \tag{8}$$

with equality iff $\mathrm{sgn}(f-g)$ is constant with respect to p. Integrating now (7) with respect to x and using the periodicity of f and g imply

$$-\frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g) \,\mathrm{d}p \,\mathrm{d}x = 0.$$

Therefore for a.a. $x \in \mathbb{T}$ we have $-\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g)\,\mathrm{d}p = 0$ and thus $\mathrm{sgn}(f-g) = \mathrm{sgn}\langle f-g\rangle$. Eventually (5) can be written now

$$v(p)\partial_x|f-g| + F(x)\partial_p|f-g| = \frac{1}{\tau}Q(|f-g|),\ \ (x,p)\in\mathbb{T}\times\mathbb{R}$$

implying that

$$\frac{d}{dx}\int_{\mathbb{R}} v(p)|f-g|\,\mathrm{d}p = 0,\ \ x\in\mathbb{T}$$

but this is not enough in order to guarantee the uniqueness of the periodic weak solution. Actually we will see that, in the particular case of electric fields satisfying $\langle E\rangle := \int_{\mathbb{T}} E(x)\,\mathrm{d}x = 0$, the above arguments allow to determine all the periodic solutions. Indeed, if $\langle E\rangle = 0$, the potential $\phi(x) = -\int_0^x E(y)\,\mathrm{d}y$ is also L-periodic and since for any $c\in\mathbb{R}$ the function $c\mathcal{M}_{\theta,\phi}(x,p)$ solves (1) we can replace (5) by

$$v(p)\partial_x(f-g-c\mathcal{M}_{\theta,\phi}(x,p))+F(x)\partial_p(f-g-c\mathcal{M}_{\theta,\phi}(x,p)) = \frac{1}{\tau}Q(f-g-c\mathcal{M}_{\theta,\phi}(x,p)). \tag{9}$$

Following the same steps as before we find for any $c\in\mathbb{R}$

$$-\int_{\mathbb{R}} \mathrm{sgn}(f-g-c\mathcal{M}_{\theta,\phi})Q(f-g-c\mathcal{M}_{\theta,\phi})\,\mathrm{d}p = 0,\ \ \text{a.e. } x\in\mathbb{T}. \tag{10}$$

Notice that the periodicity of the potential is crucial when writing

$$\int_{\mathbb{T}}\frac{d}{dx}\int_{\mathbb{R}} v(p)|f-g-c\mathcal{M}_{\theta,\phi}|\,\mathrm{d}p\,\mathrm{d}x = 0.$$

Therefore one gets for a.a. $x\in\mathbb{T}$ and any $c\in\mathbb{R}$

$$0 = \int_{\mathbb{R}}\{f-g-c\mathcal{M}_{\theta,\phi} - \langle f-g-c\mathcal{M}_{\theta,\phi}\rangle M_\theta\}\,\mathrm{sgn}(f-g-c\mathcal{M}_{\theta,\phi})\,\mathrm{d}p.$$

Taking $c = c(x)$ such that $\langle f-g-c\mathcal{M}_{\theta,\phi}\rangle = 0$ we deduce that

$$f(x,p) - g(x,p) = \langle f-g\rangle(x)M_\theta(p),\ \ (x,p)\in\mathbb{T}\times\mathbb{R}. \tag{11}$$

Replacing now (11) in (5) we deduce easily that

$$f(x,p) - g(x,p) = k\mathcal{M}_{\theta,\phi}(x,p),\ \ (x,p)\in\mathbb{T}\times\mathbb{R}$$

for some real constant k. Since the above conclusion holds for every two periodic solutions, taking $g = 0$ we deduce that all the periodic solutions for (1) when $\langle E \rangle = 0$ are $k\mathcal{M}_{\theta,\phi}$, $k \in \mathbb{R}$. Observe also that these solutions have the same current since $\int_{\mathbb{R}} v(p)\mathcal{M}_{\theta,\phi} \; \mathrm{d}p = 0$ and that they remain bounded.

Let us analyze the case of electric fields with non vanishing average. This time $\mathcal{M}_{\theta,\phi}$ is not periodic and we will see that for a given current there is at most one periodic solution. The idea is to exploit new dissipation properties of the relaxation operator Q. We have seen that the inequality

$$-\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g) \; \mathrm{d}p \geq 0$$

is not strong enough for our purposes. Actually a better minoration for the dissipation term $-\int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g) \; \mathrm{d}p$ is available.

