Finite speed propagation of the solutions for the relativistic Vlasov-Maxwell system

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Abstract

In this paper we investigate the continuous dependence with respect to the initial data of the solutions for the 1D and 1.5D relativistic Vlasov-Maxwell system. More precisely we prove that these solutions propagate with finite speed. We formulate our results in the framework of mild solutions, *i.e.*, the particle densities are solutions by characteristics and the electro-magnetic fields are Lipschitz continuous functions.

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1 Introduction

Consider a population of charged particles with mass m and charge q interacting

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through their self-consistent electro-magnetic field. We assume that the collisions are so rare such that we can neglect them. Let us denote by $f(t,x,p)$ the particle density, depending on the time $t \in [0,+\infty[$, position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$ meaning that at any time t the number of particles having the position and momentum inside the phase space infinitesimal volume $dxdp$ around (x,p) is $f(t,x,p)\ dxdp$. The particle density f satisfies the Vlasov equation

$$\partial_t f + v(p) \cdot \nabla_x f + q(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \tag{1}$$

where $v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2c^2}\right)^{-1/2}$ is the relativistic velocity associated to the momentum p and c is the light speed in the vacuum. Notice that $v(p) = \nabla_p \mathcal{E}(p)$ where $\mathcal{E}(p) = mc^2\left(\left(1 + \frac{|p|^2}{m^2c^2}\right)^{1/2} - 1\right)$ is the relativistic kinetic energy. The Vlasov equation expresses formally the invariance of the density f along the trajectories $(X(s), P(s))$ in the phase space

$$\frac{d}{ds}\{f(s, X(s), P(s))\} = 0,$$

where (X,P) are given by the motion equations under the action of the electro-magnetic field (E,B)

$$\frac{dX}{ds} = v(P(s)),\ \ \frac{dP}{ds} = q(E(s,X(s)) + v(P(s)) \wedge B(s,X(s))).$$

We introduce the charge and current densities ρ and j given by

$$\rho(t,x) = q\int_{\mathbb{R}^3} f(t,x,p)\ dp,\ \ j(t,x) = q\int_{\mathbb{R}^3} v(p)f(t,x,p)\ dp.$$

The self-consistent electro-magnetic field (E,B) satisfies the Maxwell equations

$$\partial_t E - c^2 \mathrm{rot} B = -\frac{j(t,x)}{\varepsilon_0},\ \ \partial_t B + \mathrm{rot} E = 0, \tag{2}$$

$$\mathrm{div} E = \frac{\rho(t,x) + \rho_{\mathrm{ext}}(x)}{\varepsilon_0},\ \ \mathrm{div} B = 0. \tag{3}$$

The system (1), (2), (3) is called the tri-dimensional Vlasov-Maxwell model. It plays a central role in plasma physics and the study of charged particle beam propagation.

Here ε_0 stands for the dielectric permittivity of the vacuum and $\rho_{\rm ext}$ is the charge density of a background distribution of opposite sign particles. We prescribe initial data

$$f(0,x,p) = f_0(x,p), \;\; (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3, \;\; (E,B)(0,x) = (E_0,B_0)(x), \;\; x \in \mathbb{R}^3, \quad (4)$$

satisfying the compatibility constraints

$${\rm div} E_0 = \frac{\rho_0(x) + \rho_{\rm ext}(x)}{\varepsilon_0}, \;\; {\rm div} B_0 = 0, \;\; x \in \mathbb{R}^3, \quad (5)$$

and the global neutrality condition

$$q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(x,p)\; dp\; dx + \int_{\mathbb{R}^3} \rho_{\rm ext}(x)\; dx = 0. \quad (6)$$

It is easily seen, by using the continuity equation $\partial_t \rho + {\rm div} j = 0$ and (6), that the global neutrality condition holds at any time $t > 0$. By taking the divergence in (2) and by using one more time the continuity equation, notice that (3) are consequences of (5).

The main global existence result of weak solution for the tri-dimensional Vlasov-Maxwell model was obtained in [7]. One of the crucial points here was the smoothing effect by velocity averaging, see also [13]. The boundary value problems were studied as well [19], [16], [1]. The global existence of classical solutions is still an open problem. For a conditional result we can refer to the Glassey-Strauss theorem [12]: the global existence of smooth solution holds provided that the particle density is compactly supported in momentum. The same problem has been investigated by other authors using different approaches [18], [4]. For results in lower dimensions we can refer to [2], [6], [9], [10], [11].

Recently a reduced Vlasov-Maxwell system was introduced by physicists for studying the laser-plasma interaction [17], [5], [3]. Other reduced models are obtained by considering asymptotic regimes as the intensity of the external magnetic field tends to infinity, leading to the "guiding center approximation" [14], [15].

Let us come back to the tri-dimensional relativistic Vlasov-Maxwell system. We consider physical units such that $m = 1$, $q = -1$ (f is a density of negative particles),

$\varepsilon_0 = 1, c = 1$. Assume for the moment that (f, E, B) is a smooth solution of (1), (2). Multiplying (1) by $\mathcal{E}(p)$ and (2) by (E, B) one gets the formula

$$\partial_t \left\{ \int_{\mathbb{R}^3} \mathcal{E}(p) f \; dp + \frac{1}{2}(|E|^2 + |B|^2) \right\} + \mathrm{div} \int_{\mathbb{R}^3} v(p)\mathcal{E}(p) f \; dp + \mathrm{rotE} \cdot B - \mathrm{rotB} \cdot E = 0. \tag{7}$$

For any $R > 0$ consider the set $K_R = \{(s,x) : s \in [0,R], |x| \leq R - s\}$ and denote by (n_t, n_x) the outward unit normal on ∂K_R. As usually, integrating (7) over $K_R(t) = \{(s,x) \in K_R \; : \; s \leq t\}$ yields for any $t \in [0,R]$

$$\begin{aligned}
\int_{B_{R-t}} \left\{ \int_{\mathbb{R}^3} \mathcal{E}(p) f dp + \frac{1}{2}(|E|^2 + |B|^2) \right\} dx &+ \int_{\Sigma_R(t)} \int_{\mathbb{R}^3} (n_t + n_x \cdot v(p)) \mathcal{E}(p) f dp \; d\sigma(s,x) \\
&+ \int_{\Sigma_R(t)} \left\{ \frac{n_t}{2}(|E|^2 + |B|^2) + (n_x \wedge E) \cdot B \right\} d\sigma \\
&= \int_{B_R} \left\{ \int_{\mathbb{R}^3} \mathcal{E}(p) f_0 dp + \frac{1}{2}(|E_0|^2 + |B_0|^2) \right\} dx,
\end{aligned}$$

where $\Sigma_R(t) = \{(s,x) : s \in [0,t], |x| = R - s\}$ and $B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Taking into account that $|n_x| = n_t$ on $\Sigma_R(t)$ and $|v(p)| < 1$, we deduce the well known inequality

$$\int_{B_{R-t}} \left\{ \int_{\mathbb{R}^3} \mathcal{E}(p) f dp + \frac{1}{2}(|E|^2 + |B|^2) \right\} dx \leq \int_{B_R} \left\{ \int_{\mathbb{R}^3} \mathcal{E}(p) f_0 dp + \frac{1}{2}(|E_0|^2 + |B_0|^2) \right\} dx. \tag{8}$$

In particular if $f_0|_{B_R \times \mathbb{R}^3} = 0$, $(E_0, B_0)|_{B_R} = (0,0)$ then $f|_{K_R \times \mathbb{R}^3} = 0$ and $(E, B)|_{K_R} = (0,0)$. Motivated by this standard result we inquire about a more general property whose statement, in a simplified form, could be

Property 1.1 *Assume that $(f^k, E^k, B^k)_{k \in \{1,2\}}$ are two solutions of the relativistic Vlasov-Maxwell equations satisfying $(f^1(0) - f^2(0))|_{B_R \times \mathbb{R}^3} = 0$, $(E^1(0) - E^2(0), B^1(0) - B^2(0))|_{B_R} = (0,0)$. Then we have $(f^1 - f^2)|_{K_R \times \mathbb{R}^3} = 0$, $(E^1 - E^2, B^1 - B^2)|_{K_R} = (0,0)$.*

We recognize here the finite speed propagation for the solution of the Vlasov-Maxwell equations. The propagation speed do not exceed the light speed, here normalized to the unity. The purpose in this paper is to establish this property for the relativistic Vlasov-Maxwell equations in the one dimensional case (1D) (see Theorem 2.1) and the one and one half dimensional case (1.5D) (see Theorem 3.1). Obviously this feature inherits from the hyperbolic structure of the Maxwell equations combined with the relativistic character of the particle dynamics. Although this property seems very natural we think that it is important to perform a rigorous analysis of it. This leads to a better understanding of the transport of relativistic charged particles: we justify the existence of a dependence domain in space. For the numerical point of view this property has important consequences: it shows that the numerical approximation of these equations can be localized with respect to the space variable.

