Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential

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Abstract

We consider a stationary nonlinear Schrödinger equation with a repulsive delta-function impurity in one space dimension. This equation admits a unique positive solution and this solution is even. We prove that it is a minimizer of the associated energy on the subspace of even functions of $H^1(\mathbb{R}, \mathbb{C})$, but not on all $H^1(\mathbb{R}, \mathbb{C})$, and study its orbital stability.

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1 Introduction

This work deals with the following stationary problem,

$$-D^2\phi + \omega\phi - \gamma\delta(x)\phi = |\phi|^{p-1}\phi, \quad \phi \in H^1(\mathbb{R}, \mathbb{C}), \tag{1}$$

where $\omega > 0$, $\gamma \in \mathbb{R}$, $D = d/dx$, $p > 1$ and δ is the Dirac measure at the origin.

This problem arises in the study of standing wave solutions

$$u_\omega(t, x) = e^{i\omega t}\phi_\omega(x),$$

for the nonlinear Schrödinger equation with a delta function

$$i\partial_t u + D^2 u + \gamma\delta(x)u = -|u|^{p-1}u. \tag{2}$$

Here u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$. In order for $u_\omega(t, x)$ to satisfy (2), $\phi_\omega(x)$ must be a solution of (1).

These last years equation (2) appeared in several physical models. The Dirac measure is used to model an impurity, or defect, localized at the origin (see, for example, [13, 17, 22] and the references therein). These models typically deal with the interaction between a travelling wave and the defect and studying if the ground state solutions of (2) are orbitally stable or unstable is often a key preliminary step. A main purpose of this paper is to derive some mathematically rigourous results on this issue.

Theorem 3.7.1 of [6] gives the following result for any $p > 1$ and $\gamma \in \mathbb{R}$.

Proposition 1 *For any $u_0 \in H^1(\mathbb{R}, \mathbb{C})$, there exist $T = T(u_0) > 0$ and a unique solution $u(\cdot) \in C([0, T), H^1(\mathbb{R}, \mathbb{C}))$ of (2) with $u(0) = u_0$ such that either $T = \infty$ or $T < \infty$ and $\lim_{t\to T} \|Du\|_2 = \infty$. Moreover, the energy E and the charge Q are conserved :*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0, T),$$

where

$$E(v) = \frac{1}{2}\|Dv\|_2^2 - \frac{\gamma}{2}\int_{\mathbb{R}} \delta(x)|v(x)|^2 dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad Q(v) = \frac{1}{2}\|v\|_2^2.$$

Remark 2 From the uniqueness result of Proposition 1 it follows that if an initial data u_0 is even the solution $u(t)$ is also even.

Here, as elsewhere, $H^1(\mathbb{R},\mathbb{C})$ is equipped with the real inner product and $\|\cdot\|_p$ denotes the $L^p(\mathbb{R},\mathbb{C})$ norm for $p > 1$. Also for simplicity we set $H^1 = H^1(\mathbb{R},\mathbb{C})$.

Remark 3 If $1 < p < 5$, the Cauchy problem in H^1 associated to (2) is globally well posed. From Proposition 1 this can be proved using the conservation laws and the Gagliardo-Nirenberg inequality (see [6] for such results).

In [13] Goodman, Holmes and Weinstein study the strong interactions between solitons, namely nonlinear bound states associated with the unperturbed, $\gamma = 0$, equation (2) and the defect created by the Dirac measure. They are led to consider the orbital stability of the solution of (1) given, for $2\sqrt{\omega} > |\gamma|$, by

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right\}^{\frac{1}{p-1}}. \tag{3}$$

This solution is the unique positive solution of (1), as we shall see in Section 3. It is constructed from the corresponding solution with $\gamma = 0$ on each side of the defect. At $x = 0$ one seeks to satisfy the continuity and the jump condition in the first derivative, $Du(0+) - Du(0-) = -\gamma u(0)$. Actually in [13] only the case $\gamma > 0$, namely of a delta-well defect, with $p = 3$ is considered.

The orbital stability when $\gamma = 0$ has been extensively studied (see the classical papers [4, 7, 14, 30]). In particular, Cazenave and Lions [7] proved that $e^{i\omega t}\phi_\omega(x)$ is stable for any $\omega > 0$ if $p < 5$. On the other hand, it was shown that $e^{i\omega t}\phi_\omega(x)$ is unstable for any $\omega > 0$ if $p \geq 5$ (see Berestycki and Cazenave [4] for $p > 5$, and Weinstein [30] for $p = 5$).

In [13] the authors claimed that $\phi_\omega(x)$ is orbitally stable in the case $\gamma > 0$ and $p = 3$. Their argument is based on a variational characterization of $\phi_\omega(x)$ and the use of bifurcation theory in the spirit of Rose and Weinstein [25] but no details are given. They also mention that, as $\omega \to \infty$, $\phi_\omega(x)$ looks more and more like the solitary wave corresponding to the case $\gamma = 0$ with $\omega = 1$. Subsequently in [12] the orbital stability of $\phi_\omega(x)$ was studied for any $1 < p < \infty$ and $\gamma > 0$ and it was shown, in particular, that $\phi_\omega(x)$ is stable when $\gamma > 0$ and $p = 5$. This is in sharp contrast with the case $\gamma = 0$ and $p = 5$ and indicate that one should be cautious with heuristic arguments to deduce informations on the case $\gamma \neq 0$ from the case $\gamma = 0$.

The notions of stability and instability are defined as follows.

Definition 4 For $\eta > 0$, let

$$U_\eta(\phi_\omega) := \left\{ v \in H^1 : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta}\phi_\omega\|_{H^1} < \eta \right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (2) is stable in H^1 if for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $u_0 \in U_\eta(\phi_\omega)$, the solution $u(t)$ of (2) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in H^1.

In [12], the authors proved the following.

Proposition 5 *Let $\gamma > 0$ and $\omega > \gamma^2/4$.*

(i) *Let $1 < p \leq 5$. Then $e^{i\omega t}\phi_\omega(x)$ is stable in H^1 for any $\omega \in (\gamma^2/4, \infty)$.*

(ii) *Let $p > 5$. Then there exists a $\omega_1 = \omega_1(p, \gamma) > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is stable in H^1 for any $\omega \in (\gamma^2/4, \omega_1)$, and unstable in H^1 for any $\omega \in (\omega_1, \infty)$. Here ω_1 is defined as follows:*

$$\frac{p-5}{p-1} J(\omega_1) = \frac{\gamma}{2\sqrt{\omega_1}} \left(1 - \frac{\gamma^2}{4\omega_1}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_1) = \int_{A(\omega_1,\gamma)}^{\infty} \operatorname{sech}^{4/(p-1)} y \, dy, \quad A(\omega_1, \gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_1}}\right).$$

The proof of Proposition 5 in [12] borrows ingredients from [27, 28]. It does not require any investigations of the spectrum of the linearized operator around $e^{i\omega t}\phi_\omega(x)$. The standard spectral requirements (as formulated in [14]) are replaced by the fact that $\phi_\omega(x)$ can be characterized as a minimizer of some constrained functionals. To give a full proof of Proposition 5 along these lines is quite long and actually only a sketch of proof is provided in [12]. In Remark 33 we give an, alternative, complete proof of Proposition 5.