Lemma 3.1 *Let $h = h(p)$ be a function of $L^1(\mathbb{R})$ with vanishing current $\int_{\mathbb{R}} v(p)h(p) \; \mathrm{d}p = 0$. Then we have the inequality*

$$-\int_{\mathbb{R}} \mathrm{sgn}(h(p))Q(h)(p) \; \mathrm{d}p \geq \frac{1}{c_0} \left| \int_{\mathbb{R}} v(p)|h(p)| \; \mathrm{d}p \right|. \tag{12}$$

Proof. We consider the sets

$$A_+ = \{p \in \mathbb{R} \; : \; h(p) \geq 0\}, \quad A_- = \{p \in \mathbb{R} \; : \; h(p) < 0\}.$$

Since $\int_{\mathbb{R}} v(p)h(p) \; \mathrm{d}p = 0$ then we have

$$\int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_+}(p) \; \mathrm{d}p = \int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_-}(p) \; \mathrm{d}p = \frac{1}{2}\int_{\mathbb{R}} v(p)|h(p)| \; \mathrm{d}p.$$

Observe that

$$\begin{aligned} -\int_{\mathbb{R}} \mathrm{sgn}(h(p))Q(h)(p) \; \mathrm{d}p &= \int_{\mathbb{R}} (h(p) - \langle h \rangle M_\theta(p))\mathrm{sgn}(h(p)) \; \mathrm{d}p \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(p)M_\theta(p')(\mathrm{sgn}(h(p)) - \mathrm{sgn}(h(p'))) \; \mathrm{d}p \; \mathrm{d}p'. \end{aligned}$$

But for any $(p, p') \in \mathbb{R}^2$ we have

$$h(p)M_\theta(p')(\mathrm{sgn}(h(p)) - \mathrm{sgn}(h(p'))) \geq 0$$

and thus we can write for any $(p,p') \in \mathbb{R}^2$, $\beta \in \{-1,+1\}$

$$h(p)M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p'))) \geq \beta \frac{v(p)}{c_0} h(p)M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p'))).$$

Combining these computations yields

$$\begin{aligned}
-\int_{\mathbb{R}} \operatorname{sgn}(h) Q(h)\ \mathrm{d}p &\geq \frac{\beta}{c_0}\int_{\mathbb{R}}\int_{\mathbb{R}} v(p)h(p)M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p')))\ \mathrm{d}p\ \mathrm{d}p' \\
&= \frac{2\beta}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_+}(p)\ \mathrm{d}p \int_{\mathbb{R}} M_\theta(p')\mathbf{1}_{A_-}(p')\ \mathrm{d}p' \\
&+ \frac{2\beta}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_-}(p)\ \mathrm{d}p \int_{\mathbb{R}} M_\theta(p')\mathbf{1}_{A_+}(p')\ \mathrm{d}p' \\
&= \frac{\beta}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\ \mathrm{d}p \int_{\mathbb{R}} M_\theta(p')(\mathbf{1}_{A_-} + \mathbf{1}_{A_+})(p')\ \mathrm{d}p' \\
&= \frac{\beta}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\ \mathrm{d}p, \quad \beta \in \{-1,+1\}.
\end{aligned}$$

□

Remark 3.1 *When the light speed c_0 becomes very large, the inequality (12) degenerates to (8). In particular in the non relativistic case the conclusion of Lemma 3.1 reduces to the well-known inequality (8) which is not enough for the uniqueness of the periodic weak solution.*

Proposition 3.1 *Assume that $E \in L^\infty(\mathbb{T})$ such that $\langle E\rangle \neq 0$ and let $f, g \in L^1(\mathbb{T} \times \mathbb{R})$ be two periodic weak solutions for (1) with the same current*

$$q\int_{\mathbb{R}} v(p) f\ \mathrm{d}p = q\int_{\mathbb{R}} v(p) g\ \mathrm{d}p.$$

Then we have $f = g$.

Proof. Consider the function $h = f - g - c\mathcal{M}_{\theta,\phi}$ with $c \in \mathbb{R}$. This function belongs to $L^1([a,b]\times\mathbb{R})$ for any $a < b$, has vanishing current $\int_{\mathbb{R}} v(p)h(x,p)\ \mathrm{d}p = 0, x \in \mathbb{R}$ and satisfies in $\mathcal{D}'(\mathbb{R}^2)$

$$v(p)\partial_x h + F(x)\partial_p h = \frac{1}{\tau}Q(h).$$

As usual we obtain

$$v(p)\partial_x|h| + F(x)\partial_p|h| - \frac{1}{\tau}\mathrm{sgn}(h)Q(h) = 0. \tag{13}$$

Integrating with respect to $p \in \mathbb{R}$ and combining with Lemma 3.1 yield

$$\frac{d}{dx}\int_{\mathbb{R}} v(p)|h|\ \mathrm{d}p + \frac{1}{\tau c_0}\left|\int_{\mathbb{R}} v(p)|h|\ \mathrm{d}p\right| \le \frac{d}{dx}\int_{\mathbb{R}} v(p)|h|\ \mathrm{d}p - \frac{1}{\tau}\int_{\mathbb{R}} \mathrm{sgn}(h)Q(h)\ \mathrm{d}p = 0. \tag{14}$$

Let us denote by u the function $u(x) = \int_{\mathbb{R}} v(p)|h(x,p)|\ \mathrm{d}p$, $x \in \mathbb{R}$. This function is not periodic but satisfies the bounds

$$\sup_{n\in\mathbb{Z}} |u(x+nL)| < +\infty, \quad \text{a.e. } x \in \mathbb{R}. \tag{15}$$