An interesting question concerns the validity of this result in the general framework of the tri-dimensional relativistic Vlasov-Maxwell system as suggested by (8). We expect that this holds true at least for solutions compactly supported in momentum. Probably combining our techniques with the representation formula for the electro-magnetic field obtained in [12] would provide the desired result.

The paper is organized as follows. In Section 2 we recall a basic existence and uniqueness result for the mild solution of the relativistic 1D Vlasov-Maxwell equations. The main tool here is the formulation by characteristics. Adapting the above method yields also a continuous dependence result with respect to the initial data. In particular we establish the finite speed propagation property. In Section 3 the same program is carried out for the relativistic 1.5D Vlasov-Maxwell equations. Basically we follow the same steps but some of the computations are much more difficult in this case.

2 The 1D relativistic Vlasov-Maxwell equations

We assume that the unknowns depend on time, on one spatial coordinate $x \in \mathbb{R}$ and one momentum coordinate $p \in \mathbb{R}$. In this case we obtain the system

$$\partial_t f + v(p)\partial_x f - E(t,x)\partial_p f = 0, \quad (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{9}$$

$$\partial_t E = j(t,x), \quad \partial_x E = \rho_{\text{ext}}(x) - \rho(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{10}$$

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \mathbb{R}^2, \quad E(0,x) = E_0(x), \quad x \in \mathbb{R}, \tag{11}$$

where $\rho = \int_{\mathbb{R}} f \, dp$, $j = \int_{\mathbb{R}} v(p) f \, dp$. We assume that the initial conditions and ρ_{ext} verify the hypotheses

H1) there is a function $g_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ non decreasing on $\mathbb{R}^-$ and non increasing on $\mathbb{R}^+$ such that $0 \leq f_0(x,p) \leq g_0(p)$, $\forall \, (x,p) \in \mathbb{R}^2$;
H2) E_0 belongs to $L^\infty(\mathbb{R})$ such that $E_0' = \rho_{\text{ext}} - \rho_0$, where $\rho_0 = \int_{\mathbb{R}} f_0 \, dp$;
H3) $\rho_{\text{ext}} \geq 0$, ρ_{ext} belongs to $L^\infty(\mathbb{R})$.

Observe that H1 implies $\rho_0 \in L^\infty(\mathbb{R})$ and therefore $E_0 \in W^{1,\infty}(\mathbb{R})$. Under the above hypotheses there is a unique mild solution (f, E) (*i.e.*, E is Lipschitz continuous and f is solution by characteristics) for (9), (10), (11), cf. [6], [2]. We recall here some bounds for E and its derivatives which will be useful in our further computations. Let us introduce the system of characteristics for (9)

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = -E(s, X(s)), \tag{12}$$

with the conditions

$$X(t) = x, \quad P(t) = p. \tag{13}$$

The solution of (12), (13) is denoted by $(X(s;t,x,p), P(s;t,x,p))$. Saying that f is solution by characteristics for (9) means that $f(t,x,p) = f_0(X(0;t,x,p), P(0;t,x,p))$, $\forall \, (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2$. For any test function $\varphi \in L^1(\mathbb{R})$ one gets by (10)

$$\begin{aligned} \int_{\mathbb{R}} (E(t,x) - E_0(x)) \varphi(x) \, dx &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f(s,x,p) v(p) \varphi(x) \, dp \, dx \, ds \\ &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(X(0;s,x,p), P(0;s,x,p)) v(p) \varphi(x) \, dp \, dx \, ds. \end{aligned}$$

Note that $\det \left(\frac{\partial (X(s;t,x,p), P(s;t,x,p))}{\partial (x,p)} \right) = 1$, for any $(s,t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2$ and thus, after change of variables along the characteristics we obtain

$$
\begin{aligned}
\int_{\mathbb{R}}(E(t,x)-E_0(x))\varphi(x)\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_0^t \frac{dX}{ds}\varphi(X(s))\,ds\,dp\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx. \qquad (14)
\end{aligned}
$$

Observe that $|X(t;0,x,p)-x| \le t$ and therefore we can write

$$
\begin{aligned}
\left|\int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_x^{X(t;0,x,p)}\varphi(u)\,du\,dp\,dx\right| &\le \int_{\mathbb{R}}|\varphi(u)|\int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \mathbf{1}_{\{|u-x|\le|X(t;0,x,p)-x|\}}\,dp\,dx\,du \\
&\le \int_{\mathbb{R}}|\varphi(u)|\int_{\mathbb{R}}\int_{\mathbb{R}} g_0(p)\mathbf{1}_{\{|u-x|\le t\}}\,dp\,dx\,du \\
&\le 2t\|g_0\|_{L^1}\|\varphi\|_{L^1}.
\end{aligned}
$$

We deduce that

$$
\|E(t)\|_{L^\infty} \le \|E_0\|_{L^\infty} + 2t\|g_0\|_{L^1} =: a(t). \qquad (15)
$$

For estimating $\partial_x E$ we denote by g_0^R the function given by $g_0^R(p) = g_0(p+R)$ if $p<-R$, $g_0^R(p) = g_0(p-R)$ if $p>R$ and $g_0^R(p) = g_0(0)$ if $|p|\le R$. Observe that for any $(t,x,p)\in[0,T]\times\mathbb{R}^2$ we have

$$
|P(0;t,x,p)-p| \le \int_0^t \|E(s)\|_{L^\infty}\,ds \le Ta(T) =: R,
$$

and thus we deduce by using the monotonicity of g_0 that $g_0(P(0;t,x,p)) \le g_0^R(p)$, $\forall\,(t,x,p)\in[0,T]\times\mathbb{R}^2$. We have

$$
\begin{aligned}
\rho(t,x) &= \int_{\mathbb{R}} f_0(X(0;t,x,p),P(0;t,x,p))\,dp \le \int_{\mathbb{R}} g_0(P(0;t,x,p))\,dp \le \int_{\mathbb{R}} g_0^R(p)\,dp \\
&= \|g_0\|_{L^1} + 2Ta(T)\|g_0\|_{L^\infty},
\end{aligned}
$$

implying that for any $T>0$ we have

$$
\max\{\|\partial_x E\|_{L^\infty(]0,T[\times\mathbb{R})}, \|\partial_t E\|_{L^\infty(]0,T[\times\mathbb{R})}\} \le \|\rho_{\text{ext}}\|_{L^\infty} + \|g_0\|_{L^1} + 2\|g_0\|_{L^\infty}Ta(T) =: b(T). \qquad (16)
$$

Theorem 2.1 *Assume that $(f_0^k, E_0^k)_{k\in\{1,2\}}$ satisfy the hypotheses H1-H3 and denote by $(f^k,E^k)_{k\in\{1,2\}}$ the global mild solutions of the 1D relativistic Vlasov-Maxwell system corresponding to the initial conditions $(f_0^k,E_0^k)_{k\in\{1,2\}}$. Then for any $R>0$*

there is a constant C_R depending on R, $\max_{k\in\{1,2\}} b^k(R)$ such that for all $t\in[0,R]$, $|x|\leq R-t$, $p\in\mathbb{R}$ we have

$$\begin{aligned}|E^1-E^2|(t,x)+(|X^1-X^2|+|P^1-P^2|)(0;t,x,p) &\leq C_R(\|f_0^1-f_0^2\|_{L^1(]-R,R[\times\mathbb{R})}\\ &+ \|E_0^1-E_0^2\|_{L^\infty(]-R,R[)}),\end{aligned}$$

where (X^k,P^k) are the characteristics associated to $E^k, k\in\{1,2\}$. In particular if $f_0^1(x,p)=f_0^2(x,p),\ \forall\ (x,p)\in[-R,R]\times\mathbb{R}$ and $E_0^1(x)=E_0^2(x),\ \forall\ x\in[-R,R]$ for some $R>0$ then for any $t\in[0,R]$ we have

$$f^1(t,x,p)=f^2(t,x,p),\ \forall\ (x,p)\in[-(R-t),R-t]\times\mathbb{R},$$

$$E^1(t,x)=E^2(t,x),\ \forall\ x\in[-(R-t),R-t].$$

Proof. Take $t\in[0,R]$ and $\varphi\in C_c^0([-(R-t),R-t])$. By using formula (14) one gets

$$\begin{aligned}\int_{\mathbb{R}}(E^1(t,x)-E^2(t,x))\varphi(x)\,dx &= \int_{\mathbb{R}}(E_0^1(x)-E_0^2(x))\varphi(x)\,dx \\ &- \sum_{k=1}^{2}(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}}f_0^k(x,p)\int_x^{X^k(t;0,x,p)}\varphi(u)\,du\,dp\,dx.\end{aligned} \tag{17}$$