Definition 6 For $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$, we define on H^1 the following functionals of class C^2 :

$$\begin{aligned}
S_\omega(v) &= E(v) + \omega Q(v) \\
&= \frac{1}{2}\|Dv\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}\int_{\mathbb{R}} \delta(x)|v(x)|^2 dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1} \\
&= \frac{1}{2}\|Dv\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \\
I_\omega(v) &= \|Dv\|_2^2 + \omega\|v\|_2^2 - \gamma\int_{\mathbb{R}} \delta(x)|v(x)|^2 dx - \|v\|_{p+1}^{p+1} \\
&= \|Dv\|_2^2 + \omega\|v\|_2^2 - \gamma|v(0)|^2 - \|v\|_{p+1}^{p+1}.
\end{aligned}$$

We consider the set of minimizers for the minimization problem

$$d(\omega) = \inf\{S_\omega(v) : \ v \in H^1 \setminus \{0\}, \ I_\omega(v) = 0\}. \tag{4}$$

Remark 7 The set $\{v \in H^1 \setminus \{0\}, \ I_\omega(v) = 0\}$ is called the natural constraint (sometimes also the Nehari manifold) associated to S_ω. Since $S'_\omega(v)v = I_\omega(v)$ for any $v \in H^1$, it clearly contains all the nontrivial critical points of S_ω. It is standard to show that a minimizer of $d(\omega)$ corresponds to a solution of (1) (see [31] as a reference on this subject).

Remark 8 In [12], for $\gamma > 0$ and $\omega > \gamma^2/4$, it is proved that $d(\omega)$ is reached by a positive, even function. Also it is claimed that (1) has a unique positive even solution. This implies, by uniqueness, that this minimizer is $\phi_\omega(x)$. This variational characterization of $\phi_\omega(x)$ is essential in the proof of Proposition 5. Here, for any $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$, we indeed prove that $\phi_\omega(x)$ is the unique nonnegative non trivial solution of (1), see Lemma 26. As a consequence we obtain that the set of solutions of (1)

$$\{v \in H^1 \setminus \{0\} : \ -D^2 v + \omega v - \gamma \delta v = |v|^{p-1} v\}$$

is given by $\{e^{i\theta}\phi_\omega : \ \theta \in \mathbb{R}\}$.

The main aim of this paper is to extend the existence and stability results of [12, 13] by considering also the case $\gamma < 0$. Physically this corresponds to a repulsive (or barrier) defect instead of a well-defect.

In contrast to [12] we derive our stability result using the general approach to orbital stability laid down in [14]. We shall be more precise later but in order to satisfy the assumptions of [14] the key point to check is that the self-adjoint operator L_1 defined on $L^2(\mathbb{R})$, by

$$L_1 v = -D^2 v + \omega v - p\phi_\omega^{p-1} v$$

with domain

$$\mathrm{Dom}(L_1) = \{v \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}) \ : \ Dv(0+) - Dv(0-) = -\gamma v(0)\}$$

has exactly one negative eigenvalue and its kernel is zero. Here $H^m(\mathbb{R})$, $m = 1, 2$, denotes the usual real Hilbert space.

We prove in Lemma 31 (see also Lemma 28) that the kernel is $\{0\}$. Now, if $\phi_\omega(x)$ could be characterized as a minimizer of S_ω on the Nehari manifold, which is of codimension one, we could deduce that L_1 has at most one negative eigenvalue. However when $\gamma < 0$, in contrast to [12], $d(\omega)$ is not reached anymore (see Remark 14). More globally we did not manage to characterize $\phi_\omega(x)$ as a minimizer, even if being a local one would be sufficient, of S_ω on a manifold of codimension one. In that direction we nevertheless show, in Lemma 32, that $\phi_\omega(x)$ is a minimizer of S_ω on a manifold of codimension two but this leaves open the possibility that there are two negative eigenvalues. Because of this difficulty we restrict ourselves to study the orbital stability of $\phi_\omega(x)$ within the subspace of even functions. In this subspace we show that $\phi_\omega(x)$ has a minimizing character. More precisely let

$$d_r(\omega) = \inf\{S_\omega(v) : v \in H^1_r \setminus \{0\},\ I_\omega(v) = 0\},$$

where

$$H^1_r := \{v \in H^1 : v(-x) = v(x),\ x \in \mathbb{R}\}.$$

Alternatively we can write

$$d_r(\omega) = \frac{p-1}{2(p+1)} \inf\{\|v\|^{p+1}_{p+1} :\ v \in H^1_r \setminus \{0\},\ I_\omega(v) = 0\}. \tag{5}$$

First we show

Theorem 1 *Let $\gamma < 0$. There exists a nonnegative minimizer of $d_r(\omega)$ for any $\omega > \gamma^2/4$.*

Then, in Lemma 19, we show that a minimizer of $d_r(\omega)$ is not only a critical point of S_ω restricted to H^1_r but also of S_ω on all H^1. Thus, from our uniqueness result of nonnegative solutions of (1), we deduce that $\phi_\omega(x)$ corresponds to the minimizer obtained in Theorem 1.

Let us point out that in this minimization problem we cannot benefit from the compact embedding $H^1_r(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, $q > 2$ which holds when $n \geq 2$. To obtain a minimizer we use a variational approach which relies on the decomposition of Palais-Smale sequences associated to S_ω. This type of decomposition was first introduced by P.L. Lions in [21] and is closely linked to his concentration compactness principle. Roughly speaking we prove that $d_r(\omega)$ is strictly below the level of loss of compactness for the Palais-Smale sequences associated to S_ω on H^1_r.

Since we now work in H^1_r we manage to prove in Lemma 29 that the restriction of L_1 in this subspace has only one negative eigenvalue. Having established the spectral assumptions of [14] it follows (see Theorem 3 in [14]) that

Proposition 9 *Let $p > 1$, $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Assume that $\omega \mapsto \phi_\omega$ is a C^1 mapping. Then*
(i) *if $\partial_\omega \|\phi_\omega\|_2^2 > 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t}\phi_{\omega_0}(x)$ is stable in H^1_r.*
(ii) *if $\partial_\omega \|\phi_\omega\|_2^2 < 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t}\phi_{\omega_0}(x)$ is unstable in H^1.*

We see from the explicit form of (3) that the mapping $\omega \mapsto \phi_\omega$ is C^1. Accordingly to establish the stability or instability of $\phi_\omega(x)$ it suffices to check the increase or decrease of the L^2 norm of $\phi_\omega(x)$. This is done by the same type of calculations as in [12] and leads to the following.

Theorem 2 *Let $\gamma < 0$ and $\omega > \gamma^2/4$.*

(i) *Let $1 < p \leq 3$. Then $e^{i\omega t}\phi_\omega(x)$ is stable in H^1_r for any $\omega \in (\gamma^2/4, \infty)$.*

(ii) *Let $3 < p < 5$. Then there exists a $\omega_2 = \omega_2(p,\gamma) > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is unstable in H^1 for any $\omega \in (\gamma^2/4, \omega_2)$ and stable in H^1_r for any $\omega \in (\omega_2, \infty)$. Here ω_2 is defined as follows:*

$$\frac{p-5}{p-1}J(\omega_2) = \frac{\gamma}{2\sqrt{\omega_2}}\left(1 - \frac{\gamma^2}{4\omega_2}\right)^{-(p-3)/(p-1)},$$

$$J(\omega_2) = \int_{A(\omega_2,\gamma)}^{\infty} \operatorname{sech}^{4/(p-1)} y dy, \quad A(\omega_2,\gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega_2}}\right).$$

(iii) *Let $p \geq 5$. Then $e^{i\omega t}\phi_\omega(x)$ is unstable in H^1 for any $\omega \in (\gamma^2/4, \infty)$.*

Remark 10 In the case $\gamma = 0$, we have $\|\psi_\omega\|_2^2 = \omega^{2/(p-1)-1/2}\|\psi_1\|_2^2$ by the scaling invariance. Here $\psi_\omega(x)$ denotes the solution of (1) with $\gamma = 0$ (see (10)). When the problem is non-autonomous this scale invariance is lost in general. However, in the present one-dimensional case, with our Dirac-delta potential, we can compute the increase or decrease of the L^2 norm.