Indeed we have for any $n \in \mathbb{Z}$

$$\begin{aligned}
|u(x+n)| &= \left|\int_{\mathbb{R}} v(p)\{|h(x+nL,p)| - |c\mathcal{M}_{\theta,\phi}(x+nL,p)|\}\ \mathrm{d}p\right| \\
&\le \int_{\mathbb{R}} |v(p)|\ \left|\ |h(x+nL,p)| - |c\mathcal{M}_{\theta,\phi}(x+nL,p)|\ \right|\ \mathrm{d}p \\
&\le c_0 \int_{\mathbb{R}} |h(x+nL,p) + c\mathcal{M}_{\theta,\phi}(x+nL,p)|\ \mathrm{d}p \\
&= c_0 \int_{\mathbb{R}} |f(x+nL,p) - g(x+nL,p)|\ \mathrm{d}p \\
&\le c_0 \int_{\mathbb{R}} |f(x,p)|\ \mathrm{d}p + c_0 \int_{\mathbb{R}} |g(x,p)|\ \mathrm{d}p.
\end{aligned}$$

By (14) we know that $u'(x) + \frac{\beta}{\tau c_0}u(x) \le 0$, $x \in \mathbb{R}, \beta \in \{-1,+1\}$ implying that

$$\frac{d}{dx}\left\{u(x)\exp\left(\frac{\beta x}{\tau c_0}\right)\right\} \le 0, \quad x \in \mathbb{R},\ \beta \in \{-1,+1\}. \tag{16}$$

Consider $\beta = +1$ and let us integrate (16) between $x - nL$ and x with $n \in \mathbb{N}$. We deduce that

$$u(x) \le u(x-nL)\exp\left(\frac{-nL}{\tau c_0}\right)$$

which implies by (15) and by letting $n \to +\infty$ that $u(x) \le 0$ for a.a. $x \in \mathbb{R}$. Similarly, taking $\beta = -1$, integrating over $[x, x+nL]$ with $n \in \mathbb{N}$ and letting $n \to +\infty$ one gets $u(x) \ge 0$ for a.a. $x \in \mathbb{R}$. Therefore we have $u = 0$ and coming back in (14) we deduce that

$$\int_{\mathbb{R}} \mathrm{sgn}(h)Q(h)\ \mathrm{d}p = 0, \quad \text{a.e. } x \in \mathbb{R}. \tag{17}$$

At this stage let us point out that one can not obtain (17) as in the case of periodic potentials, by integrating (13) over $\mathbb{T} \times \mathbb{R}$. Indeed, in this case h is not periodic and thus

$$\int_{\mathbb{T}} \frac{d}{dx} \int_{\mathbb{R}} v(p)|h(x,p)| \, \mathrm{d}p \, \mathrm{d}x \neq 0.$$

Therefore Lemma 3.1 is crucial when establishing (17) for non periodic potentials. From now on we follow the same steps as for periodic potentials. We deduce that there is a constant $k \in \mathbb{R}$ such that

$$f(x,p) - g(x,p) = k\mathcal{M}_{\theta,\phi}(x,p), \quad (x,p) \in \mathbb{R}^2.$$

Since f and g are periodic and $\mathcal{M}_{\theta,\phi}$ is not periodic we must have $k = 0$ and therefore $f = g$. Indeed, the formulation (3) applied to $f - g$ implies

$$k \int_{\mathbb{T}} \int_{\mathbb{R}} \mathcal{M}_{\theta,\phi}(x,p)(v(p)\partial_x \varphi + F(x)\partial_p \varphi) \, \mathrm{d}p \, \mathrm{d}x = 0, \quad \varphi \in C_b^1(\mathbb{T} \times \mathbb{R})$$

and after integration by parts one gets

$$k \int_{\mathbb{R}} v(p)\varphi(0,p)(\mathcal{M}_{\theta,\phi}(L,p) - \mathcal{M}_{\theta,\phi}(0,p)) \, \mathrm{d}p = 0.$$

Since the potential is not periodic we obtain

$$k \int_{\mathbb{R}} v(p)\varphi(0,p) M_\theta(p) \, \mathrm{d}p = 0, \quad \varphi \in C_b^1(\mathbb{T} \times \mathbb{R}).$$

Actually the above equality holds also for $\varphi(x,p) = p$ and we deduce that

$$k \int_{\mathbb{R}} v(p) p M_\theta(p) \, \mathrm{d}p = 0$$

saying that $k = 0$. $\square$

4 Existence of periodic weak solution

By standard approximation arguments it is sufficient to establish first the existence of periodic weak solution for smooth electric fields. We start our analysis by investigating the transport periodic problem with source term in $L^1(\mathbb{T} \times \mathbb{R})$.