Observe that for any $s\in[0,t]$ we have

$$|X^k(s;0,x,p)-x|=\left|\int_0^s v(P^k(s;0,x,p))\,ds\right|\leq s,\quad k\in\{1,2\},$$

which implies that $X^k(t;0,x,p)\geq R-t$ for any $(x,p)\in[R,+\infty[\times\mathbb{R}, k\in\{1,2\}$ and $X^k(t;0,x,p)\leq -R+t$ for any $(x,p)\in]-\infty,-R]\times\mathbb{R}, k\in\{1,2\}$. Since the function φ has compact support in $[-(R-t),R-t]$, we obtain

$$\int_x^{X^k(t;0,x,p)}\varphi(u)\,du=\int_x^{X^k(t;0,x,p)}\varphi(u)\,du\,\mathbf{1}_{\{|x|<R\}},\quad k\in\{1,2\},$$

and therefore formula (17) yields

$$\begin{aligned}\left|\int_{\mathbb{R}}(E^1(t,x)-E^2(t,x))\varphi(x)\,dx\right| &\leq \left|\int_{\mathbb{R}}(E_0^1(x)-E_0^2(x))\varphi(x)\,dx\right| \\ &+ \left|\int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p)\int_{X^2(t;0,x,p)}^{X^1(t;0,x,p)}\varphi(u)\,du\,dp\,dx\right| \\ &+ \left|\int_{-R}^{R}\int_{\mathbb{R}}(f_0^1(x,p)-f_0^2(x,p))\int_x^{X^2(t;0,x,p)}\varphi(u)\,du\,dp\,dx\right| \\ &\leq \left(\|f_0^1-f_0^2\|_{L^1(]-R,R[\times\mathbb{R})}+\|E_0^1-E_0^2\|_{L^\infty(]-R,R[)}\right) \\ &\times \|\varphi\|_{L^1(\mathbb{R})}+\left|\int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p)h(t,x,p)\,dp\,dx\right|,\end{aligned} \tag{18}$$

where $h(t,x,p)=\int_{X^2(t;0,x,p)}^{X^1(t;0,x,p)}\varphi(u)\,du$, $\forall\,(t,x,p)\in[0,R]\times\mathbb{R}^2$. We consider the set

$$K(t)=\{(x,p)\in\mathbb{R}^2\ :\ \exists\,\lambda(t)\in[0,1],\ \ |\lambda(t)X^1+(1-\lambda(t))X^2|(t;0,x,p)\leq R-t\}.$$

We claim that $K(t)\subset K(s)$ or $\complement K(s)\subset\complement K(t)$, for any $s\leq t$. Indeed, assume that $(x,p)\notin K(s)$ for some $s<t$. Therefore $\min\{X^1(s;0,x,p),X^2(s;0,x,p)\}>R-s$ or $\max\{X^1(s;0,x,p),X^2(s;0,x,p)\}<-(R-s)$. In the first case we deduce that

$$X^k(t;0,x,p)\geq X^k(s;0,x,p)-(t-s)>R-t,\ \ k\in\{1,2\},$$

whereas in the second case we have

$$X^k(t;0,x,p)\leq X^k(s;0,x,p)+(t-s)<-(R-t),\ \ k\in\{1,2\}.$$

Therefore in both cases $(x,p)\notin K(t)$. Notice also that for any $(x,p)\notin K(t)$ the segment between $X^1(t;0,x,p)$ and $X^2(t;0,x,p)$ has void intersection with the support of φ which implies that $h(t,x,p)=0$. We have proved that

$$h(t,x,p)=h(t,x,p)\,\mathbf{1}_{\{(x,p)\in K(t)\}},\ \forall\,(t,x,p)\in[0,R]\times\mathbb{R}^2. \tag{19}$$

Thus, when estimating $h(t,x,p)$, it is sufficient to consider $(x,p)\in K(t)$. For such (x,p) denote by $\lambda(s)\in[0,1]$, $\forall\,s\in[0,t]$, a number satisfying

$$Y(s):=\lambda(s)X^1(s;0,x,p)+(1-\lambda(s))X^2(s;0,x,p)\in[-(R-s),R-s].$$

By using the characteristic equations we obtain

$$\frac{d}{ds}|X^1(s) - X^2(s)| \leq |P^1(s) - P^2(s)|, \; \forall \; s \in [0, t], \tag{20}$$

$$\begin{aligned}
\frac{d}{ds}|P^1(s) - P^2(s)| &\leq |E^1(s, X^1(s)) - E^1(s, Y(s))| + |E^1(s, Y(s)) - E^2(s, Y(s))| \\
&+ |E^2(s, Y(s)) - E^2(s, X^2(s))| \\
&\leq b^1(t)\; |X^1(s) - Y(s)| + b^2(t)\; |Y(s) - X^2(s)| \\
&+ \|E^1(s) - E^2(s)\|_{L^\infty(]-(R-s), R-s[)}, \; \forall \; s \in [0, t],
\end{aligned} \tag{21}$$

where $b^k(t) = \|\rho_{\rm ext}\|_{L^\infty(\mathbb{R})} + \|g_0^k\|_{L^1(\mathbb{R})} + 2ta^k(t)\|g_0^k\|_{L^\infty(\mathbb{R})}$, $a^k(t) = \|E_0^k\|_{L^\infty(\mathbb{R})} + 2t\|g_0^k\|_{L^1(\mathbb{R})}$, $k \in \{1, 2\}$. From (20), (21) one gets

$$\begin{aligned}
\frac{d}{ds}\left\{|X^1 - X^2| + |P^1 - P^2|\right\} &\leq (1 + b(t))(|X^1(s) - X^2(s)| + |P^1(s) - P^2(s)|) \\
&+ \|E^1(s) - E^2(s)\|_{L^\infty(]-(R-s), R-s[)},
\end{aligned} \tag{22}$$

where $b(t) = \max\{b^1(t), b^2(t)\}$. By Gronwall lemma we obtain

$$\begin{aligned}
|X^1(t; 0, x, p) - X^2(t; 0, x, p)| &+ |P^1(t; 0, x, p) - P^2(t; 0, x, p)| \leq \exp(t(1 + b(t))) \\
&\times \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(]-(R-s), R-s[)} ds \\
&=: Q(t),
\end{aligned} \tag{23}$$

and therefore, by using (19) one gets as before

$$\begin{aligned}
\left|\int_{-R}^R \int_{\mathbb{R}} f_0^1 h \; dp \; dx\right| &\leq \int_{\mathbb{R}} |\varphi(u)| \int_{-R}^R \int_{\mathbb{R}} f_0^1 \mathbf{1}_{\{(x,p) \in K(t)\}} \mathbf{1}_{\{|u - X^1(t;0,x,p)| \leq Q(t)\}} \; dp \; dx \; du \\
&\leq \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}} \int_{\mathbb{R}} f^1(t, X^1, P^1) \mathbf{1}_{\{|u - X^1| \leq Q(t)\}} dP^1 \; dX^1 \; du \\
&= \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}} \rho^1(t, X^1) \mathbf{1}_{\{|u - X^1| \leq Q(t)\}} dX^1 \; du \\
&\leq 2b(t)Q(t)\|\varphi\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{24}$$

Finally combining (18), (24) yields for any $t \in [0, R]$

$$\begin{aligned}
\|E^1(t) - E^2(t)\|_{L^\infty(]-(R-t), R-t[)} &\leq \|f_0^1 - f_0^2\|_{L^1(]-R,R[\times\mathbb{R})} + \|E_0^1 - E_0^2\|_{L^\infty(]-R,R[)} \\
&+ C_1 \int_0^t \|E^1(s) - E^2(s)\|_{L^\infty(]-(R-s), R-s[)} ds,
\end{aligned}$$

where $C_1(R) = 2b(R)\exp((1+b(R))R)$. We obtain

$$\begin{aligned}\|E^1(t)-E^2(t)\|_{L^\infty(]-(R-t),R-t[)} &\leq \left(\|f_0^1-f_0^2\|_{L^1(]-R,R[\times\mathbb{R})}+\|E_0^1-E_0^2\|_{L^\infty(]-R,R[)}\right)\\ &\times\ C_2(R), \end{aligned}\tag{25}$$

where $C_2(R) = \exp(R\,C_1(R))$. Observe that for any $(x,p) \in [-(R-t), R-t] \times \mathbb{R}$ we have $|X^k(s;t,x,p)| \leq |x| + t - s \leq R - s, \quad s \in [0,t], \quad k \in \{1,2\}$. Therefore we can prove, by performing a similar decomposition as in (21), that

$$\begin{aligned}(|X^1-X^2|+|P^1-P^2|)(0;t,x,p) &\leq \exp(t(1+b(t)))\int_0^t \|(E^1-E^2)(s)\|_{L^\infty(]-(R-s),R-s[)}\,ds\\ &\leq \exp(R(1+b(R)))\ R\,C_2(R)\left(\|f_0^1-f_0^2\|_{L^1(]-R,R[\times\mathbb{R})}+\|E_0^1-E_0^2\|_{L^\infty(]-R,R[)}\right),\end{aligned}$$

and the first statement of our theorem holds with $C_R = C_2(R)(1 + R\exp(R(1 + b(R))))$. The second statement follows immediately by taking into account that for any $t \in [0,R]$, $|x| \leq R - t$, $p \in \mathbb{R}$ we have $|X^k(0;t,x,p)| \leq R, k \in \{1,2\}$ and therefore

$$f^1(t,x,p) = f_0^1((X^1,P^1)(0;t,x,p)) = f_0^2((X^2,P^2)(0;t,x,p)) = f^2(t,x,p).$$