Remark 11 By Proposition 5 and Theorem 2, we see that $\phi_\omega(x)$ tends to be more stable in the case $\gamma > 0$, in comparison with the case $\gamma = 0$. On the contrary it tends to be more unstable when $\gamma < 0$.

Remark 12 Concerning the critical case $\partial_\omega||\phi_\omega||_2^2 = 0$ we conjecture that $e^{i\omega_2 t}\phi_{\omega_2}(x)$ is unstable in view of the result of Comech and Pelinovsky [9]. However we have not pursued in that direction.

Remark 13 As we already mentioned, in Theorem 2 our stability results are restricted to the space H^1_r because in our proof we need to use the fact that $\phi_\omega(x)$ is a minimizer on a manifold of codimension one. We do not know if this is the case in all H^1 and thus it is unclear if our stability results still hold in H^1. For results on a somehow related problem, we refer to [11, 26]. In [11] it is proved that, in some cases, one has stability in H^1_r but instability in H^1. In our situation we conjecture that we still have stability in H^1 when it holds in H^1_r. In particular we suspect that $\phi_\omega(x)$ is a local minimizer of S_ω on the natural constraint.

This paper is organized as follows. In Section 2, we prove the existence of a nonnegative minimizer for $d_r(\omega)$. Section 3 is devoted to the proof of Theorem 2. Finally, in Section 4, we give some partial results concerning the stability of $\phi_\omega(x)$ in all H^1 and an alternative proof of Proposition 5.

2 Existence of a nonnegative minimizer for $d_r(\omega)$

The aim of this section is to prove Theorem 1. First we observe

Remark 14 Let us show that $d(\omega)$ has no minimizer when $\gamma < 0$. From the definition of $d(\omega)$ it is easy to see that we also have

$$d(\omega) = \frac{p-1}{2(p+1)} \inf\{\|v\|_{p+1}^{p+1} : \ v \in H^1 \setminus \{0\}, \ I_\omega(v) = 0\}. \tag{6}$$

Suppose, by contradiction, that $v_\omega(x)$ is a minimizer of $d(\omega)$ and let $\tau_y v_\omega(x) = v_\omega(x-y)$ for any $y \in \mathbb{R}$. We note that $\lim_{|x|\to\infty} |v_\omega(x)| = 0$ since $v_\omega \in H^1(\mathbb{R})$. Also $|v_\omega(0)| > 0$. Indeed reasoning as in Lemma 15 below, we see that $|v_\omega(x)|$ is also a minimizer and thus satisfies $S'_\omega(|v_\omega|) = 0$. Now by Lemma 26 we conclude that $|v_\omega(x)| = \phi_\omega(x)$ and in particular $|v_\omega(x)| > 0$ on all $\mathbb{R}$. This shows that $I_\omega(\tau_y v_\omega) < I_\omega(v_\omega) = 0$ for $|y|$ sufficiently large and thus there exists $\lambda^* < 1$ such that $I_\omega(\lambda^* \tau_y v_\omega) = 0$. By the definition of $d(\omega)$ given in (6) we then have

$$d(\omega) \le \frac{p-1}{2(p+1)} \|\lambda^* \tau_y v_\omega\|_{p+1}^{p+1} < \frac{p-1}{2(p+1)} \|\tau_y v_\omega\|_{p+1}^{p+1} = \frac{p-1}{2(p+1)} \|v_\omega\|_{p+1}^{p+1} = d(\omega),$$

which is a contradiction.

Lemma 15 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Then if $u \in H^1$ is a minimizer of $d_r(\omega)$, $|u| \in H^1(\mathbb{R}, \mathbb{R})$ is also a minimizer. In particular we can search for a minimizer of $d_r(\omega)$ inside the subset of real valued functions of H^1.*

Proof. It is classical (see, for example, Proposition 2.2 in [15] for a proof) that if $u \in H^1(\mathbb{R}, \mathbb{C})$ then $|u| \in H^1(\mathbb{R}, \mathbb{R})$ and that $||\nabla |u| \,||_2 \leqslant ||\nabla u||_2$. Clearly also $||\,|u|\,||_{p+1} = ||u||_{p+1}$. We deduce that if $u \in H^1(\mathbb{R}, \mathbb{C})$ then $I_\omega(|u|) \leqslant I_\omega(u)$. Thus there exists a $\lambda^* \leqslant 1$ such that $I_\omega(\lambda^* |u|) = I_\omega(u)$ and $||\lambda^* \,|u|\,||_{p+1} \leqslant ||u||_{p+1}$. We now conclude using the definition of $d_r(\omega)$ given in (5). □

As a consequence of Lemma 15 we can develop our variational approach in the space $H^1(\mathbb{R}, \mathbb{R})$ instead of H^1. We denote $H^1(\mathbb{R}, \mathbb{R})$ by $H^1(\mathbb{R})$ and the subset of real valued functions of H^1_r by $H^1_r(\mathbb{R})$. In order to prove Theorem 1 we need a detailed study of the Palais-Smale sequences of S_ω in $H^1(\mathbb{R})$ at the level $d_r(\omega)$.

Definition 16 *For $c \in \mathbb{R}$ we say that $\{u_n\} \subset H^1(\mathbb{R})$ is a Palais-Smale sequence for S_ω at the level c (a $(PS)_c$ sequence for short), if and only if it satisfies, as $n \to \infty$,*

$$S_\omega(u_n) \to c, \quad S'_\omega(u_n) \to 0 \ \ \text{in} \ \ H^{-1}(\mathbb{R}).$$

By continuity of S_ω and S'_ω, if a Palais-Smale sequence for S_ω at the level $d_r(\omega)$, which consists of elements of $H^1_r(\mathbb{R})$ converge, its limit is a minimizer of $d_r(\omega)$. As a first step in that direction we have

Lemma 17 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. There exists a bounded Palais-Smale sequence for S_ω restricted to $H^1_r(\mathbb{R})$ at the level $d_r(\omega)$. Namely a bounded sequence $\{v_n\} \subset H^1_r(\mathbb{R})$ such that, as $n \to \infty$,*

$$S_\omega(v_n) \to d_r(\omega), \quad S'_\omega(v_n) \to 0 \ \ \text{in} \ \ H^{-1}_r(\mathbb{R}).$$

Proof. The existence of a sequence $\{v_n\} \subset H^1_r(\mathbb{R})$ which is a $(PS)_{d_r(\omega)}$ sequence for $S_\omega|_{H^1_r(\mathbb{R})}$ is based on the Ekeland's variational principle (see [10]). In addition one can assume that $\{v_n\} \subset H^1_r(\mathbb{R})$ is a minimizing sequence for $d_r(\omega)$ (namely that $I_\omega(v_n) = 0$ for any $n \in \mathbb{N}$). Such statements can be proved along the same lines as Lemma 3.4 in [29]. Clearly the minimizing sequences for $d_r(\omega)$ are bounded.