Proposition 4.1 *Assume that* $S \in L^1(\mathbb{T} \times \mathbb{R}), E \in W^{1,\infty}(\mathbb{T})$ *and* $\alpha > 0$. *Then there is a unique periodic weak solution* $f \in L^1(\mathbb{T} \times \mathbb{R})$ *for the problem*

$$\alpha f(x,p) + v(p)\partial_x f + F(x)\partial_p f = S(x,p), \quad (x,p) \in \mathbb{T} \times \mathbb{R} \tag{18}$$

satisfying

$$\|f\|_{L^1(\mathbb{T}\times\mathbb{R})} \leq \frac{1}{\alpha}\|S\|_{L^1(\mathbb{T}\times\mathbb{R})}. \tag{19}$$

Moreover if $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ *then*

$$\left\| \int_{\mathbb{R}} v(p)|f(\cdot,p)| \,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \leq \frac{c_0}{\alpha}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))} \tag{20}$$

and if $S \in L^\infty(\mathbb{T} \times \mathbb{R})$ *then*

$$-\frac{1}{\alpha}\|S_-\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq f(x,p) \leq \frac{1}{\alpha}\|S_+\|_{L^\infty(\mathbb{T}\times\mathbb{R})}, \quad a.e.\ (x,p) \in \mathbb{T} \times \mathbb{R} \tag{21}$$

where $S_\pm = \max(0, \pm S)$. *In particular if* $S \geq 0$ *then* $f \geq 0$.

Proof. The uniqueness follows immediately since we have for any two solutions f, g

$$\alpha|f-g| + v(p)\partial_x|f-g| + F(x)\partial_p|f-g| = 0$$

and therefore $\alpha \int_{\mathbb{T}}\int_{\mathbb{R}} |f-g| \,\mathrm{d}p\,\mathrm{d}x = 0$ implying that $f = g$. Let us consider the characteristics (X, P) associated to the electric field E

$$\frac{dX}{ds} = v(P(s;x,p)), \quad \frac{dP}{ds} = qE(X(s;x,p)),$$

with the conditions

$$X(0;x,p) = x, \quad P(0;x,p) = p.$$

Formally (18) can be written $\frac{d}{ds}\{e^{\alpha s} f(X(s;x,p), P(s;x,p))\} = e^{\alpha s} S(X(s;x,p), P(s;x,p))$ and we check easily that the function

$$f(x,p) = \int_{-\infty}^{0} e^{\alpha s} S(X(s;x,p), P(s;x,p)) \,\mathrm{d}s \tag{22}$$

is a weak solution for (18). By the periodicity of the electric field and the uniqueness of the characteristic curves we have

$$X(s; x+L, p) = X(s;x,p) + L, \quad P(s; x+L, p) = P(s;x,p)$$

and therefore f is also L-periodic. The L^1 bound (19) comes easily by integrating over $\mathbb{T} \times \mathbb{R}$ the inequality

$$\alpha|f(x,p)| + v(p)\partial_x|f| + F(x)\partial_p|f| \leq |S(x,p)|$$

and the L^∞ bounds (21) follow immediately from the explicit formula (22). It remains to justify the current bound (20). Assume that $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ and observe that

$$\alpha \int_{\mathbb{R}} |f(x,p)| \, \mathrm{d}p + \frac{d}{dx} \int_{\mathbb{R}} v(p)|f(x,p)| \, \mathrm{d}p \leq \int_{\mathbb{R}} |S(x,p)| \, \mathrm{d}p.$$

With the notation $u(x) = \int_{\mathbb{R}} v(p)|f(x,p)| \, \mathrm{d}p$ one gets, by observing that $c_0 \int_{\mathbb{R}} |f(x,p)| \, \mathrm{d}p \geq |u(x)|$

$$\frac{\alpha\beta}{c_0} u(x) + u'(x) \leq \int_{\mathbb{R}} |S(x,p)| \, \mathrm{d}p, \quad x \in \mathbb{R}, \ \beta \in \{-1, +1\}.$$

Taking $\beta = 1$ and integrating between $x - nL$ and x, with $n \in \mathbb{N}$, we deduce that

$$u(x) \leq u(x - nL) \exp\left(-\frac{\alpha n L}{c_0}\right) + \|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))} \int_{x-nL}^{x} \exp\left(-\frac{\alpha(x-y)}{c_0}\right) \, \mathrm{d}y$$

and by letting $n \to +\infty$ we obtain $u(x) \leq \frac{c_0}{\alpha} \|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$. Similarly, by taking $\beta = -1$ and integrating between x and $x + nL$ with $n \to +\infty$ we deduce that $u(x) \geq -\frac{c_0}{\alpha} \|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$. □

Based on the previous Proposition, we construct periodic solutions for the linear Boltzmann equation with source term in $L^1(\mathbb{T} \times \mathbb{R})$. Moreover we establish bounds in L^∞, uniformly with respect to the penalization parameter $\alpha > 0$.

Proposition 4.2 *Assume that $S \in L^1(\mathbb{T} \times \mathbb{R}), E \in W^{1,\infty}(\mathbb{T})$ and $\alpha > 0$. Then there is a unique periodic solution for the problem*

$$\alpha f(x,p) + v(p)\partial_x f + F(x)\partial_p f = \frac{1}{\tau} Q(f) + S(x,p), \quad (x,p) \in \mathbb{T} \times \mathbb{R} \qquad (23)$$

satisfying

$$\|f\|_{L^1(\mathbb{T}\times\mathbb{R})} \leq \frac{1}{\alpha} \|S\|_{L^1(\mathbb{T}\times\mathbb{R})}.$$

Moreover

a) if $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ *then*

$$\left\| \int_{\mathbb{R}} v(p)|f(\cdot,p)| \,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \le \frac{c_0}{\alpha} \|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$$

b) if $S \ge 0$ *then* $f \ge 0$*;*

c) if $S \ge 0,\ S \in L^\infty(\mathbb{T}\times\mathbb{R})$ *and* $\int_{\mathbb{R}} v(p) p f(\cdot,p) \,\mathrm{d}p \in L^\infty(\mathbb{T})$ *then* $f \in L^\infty(\mathbb{T}\times\mathbb{R})$, $\langle f \rangle \in L^\infty(\mathbb{T})$.