$\square$

3 The relativistic Vlasov-Maxwell system in one and one half dimension

We assume that the electron density f depends on the time $t \geq 0$, one space coordinate $x \in \mathbb{R}$ and two momentum coordinates $p = (p_1, p_2) \in \mathbb{R}^2$. We suppose also that the electro-magnetic field is of the form $\mathbf{E} = (E_1(t,x), E_2(t,x), 0)$, $\mathbf{B} = (0, 0, B(t,x))$ for any $(t,x) \in [0, +\infty[\times\mathbb{R}$. In this case we obtain the equations

$$\partial_t f + v_1(p)\partial_x f - (E_1(t,x) + v_2(p)B(t,x))\partial_{p_1} f - (E_2(t,x) - v_1(p)B(t,x))\partial_{p_2} f = 0, \tag{26}$$

$$\partial_t E_1 = j_1(t,x), \quad \partial_x E_1 = \rho_{\text{ext}} - \rho(t,x), \quad (t,x) \in]0, +\infty[\times\mathbb{R}, \tag{27}$$

$$\partial_t E_2 + \partial_x B = j_2(t,x), \quad (t,x) \in]0,+\infty[\times\mathbb{R}, \tag{28}$$

$$\partial_t B + \partial_x E_2 = 0, \quad (t,x) \in]0,+\infty[\times\mathbb{R}, \tag{29}$$

where $\rho_{\text{ext}} \geq 0$ is the charge density of the background ion population and ρ, j are the charge and current densities of the electrons

$$\rho(t,x) = \int_{\mathbb{R}^2} f(t,x,p)\, dp, \quad j(t,x) = \int_{\mathbb{R}^2} v(p) f(t,x,p)\, dp, \quad (t,x) \in [0,+\infty[\times\mathbb{R}.$$

We supplement the above equations with the initial conditions

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \mathbb{R}^3, \quad (E_1,E_2,B)(0,x) = (E_{0,1},E_{0,2},B_0)(x), \quad x \in \mathbb{R}. \tag{30}$$

We assume that the initial conditions and ρ_{ext} satisfy the hypotheses H3 and

H4) there is a function $g_0 \in L^1(\mathbb{R}^+; u^2\, du) \cap L^\infty(\mathbb{R}^+)$ non increasing on $\mathbb{R}^+$ such that $f_0(x,p) \leq g_0(|p|)$, $\forall\, (x,p) \in \mathbb{R}^3$;

H5) $(E_{0,1}, E_{0,2}, B_0) \in L^\infty(\mathbb{R})^3$;

H6) $E'_{0,1} = \rho_{\text{ext}} - \int_{\mathbb{R}^2} f_0\, dp$, $(E'_{0,2}, B'_0) \in L^\infty(\mathbb{R})^2$.

Notice that H4 implies that $\int_{\mathbb{R}^2} (1+|p|) f_0(\cdot,p)\, dp \in L^\infty(\mathbb{R})$. In particular we have $E'_{0,1} \in L^\infty(\mathbb{R})$. Under the hypotheses H3-H6, by using the method of [3] we prove the existence of a unique mild solution (f,E,B) for the 1.5D relativistic Vlasov-Maxwell system, satisfying $(1+|p|)f \in L^\infty(]0,T[\times\mathbb{R}; L^1(\mathbb{R}^2))$, $(E,B) \in W^{1,\infty}(]0,T[\times\mathbb{R})^3$, $\forall\, T > 0$. The system (26), (27), (28), (29), (30) was studied in [9], [8]. Let us recall here the main steps for estimating the electro-magnetic field and its derivatives. We denote by $(X(s;t,x,p), P(s;t,x,p))$ the characteristics of (26) given by

$$\frac{dX}{ds} = v_1(P(s;t,x,p)),$$

$$\frac{dP_1}{ds} = -(E_1(s,X(s;t,x,p)) + v_2(P(s;t,x,p))B(s,X(s;t,x,p))),$$

$$\frac{dP_2}{ds} = -(E_2(s,X(s;t,x,p)) - v_1(P(s;t,x,p))B(s,X(s;t,x,p))),$$

satisfying the conditions $X(t;t,x,p) = x$, $P(t;t,x,p) = p$.

Lemma 3.1 *(L^∞ bounds for the electro-magnetic field) Assume that the hypotheses $H3-H6$ hold and let (f,E,B) be the global mild solution for the 1.5D relativistic Vlasov-Maxwell system. Then for any $t>0$ we have the estimates*

$$\|E_1(t)\|_{L^\infty} \le \|E_{0,1}\|_{L^\infty} + 2t\int_{\mathbb{R}^2} g_0(|p|)\,dp =: a_1(t). \tag{31}$$

$$\begin{aligned}\max\{\|E_2(t)\|_{L^\infty}, \|B(t)\|_{L^\infty}\} &\le \|E_{0,2}\|_{L^\infty} + \|B_0\|_{L^\infty} + t\int_{\mathbb{R}^2}(1+|p|^2)^{\frac{1}{2}} g_0(|p|)\,dp \\ &+ \frac{t}{2}\,(\|E_0\|^2_{L^\infty} + \|B_0\|^2_{L^\infty}) =: a_2(t).\end{aligned} \tag{32}$$

Proof. Solving (27), (28), (29) with respect to E_1, E_2, B yields

$$E_1(t,x) = E_{0,1}(x) + J_1(t,x), \tag{33}$$

$$E_2(t,x) = \frac{1}{2}(E_{0,2}+B_0)(x-t) + \frac{1}{2}(E_{0,2}-B_0)(x+t) + \frac{1}{2}J_2^+(t,x) + \frac{1}{2}J_2^-(t,x), \tag{34}$$

$$B(t,x) = \frac{1}{2}(E_{0,2}+B_0)(x-t) - \frac{1}{2}(E_{0,2}-B_0)(x+t) + \frac{1}{2}J_2^+(t,x) - \frac{1}{2}J_2^-(t,x), \tag{35}$$

where $J_1(t,x) = \int_0^t j_1(s,x)\,ds$, $J_2^\pm(t,x) = \int_0^t j_2(s, x\mp(t-s))\,ds$. Multiplying (33) by a test function $\varphi \in L^1(\mathbb{R})$, integrating with respect to $x\in\mathbb{R}$ and changing the variables along the characteristics yields, as in the 1D case (see (15)), the bound

$$\|E_1(t)\|_{L^\infty} \le \|E_{0,1}\|_{L^\infty} + 2t\int_{\mathbb{R}^2} g_0(|p|)\,dp = a_1(t).$$

For estimating (E_2, B) we follow the ideas in [9]. Multiplying (26) by $(1+|p|^2)^{\frac{1}{2}}$, the first equation of (27) by E_1, (28) by E_2, (29) by B and integrating with respect to $p\in\mathbb{R}^2$ implies

$$\partial_t\left\{\int_{\mathbb{R}^2}(1+|p|^2)^{\frac{1}{2}} f dp + \frac{1}{2}(|E|^2+B^2)\right\} + \partial_x\left\{\int_{\mathbb{R}^2} v_1(p)(1+|p|^2)^{\frac{1}{2}} f dp + E_2 B\right\} = 0. \tag{36}$$

Integrating (36) on $\{(s,y) : s\in[0,t], |x-y|\le t-s\}$ we deduce that

$$\begin{aligned}&\sum_{k=1}^{2}\int_0^t\int_{\mathbb{R}^2}(1+|p|^2)^{\frac{1}{2}}(1+(-1)^k v_1(p)) f(s, x+(-1)^k(t-s), p)\,dp\,ds \\ &\le \int_{x-t}^{x+t}\int_{\mathbb{R}^2}(1+|p|^2)^{\frac{1}{2}} f_0(y,p)\,dp\,dy + \frac{1}{2}\int_{x-t}^{x+t}(|E_0(y)|^2 + B_0(y)^2)\,dy.\end{aligned}$$