Remark 18 An alternative, longer but more classical, proof of Lemma 17 is to show that S_ω admits in $H^1_r(\mathbb{R})$ a mountain pass geometry (see [3, 19]) and that the mountain pass level corresponds to $d_r(\omega)$. This, as well as the boundedness of the Palais-Smale sequences, is true because of the simple form (of power type) of S_ω. Then it is standard that the Ekeland's principle yields the existence of a $(PS)_{d_r(\omega)}$ sequence in $H^1_r(\mathbb{R})$ (see [10, 31] for example).

Our next result shows that the sequence obtained in Lemma 17 is also a Palais-Smale sequence in $H^1(\mathbb{R})$.

Lemma 19 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Any Palais-Smale sequence of S_ω considered on $H^1_r(\mathbb{R})$ is also a Palais-Smale sequence of S_ω on $H^1(\mathbb{R})$. In particular a critical point of S_ω considered on $H^1_r(\mathbb{R})$ is also a critical point of S_ω on $H^1(\mathbb{R})$.*

Proof. This result is related to the principle of symmetric criticality of Palais and we adapt here a proof given in [16]. Clearly it suffices to show that if $u \in H^1_r$ then $S'_\omega(u)h = 0$ for any $h \in \tilde{H}^1_r$ with

$$\tilde{H}^1_r = \{h \in H^1(\mathbb{R}), \langle h, \phi\rangle = 0 \text{ for all } \phi \in H^1_r(\mathbb{R})\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual scalar product on $H^1(\mathbb{R})$. Let $u \in H^1_r(\mathbb{R})$, by Riesz's Theorem there exists a unique $\psi_0 \in H^1(\mathbb{R})$ such that $\langle \psi_0, h \rangle = \phi(h)$ where ϕ belongs to the dual of $H^1(\mathbb{R})$ and is defined by

$$\phi(h) = (\omega - 1) \int_{\mathbb{R}} u(x)h(x)dx - \gamma u(0)h(0) - \int_{\mathbb{R}} |u(x)|^{p-1}u(x)h(x)dx.$$

Let $\tilde{\psi}_0$ be given by $\tilde{\psi}_0(x) = \psi_0(-x)$. Direct calculations show that, for any $h \in H^1(\mathbb{R})$, $\langle \tilde{\psi}_0, h \rangle = \langle \psi_0, \tilde{h} \rangle$ where $\tilde{h}$ is such that $\tilde{h}(x) = h(-x)$. Now, since $u \in H^1_r(\mathbb{R})$, we have $\phi(\tilde{h}) = \phi(h)$ and thus $\langle \psi_0, \tilde{h} \rangle = \langle \psi_0, h \rangle$ for any $h \in H^1(\mathbb{R})$. It shows that $\psi_0 \in H^1_r(\mathbb{R})$. Now, for any $h \in \tilde{H}^1_r$ we indeed have,

$$S'_\omega(u)h = \langle u, h \rangle + \phi(h) = \phi(h) = \langle \psi_0, h \rangle = 0 \tag{7}$$

since $u \in H^1_r(\mathbb{R})$ and $\psi_0 \in H^1_r(\mathbb{R})$. □

Before stating our next lemma we recall some results concerning the case $\gamma = 0$ which plays the role of a problem at infinity for (1).

It is known that the set of solutions of

$$-D^2\psi + \omega\psi = |\psi|^{p-1}\psi, \quad x \in \mathbb{R}, \quad \omega > 0, \quad \psi \in H^1(\mathbb{R}) \tag{8}$$

is exactly given by $\{\pm\psi_\omega(x - y);\ y \in \mathbb{R}\}$ where $\psi_\omega(x)$ is a positive even solution. Moreover $\psi_\omega(x)$ corresponds to a minimizer of the problem

$$\begin{aligned} d^\infty(\omega) &= \inf\{S^\infty_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\},\ I^\infty_\omega(v) = 0\} \\ &= \frac{p-1}{2(p+1)} \inf\{||v||^{p+1}_{p+1} : v \in H^1(\mathbb{R}) \setminus \{0\},\ I^\infty_\omega(v) = 0\} \end{aligned} \tag{9}$$

where

$$\begin{aligned} S^\infty_\omega(v) &= \frac{1}{2}\|Dv\|^2_2 + \frac{\omega}{2}\|v\|^2_2 - \frac{1}{p+1}\|v\|^{p+1}_{p+1}, \\ I^\infty_\omega(v) &= \|Dv\|^2_2 + \omega\|v\|^2_2 - \|v\|^{p+1}_{p+1}. \end{aligned}$$

Also $d^\infty(\omega) > 0$. For a proof of such statements we refer to [19] (see also [5, 6]). Now setting $\gamma = 0$ in (3) we deduce, because of the uniqueness of positive even solutions, that

$$\psi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \text{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} |x| \right) \right\}^{\frac{1}{p-1}}. \tag{10}$$

The following lemma is in the spirit of the work of P. L. Lions [21]. For a proof we refer to Theorem 5.1 of [20].

Lemma 20 *Let $\{u_n\} \subset H^1(\mathbb{R})$ be a bounded $(PS)_c$ sequence for S_ω. Then there exists a subsequence still denoted by $\{u_n\}$ for which the following holds: there exist a solution u_0 of (1), an integer $k \geq 0$, for $i = 1, \cdots, k$, sequences of points $\{x_n^i\} \subset \mathbb{R}$ and nontrivial solutions $\nu_i(x)$ of the limit equation (8) satisfying*

$$\begin{array}{l} u_n \rightharpoonup u_0 \quad \textit{weakly in} \quad H^1(\mathbb{R}), \\ S_\omega(u_n) \to c = S_\omega(u_0) + \Sigma_{i=1}^k S_\omega^\infty(\nu_i) \\ u_n - \big(u_0 + \Sigma_{i=1}^k \nu_i(x - x_n^i)\big) \to 0 \quad \textit{strongly in} \quad H^1(\mathbb{R}), \\ |x_n^i| \to \infty, \quad |x_n^i - x_n^j| \to \infty \quad \textit{for} \quad 1 \leq i \neq j \leq k, \end{array}$$

where we agree that in the case $k = 0$, the above holds without $\nu_i(x)$ and x_n^i.

From Lemma 20 and somehow inspired by the work of Adachi [2] we deduce that

Lemma 21 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Assume that*

$$d_r(\omega) < 2d^\infty(\omega). \tag{11}$$

Then, the $(PS)_{d_r(\omega)}$ sequence $\{v_n\} \subset H_r^1(\mathbb{R})$ given by Lemma 17 admits a strongly convergent subsequence.

Proof. By Lemma 19 we know that $\{v_n\} \subset H_r^1(\mathbb{R})$ is a Palais-Smale sequence for S_ω in $H^1(\mathbb{R})$. Assume, by contradiction, that it does not admit any convergent subsequence. Then we see from Lemma 20 that the case $k = 0$ cannot occurs. Clearly, the case $k = 1$ is also impossible because each $v_n(x)$ is an even function and thus the $\nu_i(x)$ must be present in an even number. Thus, since $d^\infty(\omega) > 0$, we have

$$\begin{aligned} d_r(\omega) &= \liminf_{n\to\infty} S_\omega(v_n) \\ &\geq S_\omega(u_0) + \sum_{i=1}^{2} S_\omega^\infty(\nu_i) \\ &= S_\omega(u_0) + 2d^\infty(\omega). \end{aligned} \tag{12}$$

Now necessarily $S_\omega(u_0) \geq 0$ since $I_\omega(u_0) = 0$ and thus (12) contradicts (11). □

Remark 22 In the case $\gamma > 0$ we see from [12] that $d_r(\omega) = d(\omega)$ for any $\omega > \gamma^2/4$ (as $d(\omega)$ is reached by an even function). Also, if $\gamma > 0$ one has $d(\omega) < d^\infty(\omega)$. Indeed we have $S_\omega(\psi_\omega) < S_\omega^\infty(\psi_\omega) = d^\infty(\omega)$ and $I_\omega(\psi_\omega) < 0$.