Proof. We consider the sequence of periodic weak solutions $(f_\pm^{(n)})_{n\in\mathbb{N}}$ defined by

$$\frac{1}{\tau} f_\pm^{(0)}(x,p) + \alpha f_\pm^{(0)}(x,p) + v(p)\partial_x f_\pm^{(0)} + F(x)\partial_p f_\pm^{(0)} = S_\pm(x,p), \ \ (x,p) \in \mathbb{T}\times\mathbb{R}$$

and for any $n \in \mathbb{N}$

$$\frac{1}{\tau} f_\pm^{(n+1)} + \alpha f_\pm^{(n+1)} + v(p)\partial_x f_\pm^{(n+1)} + F(x)\partial_p f_\pm^{(n+1)} = \frac{1}{\tau}\langle f_\pm^{(n)}\rangle M_\theta + S_\pm, \ \ (x,p) \in \mathbb{T}\times\mathbb{R}$$

where $S_\pm$ are the positive/negative parts of S. Thanks to Proposition 4.1 the sequence $(f_\pm^{(n)})_{n\in\mathbb{N}}$ is well defined. We have $f_\pm^{(0)} \ge 0$ and we check recursively that $0 \le f_\pm^{(n)} \le f_\pm^{(n+1)}$ for any $n \in \mathbb{N}$. Integrating over $\mathbb{T}\times\mathbb{R}$ one gets

$$\begin{aligned}\left(\tau^{-1}+\alpha\right) \int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm^{(n+1)}(x,p) \,\mathrm{d}p \,\mathrm{d}x &= \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} \langle f_\pm^{(n)}\rangle M_\theta(p) \,\mathrm{d}p \,\mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm \,\mathrm{d}p \,\mathrm{d}x \\ &\le \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm^{(n+1)} \,\mathrm{d}p \,\mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm \,\mathrm{d}p \,\mathrm{d}x\end{aligned}$$

implying that $\sup_{n\in\mathbb{N}} \int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm^{(n)} \,\mathrm{d}p \,\mathrm{d}x \le \alpha^{-1}\int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm \,\mathrm{d}p \,\mathrm{d}x$. By the monotone convergence theorem we deduce that $(f_\pm^{(n)})_n$ converge in $L^1(\mathbb{T}\times\mathbb{R})$. Let $f_\pm = \lim_{n\to+\infty} f_\pm^{(n)}$. For any test function $\varphi \in C_c^1(\mathbb{T}\times\mathbb{R})$ we have

$$\int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm^{(n+1)} \left((\tau^{-1}+\alpha)\varphi - v(p)\partial_x\varphi - F\partial_p\varphi\right) \,\mathrm{d}p \,\mathrm{d}x = \int_{\mathbb{T}}\int_{\mathbb{R}} \left(\langle f_\pm^{(n)}\rangle \frac{M_\theta}{\tau} + S_\pm\right)\varphi \,\mathrm{d}p \,\mathrm{d}x.$$

Passing to the limit for $n \to +\infty$ we deduce that $f_\pm$ are periodic weak solutions for

$$\alpha f_\pm + v(p)\partial_x f_\pm + F(x)\partial_p f_\pm = \frac{1}{\tau} Q(f_\pm) + S_\pm, \ \ (x,p) \in \mathbb{T}\times\mathbb{R}$$

satisfying $\int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm \,\mathrm{d}p \,\mathrm{d}x = \alpha^{-1}\int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm \,\mathrm{d}p \,\mathrm{d}x$ and therefore $f = f_+ - f_-$ is a periodic weak solution for (23) satisfying $\int_{\mathbb{T}}\int_{\mathbb{R}} |f| \,\mathrm{d}p \,\mathrm{d}x \le \alpha^{-1}\int_{\mathbb{T}}\int_{\mathbb{R}} |S| \,\mathrm{d}p \,\mathrm{d}x$. Assume now that S belongs to $L^\infty(\mathbb{T}; L^1(\mathbb{R}))$. The estimate of $\left\| \int_{\mathbb{R}} v(p)|f(\cdot,p)| \,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})}$ follows exactly as in the proof of Proposition 4.1 since we have

$$\alpha \int_{\mathbb{R}} |f| \,\mathrm{d}p + \frac{d}{dx}\int_{\mathbb{R}} v(p)|f| \,\mathrm{d}p = \frac{1}{\tau}\int_{\mathbb{R}} Q(f)\mathrm{sgn}(f) \,\mathrm{d}p + \int_{\mathbb{R}} S \,\mathrm{sgn}(f) \,\mathrm{d}p \leq \int_{\mathbb{R}} |S| \,\mathrm{d}p.$$