Observing that $(1+|p|^2)^{\frac{1}{2}}(1-|v_1(p)|) \geq |v_2(p)|$ we obtain

$$|J_2^+(t,x)| + |J_2^-(t,x)| \leq 2t \int_{\mathbb{R}^2} (1+|p|^2)^{\frac{1}{2}} g_0(|p|)\ dp + t\ (\|E_0\|_{L^\infty}^2 + \|B_0\|_{L^\infty}^2),$$

implying that

$$\begin{aligned} \max\{\|E_2(t)\|_{L^\infty}, \|B(t)\|_{L^\infty}\} &\leq \|E_{0,2}\|_{L^\infty} + \|B_0\|_{L^\infty} + t \int_{\mathbb{R}^2} (1+|p|^2)^{\frac{1}{2}} g_0(|p|)\ dp \\ &+ \frac{t}{2}\ (\|E_0\|_{L^\infty}^2 + \|B_0\|_{L^\infty}^2) = a_2(t). \end{aligned}$$

$\square$

Lemma 3.2 *(L^∞ bounds for the derivatives of the electro-magnetic field) With the notations of Lemma 3.1 we denote by $a(\cdot)$ the function $a(t) = (a_1(t)^2 + a_2(t)^2)^{1/2}$. Then under the hypotheses of Lemma 3.1 we have the inequalities*

$$\left\| \int_{\mathbb{R}^2} f(t,\cdot,p)\ dp \right\|_{L^\infty(\mathbb{R})} \leq \pi (ta(t))^2 \|g_0\|_{L^\infty} + 2\pi \int_0^{+\infty} g_0(u)(u+ta(t))\ du. \qquad (37)$$

$$\left\| \int_{\mathbb{R}^2} |p| f(t,\cdot,p)\ dp \right\|_{L^\infty(\mathbb{R})} \leq \frac{2\pi}{3} (ta(t))^3 \|g_0\|_{L^\infty} + 2\pi \int_0^{+\infty} (u+ta(t))^2 g_0(u)\ du =: d(t).$$

$$\begin{aligned} \|\partial_x E_1\|_{L^\infty(]0,T[\times\mathbb{R})} &\leq \max\{\|\rho_{\text{ext}}\|_{L^\infty}, \pi(Ta(T))^2 \|g_0\|_{L^\infty} + 2\pi \int_0^{+\infty} g_0(u)(u+Ta(T))\ du\} \\ &= : b_1(T) \end{aligned} \qquad (38)$$

$$\begin{aligned} \max\{\|\partial_x E_2\|_{L^\infty(]0,T[\times\mathbb{R})}\ ,\ \|\partial_x B\|_{L^\infty(]0,T[\times\mathbb{R})}\} &\leq \|E_{0,2}'\|_{L^\infty(\mathbb{R})} + \|B_0'\|_{L^\infty(\mathbb{R})} \qquad (39) \\ &+ (b_1(T)+d(T))(2+T(a_1(T)+6a_2(T))) =: b_2(T). \end{aligned}$$

Proof. The charge density can be estimated as in the 1D case by using H4. Indeed, by the characteristic equations we deduce that $|\ |P(0;t,x,p)| - |p|\ | \leq ta(t)$ and therefore we obtain

$$\begin{aligned} \rho(t,x) &= \int_{\mathbb{R}^2} f_0(X(0;t,x,p), P(0;t,x,p))\ dp \leq \int_{\mathbb{R}^2} g_0(|P(0;t,x,p)|)\ dp \\ &\leq \pi(ta(t))^2 \|g_0\|_{L^\infty} + 2\pi \int_0^{+\infty} g_0(u)(u+ta(t))\ du. \end{aligned}$$

By the second equation in (27) one gets

$$\begin{aligned}
\|\partial_x E_1\|_{L^\infty(]0,T[\times\mathbb{R})} &\leq \max\{\|\rho_{\text{ext}}\|_{L^\infty}, \pi(Ta(T))^2\|g_0\|_{L^\infty} + 2\pi\int_0^{+\infty} g_0(u)(u+Ta(T))\,du\} \\
&= b_1(T).
\end{aligned}$$

In a similar way we can estimate $k(t,x) = \int_{\mathbb{R}^2}|p|f(t,x,p)\,dp$. We obtain

$$k(t,x) \leq \frac{2\pi}{3}(ta(t))^3\|g_0\|_{L^\infty} + 2\pi\int_0^{+\infty}(u+ta(t))^2 g_0(u)\,du = d(t).$$

We estimate now the x derivatives of $J_2^\pm$. The idea is to proceed by duality approach: we will check that for any $t>0$ there is a constant $C = C(t)$ such that $|\int_{\mathbb{R}} J_2^\pm(t,x)\varphi'(x)\,dx| \leq C(t)\|\varphi\|_{L^1(\mathbb{R})}$ for any $\varphi \in C_c^1(\mathbb{R})$ which would imply that $\partial_x J_2^\pm(t) \in L^\infty(\mathbb{R})$ and $\|\partial_x J_2^\pm(t)\|_{L^\infty(\mathbb{R})} \leq C(t)$. For any test function $\varphi \in C_c^1(\mathbb{R})$ we write

$$\begin{aligned}
\int_{\mathbb{R}} J_2^\pm(t,x)\varphi'(x)\,dx &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} v_2(p)\varphi'(x\pm(t-s))f(s,x,p)\,dp\,dx\,ds \qquad (40) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0(x,p)\int_0^t G^\pm(P(s))\frac{d}{ds}\varphi(X(s)\pm(t-s))\,ds\,dp\,dx,
\end{aligned}$$

where $G^\pm(p) = \frac{p_2(-p_1\mp(1+|p|^2)^{1/2})}{1+p_2^2}$ for any $p\in\mathbb{R}^2$. By direct computation we check that

$$\max\{|G^\pm(p)|, |\nabla_p G^\pm(p)|, |\nabla_p^2 G^\pm(p)|\} \leq C(1+|p|),\ \forall\, p\in\mathbb{R}^2,$$

for some constant C. Integrating by parts with respect to s in (40) yields

$$\begin{aligned}
\int_{\mathbb{R}} J_2^\pm(t,x)\varphi'(x)\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0 G^\pm(P(t))\varphi(X(t))\,dp\,dx - \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0 G^\pm(p)\varphi(x\pm t)\,dp\,dx \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0(x,p)\int_0^t \frac{d}{ds}\{G^\pm(P(s))\}\varphi(X(s)\pm(t-s))\,ds\,dp\,dx \\
&= T_1^\pm - T_2^\pm - T_3^\pm. \qquad (41)
\end{aligned}$$

The terms $T_1^{\pm}$ can be estimated as follows

$$
\begin{aligned}
|T_1^{\pm}| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(t,X(t),P(t))G^{\pm}(P(t))\varphi(X(t))\;dp\;dx\right| \\
&= \left|\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(t,x,p)G^{\pm}(p)\varphi(x)\;dp\;dx\right| \\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^2} f(t,x,p)(1+|p|)|\varphi(x)|\;dp\;dx \\
&\le \|\varphi\|_{L^1(\mathbb{R})}\left\|\int_{\mathbb{R}^2}(1+|p|)f(t,\cdot,p)\;dp\right\|_{L^\infty(\mathbb{R})} \\
&\le \|\varphi\|_{L^1(\mathbb{R})}(b_1(t)+d(t)).
\end{aligned} \tag{42}
$$

Similarly we obtain

$$
|T_2^{\pm}| \le \|\varphi\|_{L^1(\mathbb{R})}(b_1(0)+d(0)). \tag{43}
$$

In order to estimate $T_3^{\pm}$ notice that for any $s\in[0,t]$ we have

$$
\left|\frac{d}{ds}\{G^{\pm}(P(s))\}\right| \le \left|\frac{dP_1}{ds}\right| + \left(\frac{1}{2}+2(1+|P(s)|)\right)\left|\frac{dP_2}{ds}\right| \le (a_1(t)+6a_2(t))(1+|P(s)|),
$$

and therefore one gets

$$
\begin{aligned}
|T_3^{\pm}| &\le \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0\int_0^t (a_1(t)+6a_2(t))(1+|P(s)|)|\varphi(X(s)\pm(t-s))|\;ds\;dp\;dx \\
&= (a_1(t)+6a_2(t))\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(s,x,p)(1+|p|)|\varphi(x\pm(t-s))|\;dp\;dx\;ds \\
&= (a_1(t)+6a_2(t))\int_0^t\int_{\mathbb{R}} |\varphi(x\pm(t-s))|(\rho(s,x)+k(s,x))\;dx\;ds \\
&\le (a_1(t)+6a_2(t))\;t\;\|\varphi\|_{L^1(\mathbb{R})}(b_1(t)+d(t)).
\end{aligned} \tag{44}
$$