Thus there exists $\lambda^* < 1$ such that $I_\omega(\lambda^*\psi_\omega) = 0$ and $S_\omega(\lambda^*\psi_\omega) = S_\omega^\infty(\lambda^*\psi_\omega) - \frac{\gamma}{2}|\lambda^*\psi_\omega(0)|^2 < S_\omega^\infty(\psi_\omega) - \frac{\gamma}{2}|\lambda^*\psi_\omega(0)|^2 < S_\omega^\infty(\psi_\omega) = d^\infty(\omega)$. This shows that $d(\omega) < d^\infty(\omega)$. On the contrary when $\gamma < 0$ we have $d(\omega) = d^\infty(\omega)$. To see this let $u \in H_r^1(\mathbb{R}) \setminus \{0\}$ be such that $I_\omega^\infty(u) = 0$. Then there exists a unique $t \in \mathbb{R}$ such that $I_\omega(tu) = 0$. In addition, since $\gamma < 0$ we have $t > 1$. From the definition of $d(\omega)$ and $d^\infty(\omega)$ given in (6) and (9) we then deduce that $d(\omega) \geqslant d^\infty(\omega)$. Now let $\tau_y\psi_\omega(x) = \psi_\omega(x-y)$ for $y \in \mathbb{R}$. Clearly $S_\omega(\tau_y\psi_\omega) \to S_\omega^\infty(\tau_y\psi_\omega) = S_\omega^\infty(\psi_\omega) = d^\infty(\omega)$ and $I_\omega(\tau_y\psi_\omega) \to I_\omega^\infty(\tau_y\psi_\omega) = I_\omega^\infty(\psi_\omega) = 0$ as $|y| \to \infty$. At this point we conclude easily that $d(\omega) \leqslant d^\infty(\omega)$ and thus $d(\omega) = d^\infty(\omega)$. Finally $d(\omega) < d_r(\omega)$ since otherwise $d(\omega)$ would be reached.

Remark 23 When $\gamma > 0$ we have just observe that $d(\omega) < d^\infty(\omega)$. Also it is readily seen that S_ω admits a Palais-Smale sequence at the level $d(\omega)$ in $H^1(\mathbb{R})$. Thus we can deduce, using Lemma 20, that $d(\omega)$ has a minimizer $u_\omega(x)$. Then $|u_\omega(x)|$ is also a minimizer and thus a nonnegative non trivial solution of (1). By Lemma 26 we identify $|u_\omega(x)|$ with $\phi_\omega(x)$.

Let us now determine when the condition (11) holds.

Lemma 24 *Let $\gamma < 0$ and $\omega > \gamma^2/4$. The inequality (11) is satisfied for any $\omega > \gamma^2/4$.*

Proof. Because $\psi_\omega(x)$ is a minimizer of (9) and since $\phi_\omega(x) \in H_r^1(\mathbb{R})$ and $I_\omega(\phi_\omega) = 0$, to check that (11) holds it suffices to verify that

$$\|\phi_\omega\|_{p+1}^{p+1} < 2\|\psi_\omega\|_{p+1}^{p+1}.$$

We put $A(\omega, \gamma) = \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)$ and note that $A(\omega,\gamma) < 0$ if $\gamma < 0$. We directly calculate each $L^{p+1}(\mathbb{R})$ norm and we obtain

$$\begin{aligned}
\|\phi_\omega\|_{p+1}^{p+1} &= 2\int_0^\infty \phi_\omega^{p+1}(r)dr \\
&= 2C_0C_1\int_{A(\omega,\gamma)}^\infty \operatorname{sech}^{2(p+1)/(p-1)}(y)dy \\
&= 2C_0C_1\left\{\int_{A(\omega,\gamma)}^0 \operatorname{sech}^{2(p+1)/(p-1)}(y)dy + \int_0^\infty \operatorname{sech}^{2(p+1)/(p-1)}(y)dy\right\} \\
&< 4C_0C_1\int_0^\infty \operatorname{sech}^{2(p+1)/(p-1)}(y)dy \\
&= 4C_0\int_0^\infty \operatorname{sech}^{2(p+1)/(p-1)}\left(\frac{(p-1)\sqrt{\omega}}{2}r\right)dr = 2\|\psi_\omega\|_{p+1}^{p+1},
\end{aligned}$$

where $C_0 = \left(\frac{(p+1)\omega}{2} \right)^{(p+1)/(p-1)}$ and $C_1 = \frac{2}{(p-1)\sqrt{\omega}}$. $\square$

Proof of Theorem 1. From Lemmas 15, 17, 21 and 24 we see that $d_r(\omega)$ admits a nonnegative minimizer. $\square$

3 Stability and instability

In this section we first identify the nonnegative minimizer of $d_r(\omega)$ obtained in Theorem 1 with $\phi_\omega(x)$. Next we formulate the spectral assumptions which have to be satisfied for the theory of [14] to apply. Having checked these assumptions Theorem 2 follows from Proposition 9.

Lemma 25 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Then any solution $v(x) \in H^1(\mathbb{R})$ of* (1) *verifies the following:*

$$v \in C^j(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2, \tag{13}$$

$$-D^2 v + \omega v - v^p = 0, \quad x \neq 0, \tag{14}$$

$$Dv(0+) - Dv(0-) = -\gamma v(0), \tag{15}$$

$$Dv(x), v(x) \to 0, \ \text{as } |x| \to \infty. \tag{16}$$

Proof. To check (13) and (16), we make use of functions $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Clearly $\xi v(x)$ satisfies

$$-D^2(\xi v) + \omega \xi v = -(D^2\xi)v - 2(D\xi)(Dv) + \xi v^p,$$

in the sense of distributions. We employ a standard bootstrap argument (see Section 8 of [6] for details). The right hand side is in $L^2(\mathbb{R})$ and so $\xi v \in H^2(\mathbb{R})$, that is, $v \in H^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R} \setminus \{0\})$. The case of $j = 2$ is similar. The equation (14) follows from the fact that $C_0^\infty(\mathbb{R} \setminus \{0\})$ is dense in $L^2(\mathbb{R})$. Concerning (15), we integrate (1) from $-\varepsilon$ to ε,

$$-\int_{-\varepsilon}^{\varepsilon} D^2 v dx + \omega \int_{-\varepsilon}^{\varepsilon} v dx - \gamma \int_{-\varepsilon}^{\varepsilon} \delta(x) v dx = \int_{-\varepsilon}^{\varepsilon} v^p dx.$$

Then, letting $\varepsilon \to 0$, we get that $Dv(0+) - Dv(0-) = -\gamma v(0)$. $\square$

Lemma 26 *Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$. Then* (1) *has a unique nonnegative nontrivial solution. By uniqueness $\phi_\omega(x)$ is this solution and thus the set of all solutions of* (1) *is given by*

$$\{e^{i\theta}\phi_\omega : \ \theta \in \mathbb{R}\}.$$

Proof. Since $S'_\omega(v) = 0$, $\phi_\omega(x)$ satisfies (1) and thus by Lemma 25 the properties (13)-(16) hold for $\phi_\omega(x)$. Let $v \in H^1(\mathbb{R})$ be nonnegative, nontrivial and satisfying (13)-(16). We shall prove that $v(x)$ is unique. It will establish that $v(x) = \phi_\omega(x)$.