Obviously, when $S \geq 0$ we have $S_- = 0, f_- = 0$ and thus $f = f_+ \geq 0$. Assume now that $S \in L^\infty(\mathbb{T}\times\mathbb{R}), S \geq 0$ and that there is $\mathcal{K} > 0$ such that

$$\int_{\mathbb{R}} v(p) p f_+^{(n)} \,\mathrm{d}p \leq \int_{\mathbb{R}} v(p) p f_+ \,\mathrm{d}p = \int_{\mathbb{R}} v(p) p f \,\mathrm{d}p \leq \mathcal{K}, \quad \text{a.e. } x \in \mathbb{T}.$$

Let $\tilde{\mathcal{K}}$ be a non negative constant, which we will precise later on, such that $\tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}$. By Proposition 4.1 we know that

$$\|f_+^{(0)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \frac{1}{\tau^{-1}+\alpha}\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}.$$

Assume that $f_+^{(n)} \in L^\infty(\mathbb{T}\times\mathbb{R})$ and that $\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}$. Applying standard interpolation inequalities yields

$$\begin{aligned}
\langle f_+^{(n)} \rangle &= \int_{\mathbb{R}} f_+^{(n)} \mathbf{1}_{\{|p|<R\}} \,\mathrm{d}p + \int_{\mathbb{R}} f_+^{(n)} \mathbf{1}_{\{|p|\geq R\}} \,\mathrm{d}p \\
&\leq 2R\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \frac{1}{Rv(R)}\int_{\mathbb{R}} v(p) p f_+^{(n)} \mathbf{1}_{\{|p|\geq R\}} \,\mathrm{d}p \\
&\leq 2R\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \frac{1}{Rv(R)}\mathcal{K} \\
&\leq 2R\tilde{\mathcal{K}} + \frac{1}{Rv(R)}\mathcal{K}.
\end{aligned}$$

We take $R = R(\tilde{\mathcal{K}})$ such that $2R\tilde{\mathcal{K}} = \frac{\mathcal{K}}{Rv(R)}$ which is equivalent to $2R^2 v(R) = \mathcal{K}/\tilde{\mathcal{K}}$. Then we obtain the inequality

$$\|\langle f_+^{(n)} \rangle\|_{L^\infty(\mathbb{T})} \leq 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}.$$

Applying one more time Proposition 4.1 one gets

$$\begin{aligned}
\|f_+^{(n+1)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} &\leq \left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(q)}{\theta}\right) \,\mathrm{d}q\right)^{-1} \|\langle f_+^{(n)} \rangle\|_{L^\infty(\mathbb{T})} + \tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \\
&\leq 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}} \left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(q)}{\theta}\right) \,\mathrm{d}q\right)^{-1} + \tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})}.
\end{aligned}$$

Therefore we can find uniform estimates for $\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})}$ if there is a constant $\tilde{\mathcal{K}}$ such that $\tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}$ and

$$4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}\left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(q)}{\theta}\right) \,\mathrm{d}q\right)^{-1} + \tau\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}.$$

This is obviously possible for $\tilde{\mathcal{K}}$ large enough since $\lim_{\tilde{\mathcal{K}}\to+\infty} R(\tilde{\mathcal{K}}) = 0$. Therefore we obtain the bounds

$$\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}, \quad \|\langle f_+^{(n)}\rangle\|_{L^\infty(\mathbb{T})} \leq 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}, \quad n \in \mathbb{N}.$$

By the pointwise convergence $\lim_{n\to+\infty} f_+^{(n)}(x,p) = f_+(x,p) = f(x,p)$ for a.a. $(x,p) \in \mathbb{T}\times\mathbb{R}$ we deduce that $\|f\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq \tilde{\mathcal{K}}$. Using the monotone convergence theorem one gets also the pointwise convergence $\lim_{n\to+\infty}\langle f_+^{(n)}\rangle(x) = \langle f\rangle(x)$ for a.a. $x \in \mathbb{T}$ and thus

$$\|\langle f\rangle\|_{L^\infty(\mathbb{T})} \leq 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}.$$

Notice that the L^∞ bounds for f and $\langle f\rangle$ depend on $\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})}$ and $\left\|\int_{\mathbb{R}} v(p)pf(\cdot,p)\,\mathrm{d}p\right\|_{L^\infty(\mathbb{T})}$ but not on the parameter $\alpha > 0$. $\square$

The uniform estimates established above allow us to construct a periodic solution for the linear Boltzmann equation by taking a limit point of the family $(f_\alpha)_{\alpha>0}$ with respect to the weak $\star$ topology of $L^\infty(\mathbb{T}\times\mathbb{R})$.