Collecting the inequalities (42), (43), (44) yields

$$
\left|\int_{\mathbb{R}} J_2^{\pm}(t,x)\varphi'(x)\;dx\right| \le \|\varphi\|_{L^1(\mathbb{R})}(b_1(t)+d(t))(2+t(a_1(t)+6a_2(t))),
$$

saying that $\|\partial_x J_2^{\pm}(t)\|_{L^\infty(\mathbb{R})} \le (b_1(t)+d(t))(2+t(a_1(t)+6a_2(t)))$. Finally we deduce by (34), (35)

$$
\begin{aligned}
\max\{\|\partial_x E_2\|_{L^\infty(]0,T[\times\mathbb{R})}\;,\;\|\partial_x B\|_{L^\infty(]0,T[\times\mathbb{R})}\} &\le \|E_{0,2}'\|_{L^\infty(\mathbb{R})}+\|B_0'\|_{L^\infty(\mathbb{R})} \\
&\quad + (b_1(T)+d(T))(2+T(a_1(T)+6a_2(T))) = b_2(T).
\end{aligned}
$$

□

For any $R > 0$ and $t \in [0, R]$ let us introduce

$$\tilde{K}(t) = \{(x, p) \in \mathbb{R}^3 \ : \ \exists\, \lambda(t) \in [0, 1],\ |\lambda(t)X^1 + (1 - \lambda(t))X^2|(t; 0, x, p) \le R - t\},$$

where $(X^k, P^k)_{k\in\{1,2\}}$ are the characteristics associated to some smooth electro-magnetic fields $(E^k, B^k)_{k\in\{1,2\}}$. We intend to establish a continuous dependence result with respect to the electro-magnetic field for characteristics starting from $\tilde{K}(t)$ at $s = 0$.

Lemma 3.3 *(Continuous dependence for the characteristics) Consider* $(E^k, B^k)_{k\in\{1,2\}} \subset L^\infty(]0, R[; W^{1,\infty}(\mathbb{R}))$ *for some* $R > 0$ *and let us denote by* $(X^k, P^k)_{k\in\{1,2\}}$ *the corresponding characteristics. Then for any* $0 \le s \le t \le R$, $(x, p) \in \tilde{K}(t)$ *we have the inequality*

$$|X^1(s) - X^2(s)| + |P^1(s) - P^2(s)| \le 2\exp(t(2 + \max_{s\in[0,t]} C(s))) \int_0^t D^R(s)\, ds, \quad (45)$$

where $D^R(s) = \|E^1(s) - E^2(s)\|_{L^\infty(]-(R-s),R-s[)} + \|B^1(s) - B^2(s)\|_{L^\infty(]-(R-s),R-s[)}$ *and* $C(s) = 2\max_{k\in\{1,2\}} \|\partial_x E^k(s)\|_{L^\infty} + 2\max_{k\in\{1,2\}} \|\partial_x B^k(s)\|_{L^\infty} + 4\max_{k\in\{1,2\}} \|B^k(s)\|_{L^\infty}$.

Proof. As before we have $\tilde{K}(t) \subset \tilde{K}(s)$ for all $s \in [0, t]$ and thus for any $(s, x, p) \in [0, t] \times \tilde{K}(t)$, the segment between $X^1(s; 0, x, p), X^2(s; 0, x, p)$ has non void intersection with $[-(R - s), R - s]$

$$\forall\, s \in [0, t], \exists\, \lambda(s) \in [0, 1], Y(s) := (\lambda(s)X^1 + (1 - \lambda(s))X^2)(s; 0, x, p) \in [-(R-s), R-s].$$

Using the characteristic equations yields

$$\frac{d}{ds}|X^1 - X^2| \le 2|P^1(s) - P^2(s)|,\ \forall\, s \in [0, t],$$

$$\frac{d}{ds}|P^1 - P^2| \le 2D^R(s) + C(s)\{|X^1(s) - X^2(s)| + |P^1(s) - P^2(s)|\},\ \forall\, s \in [0, t].$$

By Gronwall lemma we deduce that for any $s \in [0, t]$

$$|X^1(s) - X^2(s)| + |P^1(s) - P^2(s)| \le 2\exp(t(2 + \max_{s\in[0,t]} C(s))) \int_0^t D^R(s)\, ds.$$

$\square$

We formulate now our main result for the relativistic 1.5D Vlasov-Maxwell system, concerning the continuous dependence of the mild solution with respect to the initial conditions. A straightforward consequence will be the finite speed propagation property.

Theorem 3.1 *Assume that* $(f_0^k, E_0^k, B_0^k)_{k\in\{1,2\}}$ *satisfy the hypotheses H3-H6 and denote by* $(f^k, E^k, B^k)_{k\in\{1,2\}}$ *the global mild solutions of the 1.5D relativistic Vlasov-Maxwell system corresponding to the initial conditions* $(f_0^k, E_0^k, B_0^k)_{k\in\{1,2\}}$. *Then for any* $R > 0$ *there is a constant* C_R *depending on* R, $\max_{k\in\{1,2\}}\{\|E_0^k\|_{W^{1,\infty}} + \|B_0^k\|_{W^{1,\infty}}\}$, $\max_{k\in\{1,2\}}\{\|g_0^k\|_{L^1(\mathbb{R}^+;u^2du)} + \|g_0^k\|_{L^\infty}\}$ *such that for all* $t \in [0, R]$ *we have*

$$\max_{|x|\leq R-t, p\in\mathbb{R}^2}(|E^1 - E^2| + |B^1 - B^2|)(t, x) + (|X^1 - X^2| + |P^1 - P^2|)(0; t, x, p) \leq C_R D_0^R,$$

where (X^k, P^k) *are the characteristics associated to* (E^k, B^k), $k \in \{1, 2\}$ *and*

$$D_0^R = \int_{-R}^{R}\int_{\mathbb{R}^2}(1 + |p|)|f_0^1 - f_0^2| \, dp \, dx + \|E_0^1 - E_0^2\|_{L^\infty(]-R,R[)} + \|B_0^1 - B_0^2\|_{L^\infty(]-R,R[)}.$$

In particular if $f_0^1(x, p) = f_0^2(x, p)$, $\forall\ (x, p) \in [-R, R] \times \mathbb{R}$ *and* $(E_0^1, B_0^1)(x) = (E_0^2, B_0^2)(x)$, $\forall\, x \in [-R, R]$ *for some* $R > 0$ *then for any* $t \in [0, R]$ *we have*

$$f^1(t, x, p) = f^2(t, x, p), \ \ (x, p) \in [-(R - t), R - t] \times \mathbb{R},$$

$$(E^1, B^1)(t, x) = (E^2, B^2)(t, x), \ \ x \in [-(R - t), R - t].$$

Proof. From now on the notation C_R stands for any constant as in the statement of the above theorem. By the previous computations we know that

$$\max\{\|E^k\|_{L^\infty(]0,T[\times\mathbb{R})}, \|B^k\|_{L^\infty(]0,T[\times\mathbb{R})}\} \leq a^k(T), \ \ k \in \{1, 2\},$$

$$\max\{\|\partial_x E^k\|_{L^\infty(]0,T[\times\mathbb{R})}, \|\partial_x B^k\|_{L^\infty(]0,T[\times\mathbb{R})}\} \leq b^k(T), \ \ k \in \{1, 2\},$$

where $a^k = ((a_1^k)^2 + (a_2^k)^2)^{1/2}$, $b^k = ((b_1^k)^2 + (b_2^k)^2)^{1/2}$, the coefficients $a_1^k(T), a_2^k(T)$, $b_1^k(T), b_2^k(T)$ being defined as in (31), (32), (38), (39). As in the one dimensional case, the main idea is to get estimates by duality approach. The computations are long, using heavily integration along the characteristics associated to the electromagnetic field. We split them into two steps. We start by estimating the difference of the electric first components $E_1^2 - E_1^1$.