Let $f(s) = -\omega s + s^p$ and $F(s) = \displaystyle\int_0^s f(t)dt$. Multiplying (14) by $Dv(x)$ and integrating from $x = 0$ to $x = R$, we have

$$-\frac{1}{2}\{Dv(R)\}^2 + \frac{1}{2}\{Dv(0+)\}^2 - F(v(R)) + F(v(0+)) = 0, \tag{17}$$

for any $R > 0$. Letting $R \to \infty$, we get

$$\frac{1}{2}\{Dv(0+)\}^2 + F(v(0+)) = 0. \tag{18}$$

Similarly, we see that

$$\frac{1}{2}\{Dv(0-)\}^2 + F(v(0-)) = 0.$$

Thus to insure the continuity of $v(x)$ at $x = 0$ we must have $|Dv(0-)| = |Dv(0+)|$. If $Dv(0-) = Dv(0+)$ then $v(0) = 0$ by (15). If $Dv(0-) = Dv(0+) = 0$ then by Cauchy Uniqueness Principle we get that $v \equiv 0$ on $\mathbb{R}$. If $Dv(0-) = Dv(0+) \neq 0$ then necessarily $v(x_0) < 0$ for some $x_0 \in \mathbb{R}$ close to 0 and this contradicts the fact that $v(x)$ in nonnegative. Thus necessarily $Dv(0-) = -Dv(0+)$ and we have

$$Dv(0+) = -\frac{\gamma}{2}v(0). \tag{19}$$

Now we set $v(0) = c > 0$ where $c > 0$ is a parameter and consider the function

$$P(c) = \frac{\gamma^2 - 4\omega}{8}c^2 + \frac{1}{p+1}c^{p+1}.$$

For $c > 0$, this function has a unique zero that we denote by c_0. We remark that (18) and (19) imply $P(v(0)) = 0$. Thus necessarily

$$v(0) = c_0. \tag{20}$$

Consider now the initial value problem on $(0, \infty)$ given by (14) with (19) and (20) as initial conditions. It has a unique solution. Indeed, the solution is unique for $x > 0$ close to 0 since $f(s)$ is Lipschitz. This uniqueness holds for all $x > 0$ since, by (13) and (16), $v(x)$ is bounded. The uniqueness for

$x \in (-\infty, 0)$ is proved in the same way. The proof of the last statement follows from this uniqueness as in the proof of Theorem 8.1.6 in [6]. $\square$

To derive our stability results we use Theorem 3 in [14]. It is clear that in our situation Assumptions 1 and 2 of Theorem 3 in [14] hold and thus we shall concentrate on proving Assumption 3 that we have referred to so far as the spectral requirement of [14].

For $u \in H^1_r$ we write $u = u_1 + iu_2$. Let H_ω be defined by

$$H_\omega u = L_1 u_1 + i L_2 u_2$$

where, for $v \in \mathrm{Dom}(L_1) = \mathrm{Dom}(L_2) = \{v \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1_r(\mathbb{R}) :\ Dv(0+) - Dv(0-) = -\gamma v(0)\}$,

$$L_1 v = -D^2 v + \omega v - p\phi_\omega^{p-1} v,$$

$$L_2 v = -D^2 v + \omega v - \phi_\omega^{p-1} v.$$

It follows from the explicit formula of the resolvent of $-D^2$ with domain $\mathrm{Dom}(L_1)$ (see Theorem 3.1.2 in [1]) that the operator $-D^2$ with $\mathrm{Dom}(L_1)$ is a self-adjoint operator on $L^2(\mathbb{R})$. Since the perturbation $\omega - p\phi_\omega^{p-1}$ is real valued and not too large compared to $-D^2$, the operators L_1 and L_2 are also self-adjoint on $L^2(\mathbb{R})$ with domains $\mathrm{Dom}(L_1)$ and $\mathrm{Dom}(L_2)$. The operator H_ω corresponds to the linearization of S''_ω at $\phi_\omega(x)$. Indeed we have for L_1 (and a similar result holds for L_2) that

Lemma 27 Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Then

$$\langle L_1 u, v\rangle = \langle S''_\omega(\phi_\omega) u, v\rangle\,,\ \text{when } u \in \mathrm{Dom}(L_1) \text{ and } v \in H^1.$$

Proof. Let $u \in \mathrm{Dom}(L_1)$ and $v \in H^1$. We have

$$\begin{aligned}
\langle L_1 u, v\rangle &= -\int_{\mathbb{R}} (D^2u)\overline{v}\,dx + \omega \int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1} u\overline{v}\,dx \\
&= \lim_{\varepsilon\downarrow 0}\{-\int_{\varepsilon}^{\infty} D(Du\cdot\overline{v})\,dx - \int_{-\infty}^{-\varepsilon} D(Du\cdot\overline{v})\,dx\} + \int_{\mathbb{R}} Du\cdot D\overline{v}\,dx \\
&+ \omega\int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1}u\overline{v}\,dx \\
&= \lim_{\varepsilon\downarrow 0}\{Du(\varepsilon)\overline{v}(\varepsilon) - Du(-\varepsilon)\overline{v}(\varepsilon)\} + \int_{\mathbb{R}} Du\cdot D\overline{v}\,dx \\
&+ \omega\int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1}u\overline{v}\,dx \\
&= (Du(0^+) - Du(0^-))\overline{v}(0) + \int_{\mathbb{R}} Du\cdot D\overline{v}\,dx \\
&+ \omega\int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1}u\overline{v}\,dx \\
&= -\gamma u(0)\overline{v}(0) + \int_{\mathbb{R}} Du\cdot D\overline{v}\,dx + \omega\int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1}u\overline{v}\,dx \\
&= -\gamma\int_{\mathbb{R}}\delta(x)u\overline{v}\,dx + \int_{\mathbb{R}} Du\cdot D\overline{v}\,dx + \omega\int_{\mathbb{R}} u\overline{v}\,dx - p\int_{\mathbb{R}} \phi_\omega^{p-1}u\overline{v}\,dx \\
&= \langle S''_\omega(\phi_\omega)u, v\rangle\,.
\end{aligned}$$

□

In our notations the Assumption 3 of [14] is reduced to show that for each $\omega > \gamma^2/4$, H_ω has exactly one negative simple eigenvalue and has its kernel spanned by $i\phi_\omega(x)$. Observe that $L_2\phi_\omega = 0$, therefore $\phi_\omega(x)$ is in the kernel of L_2. Moreover $\phi_\omega(x)$ being positive it corresponds to the first eigenvalue of L_2 which is simple. Thus the kernel of L_2 is spanned by $\phi_\omega(x)$ and to satisfy Assumption 3 it is sufficient to show that L_1 has exactly one negative eigenvalue and only 0 in its kernel.