Proposition 4.3 *Assume that $E \in W^{1,\infty}(\mathbb{T})$. Then there is a non trivial periodic weak solution for (1) satisfying*

$$f \geq 0, \quad \int_0^L \int_{\mathbb{R}} f(x,p)\,\mathrm{d}p\,\mathrm{d}x = L, \quad \frac{d}{dx}\int_{\mathbb{R}} v(p)f\,\mathrm{d}p = 0$$

$$\frac{q}{\theta}\langle E\rangle \int_{\mathbb{R}} v(p)f\,\mathrm{d}p = \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} (f - \langle f\rangle M_\theta(p)) \ln\left(\frac{f}{\langle f\rangle M_\theta(p)}\right)\,\mathrm{d}p\,\mathrm{d}x.$$

Proof. By Proposition 4.2 we know that for any $\alpha > 0$ there is a unique periodic weak solution f_α for the problem

$$\alpha f_\alpha(x,p) + v(p)\partial_x f_\alpha + F(x)\partial_p f_\alpha = \frac{1}{\tau}Q(f_\alpha) + \alpha M_\theta(p), \quad (x,p) \in \mathbb{T}\times\mathbb{R}. \tag{24}$$

These solutions are non negative and satisfy

$$\int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)\,\mathrm{d}p\,\mathrm{d}x = L,\quad \left\| \int_{\mathbb{R}} v(p) f_\alpha(\cdot,p)\,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \leq c_0,\quad \alpha > 0.$$

By applying the weak formulation of (24) to the test function $\mathcal{E}(p) + q\phi(x)$ one gets

$$\begin{aligned} \alpha \int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x &+ q\phi(L)\int_{\mathbb{R}} v(p) f_\alpha(L,p)\,\mathrm{d}p - q\phi(0)\int_{\mathbb{R}} v(p) f_\alpha(0,p)\,\mathrm{d}p \\ &+ \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}} (f_\alpha - \langle f_\alpha\rangle M_\theta(p))(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x \\ &= \alpha \int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x \end{aligned}$$

and therefore we deduce that

$$\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(x,p)\mathcal{E}(p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

Multiplying now (24) by p one gets

$$(\alpha + \tau^{-1})\int_{\mathbb{R}} p f_\alpha\,\mathrm{d}p + \frac{d}{dx}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p = F(x)\langle f_\alpha\rangle. \tag{25}$$

Using the inequality $mc_0^2 + \mathcal{E}(p) \geq v(p)p$ we have

$$\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x \leq \sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} (mc_0^2 + \mathcal{E}(p)) f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

Therefore there is $x_\alpha \in \mathbb{T}$ such that

$$\sup_{0<\alpha<1}\int_{\mathbb{R}} v(p) p f_\alpha(x_\alpha,p)\,\mathrm{d}p \leq \frac{1}{L}\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

We integrate now (25) between x_α and x for any $x \in [x_\alpha, x_\alpha + L]$. Taking into account that $c_0|p| \leq mc_0^2 + \mathcal{E}(p)$ we deduce that there is a constant C (not depending on α) such that

$$\sup_{0<\alpha<1}\left\| \int_{\mathbb{R}} v(p) p f_\alpha(\cdot,p)\,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \leq C \sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} (mc_0^2 + \mathcal{E}(p)) f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x < +\infty$$

and thus applying the last statement in Proposition 4.2 we obtain uniform L^∞ bounds for f_α and $\langle f_\alpha\rangle$. Therefore there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converging towards zero such that $\lim_{n\to+\infty} f_{\alpha_n} = f$ weakly $\star$ in $L^\infty(\mathbb{T}\times\mathbb{R})$, $\lim_{n\to+\infty}\langle f_{\alpha_n}\rangle = \rho$ weakly

$\star$ in $L^\infty(\mathbb{T})$ and $\lim_{n\to+\infty}\int_{\mathbb{R}} v(p) f_{\alpha_n}\,\mathrm{d}p = j$ weakly $\star$ in $L^\infty(\mathbb{T})$. It is easily seen that $(1+\mathcal{E}(p))f \in L^1(\mathbb{T}\times\mathbb{R})$, $f\in L^\infty(\mathbb{T}\times\mathbb{R})$, $f\geq 0$, $\int_{\mathbb{T}}\int_{\mathbb{R}} f\,\mathrm{d}p\,\mathrm{d}x = L$, $\rho\in L^1(\mathbb{T})\cap L^\infty(\mathbb{T})$ and we check immediately that $\rho = \langle f\rangle$ and $j = \int_{\mathbb{R}} v(p) f\,\mathrm{d}p \in [-c_0, c_0]$. Now passing to the limit for $n\to+\infty$ in the weak formulation (24) satisfied by f_{α_n} we deduce that f is a periodic weak solution for

$$v(p)\partial_x f + F(x)\partial_p f = \frac{1}{\tau} Q(f), \quad (x,p)\in\mathbb{T}\times\mathbb{R}$$

satisfying

$$f\geq 0, \quad \int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p)\,\mathrm{d}p\,\mathrm{d}x = L, \quad \left|\int_{\mathbb{R}} v(p) f\,\mathrm{d}p\right| \leq c_0, \quad \frac{d}{dx}\int_{\mathbb{R}} v(p) f\,\mathrm{d}p = 0.$$