Step 1. (*Continuous dependence for the electric first component*) For any $0 \leq t \leq R$ there is a constant C_R such that

$$\begin{aligned} \|E_1^2 - E_1^1\|_{L^\infty(]-(R-t),R-t[)} &\leq \|E_{0,1}^2 - E_{0,1}^1\|_{L^\infty(]-R,R[)} + \|f_0^2 - f_0^1\|_{L^1(]-R,R[\times\mathbb{R})} \\ &+ C_R \int_0^t D^R(s)\, ds, \quad s \in [0,t]. \end{aligned} \tag{46}$$

Proof. (of Step 1.) Take $t \in [0,R]$ and $\varphi \in C^0(\mathbb{R})$ with compact support in $[-(R-t), R-t]$. Multiplying (33) by φ and integrating with respect to x implies

$$\begin{aligned} \left|\int_{\mathbb{R}} \varphi(x)(E_1^2 - E_1^1)\, dx\right| \leq \left|\int_{\mathbb{R}} \varphi(x)(E_{0,1}^2 - E_{0,1}^1)\, dx\right| + \left|\int_{\mathbb{R}}\int_{\mathbb{R}^2} (f_0^1 - f_0^2) \int_x^{X^2(t;0,x,p)} \varphi(u)\, du\, dp\, dx\right| \\ + \left|\int_{\mathbb{R}}\int_{\mathbb{R}^2} f_0^1 \int_{X^1(t;0,x,p)}^{X^2(t;0,x,p)} \varphi(u)\, du\, dp\, dx\right|. \end{aligned} \tag{47}$$

Notice that if $x \leq -R$ then $X^k(t;0,x,p) \leq -R+t$ and if $x \geq R$ then $X^k(t;0,x,p) \geq R-t$. Taking into account that $\operatorname{supp}\varphi \subset [-(R-t), R-t]$ we can restrict the integrations with respect to x in the right hand side of (47) to $[-R,R]$. Therefore we obtain

$$\left|\int_{\mathbb{R}} \varphi(E_1^2 - E_1^1)\, dx\right| \leq \|\varphi\|_{L^1}(\|E_{0,1}^2 - E_{0,1}^1\|_{L^\infty(]-R,R[)} + \|f_0^2 - f_0^1\|_{L^1(]-R,R[\times\mathbb{R}^2)}) + |T_4|, \tag{48}$$

where $T_4 = \int_{-R}^{R}\int_{\mathbb{R}^2} f_0^1(x,p) \int_{X^1(t;0,x,p)}^{X^2(t;0,x,p)} \varphi(u)\, du\, dp\, dx$. Observe that if $(x,p) \notin \tilde{K}(t)$ then $\int_{X^1(t;0,x,p)}^{X^2(t;0,x,p)} \varphi(u)\, du = 0$ and if $(x,p) \in \tilde{K}(t)$ then we have by (45)

$$(|X^1 - X^2| + |P^1 - P^2|)(t;0,x,p) \leq 2\exp(t(2 + 4a(t) + 4b(t))) \int_0^t D^R(s)\, ds =: \tilde{Q}(t), \tag{49}$$

where $a(t) = \max_{k\in\{1,2\}} a^k(t)$, $b(t) = \max_{k\in\{1,2\}} b^k(t)$. Therefore we can estimate T_4

as in the one dimensional case

$$
\begin{aligned}
|T_4| &= \left| \int_{-R}^{R}\int_{\mathbb{R}^2} f_0^1(x,p)\mathbf{1}_{\{(x,p)\in \tilde{K}(t)\}} \int_{X^1(t;0,x,p)}^{X^2(t;0,x,p)} \varphi(u)\;du\;dp\;dx \right| \\
&\leq \int_{\mathbb{R}} |\varphi(u)| \int_{-R}^{R}\int_{\mathbb{R}^2} f_0^1(x,p)\mathbf{1}_{\{(x,p)\in \tilde{K}(t)\}}\mathbf{1}_{\{|u-X^1(t;0,x,p)|\leq \tilde{Q}(t)\}}\;dp\;dx\;du \\
&\leq C_R\|\varphi\|_{L^1}\int_0^t D^R(s)\;ds. \qquad (50)
\end{aligned}
$$

We deduce from (48), (50) that

$$
\begin{aligned}
\|E_1^2 - E_1^1\|_{L^\infty(]-(R-t),R-t[)} &\leq \|E_{0,1}^2 - E_{0,1}^1\|_{L^\infty(]-R,R[)} + \|f_0^2 - f_0^1\|_{L^1(]-R,R[\times\mathbb{R})} \\
&+ C_R\int_0^t D^R(s)\;ds, \quad s\in[0,t].
\end{aligned}
$$

$\square$

We estimate now the differences between the electric second components $E_2^2 - E_2^1$ and the magnetic components $B^2 - B^1$.

Step 2. (*Continuous dependence for the electric second component and the magnetic component*) For any $0\leq t\leq R$ there is a constant C_R such that

$$
\begin{aligned}
&\max\{\|E_2^2(t) - E_2^1(t)\|_{L^\infty(]-(R-t),R-t[)}, \|B^2(t) - B^1(t)\|_{L^\infty(]-(R-t),R-t[)}\} \\
\leq\;& \|E_{0,2}^2(t) - E_{0,2}^1(t)\|_{L^\infty(]-R,R[)} + \|B_0^2(t) - B_0^1(t)\|_{L^\infty(]-R,R[)} \\
+\;& C_R\int_{-R}^{R}\int_{\mathbb{R}^2}(1+|p|)|f_0^2 - f_0^1|\;dp\;dx + C_R\int_0^t D^R(s)\;ds. \qquad (51)
\end{aligned}
$$

Proof. (of Step 2.) By (34), (35) it is easily seen that

$$
\begin{aligned}
&\max\{\|E_2^2(t) - E_2^1(t)\|_{L^\infty(]-(R-t),R-t[)}, \|B^2(t) - B^1(t)\|_{L^\infty(]-(R-t),R-t[)}\} \\
\leq\;& \|E_{0,2}^2(t) - E_{0,2}^1(t)\|_{L^\infty(]-R,R[)} + \|B_0^2(t) - B_0^1(t)\|_{L^\infty(]-R,R[)} \qquad (52) \\
+\;& \frac{1}{2}\|J_2^{+,2}(t) - J_2^{+,1}(t)\|_{L^\infty(]-(R-t),R-t[)} + \frac{1}{2}\|J_2^{-,2}(t) - J_2^{-,1}(t)\|_{L^\infty(]-(R-t),R-t[)},
\end{aligned}
$$

and therefore we need to estimate $\|J_2^{\pm,2}(t) - J_2^{\pm,1}(t)\|_{L^\infty(]-(R-t),R-t[)}$. Multiplying by

φ and integrating with respect to x yields as in (40)

$$\begin{aligned}
\left|\int_{\mathbb{R}}(J_2^{\pm,2}-J_2^{\pm,1})\varphi\,dx\right| &= \left|\sum_{k=1}^{2}(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^k\int_0^t v_2(P^k(s))\varphi(X^k(s)\pm(t-s))\,ds\,dp\,dx\right| \\
&= \left|\sum_{k=1}^{2}(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^k\int_0^t G^{\pm}(P^k(s))\frac{d}{ds}\int_{X^k(t)}^{X^k(s)\pm(t-s)}\varphi(u)\,du\,ds\,dp\,dx\right| \\
&\leq \left|\sum_{k=1}^{2}(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^k G^{\pm}(p)\int_{x\pm t}^{X^k(t)}\varphi(u)\,du\,dp\,dx\right| \\
&+ \left|\sum_{k=1}^{2}(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^k\int_0^t\frac{d}{ds}\{G^{\pm}(P^k(s))\}\int_{X^k(t)}^{X^k(s)\pm(t-s)}\varphi(u)\,du\,ds\,dp\,dx\right| \\
&= |T_5|+|T_6|. \qquad (53)
\end{aligned}$$

It is easily seen that for any $s\in[0,t]$, $|x|>R$, $p\in\mathbb{R}^2$, $k\in\{1,2\}$ the segment between $X^k(t)$ and $X^k(s)\pm(t-s)$ has void intersection with the support of φ and thus the integrations with respect to x in the terms T_5, T_6 can be restricted over $[-R,R]$. Performing similar computations as those in (24) and taking into account that $|G^{\pm}(p)|\leq 1+|p|$ and $|p|\leq|P^1(t;0,x,p)|+ta(t)$ yields

$$\begin{aligned}
|T_5| &\leq \|\varphi\|_{L^1}\int_{-R}^{R}\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx+\int_{-R}^{R}\int_{\mathbb{R}^2}(1+|p|)f_0^1\left|\int_{X^1(t;0,x,p)}^{X^2(t;0,x,p)}\varphi(u)\,du\right|\,dp\,dx \\
&\leq \|\varphi\|_{L^1}\int_{-R}^{R}\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx+\int_{\mathbb{R}}|\varphi(u)|\int_{\mathbb{R}}\int_{\mathbb{R}^2}f^1(t,X^1,P^1) \\
&\times\ (1+ta(t)+|P^1(t;0,x,p)|)\mathbf{1}_{\{|u-X^1(t;0,x,p)|\leq\tilde{Q}(t)\}}\,dp\,dx\,du \\
&\leq \|\varphi\|_{L^1}\int_{-R}^{R}\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx+C_R\|\varphi\|_{L^1}\int_0^t D^R(s)\,ds. \qquad (54)
\end{aligned}$$