Lemma 28 *Let $\gamma \in \mathbb{R}\setminus\{0\}$ and $\omega > \gamma^2/4$. Then the kernel of L_1 is zero.*

Proof. It is classical to show (see for example Theorem 8.1 in [8]) that the subspace of L^2 solutions of $L_1 v = 0$, $x > 0$ is of dimension one. Also we can show that $D\phi_\omega$ satisfies $L_1(D\phi_\omega) = 0$ for $x > 0$. As a consequence to show that the kernel of L_1 is $\{0\}$ it suffices to prove that $D\phi_\omega$ does not satisfy the condition $Dv(0+) - Dv(0-) = -\gamma v(0)$ which since we work in $H^1_r(\mathbb{R})$ is reduced to $Dv(0+) = (-\gamma/2)v(0)$.

We have seen in the proof of Lemma 26 that $\phi_\omega(0) = c_0$, where c_0 is the unique zero of

$$-\frac{4\omega - \gamma^2}{8}c^2 + \frac{1}{p+1}c^{p+1} = 0$$

(see (20)). More precisely,

$$c_0 = \left(\frac{p+1}{8}(4\omega - \gamma^2)\right)^{1/(p-1)}.$$

Since $D\phi_\omega(0+) = -\frac{\gamma}{2}\phi_\omega(0)$ (see (19)) we have $-\frac{\gamma}{2}(D\phi_\omega)(0+) = \frac{\gamma^2}{4}c_0$. On the other hand, we see from (1) that $\phi_\omega(x)$ satisfies,

$$\lim_{x\to 0+} D^2\phi_\omega(x) = \omega c_0 - c_0^p.$$

Let z_0 be the unique zero of

$$\frac{\gamma^2}{4}z = \omega z - z^p,$$

namely,

$$z_0 = \left(\frac{4\omega - \gamma^2}{4}\right)^{1/(p-1)}.$$

It is easily seen that $z_0 < c_0$, which concludes the proof. □

Lemma 29 *Let $\gamma < 0$ and $\omega > \gamma^2/4$. Then L_1 has exactly one negative eigenvalue.*

Proof. Since $\langle L_1\phi_\omega, \phi_\omega\rangle = -(p-1)\|\phi_\omega\|_{p+1}^{p+1} < 0$, the first eigenvalue λ_1 is negative. From Weyl's theorem, we see that the essential spectrum of L_1 is in $[\omega, +\infty)$. Thanks to Lemma 28 the kernel of L_1 is $\{0\}$. Therefore to prove the lemma it just remains to show that L_1 cannot have two negative eigenvalues. Assume, by contradiction, that there exists two distinct negative eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and denote by v_1 and v_2 two eigenvectors associated to λ_1 and λ_2 respectively.

By Lemma 27 we know that $\langle S''_\omega(\phi_\omega)u, v\rangle = \langle L_1 u, v\rangle$ for any $u, v \in \mathrm{Dom}(L_1)$. Also since $S''_\omega(\phi_\omega)$ is self adjoint we have $\langle v_1, v_2\rangle = 0$. Now let $\beta_1, \beta_2 \in \mathbb{R}$ be arbitrary. Since $\langle v_1, v_2\rangle = \langle S''_\omega(\phi_\omega)v_1, v_2\rangle = \langle S''_\omega(\phi_\omega)v_2, v_1\rangle = 0$ we have

$$\langle S''_\omega(\phi_\omega)(\beta_1 v_1 + \beta_2 v_2), \beta_1 v_1 + \beta_2 v_2\rangle = \lambda_1\beta_1^2||v_1||_2^2 + \lambda_2\beta_2^2||v_2||_2^2 < 0.$$

Thus $S''_\omega(\phi_\omega)$ is negative definite on a subspace of dimension two. To conclude we shall prove that the variational characterization of $\phi_\omega(x)$ implies that $S''_\omega(\phi_\omega)$ is positive on a subspace of co-dimension one.

We recall that $\phi_\omega(x)$ is a minimizer of S_ω on the natural constraint $\{v \in H^1_r(\mathbb{R}) \setminus \{0\}, I_\omega(v) = 0\}$. Let us show that $\langle S''_\omega(\phi_\omega)v, v\rangle \geqslant 0$ on the subspace of codimension one $\{v \in H^1_r(\mathbb{R}), I'(\phi_\omega)(v) = 0\}$. For this we borrow some elements from [18] (see also [23]). Let $v \in H^1_r(\mathbb{R})$ be such $I'_\omega(\phi_\omega)v = 0$. Using the Implicit Function Theorem, we see that there exist $\varepsilon > 0$ and a $\mathcal{C}^2$-curve $\Lambda : (-\varepsilon, \varepsilon) \to H^1_r(\mathbb{R})$ such that

$$\Lambda(0) = \phi_\omega,\ \Lambda'(0) = v \ \text{ and } \ I_\omega(\Lambda(s)) = 0, \quad s < |\varepsilon|.$$

Thanks to the variational characterization of ϕ_ω, 0 is a local minimum of $s \mapsto S_\omega(\Lambda(s))$, and therefore $\frac{d^2}{d^2s} S_\omega(\Lambda(s))|_{s=0} \geqslant 0$. But, since $S'_\omega(\phi_\omega) = 0$, we have

$$0 \leqslant \frac{d^2}{d^2s} S_\omega(\Lambda(s))|_{s=0} = \langle S''_\omega(\phi_\omega)v, v\rangle$$

which ends our proof. $\square$

At this point the Assumption 3 of [14] is established and thus to prove Theorem 2 we can use Proposition 9.

End of Proof of Theorem 2. We put $\alpha = \omega^{-1/2}$. Then it follows from (3) that

$$\frac{\partial}{\partial\omega}\|\phi_\omega\|_2^2 = -\frac{\alpha^3}{2}\frac{\partial}{\partial\alpha}\|\phi_\alpha\|_2^2, \quad \frac{\partial}{\partial\alpha}\|\phi_\alpha\|_2^2 = C_p \alpha^{-4/(p-1)} g(\alpha),$$
$$g(\alpha) = \frac{p-5}{p-1} J(\alpha) - \frac{\alpha\gamma}{2}(1 - C^2_{\alpha,\gamma})^{-(p-3)/(p-1)},$$
$$J(\alpha) = \int_{A(\alpha,\gamma)}^{\infty} \operatorname{sech}^{4/(p-1)} y dy, \quad A(\alpha,\gamma) = \tanh^{-1} C_{\alpha,\gamma}.$$

where $C_{\alpha,\gamma} = \gamma\alpha/2$ and C_p is a positive constant depending only on p. It suffices to check the sign of $g(\alpha)$. Here, we have

$$\frac{\partial}{\partial\alpha} A(\alpha,\gamma) = \frac{\gamma}{2(1 - C^2_{\alpha,\gamma})}, \quad \operatorname{sech}^2 A(\alpha,\gamma) = 1 - C^2_{\alpha,\gamma},$$

hence,

$$J'(\alpha) = -\frac{\gamma}{2(1 - C^2_{\alpha,\gamma})^{-(p-3)/(p-1)}}. \tag{21}$$

After some calculations we obtain, using (21),

$$g'(\alpha) = -\frac{\gamma(p-3)}{p-1}(1-C^2_{\alpha,\gamma})^{-2(p-2)/(p-1)},$$
$$g''(\alpha) = -\frac{\alpha\gamma^3(p-3)(p-2)}{(p-1)^2}(1-C^2_{\alpha,\gamma})^{-(3p-5)/(p-1)}.$$