For the last statement we combine the equalities

$$(v(p)\partial_x + F(x)\partial_p)(f\ln f) = \frac{1}{\tau} Q(f)(1+\ln f)$$

$$\frac{1}{\theta}(v(p)\partial_x + F(x)\partial_p)(f(\mathcal{E}(p) + q\phi(x))) = \frac{1}{\tau\theta} Q(f)(\mathcal{E}(p)+q\phi(x))$$

and we deduce that

$$\begin{aligned}(v(p)\partial_x + F\partial_p)(f(\ln f + \theta^{-1}(\mathcal{E}(p)+q\phi(x)))) \quad &= \quad \frac{1}{\tau}(\langle f\rangle M_\theta(p) - f)\ln\frac{f}{\langle f\rangle M_\theta(p)} \\ &+ \quad \frac{Q(f)}{\tau}\left(1 + \ln\frac{\langle f\rangle}{\int_{\mathbb{R}}\exp(-\frac{\mathcal{E}}{\theta})\mathrm{d}q} + q\frac{\phi}{\theta}\right).\end{aligned}$$

After integration over $\mathbb{T}\times\mathbb{R}$ one gets

$$\frac{q}{\theta}(\phi(L)-\phi(0))\int_{\mathbb{R}} v(p) f\,\mathrm{d}p + \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}}(f - \langle f\rangle M_\theta(p))\ln\frac{f}{\langle f\rangle M_\theta(p)}\,\mathrm{d}p\,\mathrm{d}x = 0$$

and the last statement follows. $\square$

Remark 4.1 *Notice that all the bounds of the above solution depend on m, c_0, q, τ, θ and $\|E\|_{L^\infty(\mathbb{T})}$ but not on $\|E'\|_{L^\infty(\mathbb{T})}$.*

Corollary 4.1 *Assume that $E\in L^\infty(\mathbb{T})$. Then there is a non trivial periodic weak solution for (1) satisfying all the statements of Proposition 4.3.*

Proof. It is a direct consequence of Proposition 4.3. Consider $(E_n)_{n\in\mathbb{N}}$ a sequence of smooth fields - for each n, $E_n \in W^{1,\infty}(\mathbb{T})$ - which converges a.e. towards $E \in L^\infty(\mathbb{T})$, with $\|E_n\|_{L^\infty(\mathbb{T})} \leq \|E\|_{L^\infty(\mathbb{T})}$. For any n denote by f_n the periodic weak solution associated to E_n, constructed in Proposition 4.3. We have the uniform bounds (see Remark 4.1)

$$\sup_{n\in\mathbb{N}}\|f_n\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \sup_{n\in\mathbb{N}}\int_{\mathbb{T}}\int_{\mathbb{R}}(1+v(p)p)f_n(x,p)\ \mathrm{d}p\ \mathrm{d}x < +\infty$$

$$\sup_{n\in\mathbb{N}}\|\langle f_n\rangle\|_{L^\infty(\mathbb{T})} < +\infty,\quad \int_{\mathbb{T}}\int_{\mathbb{R}} f_n(x,p)\ \mathrm{d}p\ \mathrm{d}x = L,\quad \left|\int_{\mathbb{R}} v(p)f_n\ \mathrm{d}p\right| \leq c_0.$$

Our conclusion follows easily after extraction of a weak $\star$ convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$ in $L^\infty(\mathbb{T}\times\mathbb{R})$. $\square$

Now we are ready to prove our main result concerning the well posedness of the periodic linear Boltzmann problem.

Proof. (of Theorem 1.1) The statement a) was clarified at the begining of Section 3. The uniqueness part of the statement b) was proved in Proposition 3.1. It remains to prove the existence of periodic solution for any given current $j \in \mathbb{R}$, when the electric field has no vanishing average. Let $f \geq 0$ be the periodic solution constructed in Corollary 4.1. If the current of this solution vanishes then

$$\frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}}(f - \langle f\rangle M_\theta(p))\ln\left(\frac{f}{\langle f\rangle M_\theta(p)}\right)\ \mathrm{d}p\ \mathrm{d}x = \frac{q}{\theta}\langle E\rangle\int_{\mathbb{R}} v(p)f\ \mathrm{d}p = 0, \tag{26}$$

implying that $f = \langle f\rangle M_\theta$ and also

$$(v(p)\partial_x + F(x)\partial_p)(\langle f\rangle M_\theta) = \frac{1}{\tau}Q(\langle f\rangle M_\theta) = 0.$$

Finally one gets $f = \langle f\rangle M_\theta = k\mathcal{M}_{\theta,\phi}$ which is periodic only if $k = 0$ (since $\langle E\rangle \neq 0$). But this is not possible since we know that $\int_{\mathbb{T}}\int_{\mathbb{R}} f\ \mathrm{d}p\ \mathrm{d}x = L$. Therefore we have $\int_{\mathbb{R}} v(p)f\ \mathrm{d}p \neq 0$ (actually by (26) we deduce that $\mathrm{sgn}(q\int_{\mathbb{R}} v(p)f\ \mathrm{d}p) = \mathrm{sgn}(\langle E\rangle)$) and thus we can consider $\tilde{f} = \frac{j}{q\int_{\mathbb{R}} v(p)f\ \mathrm{d}p}f$ which is a periodic weak solution with current equal to j and sign given by $\mathrm{sgn}\tilde{f} = \mathrm{sgn}\frac{j}{\langle E\rangle}$. $\square$

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