The treatment of the term T_6 requires a careful analysis of the characteristics. It is convenient to introduce some notations. For any $(x,p)\in[-R,R]\times\mathbb{R}^2$, $k\in\{1,2\}$ we define

$$s^k(t,x,p)=\sup\{s\in[0,t]\ :\ |X^k(s;0,x,p)|\leq R-s\}.$$

Observing that $s\to|X^k(s;0,x,p)|-(R-s)$ is strictly increasing we deduce that $|X^k(s;0,x,p)|\leq R-s$, $s\in[0,s^k(t,x,p)]$ and $|X^k(s;0,x,p)|>R-s$, $s\in]s^k(t,x,p),t]$. In particular for any $(x,p)\in[-R,R]\times\mathbb{R}^2$ we have $(x,p)\in\tilde{K}(s(t,x,p))$, where

$s(t,x,p) = \max_{k\in\{1,2\}} s^k(t,x,p)$. Moreover for any $(x,p) \in [-R,R]\times\mathbb{R}^2$, $s \in]s(t,x,p),t]$ the segment between $X^k(t), X^k(s)\pm(t-s)$ has void intersection with the support of φ and thus the integration with respect to s in the term T_6 can be restricted to $[0,s(t,x,p)]$. We introduce the functions $H^\pm : \mathbb{R}^5 \to \mathbb{R}$ given by

$$H^\pm(p,e,b) = -(e_1 + v_2(p)b)\frac{\partial G^\pm}{\partial p_1}(p) - (e_2 - v_1(p)b)\frac{\partial G^\pm}{\partial p_2}(p), \quad (p,e,b) \in \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}.$$

By direct computation we check that there is a continuous function $h : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\max\{|H^\pm(p,e,b)|, |\nabla_{(p,e,b)}H^\pm|\} \le (1+|p|)h(e,b), \quad (p,e,b)\in\mathbb{R}^5. \tag{55}$$

Actually the functions $H^\pm$ are the derivatives of $G^\pm$ along the characteristics

$$\frac{d}{ds}G^\pm(P^k(s)) = H^\pm(P^k(s), E^k(s,X^k(s)), B^k(s,X^k(s))) =: H^{\pm,k}(s), k\in\{1,2\}.$$

Thanks to (55) we can write

$$\begin{aligned}\left|\sum_{k=1}^2(-1)^k\frac{d}{ds}\{G^\pm(P^k(s))\}\int_{X^k(t)}^{X^k(s)\pm(t-s)}\varphi(u)\,du\right| &\le \|\varphi\|_{L^1}\left|\sum_{k=1}^2(-1)^kH^{\pm,k}(s)\right| \\ &+ C_R(1+|P^1(s)|)\left|\int_{X^1(t)}^{X^2(t)}\varphi(u)\,du\right| \\ &+ C_R(1+|P^1(s)|)\left|\int_{X^1(s)\pm(t-s)}^{X^2(s)\pm(t-s)}\varphi(u)\,du\right|.\end{aligned} \tag{56}$$

Notice also that for any $(x,p)\in[-R,R]\times\mathbb{R}^2$, $s\in[0,s(t,x,p)]$ we have $(x,p)\in\tilde{K}(s(t,x,p))\subset\tilde{K}(s)$ and thus there is $\lambda(s)\in[0,1]$ such that $Z(s) = \lambda(s)X^1(s) + (1-\lambda(s))X^2(s)\in[-(R-s),R-s]$. Using now (55), (49) one gets by intercalating $H^\pm(P^k(s), E^k(s,Z(s)), B^k(s,Z(s)))$

$$\begin{aligned}\left|H^{\pm,2}-H^{\pm,1}\right|(s) &\le C_R(1+|P^1(s)|+|P^2(s)|)(|X^1-X^2|+|P^1-P^2|+D^R(s)) \\ &\le C_R(1+|P^1(s)|+|P^2(s)|)\left(D^R(s)+\int_0^s D^R(\tau)\,d\tau\right).\end{aligned} \tag{57}$$

Combining (56), (57) yields the following bound for the term T_6

$$\begin{aligned}
|T_6| &\leq C_R\|\varphi\|_{L^1}\int_0^t\int_{-R}^R\int_{\mathbb{R}^2}(1+|P^2(s)|)|f_0^2-f_0^1|\,dp\,dx\,ds \\
&+ C_R\|\varphi\|_{L^1}\int_{-R}^R\int_{\mathbb{R}^2}f_0^1\int_0^{s(t,x,p)}(1+|P^1(s)|+|P^2(s)|)\left(D^R(s)+\int_0^s D^R\,d\tau\right)\,ds\,dp\,dx \\
&+ C_R\int_{-R}^R\int_{\mathbb{R}^2}f_0^1\int_0^{s(t,x,p)}(1+|P^1(s)|)\left|\int_{X^1(t)}^{X^2(t)}\varphi(u)\,du\right|\,ds\,dp\,dx \\
&+ C_R\int_{-R}^R\int_{\mathbb{R}^2}f_0^1\int_0^{s(t,x,p)}(1+|P^1(s)|)\left|\int_{X^1(s)\pm(t-s)}^{X^2(s)\pm(t-s)}\varphi(u)\,du\right|\,ds\,dp\,dx \\
&= |T_7|+|T_8|+|T_9|+|T_{10}|. \qquad (58)
\end{aligned}$$

We estimate now one by one the terms T_k, $k\in\{7,8,9,10\}$. Taking into account that $1+|P^2(s;0,x,p)|\leq C_R(1+|p|)$ we deduce that

$$|T_7| \leq C_R\|\varphi\|_{L^1}\int_{-R}^R\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx. \qquad (59)$$

Since $1+|P^1(s;0,x,p)|+|P^2(s;0,x,p)|\leq C_R(1+|p|)$ one gets easily that

$$\begin{aligned}
|T_8| &\leq C_R\|\varphi\|_{L^1}\int_0^t\int_{-R}^R\int_{\mathbb{R}^2}(1+|p|)f_0^1(x,p)\left(D^R(s)+\int_0^s D^R(\tau)d\tau\right)\,dp\,dx\,ds \\
&\leq C_R\|\varphi\|_{L^1}\int_0^t\left(D^R(s)+\int_0^s D^R(\tau)\,d\tau\right)\,ds\int_{-R}^R\int_{\mathbb{R}^2}(1+|p|)g_0^1(|p|)\,dp\,dx \\
&\leq C_R\|\varphi\|_{L^1}\int_0^t D^R(s)\,ds. \qquad (60)
\end{aligned}$$

The analysis of T_9, T_{10} are similar to those of T_4 in (50). Notice that we can apply (49) on $[0,s(t,x,p)]$ for any $(x,p)\in[-R,R]\times\mathbb{R}^2$. We obtain

$$|T_9|+|T_{10}|\leq C_R\|\varphi\|_{L^1}\int_0^t D^R(s)\,ds. \qquad (61)$$

Finally putting together (53), (54), (58), (59), (60), (61) we deduce that

$$\|J_2^{\pm,2}(t)-J_2^{\pm,1}(t)\|_{L^\infty(]-(R-t),R-t[)}\leq C_R\int_{-R}^R\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx+C_R\int_0^t D^R\,ds,$$

and therefore (52) implies

$$\begin{aligned}
&\max\{\|E_2^2(t)-E_2^1(t)\|_{L^\infty(]-(R-t),R-t[)},\|B^2(t)-B^1(t)\|_{L^\infty(]-(R-t),R-t[)}\} \\
\leq\ & \|E_{0,2}^2(t)-E_{0,2}^1(t)\|_{L^\infty(]-R,R[)}+\|B_0^2(t)-B_0^1(t)\|_{L^\infty(]-R,R[)} \\
+\ & C_R\int_{-R}^R\int_{\mathbb{R}^2}(1+|p|)|f_0^2-f_0^1|\,dp\,dx+C_R\int_0^t D^R(s)\,ds.
\end{aligned}$$

$\square$

Based on the above two steps we can finish the proof of Theorem 3.1. Combining (46), (51) yields

$$D^R(t) \leq C_R D_0^R + C_R \int_0^t D^R(s)\, ds,$$

which implies by Gronwall lemma that for any $t \in [0, R]$ we have $D^R(t) \leq C_R D_0^R$. Finally observe that for any $(x, p) \in [-(R-t), R-t] \times \mathbb{R}^2$ we have $|X^k(s; t, x, p)| \leq R - s$, $\forall\, s \in [0, t]$ and therefore we obtain as in the proof of (45)

$$(|X^1 - X^2| + |P^1 - P^2|)(0; t, x, p) \leq C_R \int_0^t D^R(s)\, ds \leq C_R D_0^R.$$

$\square$

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