In the case where $\gamma < 0$ and $p \leq 3$, we have $g(\alpha) < 0$ for any $\alpha \in (0, -2/\gamma)$ since $g(0) < 0$, and $g'(\alpha) \leq 0$ for any $\alpha \in (0, -2/\gamma)$. In the case where $\gamma < 0$ and $3 < p < 5$, we note that $g(0) < 0$, $g'(0) > 0$ and $g''(\alpha) > 0$ for any $\alpha \in (0, -2/\gamma)$. Thus, there exists a unique $\alpha^{**} \in (0, -2/\gamma)$ such that $g(\alpha^{**}) = 0$, $g(\alpha) < 0$ for any $\alpha \in (0, \alpha^{**})$ and $g(\alpha) > 0$ for any $\alpha \in (\alpha^{**}, -2/\gamma)$. Finally, in the case where $\gamma < 0$ and $p \geq 5$, we have $g(0) > 0$ and $g'(\alpha) > 0$ for all $\alpha \in (0, -2/\gamma)$. Thus $g(\alpha) > 0$ for any $\alpha \in (0, -2/\gamma)$. □

4 Some remarks on the stability of $\phi_\omega(x)$ in H^1

In this final section we establish some results related to the stability of $\phi_\omega(x)$ in all H^1 and give an alternative proof of Proposition 5.

Clearly Proposition 9 still holds under the spectral conditions of [14] if H^1_r is remplaced by H^1 and thus, in order to apply the theory of [14], we just need to check that the Assumption 3 of [14] holds. The operators L_1 and L_2 are now defined on $\mathrm{Dom}(L_1) = \mathrm{Dom}(L_2) = \{v \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) \ : \ Dv(0+) - Dv(0-) = -\gamma v(0)\}$ and as previously we just need to check that L_1 has exactly one negative eigenvalue and its kernel is zero. In that direction we prove three results. First we show that the eigenvectors of L_1 are either even or odd. Then we show that the kernel of L_1, denoted by $N(L_1) = \{v \in \mathrm{Dom}(L_1) : \ L_1 v = 0\}$, is still $\{0\}$. Finally we prove that L_1 has at most two negative eigenvalues.

Lemma 30 *Let $\gamma \in \mathbb{R}$ and $\omega > \gamma^2/4$. Then the eigenvectors of L_1 are either even or odd.*

Proof. Our proof is inspired by the one of Proposition 15 in [11]. Let $v \in \mathrm{Dom}(L_1)$ be an eigenfunction of L_1, namely $L_1 v = \lambda v$ for a $\lambda \in \mathbb{R}$. Then, since $\phi_\omega(x)$ is even, we also have

$$L_1 v(-x) = \lambda v(-x).$$

It follows, since the eigenvalues of L_1 are simple (see Theorem 8.1 of [8] for such result) that there exists $\beta \in \mathbb{R}$ such that

$$v(x) = \beta v(-x), \ \forall x \in \mathbb{R}. \tag{22}$$

If $v(0) \neq 0$ we see from (22) that $\beta = 1$ and thus $v(x)$ is even. If $v(0) = 0$, deriving (22) we get that $v'(0) = -\beta v(0)$ and, since by Cauchy Uniqueness Principle $v'(0) \neq 0$, we deduce that $\beta = -1$. Namely that $v(x)$ is odd. $\square$

Lemma 31 *Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$. Then $N(L_1) = \{0\}$.*

Proof. By Lemma 30 the eigenvectors of L_1 are either even or odd. Also we proved in Lemma 28 that there is no even functions in $N(L_1)$. Thus it suffices to show that there is no odd functions in $N(L_1)$.

Let $v \in N(L_1)$ be odd. Then $v(0) = 0$ and reasoning as in the proof of Lemma 28, to conclude we just need to prove that $D\phi_\omega(x)$ does not belong to $N(L_1)$. But since $D\phi_\omega(0) = -\frac{\gamma}{2}\phi_\omega(0) \neq 0$, this is immediate. $\square$

Lemma 32 *Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega > \gamma^2/4$. The operator L_1 has at most two negative eigenvalues.*

Proof. This result and its proof were given to us by M. Maris [24]. Let I^1_ω and I^2_ω be the C^2-functionals defined on $H^1(\mathbb{R})$ by

$$I^1_\omega(u) = \int_{-\infty}^{0} |Du|^2 + \omega|u|^2 - |u|^{p+1}\, dx - \frac{\gamma}{2}|u(0)|^2,$$

$$I^2_\omega(u) = \int_{0}^{+\infty} |Du|^2 + \omega|u|^2 - |u|^{p+1}\, dx - \frac{\gamma}{2}|u(0)|^2.$$

We shall prove that $\phi_\omega(x)$ minimizes S_ω in $H^1(\mathbb{R})$ under the constraint $M = \{v \in H^1(\mathbb{R}) \setminus \{0\} : \ I^1_\omega(v) = 0, I^2_\omega(v) = 0\}$. Since this constraint is of codimension two reasoning as in the proof of Lemma 29 it is easily shown that L_1 cannot have more than two negative eigenvalues. First observe that $I^1_\omega(\phi_\omega) = I^2_\omega(\phi_\omega)$ since $\phi_\omega(x)$ is even. Also $I^1_\omega(\phi_\omega) + I^2_\omega(\phi_\omega) = I_\omega(\phi_\omega) = 0$. Thus $I^1_\omega(\phi_\omega) = I^2_\omega(\phi_\omega) = 0$ and $\phi_\omega(x)$ belongs to the constraint M.

Now let $v \in H^1(\mathbb{R})$ be such that $I^1_\omega(v) = I^2_\omega(v) = 0$. We define

$$v_1(x) = \begin{cases} v(x) \text{ if } x < 0 \\ v(-x) \text{ if } x \geqslant 0 \end{cases}$$

and

$$v_2(x) = \begin{cases} v(-x) \text{ if } x < 0 \\ v(x) \text{ if } x \geqslant 0. \end{cases}$$

We have $I^1_\omega(v_1) = I^1_\omega(v) = 0$ and $I^2_\omega(v_1) = I^2_\omega(v) = 0$. Thus $I_\omega(v_1) = I^1_\omega(v_1) + I^2_\omega(v_1) = 0$ and because $v_1 \in H^1_r(\mathbb{R})$ it follows that $S_\omega(\phi_\omega) \leqslant S_\omega(v_1)$. Similarly one deduce that $S_\omega(\phi_\omega) \leqslant S_\omega(v_2)$. Making the sum it follows that

$$2S_\omega(\phi_\omega) \leqslant S_\omega(v_1) + S_\omega(v_2) = 2S_\omega(v)$$

which complete the proof. □

Remark 33 The approach developed in this paper permits to give an alternative, complete, proof of Proposition 5 which corresponds to the case $\gamma > 0$. We already know from Remark 23 that $\phi_\omega(x)$ minimizes S_ω on the manifold of codimension one $\{v \in H^1(\mathbb{R})\backslash\{0\}, I_\omega(v) = 0\}$. Also, from Lemma 31, the kernel of L_1 defined on $\{v \in H^2(\mathbb{R})\backslash\{0\} \cap H^1(\mathbb{R}) \ : \ Dv(0+) - Dv(0-) = -\gamma v(0)\}$ is zero. Finally, reasoning as in Lemma 29 the variational characterization of $\phi_\omega(x)$ implies that L_1 can only have one negative eigenvalue. Hence the Assumption 3 of [14] holds and, using Proposition 9, we conclude the proof of Proposition 5.

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